Non-abelian factorisation for next-to-leading-power threshold logarithms

Bonocore, D.; Laenen, E.; Magnea, L.; Vernazza, L.; White, C.D.

DOI
10.1007/JHEP12(2016)121

Publication date
2016

Document Version
Final published version

Published in
The Journal of High Energy Physics

Published for SISSA by Springer
Received: October 27, 2016
Accepted: December 9, 2016
Published: December 22, 2016

Non-abelian factorisation for next-to-leading-power threshold logarithms

D. Bonocore,[a,b] E. Laenen,[a,c,d,e] L. Magnea,[f] L. Vernazza[g] and C.D. White[h]

[a] *Nikhef, Science Park 105, NL-1098 XG Amsterdam, The Netherlands*
[b] *Institute for Theoretical Particle Physics and Cosmology, RWTH Aachen University, Sommerfeldstr. 16, 52074 Aachen, Germany*
[c] *ITFA, University of Amsterdam, Science Park 904, Amsterdam, The Netherlands*
[d] *ITF, Utrecht University, Leuvenlaan 4, Utrecht, The Netherlands*
[e] *Kavli Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106-4030, U.S.A.*
[f] *Dipartimento di Fisica, Università di Torino and INFN, Sezione di Torino, Via P. Giuria 1, I-10125 Torino, Italy*
[g] *Higgs Centre for Theoretical Physics, School of Physics and Astronomy, The University of Edinburgh, Edinburgh EH9 3JZ, Scotland, U.K.*
[h] *Centre for Research in String Theory, School of Physics and Astronomy, Queen Mary University of London, 327 Mile End Road, London E1 4NS, U.K.*

E-mail: bonocore@physik.rwth-aachen.de, Eric.Laenen@nikhef.nl, lorenzo.magnea@unito.it, lvernazz@staffmail.ed.ac.uk, christopher.white@qmul.ac.uk

Abstract: Soft and collinear radiation is responsible for large corrections to many hadronic cross sections, near thresholds for the production of heavy final states. There is much interest in extending our understanding of this radiation to next-to-leading power (NLP) in the threshold expansion. In this paper, we generalise a previously proposed all-order NLP factorisation formula to include non-abelian corrections. We define a non-abelian radiative jet function, organising collinear enhancements at NLP, and compute it for quark jets at one loop. We discuss in detail the issue of double counting between soft and collinear regions. Finally, we verify our prescription by reproducing all NLP logarithms in Drell-Yan production up to NNLO, including those associated with double real emission. Our results constitute an important step in the development of a fully general resummation formalism for NLP threshold effects.

Keywords: QCD Phenomenology

ArXiv ePrint: 1610.06842

doi:10.1007/JHEP12(2016)121

JHEP12(2016)121

Contents

1 Introduction

The properties of QCD radiation near partonic thresholds for the production of heavy final states have a significant impact on a wide range of phenomenologically relevant collider observables. Typically, if ξ is a dimensionless kinematic variable vanishing at the threshold, differential QCD cross-sections will contain terms of the form

$$\frac{d\sigma}{d\xi} = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \sum_{m=0}^{2n-1} \left[c_{nm}^{(-1)} \left(\frac{\log^m \xi}{\xi}\right)_+ + c_n^{(\delta)}\, \delta(\xi) + c_{nm}^{(0)} \log^m \xi + \ldots \right], \tag{1.1}$$

where the ellipsis denotes terms suppressed by further powers of ξ. The first set of terms, at leading power in ξ, originates from the singularities associated with soft and collinear gluon emission. These singularities are universal and factorising, which leads to the possibility of resumming the resulting logarithms to all orders in perturbation theory. The formalism to perform this resummation is well-known, and it has been extensively applied to a plethora of collider observables (see, for example, [1–9]). The second set of terms in eq. (1.1), which are localised at threshold, originates mostly from singular virtual corrections to the production amplitude. These terms can also be organised to all orders for processes which are electroweak at tree level [10–12], albeit with reduced predictive power. The vast amount and increased precision of LHC data, together with the lack of any striking signature for new physics, make the third set of terms in eq. (1.1), at next-to-leading power in the threshold parameter ξ, potentially relevant for precision Standard Model studies. Indeed, quite a large body of work has already been devoted to this problem.

The fact that at least some NLP contributions can be understood to all orders is well known, as a consequence of the LBKD theorem [13–15]. Further evidence for a non-trivial

JHEP12(2016)121

relation between LP and NLP logarithms came from the analysis of DGLAP splitting functions in ref. [16]. Since then, and following early studies in [17, 18], several groups have attempted to construct a systematic formalism for understanding NLP logarithms, using a variety of methods, ranging from path integral techniques [19], to diagrammatic approaches [20], physical evolution kernels [21–26], effective field theories [27, 28], and other techniques [29–31]. Interestingly, the study of next-to-soft contributions to scattering amplitudes in both gauge theory and gravity from a more formal point of view, based on asymptotic symmetries of the S matrix, has also received a great deal of attention (see for example [32–38]).

Recently, building on the results of [15], in ref. [39] we proposed a factorisation formula for the Drell-Yan scattering amplitude, valid at the accuracy needed to generate NLP logarithms in the cross section. This formula generalises the factorisation of soft and collinear divergences by including NLP effects, and contains the same universal functions as the leading-power factorisation, together with a new universal *radiative jet function*, responsible for next-to-soft emission from a collinearly enhanced configuration. Ref. [39] evaluated this quantity up to one-loop order for an external quark, and, using as a guideline the calculation performed with the method of regions [40–42] in ref. [43], succeeded in reproducing a set of NLP terms in the Drell-Yan cross section at NNLO, originally computed in [44, 45].

The factorisation formula proposed in refs. [15, 39] was, however, appropriate for an abelian theory, and could only reproduce abelian-like QCD contributions, proportional to the color factor C_F^n at $\mathcal{O}(\alpha_s^n)$. The aim of this paper is to provide a fully non-abelian NLP factorisation formula, a generalisation from previous results which is non-trivial for a number of reasons.

First, it is necessary to include the emission of colour-correlated gluons from outside the hard interaction. These diagrams were called *next-to-eikonal webs* in refs. [19, 20], where they were shown to be described by generalised Wilson line operators, obeying exponentiation properties similar to their leading-power counterparts. Second, the definition of the radiative jet function must be generalised to cope with a non-abelian operator insertion for the additional gluon. Third, one must address double counting of soft and collinear regions at NLP level, a problem which occurs also at leading power, or in effective field theory approaches [27, 28], but was easily circumvented in the abelian limit.

The structure of our paper is as follows. In section 2, we introduce a new, non-abelian definition of the radiative jet function, in terms of a non-abelian conserved current, and we use it to derive a complete factorisation formula for the Drell-Yan amplitude, with the required accuracy to reproduce all NLP effects. In section 3, we compute the new radiative jet function to one loop, which is sufficient to generate all NLP logarithms in the cross section at NNLO. In section 4, we use these results to check that the known non-abelian terms in the Drell-Yan K-factor up to NNLO are indeed reproduced. In section 5, we briefly describe how our methods can also reproduce double-real emission contributions at NNLO. In section 6 we present our conclusion and outline future work towards an effective NLP resummation formalism.

2 The non-abelian NLP factorisation formula

2.1 Leading power factorisation

Anticipating the Drell-Yan application of section 4, we consider a quark scattering amplitude involving two partons with momenta p_1 and p_2, which we write as

$$\mathcal{M}(p_1, p_2) = \bar{v}(p_2)\, \mathcal{A}(p_1, p_2)\, u(p_1)\,, \tag{2.1}$$

so that $\mathcal{A}$ has the external fermion wave functions removed. In massless QCD, $\mathcal{A}$ is affected by infrared and collinear divergences, which however factorise to all orders in the form [46]

$$\mathcal{A}(p_1, p_2) = \mathcal{H}(p_j, n_j)\, \mathcal{S}(\beta_j) \prod_{i=1}^{2} \frac{J(p_i, n_i)}{\mathcal{J}(p_i, n_i)}\,. \tag{2.2}$$

In eq. (2.2), $J(p_i, n_i)$ is a *jet function*, collecting collinear singularities associated with parton i: it depends on an auxiliary vector n_i, as described below; this dependence cancels with the other factors in eq. (2.2), so that the full scattering amplitude is independent of n_i, as expected. For a quark, the jet function is given by[1]

$$J(p, n) u(p) = \langle 0 | \Phi_n(\infty, 0) \psi(0) | p \rangle\,, \tag{2.3}$$

where the fermion field $\psi(x)$ absorbs the external parton of momentum p, and $\Phi_n(\infty, 0)$ is a Wilson line in the direction of the auxiliary four vector n_μ, guaranteeing gauge invariance, and defined according to

$$\Phi_v(\lambda_2, \lambda_1) = \mathcal{P} \exp\left[ig \int_{\lambda_1}^{\lambda_2} d\lambda\, v \cdot A(\lambda v) \right]. \tag{2.4}$$

The *soft function* $\mathcal{S}(\beta_i)$ collects soft divergences, and is a correlator of Wilson lines directed along the classical trajectories of the hard emitting particles: in fact, β_i is a dimensionless vector proportional to the four-velocity of parton i according to $p_i = Q\beta_i$, with Q a hard scale. We define then

$$\mathcal{S}(\beta_i) = \langle 0 | \Phi_{\beta_2}(\infty, 0) \Phi_{\beta_1}(0, -\infty) | 0 \rangle\,. \tag{2.5}$$

The final ingredient of eq. (2.2), the *eikonal jet* function $\mathcal{J}(p_i, n_i)$, is responsible for subtracting the double counting of soft-collinear configurations, which contribute to both the jets and the soft function. The eikonal jet is obtained by replacing the hard line of momentum p^μ in the partonic jet with a Wilson line with four-velocity β^μ, yielding

$$\mathcal{J}(\beta, n) = \langle 0 | \Phi_n(\infty, 0) \Phi_\beta(0, -\infty) | 0 \rangle\,. \tag{2.6}$$

After factorising all singular (and universal) contributions, the matching to the exact amplitude order by order yields $\mathcal{H}$, an infrared-finite, but process-dependent *hard function.*

Eq. (2.2) forms the starting point for describing radiation at leading power in the threshold expansion, where the soft function corresponds to dressing the hard function

[1] Throughout, we leave time ordering implicit for brevity.

with virtual gluons of 4-momentum $k^\mu \to 0$, whose infrared singularities cancel those associated with real emissions. In what follows, considering real radiation up to next-to-soft level, it will be convenient to generalise the soft function to include sub-leading powers of momentum in the propagators and emission vertices for virtual gluons. This can be done by defining a *next-to-soft function* as

$$\widetilde{\mathcal{S}}(p_1,p_2) = \langle 0|F_{p_2}(\infty,0)F_{p_1}(0,-\infty)|0\rangle\big|_{\rm NLP}\,, \tag{2.7}$$

where F_p is a generalised Wilson line operator, constructed in ref. [19], which generates the required next-to-soft emission vertices. A general definition of such an operator for generic trajectories can be given in a coordinate-space representation, and is presented in [19]. For straight semi-infinite trajectories one can easily transform to a momentum-space representation, given by

$$\begin{aligned} F_p(0,\infty) = \mathcal{P}\exp\Bigg[& g\int\frac{d^dk}{(2\pi)^d}A_\mu(k)\left(-\frac{p^\mu}{p\cdot k}+\frac{k^\mu}{2p\cdot k}-k^2\frac{p^\mu}{2(p\cdot k)^2}-\frac{{\rm i}k_\nu\Sigma^{\nu\mu}}{p\cdot k}\right)\\ &+\int\frac{d^dk}{(2\pi)^d}\int\frac{d^dl}{(2\pi)^d}A_\mu(k)A_\nu(l)\left(\frac{\eta^{\mu\nu}}{2p\cdot(k+l)}-\frac{p^\nu l^\mu p\cdot k+p^\mu k^\nu p\cdot l}{2(p\cdot l)(p\cdot k)[p\cdot(k+l)]}\right.\\ &\left.\left.+\frac{(k\cdot l)p^\mu p^\nu}{2(p\cdot l)(p\cdot k)[p\cdot(k+l)]}-\frac{{\rm i}\Sigma^{\mu\nu}}{p\cdot(k+l)}\right)\right]. \end{aligned} \tag{2.8}$$

Note that we have given the result for a fermion, where $\Sigma^{\mu\nu}=\frac{\rm i}{4}[\gamma^\mu,\gamma^\nu]$ is the appropriate Lorentz-group spin generator. The subscript on the r.h.s. of eq. (2.7) indicates than one should truncate the resulting expression to include at most one next-to-soft vertex. Correspondingly, one may define a next-to-soft jet function

$$\widetilde{\mathcal{J}}(p,n) = \langle 0|\Phi_n(\infty,0)F_p(0,-\infty)|0\rangle\,\big|_{\rm NLP}\,. \tag{2.9}$$

With these definitions, the non-radiative amplitude reads

$$\mathcal{A}(p_1,p_2) = \widetilde{\mathcal{H}}(p_j,n_j)\,\widetilde{\mathcal{S}}(p_j)\prod_{i=1}^{2}\frac{J(p_i,n_i)}{\widetilde{\mathcal{J}}(p_i,n_i)}\,, \tag{2.10}$$

as schematically depicted in figure 1(a). Just as in eq. (2.2), $\widetilde{\mathcal{H}}$ is obtained by matching to the full amplitude on the left-hand side. It differs from the function $\mathcal{H}$ appearing in the factorisation formula in eq. (2.2), as next-to-soft effects have now been explicitly factored out.

2.2 Real radiation at next-to-leading power

Let us now consider adding the radiation of an additional (next-to-)soft gluon to the amplitude in eq. (2.10). The emission of the extra gluon can be assigned to different factors in the non-radiative amplitude, as shown in figure 1(b,c,d). Proceeding by analogy with the treatment of the abelian case [15, 39], we will start by giving a formal definition of the contribution to the amplitude due to radiation from a jet, say $J(p_1,n_1)\equiv J_1$. We write

$$\mathcal{A}^{J_1}_{\mu,a}(p_1,p_2,k) = \widetilde{\mathcal{H}}(p_1-k,p_2,n_j)\,\frac{\widetilde{\mathcal{S}}(p_j)}{\prod_{k=1}^2\widetilde{\mathcal{J}}(p_k,n_k)}\,J_{\mu,a}(p_1,n_1,k)\,J(p_2,n_2)\,, \tag{2.11}$$

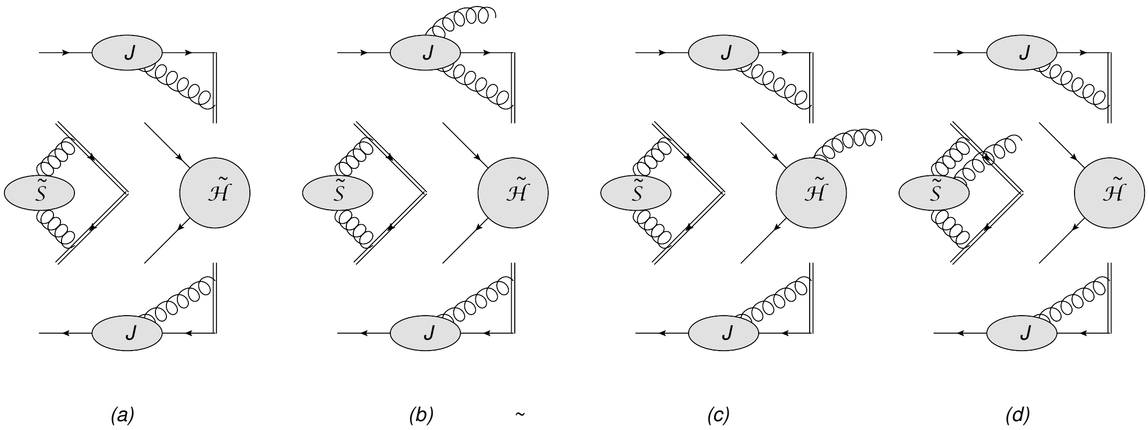

Figure 1. (a) Schematic factorisation of a two-point amplitude. $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{S}}$ are the hard and next-to-soft functions, and J is a non-radiative jet function; next-to-soft subtractions to J are omitted for simplicity. (b) Emission of a gluon from a jet (to be described by a *radiative jet function* J^μ); (c) emission from the hard function. (d) Emission through a radiative next-to-soft function $\widetilde{S}^\mu$.

where μ and a are the Lorentz and the color adjoint indices of the emitted gluon, respectively, and we have introduced the *radiative jet function*, defined, as in refs. [15, 39], by

$$J_{\mu,a}(p,n,k)u(p) = \int d^d y \, \mathrm{e}^{-\mathrm{i}(p-k)\cdot y} \, \langle 0|\Phi_n(\infty,y)\,\psi(y)\,j_{\mu,a}(0)|p\rangle \,. \tag{2.12}$$

The crucial issue in generalising the radiative jet function to the non-abelian theory is the definition of the non-abelian gauge current $j_{\mu,a}(x)$. First of all, it must be a conserved current, $\partial_\mu j_a^\mu = 0$, in order for the radiative jet to obey the Ward identity

$$k_\mu J^{\mu,a}(p,n,k) = g\,\mathbf{T}^a J(p,n)\,, \tag{2.13}$$

which is the natural generalisation of the abelian case, and is a necessary ingredient for the proof of our factorisation formula. Furthermore, we must require that the matrix element in eq. (2.12) should fully reproduce the relevant terms in the amplitude when the (next-to-)soft gluon is radiated from virtual gluons inside the jet. It turns out that the standard, textbook definition of the non-abelian Noether current (see, for example, ref. [47]) does not have this property. One must however keep in mind that Noether currents are not uniquely defined: in general, it is possible to add *improvement terms* (see, for example, ref. [48]) which do not spoil charge conservation but may improve other symmetry properties of the operator. In our case, we have found that an improvement term indeed exists which reproduces all relevant terms in diagrams where the (next-to-)soft gluon is emitted though a three-gluon vertex. Our choice for the non-abelian current is then

$$j_a^\mu(x) = g\left\{ -\overline{\psi}(x)\,\gamma^\mu\,\mathbf{T}_a\,\psi(x) + f_a{}^{bc}\left[F_c^{\mu\nu}(x)\,A_{\nu\,b}(x) + \partial_\nu\left(A_b^\mu(x)A_c^\nu(x)\right)\right]\right\}, \tag{2.14}$$

which is indeed conserved, as one can readily verify. Note that the last term in eq. (2.14) (the 'improvement') does not contribute to $\partial_\mu j_a^\mu$.

Our next step is to define an operator matrix element describing (next-to-)soft real gluon radiation from the soft factor of the non-radiative amplitude, as depicted in figure 1(d). A natural choice is

$$\epsilon_\mu^{*(\lambda)}(k)\,\widetilde{\mathcal{S}}_a^\mu(p_1,p_2,k) = \langle k,\lambda,a|F_{p_2}(\infty,0)F_{p_1}(0,-\infty)|0\rangle\big|_{\rm NLP}\,. \tag{2.15}$$

This definition is similar to the non-radiative function of eq. (2.7), but one of the gluons is now real rather than virtual. A similar quantity is defined at leading power in refs. [49–53]. The radiative next-to-soft function obeys the Ward identity

$$k_\mu \widetilde{S}^\mu(p_1,p_2,k) = 0\,. \tag{2.16}$$

Following ref. [19], the radiative next-to-soft function can be shown to exponentiate, and it can be evaluated using *next-to-soft webs*, generalising the methods used at leading power. These webs can connect all hard partons in the process, thus they are not captured by the emission of gluons from inside single-parton jets. Nevertheless, there is clearly a double counting of contributions between the jets and the next-to-soft function, which is directly analogous to the double counting of soft and collinear contributions in eq. (2.2). We may correct for this by subtracting from the jet emission contributions defined in eq. (2.11) their next-to-soft expansion, according to

$$\mathcal{A}^{J_i}_{\mu,a}(p_i,p_j,k) \to \mathcal{A}^{J_i}_{\mu,a}(p_i,p_j,k) - \mathcal{A}^{\widetilde{\mathcal{J}}_i}_{\mu,a}(p_i,p_j,k)\,, \tag{2.17}$$

where the subtraction term on the right-hand side is simply defined as the next-to-soft approximation to the full radiative jet function. Given that the overlap between the soft and jet functions must be separately gauge-invariant, the Ward identity of eq. (2.16) implies

$$k^\mu \mathcal{A}^{\widetilde{\mathcal{J}}_i}_{\mu,a}(p_i,p_j,k) = 0\,. \tag{2.18}$$

Having precisely defined the jet and next-to-soft contributions to the radiative amplitude in terms of operator matrix elements, and having taken care to subtract double counted contributions, the emissions from the hard sub-process, which we denote by $\mathcal{A}^{\widetilde{\mathcal{H}}}_{\mu,a}$ and depict in figure 1(c), are defined by matching to the full radiative amplitude $\mathcal{A}_{\mu,a}$. We will discuss their properties in the following subsection.

2.3 Derivation of the non-abelian factorisation formula

Combining the above ingredients gives a total radiative amplitude

$$\begin{aligned}\mathcal{A}_{\mu,a}(p_i,k) &= \mathcal{A}^{J}_{\mu,a}(p_i,n_i,k) - \mathcal{A}^{\widetilde{\mathcal{J}}}_{\mu,a}(p_i,n_i,k) + \mathcal{A}^{\widetilde{\mathcal{H}}}_{\mu,a}(p_i,n_i,k)\\ &\quad + \widetilde{\mathcal{S}}_{\mu,a}(p_i,k)\left(\widetilde{\mathcal{H}}(p_i,n_i)\prod_{i=1}^{2}\frac{J(p_i,n_i)}{\widetilde{\mathcal{J}}(p_i,n_i)}\right),\end{aligned} \tag{2.19}$$

where we defined

$$\mathcal{A}^{J}_{\mu,a}(p_1,n_1,p_2,n_2,k) \equiv \sum_{i=1}^{2}\mathcal{A}^{J_i}_{\mu,a}(p_i,n_i,k)\,, \tag{2.20}$$

and similarly for $\mathcal{A}^{\widetilde{\mathcal{J}}}_{\mu,a}$. The complete radiative amplitude in eq. (2.19) satisfies the Ward identity

$$k^\mu \mathcal{A}^a_\mu(p_1,p_2,k) = 0\,, \tag{2.21}$$

which, together with eqs. (2.13), (2.16), (2.18) implies the relation

$$k^\mu\, \mathcal{A}^{\widetilde{\mathcal{H}}}_{\mu,a} = -k^\mu\, \mathcal{A}^{J}_{\mu,a}\,. \tag{2.22}$$

Taylor expanding eq. (2.11) and using eq. (2.22) and colour conservation, in the form $\sum_i \mathbf{T}_i = 0$, one finds

$$\mathcal{A}^{\widetilde{\mathcal{H}}}_{\mu,a}(p_j,n_j,k) = \sum_{i=1}^{2} g\,\mathbf{T}^a_i \left[\frac{\partial}{\partial p_i^\mu}\widetilde{\mathcal{H}}(p_j,n_j)\right]\widetilde{\mathcal{S}}(p_j)\prod_{k=1}^{2}\frac{J(p_k,n_k)}{\widetilde{\mathcal{J}}(p_k,n_k)}\,. \tag{2.23}$$

In order to characterise the radiative jet functions it is convenient, as in refs. [15, 39] to introduce polarisation tensors [54]

$$\eta^{\mu\nu} = G_i^{\mu\nu} + K_i^{\mu\nu}\,, \qquad K_i^{\mu\nu} = \frac{(2p_i-k)^\nu}{2p_i\cdot k - k^2}\,k^\mu, \tag{2.24}$$

so that the total radiative amplitude is given by the sum of "K-polarised" and "G-polarised" gluons. Considering first the K-projection of the jet contribution to the amplitude, and using eqs. (2.11), (2.13), (2.18), we find

$$\begin{aligned}\sum_{i=1}^{2}\left(\mathcal{A}^{J_i}_{\nu,a} - \mathcal{A}^{\widetilde{\mathcal{J}}_i}_{\nu,a}\right) K_i^{\nu\mu} = \sum_{i=1}^{2} g\,\mathbf{T}_{a,i}\Bigg[&\frac{(2p_i-k)^\mu}{2p_i\cdot k-k^2}\,\widetilde{\mathcal{H}}(p_j,n_j)\widetilde{\mathcal{S}}(p_j)\prod_k\frac{J(p_k,n_k)}{\widetilde{\mathcal{J}}(p_k,n_k)}\\ &-\left(K_i^{\nu\mu}\frac{\partial}{\partial p_i^\nu}\,\widetilde{\mathcal{H}}(p_j,n_j)\right)\widetilde{\mathcal{S}}(p_j)\prod_k\frac{J(p_k,n_k)}{\widetilde{\mathcal{J}}(p_k,n_k)}\Bigg]\,,\end{aligned} \tag{2.25}$$

where we have again Taylor expanded in k. The emission of a G-gluon from a jet, on the other hand, is given by

$$\sum_{i=1}^{2}\mathcal{A}^{J_i}_{\nu,a}\,G_i^{\nu\mu} = \sum_{i=1}^{2} G_i^{\nu\mu}\,\widetilde{\mathcal{H}}(p_j,n_j)\,J_{\nu,a}(p_i,n_i,k)\frac{\widetilde{\mathcal{S}}(p_j)}{\prod_{k=1}^{2}\widetilde{\mathcal{J}}(p_k,n_k)}\prod_{j\neq i}J(p_j,n_j)\,, \tag{2.26}$$

where we set $k\to 0$ in the hard function, retaining only the leading term in its Taylor expansion, owing to the fact that the G tensor acts on terms proportional to p_i^μ to make them $\mathcal{O}(k)$ [15, 39]. Combining this with the K-gluon emissions, with the G-gluon contribution from the subtraction term, and with emissions from the hard function, as given by eq. (2.23), we find that the total amplitude, to the required accuracy, becomes[2]

$$\begin{aligned}\mathcal{A}_{\mu,a}(p_j,k) = \sum_{i=1}^{2}\Bigg\{&\left[\frac{1}{2}\,\widetilde{S}_{\mu,a}(p_j,k)\,\widetilde{\mathcal{H}}(p_j,n_j) + g\,\mathbf{T}^a_i G_i^{\nu\mu}\left(\frac{\partial}{\partial p_i^\nu}\widetilde{\mathcal{H}}(p_j,n_j)\right)\widetilde{\mathcal{S}}(p_j)\right]\\ &\times\prod_j\frac{J(p_j,n_j)}{\widetilde{\mathcal{J}}(p_j,n_j)} + \widetilde{\mathcal{H}}(p_j,n_j)\widetilde{\mathcal{S}}(p_j)\,\frac{J_{\mu,a}(p_i,n_i,k)}{\widetilde{\mathcal{J}}(p_i,n_i)}\prod_{j\neq i}\frac{J(p_j,n_j)}{\widetilde{\mathcal{J}}(p_j,n_j)} - \mathcal{A}^{\widetilde{\mathcal{J}}_i}_{\mu,a}\Bigg\}\,,\end{aligned} \tag{2.27}$$

[2] For convenience, in what follows, we have chosen to keep terms that vanish due to the on-shell condition for the emitted gluon, $k^2=0$, and due to the physical polarisation condition, $k_\mu\epsilon^\mu(k)=0$.

where the factor of 1/2 in the first term is due to the fact that we have placed this inside the sum over hard particles, for brevity.

Reconstructing the expression for the non-radiative amplitude given in eq. (2.10), we may express eq. (2.27) as

$$\begin{aligned}\mathcal{A}_{\mu,a}(p_j,k) = \sum_{i=1}^{2}\Bigg\{\Bigg[\frac{1}{2}\,\frac{\widetilde{\mathcal{S}}_{\mu,a}(p_j,k)}{\widetilde{\mathcal{S}}(p_j)} + g\,\mathbf{T}_{i,a}G^{\nu}_{i,\mu}\,\frac{\partial}{\partial p_i^{\nu}} + \frac{J_{\mu,a}(p_i,n_i,k)}{J(p_i,n_i)} \\ - g\,\mathbf{T}_{i,a}\,G^{\nu}_{i,\mu}\,\frac{\partial}{\partial p_i^{\nu}}\log\left(\frac{J(p_i,n_i)}{\widetilde{\mathcal{J}}(p_i,n_i)}\right)\Bigg]\mathcal{A}(p_j) - \mathcal{A}^{\widetilde{\mathcal{J}}_i}_{\mu,a}(p_j,k)\Bigg\}\,,\end{aligned} \tag{2.28}$$

where we have used eq. (2.10) to replace derivatives of the hard interaction with those acting on the full non-radiative amplitude and jet functions. Eq. (2.28) is our final non-abelian factorisation formula, capturing all NLP contributions near threshold. A few comments are in order.

- The first two terms in eq. (2.28) contain next-to-eikonal webs, composed of generalised Wilson lines [19], dressing the non-radiative amplitude, together with a derivative operator. These terms provide a non-abelian version of the original Low's theorem (in the absence of collinear enhancements).
- The remaining terms in square brackets organise emissions from jet functions, generalising to the non-abelian theory the results of [15, 39].
- The last term in eq. (2.28) corrects the radiative jet factors for the double counting of contributions between the radiative jets and next-to-soft functions.
- Eq. (2.28) describes the amplitude stripped of external wave functions for the hard partons. To build a cross section, these must be reinstated, as in eq. (2.1), noting that the derivative in eq. (2.28) does not act on the wave functions.

As discussed in refs. [39, 43], considerable simplifications occur in eq. (2.28) upon choosing the auxiliary vectors n_i to be null, $n_i^2 = 0$. In this case, one may work in a renormalisation scheme such that the non-radiative soft and jet functions are unity, to all orders in perturbation theory. Eq. (2.28) then becomes

$$\mathcal{A}_{\mu,a}(p_j,k) = \sum_{i=1}^{2}\left(\frac{1}{2}\,\widetilde{\mathcal{S}}_{\mu,a}(p_j,k) + g\,\mathbf{T}_{i,a}\,G^{\nu}_{i,\mu}\,\frac{\partial}{\partial p_i^{\nu}} + J_{\mu,a}(p_i,n_i,k)\right)\mathcal{A}(p_j) - \mathcal{A}^{\widetilde{\mathcal{J}}}_{\mu,a}(p_j,k)\,. \tag{2.29}$$

As in refs. [39, 43], in the detailed calculations below we will further make the specific choice of reference vectors

$$n_1 = p_2\,, \qquad n_2 = p_1\,, \tag{2.30}$$

whose interpretation is that n_i is the anti-collinear direction associated with p_i. This is physically motivated by the fact that p_1 and p_2 are the only momenta in the problem at hand, and it allows to make direct contact with the method-of-regions calculation of ref. [43].

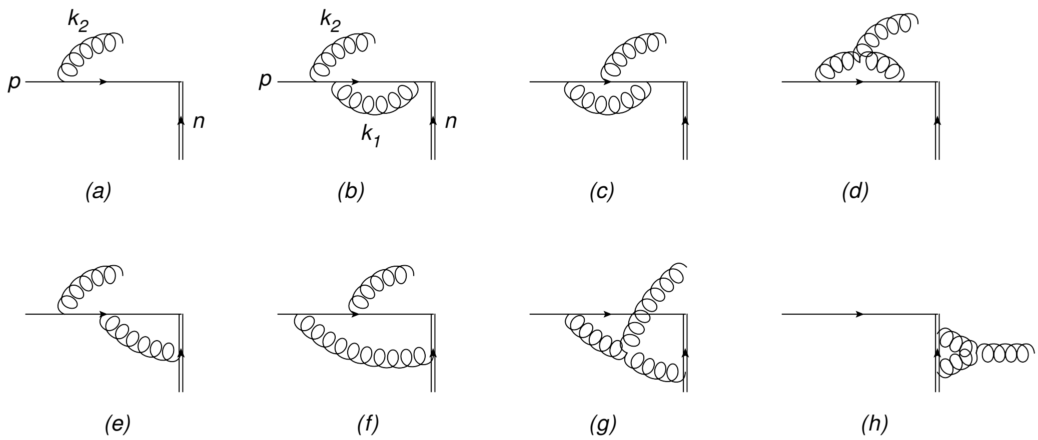

Figure 2. Diagrams contributing to radiative quark jet function: (a) tree level; (b)–(h) one-loop.

3 The non-abelian radiative jet function

Before testing eq. (2.29) in Drell-Yan production, we must first calculate the non-abelian radiative quark jet function, defined in eq. (2.12). To perform a test at NNLO, we need to compute $J_{\mu,a}$ at one loop, which we do for null n, in order to use the result in eq. (2.29). Relevant Feynman diagrams are shown in figure 2. Defining the perturbative coefficients of $J_{\mu,a}$ via

$$J_\mu^a(p,n,k) = g\,\mathbf{T}^a \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^n J_\nu^{(n)}(p,n,k)\,, \tag{3.1}$$

the diagram of figure 2(a) gives

$$J_\mu^{(0)}(p,n,k) = -\frac{p_\mu}{p\cdot k} + \frac{k_\mu}{2p\cdot k} - \frac{\mathrm{i}\,k_\alpha \Sigma_\mu^\alpha}{p\cdot k}\,. \tag{3.2}$$

One-loop diagrams are shown in figure 2(b)–(h). Notice that we are computing the bare $J_{\mu,a}$, as required in eq. (2.29), following the discussion in ref. [39]. We can then use the fact that the integral for (h) is scaleless for null n and thus vanishing. Similarly, we have omitted external-leg vacuum polarisations dressing the tree-level diagram. The result can be cast in the form

$$J_\mu^{(1)} = (-2p\cdot k)^{-\epsilon}\big[C_F J_{\mu,F}^{(1)} + C_A J_{\mu,A,\mathrm{coll.}}^{(1)}\big] + \left(\frac{2p\cdot n}{(-2p\cdot k)(-2n\cdot k)}\right)^{\epsilon} C_A\, J_{\mu,A,\mathrm{soft}}^{(1)}\,. \tag{3.3}$$

Defining the kinematic variables

$$t = -2p\cdot k\,, \qquad n_p^\mu = \frac{n^\mu}{2n\cdot p}\,, \qquad n_k^\mu = \frac{n^\mu}{2n\cdot k}\,, \qquad r = \frac{n\cdot k}{n\cdot p}\,, \tag{3.4}$$

we find that the coefficients in eq. (3.3) can be written as

$$
\begin{aligned}
J^{(1)}_{\mu,F} &= (1+2\epsilon)\frac{\not{k}}{t}\gamma^\mu + (2+6\epsilon)\not{n}_p\gamma^\mu + \left[\frac{2}{\epsilon} - 2 - \epsilon(8+\zeta_2)\right]\frac{k^\mu}{t} \\
&\quad + \left[\frac{4}{\epsilon} + 4 + 2\epsilon(2-\zeta_2)\right] n_p^\mu + \left[\frac{4}{\epsilon} + 8 - 2\epsilon(-8+\zeta_2)\right]\frac{r}{t}\, p^\mu - 4(1+3\epsilon)\frac{\not{k}\not{n}_p}{t}\, p^\mu\,; \\
J^{(1)}_{\mu,A,\mathrm{coll.}} &= -(1+2\epsilon)\frac{\not{k}}{t}\gamma^\mu + \left[\frac{1}{\epsilon} + 1 + \epsilon\left(1 - \frac{\zeta_2}{2}\right)\right]\not{n}_p\,\gamma^\mu - 4n_p^\mu \\
&\quad - \left[-\frac{2}{\epsilon} + 2 + \epsilon(-2+\zeta_2)\right]\frac{r}{t}\, p^\mu + \left[-\frac{2}{\epsilon} - 2 + \epsilon(-2+\zeta_2)\right]\frac{\not{k}\not{n}_p}{t}\, p^\mu \\
&\quad + \left[-\frac{1}{\epsilon^2} - \frac{3}{\epsilon} - 3 + \frac{\zeta_2}{2} + \epsilon\left(-4 + \frac{3}{2}\zeta_2 + \frac{7}{3}\zeta_3\right)\right]\frac{k^\mu}{t}\,; \\
J^{(1)}_{\mu,A,\mathrm{soft}} &= \left[-\frac{2}{\epsilon} - 4 + \epsilon(-8-\zeta_2)\right]\not{n}_p\,\gamma^\mu + \Bigg\{-\left[\frac{2}{\epsilon} + 4 + \epsilon(8+\zeta_2)\right] n_k^\mu \\
&\qquad + \left[\frac{2}{\epsilon} + 4 + \epsilon(8+\zeta_2)\right]\frac{p^\mu}{t} - \left[\frac{1}{\epsilon^2} + \frac{2}{\epsilon} + 4 + \frac{\zeta_2}{2} + \epsilon\left(8 + \zeta_2 - \frac{7}{3}\zeta_3\right)\right]\frac{k^\mu}{rt}\Bigg\}\not{k}\not{n}_p \\
&\quad + \left[\frac{1}{\epsilon^2} + \frac{2}{\epsilon} + 4 + \frac{\zeta_2}{2} + \epsilon\left(8 + \zeta_2 - \frac{7}{3}\zeta_3\right)\right]\frac{k^\mu}{t} + \left(\frac{1}{\epsilon^2} + \frac{\zeta_2}{2} - \frac{7}{3}\epsilon\,\zeta_3\right)\frac{\not{k}}{t}\gamma^\mu \\
&\quad + \left\{-\frac{2}{\epsilon^2} - \zeta_2 + \frac{14}{3}\epsilon\,\zeta_3 - \left[\frac{2}{\epsilon} + 4 + \epsilon(8+\zeta_2)\right] r\right\}\frac{p^\mu}{t} \\
&\quad + \left[\frac{2}{\epsilon} + 4 + \epsilon(8+\zeta_2) - \left(\frac{2}{\epsilon^2} + \zeta_2 - \frac{14}{3}\epsilon\,\zeta_3\right)\frac{1}{r}\right] n_p^\mu\,.
\end{aligned}
\tag{3.5}
$$

The first two terms in eq. (3.3) are accompanied by a factor $(2p\cdot k)^{-\epsilon}$, corresponding to the collinear scale associated with radiation from a jet [39, 43]. The third term in eq. (3.3), on the other hand, contains a different ratio of scales involving the auxiliary vector n. Note that for the choices in eq. (2.30) the ratio for both jets becomes

$$
\left(\frac{2p\cdot n}{(-2p\cdot k)(-2n\cdot k)}\right)^\epsilon \rightarrow \left(\frac{2p_1\cdot p_2}{(-2p_1\cdot k)(-2p_2\cdot k)}\right)^\epsilon. \tag{3.6}
$$

This is the same dependence arising in (next-to-)soft webs connecting both external partons (shown for example in figure 1(d)). Terms with this scale dependence thus constitute the double counting of overlapping (next-to-)soft and collinear regions for the virtual gluon momentum, to be removed by the subtraction term $\mathcal{A}^{\widetilde{\mathcal{J}}}_{\mu,a}$. In our present calculation, one may interpret this overlap diagrammatically by defining a *next-to-soft radiative jet function* $\widetilde{\mathcal{J}}_{\mu,a}(p,k,n)$. This function appears in the subtraction term $\mathcal{A}^{\widetilde{\mathcal{J}}}_{\mu,a}$ instead of the full radiative jet function used in the definition of $\mathcal{A}^{J_i}_{\mu,a}$, eq. (2.11). By analogy with eq. (2.11), we then write

$$
\mathcal{A}^{\widetilde{\mathcal{J}}_1}_{\mu,a}(p_1,p_2,k) = \widetilde{\mathcal{H}}(p_1,p_2,n_j)\,\frac{\widetilde{\mathcal{S}}(p_j)}{\prod_{k=1}^2\widetilde{\mathcal{J}}(p_k,n_k)}\,\widetilde{\mathcal{J}}_{\mu,a}(p_1,n_1,k)\,J(p_2,n_2)\,. \tag{3.7}
$$

The function $\widetilde{\mathcal{J}}_{\mu,a}$ can be obtained from the diagrams for the full radiative jet, by replacing the emission vertices on the p leg with the soft or next-to-soft Feynman rules arising from eq. (2.8), and including at most one next-to-soft vertex. At tree-level (using the

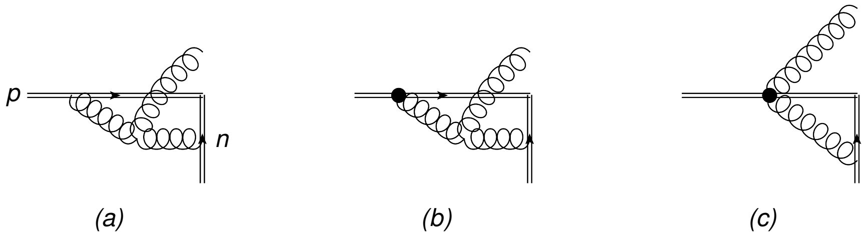

Figure 3. Example diagrams for the next-to-soft radiaive jet function, where the p leg has been replaced by a generalised Wilson line, and • denotes a next-to-soft emission vertex, arising from eq. (2.8).

normalisation of eq. (3.1)) one simply finds $\widetilde{\mathcal{J}}_\mu^{(0)}(p,n,k) = J_\mu^{(0)}(p,n,k)$. At the one-loop level, one encounters diagrams such as those in figure 3: in fact, only the diagrams in figure 3(a) and (b) are non-vanishing, By analogy with eq. (3.3), one can write the result in the form

$$\widetilde{\mathcal{J}}_\mu^{(1)} = (-2p\cdot k)^{-\epsilon}\left[C_F \widetilde{\mathcal{J}}_{\mu,F}^{(1)} + C_A \widetilde{\mathcal{J}}_{\mu,A,\text{coll.}}^{(1)}\right] + \left(\frac{2p\cdot n}{(-2p\cdot k)(-2n\cdot k)}\right)^{\epsilon} C_A\, \widetilde{\mathcal{J}}_{\mu,A,\text{soft}}^{(1)}\,, \tag{3.8}$$

and one finds that

$$\widetilde{\mathcal{J}}_{\mu,F}^{(1)} = \widetilde{\mathcal{J}}_{\mu,A,\text{coll.}}^{(1)} = 0\,, \qquad \widetilde{\mathcal{J}}_{\mu,A,\text{soft}}^{(1)} = J_{\mu,A,\text{soft}}^{(1)}\,, \tag{3.9}$$

so that the next-to-soft radiative jet function reproduces precisely the third term in eq. (3.3): subtracting it from the full jet leaves only collinear contributions, as required.

According to eq. (2.29), for the complete result one also needs the radiative next-to-soft function $\widetilde{\mathcal{S}}_\mu$ at one-loop. The relevant diagrams are similar to those entering the next-to-soft radiative jet function. The leading power soft diagrams can be obtained simply by relabelling $p \to p_1$, $n \to p_2$. For the next-to-soft contribution, there are two sets of diagrams: those where the next-to-soft emission vertex is on leg p_1, and those where it is on leg p_2. One may then write $\widetilde{\mathcal{S}}^\mu = \widetilde{\mathcal{S}}^\mu_{\text{E}} + \widetilde{\mathcal{S}}^\mu_{\text{NE}}$, with

$$\widetilde{\mathcal{S}}^\mu_{\text{E}} = \widetilde{\mathcal{J}}^\mu_{\text{E}}\big|_{p\to p_1, n\to p_2}\,, \qquad \widetilde{\mathcal{S}}^\mu_{\text{NE}} = \widetilde{\mathcal{J}}^\mu_{\text{NE}}\big|_{p\to p_1, n\to p_2} + \widetilde{\mathcal{J}}^\mu_{\text{NE}}\big|_{p\to p_2, n\to p_1}\,, \tag{3.10}$$

where the subscripts E and NE refer to eikonal and next-to-eikonal contributions respectively.

4 Application to Drell-Yan production

We now have all ingredients to verify eq. (2.29) in the Drell-Yan process

$$q(p_1) + \bar{q}(p_2) \to V^*(Q)\,, \tag{4.1}$$

where q and $\bar{q}$ denote a quark and antiquark respectively, V^* an off-shell vector boson, and arguments label 4-momenta. At cross-section level, all LP and NLP threshold logarithms are associated with real emission of soft or next-to-soft gluons; virtual gluons, however, can be hard and collinear, thus loop corrections test all ingredients in eq. (2.28). As usual, one defines the threshold variable $z = Q^2/s$, representing the fraction of available energy carried by the final state vector boson; the threshold limit then corresponds to $z \to 1$. The K-factor at fixed order in perturbation theory is defined by

$$K^{(n)}(z) = \frac{1}{\sigma^{(0)}} \frac{d\sigma^{(n)}(z)}{dz}, \tag{4.2}$$

with $\sigma^{(n)}$ the n-loop cross section. As was the case in [39, 43], the first non-trivial test of eq. (2.28) is to reproduce the real-virtual contribution to the NNLO K-factor. To do so, we need the tree-level and one-loop amplitudes with one real emission. Using eq. (2.28), we find

$$\mathcal{A}^{(0)}_{\mu,a} = g\left[-\mathbf{T}_{1,a}\,\mathcal{A}^{(0)}\left(\frac{p_{1,\mu}}{p_1\cdot k} + \frac{\mathrm{i}k_\alpha \Sigma^\alpha{}_\mu}{p_1\cdot k}\right) + \mathbf{T}_{2,a}\left(\frac{p_{2,\mu}}{p_2\cdot k} + \frac{\mathrm{i}k_\alpha \Sigma^\alpha{}_\mu}{p_2\cdot k}\right)\mathcal{A}^{(0)}\right], \tag{4.3}$$

where $\mathcal{A}^{(0)}$ is the leading order non-radiative amplitude, stripped of external spinors, and we have used the tree-level radiative jet function in eq. (3.2), as well as the physical polarisation condition $k_\mu \epsilon^\mu(k) = 0$. We have also defined colour generators $\mathbf{T}^a_{1,2}$ acting on the p_1, p_2 legs respectively. For the one-loop amplitude, we use eq. (2.29), which gives

$$\mathcal{A}^{(1)}_{\mu,a} = \sum_{i=1}^{2}\left\{\left[\frac{1}{2}\,\widetilde{\mathcal{S}}^{(0)}_{\mu,a} + g\mathbf{T}_{i,a}G^{\nu}_{i,\mu}\,\frac{\partial}{\partial p_i^\nu} + J^{(0)}_{\mu,a} - \widetilde{\mathcal{J}}^{(0)}_{\mu,a}\right]\mathcal{A}^{(1)} + \left[\frac{1}{2}\,\widetilde{\mathcal{S}}^{(1)}_{\mu,a} + J^{(1)}_{\mu,a} - \widetilde{\mathcal{J}}^{(1)}_{\mu,a}\right]\mathcal{A}^{(0)}\right\}, \tag{4.4}$$

where in the second term we have used the fact that the derivative of the non-radiative tree-level amplitude vanishes [39]. Eq. (4.4) can be further simplified by noting that, at tree level, the next-to-soft function contribution precisely cancels the next-to-soft radiative jet contribution. The only missing ingredient at this point is the derivative of the one-loop non-radiative amplitude, which was already derived in ref. [39]. For example, the contribution from the p_1 leg is given by

$$G_1^{\nu\mu}\,\frac{\partial \mathcal{A}^{(1)}}{\partial p_1^\nu} = -\frac{\epsilon}{p_1\cdot p_2}\left(-p_1^\mu + \frac{p_2\cdot k}{p_1\cdot k}\,p_2^\mu\right)\mathcal{A}^{(1)}. \tag{4.5}$$

It is straightforward to assemble all the ingredients,[3] to compute the full real-virtual contribution NNLO K-factor. We find

$$\begin{aligned}
K^{(2)}_{\mathrm{rv}}(z) = \left(\frac{\alpha_s}{4\pi}\right)^2\Bigg\{C_F^2\Bigg[&\frac{32\mathcal{D}_0(z)-32}{\epsilon^3} + \frac{-64\mathcal{D}_1(z)+48\mathcal{D}_0(z)+64L(z)-96}{\epsilon^2}\\
&+\frac{64\mathcal{D}_2(z)-96\mathcal{D}_1(z)+128\mathcal{D}_0(z)-64L^2(z)+208L(z)-196}{\epsilon} - \frac{128}{3}\mathcal{D}_3(z)\\
&+96\mathcal{D}_2(z)-256\mathcal{D}_1(z)+256\mathcal{D}_0(z)+\frac{128}{3}L^3(z)-232L^2(z)+412L(z)-408\Bigg]\\
+C_AC_F\Bigg[&\frac{8\mathcal{D}_0(z)-8}{\epsilon^3} + \frac{-32\mathcal{D}_1(z)+32L(z)-16}{\epsilon^2} + \frac{64\mathcal{D}_2(z)-64L^2(z)+64L(z)+20}{\epsilon}\\
&-\frac{256}{3}\mathcal{D}_3(z)+\frac{256}{3}L^3(z)-128L^2(z)-60L(z)+8\Bigg]\Bigg\},
\end{aligned} \tag{4.6}$$

[3] Results for the non-radiative amplitude up to one-loop, as well as parametrisations of phase space integrals in the present notation, may be found in ref. [39]. In the result we present, as was done in ref. [39], we neglect terms involving transcendental constants for brevity, and we do not include δ-function terms, which mix with the fully virtual two-loop contribution.

JHEP12(2016)121

where

$$\mathcal{D}_i(z) = \left(\frac{\log^i(1-z)}{(1-z)}\right)_+ , \qquad L(z) = \log(1-z)\,. \tag{4.7}$$

For comparison with the exact two-loop calculation, we note that the real-virtual contribution to the NNLO K-factor is not separately available in the literature [55, 56]. We have performed an independent calculation of this result, similar to the one carried out for the abelian-like contributions in ref. [39]. We find that eq. (4.6) reproduces exactly the full NNLO result, when the latter is truncated to NLP in $(1-z)$, including non-logarithmic contributions.

5 Double real emission contributions

In section 4, we have focused on a single additional gluon emission dressing the non-radiative amplitude. Although a full factorisation formula for multiple emissions is beyond the scope of this paper, we can nevertheless obtain the double-real emission contributions to Drell-Yan production at NNLO by noting that all purely real-emission near-threshold contributions are (next-to-)soft in nature, with no hard collinear terms. This fact was already exploited in ref. [20], where next-to-soft Feynman rules were employed to compute the abelian part of the NNLO K-factor for double real emission. That calculation can easily be reproduced and generalised to the full non-abelian theory in the present framework. In essence, all relevant terms can be obtained by dressing the Born amplitude with (next-to-)soft webs. Formally, by analogy with eq. (2.15), we may define a double radiative next-to-soft function according to

$$\epsilon^*_{\mu,\lambda_1}(k_1)\,\epsilon^*_{\nu,\lambda_2}(k_2)\,\widetilde{\mathcal{S}}^{\mu\nu}(p_1,p_2,k_1,k_2) = \langle k_1,\lambda_1;k_2,\lambda_2|F_{p_2}(\infty,0)F_{p_1}(0,-\infty)|0\rangle\big|_{\rm NLP}\,. \tag{5.1}$$

A sampling of soft and next-to-soft diagrams resulting from this definition are shown in figure 4. We have evaluated all diagrams using the next-to-soft Feynman rules arising from eq. (2.8), and we have integrated over the three-body phase space as in refs. [20, 55]. The result for the double real emission contribution to the NNLO K factor is

$$\begin{aligned}
K^{(2)}_{\rm rr}(z) = \left(\frac{\alpha_s}{4\pi}\right)^2 \Bigg\{ C_F^2\Bigg[&-\frac{32\mathcal{D}_0(z)-32}{\epsilon^3}+\frac{128\mathcal{D}_1(z)-128L(z)+80}{\epsilon^2}\\
&-\frac{256\mathcal{D}_2(z)-256L^2(z)+320L(z)}{\epsilon}+\frac{1024}{3}\mathcal{D}_3(z)-\frac{1024}{3}L^3(z)+640L^2(z)\Bigg]\\
+C_AC_F\Bigg[&-\frac{8\mathcal{D}_0(z)-8}{\epsilon^3}+\frac{1}{\epsilon^2}\left(32\mathcal{D}_1(z)-\frac{44}{3}\mathcal{D}_0(z)-32L(z)+\frac{92}{3}\right)\\
&+\frac{1}{\epsilon}\left(-64\mathcal{D}_2(z)+\frac{176}{3}\mathcal{D}_1(z)-\frac{268}{9}\mathcal{D}_0(z)+64L^2(z)-\frac{368}{3}L(z)+\frac{520}{9}\right)\\
&+\frac{256}{3}\mathcal{D}_3(z)-\frac{352}{3}\mathcal{D}_2(z)+\frac{1072}{9}\mathcal{D}_1(z)-\frac{1616}{27}\mathcal{D}_0(z)\\
&-\frac{256}{3}L^3(z)+\frac{736}{3}L^2(z)-\frac{2080}{9}L(z)+\frac{2912}{27}\Bigg]\\
+n_fC_F\Bigg[&\frac{8\mathcal{D}_0(z)-8}{3\epsilon^2}+\frac{1}{\epsilon}\left(-\frac{32}{3}\mathcal{D}_1(z)+\frac{40}{9}\mathcal{D}_0(z)+\frac{32}{3}L(z)-\frac{112}{9}\right)\\
&+\frac{64}{3}\mathcal{D}_2(z)-\frac{160}{9}\mathcal{D}_1(z)+\frac{224}{27}\mathcal{D}_0(z)-\frac{64}{3}L^2(z)+\frac{448}{9}L(z)-\frac{656}{27}\Bigg]\Bigg\}\,,
\end{aligned} \tag{5.2}$$

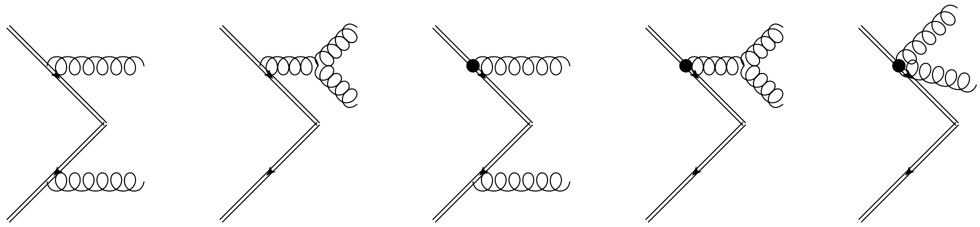

Figure 4. Sample diagrams contributing to the double radiative next-to-soft function, where • denotes a next-to-eikonal Feynman rule, and all other couplings to the external lines are eikonal.

which again agrees with an exact calculation, including non-logarithmic NLP terms. It is not in fact surprising that this happens: one may derive the next-to-soft Feynman rules by systematically expanding the exact unintegrated amplitude in the emitted gluon momenta (following the diagrammatic approach of ref. [20]). Thus, the effective approach and the full calculation agree at the amplitude level by construction. We have also checked that, upon combining our results for double real emission and real-virtual corrections with the well-known two-loop virtual corrections, and with mass factorisation, the complete result of refs. [55, 56] is reproduced, to the expected accuracy.

6 Conclusion

In this paper, we have derived an all-order factorisation formula, eq. (2.28), organising at the amplitude level all contributions which give rise to threshold logarithms up to next-to leading power. The formula has been derived for the Drell-Yan process, but we expect it to apply, with minor modifications, for all processes involving the annihilation of QCD partons into electroweak final states, such as (multiple) Higgs production via gluon fusion or multiple vector boson production. Eq. (2.28) generalises the well-known leading power soft-collinear factorisation formula described in ref. [46], as well as previous formulae that included only abelian-like contributions [15, 39]. It contains similar universal functions, namely the leading-power soft and jet functions, together with a radiative jet function. We have generalised the definition of the latter to a non-abelian theory, and calculated this quantity at one-loop order for quark jets.

We have verified our formula by reproducing known threshold logarithms at NNLO in Drell-Yan production, which is a non-trivial check at loop-level since both collinear and soft momentum regions are tested. We discussed how to remove the double counting of next-to-soft and collinear contributions via a subtraction term, by defining a *next-to-soft radiative jet function*. We note that a more general definition of this function, in particular for general values of the auxiliary vector n, deserves further study, which we postpone to future work. As was the case for the abelian-like contributions in previous work, we find that there is a non-vanishing loop-level contribution to NLP logarithms from hard collinear configurations of virtual gluons: this leads to a breaking of the next-to-soft theorems discussed for example in refs. [33, 34], at loop level.

A new feature of the present work is the role of next-to-soft web diagrams, describing the correlated emission of gluons external to the hard interaction. Using next-to-soft webs, we reproduce the double real emission contributions in Drell-Yan production, which are dominated near threshold by radiation which is always (next-to-)soft, whether or not it is collinear. We note that, as discussed in detail in refs. [19, 20], the web language is potentially much more powerful, implying formal exponentiation of next-to-soft contributions in a much more general context, as discussed for example in refs. [57–67]. We conclude that eq. (2.28) is an important step towards a general resummation procedure for NLP threshold logarithms. Further necessary ingredients include the calculation of radiative jet functions for external gluon jets, and the elucidation of subleading collinear effects in processes with final state parton jets (see, for example [68]). These developments will be the subject of future work.

Acknowledgments

This research was supported by the Research Executive Agency (REA) of the European Union through the contract PITN-GA-2012-316704 (HIGGSTOOLS), and by MIUR (Italy), under contract 2010YJ2NYW_006. DB and EL have been supported by the Netherlands Foundation for Fundamental Research of Matter (FOM) programme 156, "Higgs as Probe and Portal", and by the National Organisation for Scientific Research (NWO). This research was also supported in part by the National Science Foundation under Grant No. NSF PHY-1125915, and EL is grateful to the Kavli Institute for Theoretical Physics, where part of this work was done, for hospitality. CDW is supported by the U.K. Science and Technology Facilities Council (STFC). We are very grateful to the Higgs Centre for Theoretical Physics at the University of Edinburgh, where part of this work was carried out, for warm hospitality. We also thank Simon Caron-Huot, Einan Gardi, and Jort Sinninghe-Damsté for useful discussions.

References

[1] G.F. Sterman, *Summation of large corrections to short distance hadronic cross-sections*, *Nucl. Phys.* **B 281** (1987) 310 [INSPIRE].

[2] S. Catani and L. Trentadue, *Resummation of the QCD perturbative series for hard processes*, *Nucl. Phys.* **B 327** (1989) 323 [INSPIRE].

[3] G.P. Korchemsky and G. Marchesini, *Resummation of large infrared corrections using Wilson loops*, *Phys. Lett.* **B 313** (1993) 433 [INSPIRE].

[4] G.P. Korchemsky and G. Marchesini, *Structure function for large x and renormalization of Wilson loop*, *Nucl. Phys.* **B 406** (1993) 225 [hep-ph/9210281] [INSPIRE].

[5] S. Forte and G. Ridolfi, *Renormalization group approach to soft gluon resummation*, *Nucl. Phys.* **B 650** (2003) 229 [hep-ph/0209154] [INSPIRE].

[6] H. Contopanagos, E. Laenen and G.F. Sterman, *Sudakov factorization and resummation*, *Nucl. Phys.* **B 484** (1997) 303 [hep-ph/9604313] [INSPIRE].

[7] A. Banfi, G.P. Salam and G. Zanderighi, *Principles of general final-state resummation and automated implementation*, *JHEP* **03** (2005) 073 [hep-ph/0407286] [INSPIRE].

[8] T. Becher and M. Neubert, *Threshold resummation in momentum space from effective field theory*, *Phys. Rev. Lett.* **97** (2006) 082001 [hep-ph/0605050] [INSPIRE].

[9] G. Luisoni and S. Marzani, *QCD resummation for hadronic final states*, *J. Phys.* **G 42** (2015) 103101 [arXiv:1505.04084] [INSPIRE].

[10] G. Parisi, *Summing large perturbative corrections in QCD*, *Phys. Lett.* **B 90** (1980) 295 [INSPIRE].

[11] T.O. Eynck, E. Laenen and L. Magnea, *Exponentiation of the Drell-Yan cross-section near partonic threshold in the DIS and MS-bar schemes*, *JHEP* **06** (2003) 057 [hep-ph/0305179] [INSPIRE].

[12] V. Ahrens, T. Becher, M. Neubert and L.L. Yang, *Origin of the large perturbative corrections to Higgs production at hadron colliders*, *Phys. Rev.* **D 79** (2009) 033013 [arXiv:0808.3008] [INSPIRE].

[13] F.E. Low, *Bremsstrahlung of very low-energy quanta in elementary particle collisions*, *Phys. Rev.* **110** (1958) 974 [INSPIRE].

[14] T.H. Burnett and N.M. Kroll, *Extension of the low soft photon theorem*, *Phys. Rev. Lett.* **20** (1968) 86 [INSPIRE].

[15] V. Del Duca, *High-energy bremsstrahlung theorems for soft photons*, *Nucl. Phys.* **B 345** (1990) 369 [INSPIRE].

[16] Y.L. Dokshitzer, G. Marchesini and G.P. Salam, *Revisiting parton evolution and the large-x limit*, *Phys. Lett.* **B 634** (2006) 504 [hep-ph/0511302] [INSPIRE].

[17] M. Krämer, E. Laenen and M. Spira, *Soft gluon radiation in Higgs boson production at the LHC*, *Nucl. Phys.* **B 511** (1998) 523 [hep-ph/9611272] [INSPIRE].

[18] R. Akhoury, M.G. Sotiropoulos and G.F. Sterman, *An operator expansion for the elastic limit*, *Phys. Rev. Lett.* **81** (1998) 3819 [hep-ph/9807330] [INSPIRE].

[19] E. Laenen, G. Stavenga and C.D. White, *Path integral approach to eikonal and next-to-eikonal exponentiation*, *JHEP* **03** (2009) 054 [arXiv:0811.2067] [INSPIRE].

[20] E. Laenen, L. Magnea, G. Stavenga and C.D. White, *Next-to-eikonal corrections to soft gluon radiation: a diagrammatic approach*, *JHEP* **01** (2011) 141 [arXiv:1010.1860] [INSPIRE].

[21] G. Soar, S. Moch, J.A.M. Vermaseren and A. Vogt, *On Higgs-exchange DIS, physical evolution kernels and fourth-order splitting functions at large x*, *Nucl. Phys.* **B 832** (2010) 152 [arXiv:0912.0369] [INSPIRE].

[22] S. Moch and A. Vogt, *On non-singlet physical evolution kernels and large-x coefficient functions in perturbative QCD*, *JHEP* **11** (2009) 099 [arXiv:0909.2124] [INSPIRE].

[23] S. Moch and A. Vogt, *Threshold resummation of the structure function F_L*, *JHEP* **04** (2009) 081 [arXiv:0902.2342] [INSPIRE].

[24] A.A. Almasy, G. Soar and A. Vogt, *Generalized double-logarithmic large-x resummation in inclusive deep-inelastic scattering*, *JHEP* **03** (2011) 030 [arXiv:1012.3352] [INSPIRE].

[25] D. de Florian, J. Mazzitelli, S. Moch and A. Vogt, *Approximate N^3LO Higgs-boson production cross section using physical-kernel constraints*, *JHEP* **10** (2014) 176 [arXiv:1408.6277] [INSPIRE].

JHEP12(2016)121

[26] N.A. Lo Presti, A.A. Almasy and A. Vogt, *Leading large-x logarithms of the quark-gluon contributions to inclusive Higgs-boson and lepton-pair production*, Phys. Lett. **B 737** (2014) 120 [arXiv:1407.1553] [INSPIRE].

[27] A.J. Larkoski, D. Neill and I.W. Stewart, *Soft theorems from effective field theory*, JHEP **06** (2015) 077 [arXiv:1412.3108] [INSPIRE].

[28] D.W. Kolodrubetz, I. Moult and I.W. Stewart, *Building blocks for subleading helicity operators*, JHEP **05** (2016) 139 [arXiv:1601.02607] [INSPIRE].

[29] E. Laenen, L. Magnea and G. Stavenga, *On next-to-eikonal corrections to threshold resummation for the Drell-Yan and DIS cross sections*, Phys. Lett. **B 669** (2008) 173 [arXiv:0807.4412] [INSPIRE].

[30] G. Grunberg and V. Ravindran, *On threshold resummation beyond leading* $1-x$ *order*, JHEP **10** (2009) 055 [arXiv:0902.2702] [INSPIRE].

[31] G. Grunberg, *Large-x structure of physical evolution kernels in deep inelastic scattering*, Phys. Lett. **B 687** (2010) 405 [arXiv:0911.4471] [INSPIRE].

[32] A. Strominger, *On BMS invariance of gravitational scattering*, JHEP **07** (2014) 152 [arXiv:1312.2229] [INSPIRE].

[33] F. Cachazo and A. Strominger, *Evidence for a new soft graviton theorem*, arXiv:1404.4091 [INSPIRE].

[34] E. Casali, *Soft sub-leading divergences in Yang-Mills amplitudes*, JHEP **08** (2014) 077 [arXiv:1404.5551] [INSPIRE].

[35] Z. Bern, S. Davies and J. Nohle, *On loop corrections to subleading soft behavior of gluons and gravitons*, Phys. Rev. **D 90** (2014) 085015 [arXiv:1405.1015] [INSPIRE].

[36] A.J. Larkoski, *Conformal invariance of the subleading soft theorem in gauge theory*, Phys. Rev. **D 90** (2014) 087701 [arXiv:1405.2346] [INSPIRE].

[37] C.D. White, *Factorization properties of soft graviton amplitudes*, JHEP **05** (2011) 060 [arXiv:1103.2981] [INSPIRE].

[38] C.D. White, *Diagrammatic insights into next-to-soft corrections*, Phys. Lett. **B 737** (2014) 216 [arXiv:1406.7184] [INSPIRE].

[39] D. Bonocore et al., *A factorization approach to next-to-leading-power threshold logarithms*, JHEP **06** (2015) 008 [arXiv:1503.05156] [INSPIRE].

[40] M. Beneke and V.A. Smirnov, *Asymptotic expansion of Feynman integrals near threshold*, Nucl. Phys. **B 522** (1998) 321 [hep-ph/9711391] [INSPIRE].

[41] A. Pak and A. Smirnov, *Geometric approach to asymptotic expansion of Feynman integrals*, Eur. Phys. J. **C 71** (2011) 1626 [arXiv:1011.4863] [INSPIRE].

[42] B. Jantzen, *Foundation and generalization of the expansion by regions*, JHEP **12** (2011) 076 [arXiv:1111.2589] [INSPIRE].

[43] D. Bonocore, E. Laenen, L. Magnea, L. Vernazza and C.D. White, *The method of regions and next-to-soft corrections in Drell-Yan production*, Phys. Lett. **B 742** (2015) 375 [arXiv:1410.6406] [INSPIRE].

[44] T. Matsuura, S.C. van der Marck and W.L. van Neerven, *The order* α_s^2 *contribution to the K-factor of the Drell-Yan process*, Phys. Lett. **B 211** (1988) 171 [INSPIRE].

JHEP12(2016)121

[45] T. Matsuura, S.C. van der Marck and W.L. van Neerven, *The calculation of the second order soft and virtual contributions to the Drell-Yan cross-section*, Nucl. Phys. **B 319** (1989) 570 [INSPIRE].

[46] L.J. Dixon, L. Magnea and G.F. Sterman, *Universal structure of subleading infrared poles in gauge theory amplitudes*, JHEP **08** (2008) 022 [arXiv:0805.3515] [INSPIRE].

[47] S. Weinberg, *The quantum theory of fields. Vol. 2: Modern applications*, Cambridge University Press, Cambridge U.K. (1996) [INSPIRE].

[48] B. De Wit and J. Smith, *Field theory in particle physics. Vol. 1*, North-Holland, Amsterdam Netherlands (1986) [INSPIRE].

[49] A.V. Belitsky, *Two loop renormalization of Wilson loop for Drell-Yan production*, Phys. Lett. **B 442** (1998) 307 [hep-ph/9808389] [INSPIRE].

[50] M. Czakon, A. Mitov and G.F. Sterman, *Threshold resummation for top-pair hadroproduction to next-to-next-to-leading log*, Phys. Rev. **D 80** (2009) 074017 [arXiv:0907.1790] [INSPIRE].

[51] M. Beneke, P. Falgari and C. Schwinn, *Soft radiation in heavy-particle pair production: all-order colour structure and two-loop anomalous dimension*, Nucl. Phys. **B 828** (2010) 69 [arXiv:0907.1443] [INSPIRE].

[52] A. Idilbi, C. Kim and T. Mehen, *Factorization and resummation for single color-octet scalar production at the LHC*, Phys. Rev. **D 79** (2009) 114016 [arXiv:0903.3668] [INSPIRE].

[53] M. Czakon and P. Fiedler, *The soft function for color octet production at threshold*, Nucl. Phys. **B 879** (2014) 236 [arXiv:1311.2541] [INSPIRE].

[54] G. Grammer Jr. and D.R. Yennie, *Improved treatment for the infrared divergence problem in quantum electrodynamics*, Phys. Rev. **D 8** (1973) 4332 [INSPIRE].

[55] R. Hamberg, W.L. van Neerven and T. Matsuura, *A complete calculation of the order α_s^2 correction to the Drell-Yan K factor*, Nucl. Phys. **B 359** (1991) 343 [*Erratum ibid.* **B 644** (2002) 403] [INSPIRE].

[56] T. Matsuura, R. Hamberg and W.L. van Neerven, *The contribution of the gluon-gluon subprocess to the Drell-Yan K-factor*, Nucl. Phys. Proc. Suppl. **23B** (1991) 3 [INSPIRE].

[57] J.G.M. Gatheral, *Exponentiation of eikonal cross sections in nonabelian gauge theories*, Phys. Lett. **B 133** (1983) 90 [INSPIRE].

[58] J. Frenkel and J.C. Taylor, *Non-abelian eikonal exponentiation*, Nucl. Phys. **B 246** (1984) 231 [INSPIRE].

[59] G.F. Sterman, *Infrared divergences in perturbative QCD*, in Proceedings of the *Conference on Perturbative Quantum Chromodynamics*, Tallahassee U.S.A. (1981) [*AIP Conf. Proc.* **74** (1981) 22] [INSPIRE].

[60] A. Mitov, G. Sterman and I. Sung, *Diagrammatic exponentiation for products of Wilson lines*, Phys. Rev. **D 82** (2010) 096010 [arXiv:1008.0099] [INSPIRE].

[61] E. Gardi, E. Laenen, G. Stavenga and C.D. White, *Webs in multiparton scattering using the replica trick*, JHEP **11** (2010) 155 [arXiv:1008.0098] [INSPIRE].

[62] E. Gardi, J.M. Smillie and C.D. White, *On the renormalization of multiparton webs*, JHEP **09** (2011) 114 [arXiv:1108.1357] [INSPIRE].

[63] E. Gardi, J.M. Smillie and C.D. White, *The non-Abelian exponentiation theorem for multiple Wilson lines*, JHEP **06** (2013) 088 [arXiv:1304.7040] [INSPIRE].

JHEP12(2016)121

[64] E. Gardi and C.D. White, *General properties of multiparton webs: proofs from combinatorics*, *JHEP* **03** (2011) 079 [arXiv:1102.0756] [INSPIRE].

[65] G. Falcioni, E. Gardi, M. Harley, L. Magnea and C.D. White, *Multiple gluon exchange webs*, *JHEP* **10** (2014) 010 [arXiv:1407.3477] [INSPIRE].

[66] E. Gardi, *From webs to polylogarithms*, *JHEP* **04** (2014) 044 [arXiv:1310.5268] [INSPIRE].

[67] Ø. Almelid, C. Duhr and E. Gardi, *Three-loop corrections to the soft anomalous dimension in multileg scattering*, *Phys. Rev. Lett.* **117** (2016) 172002 [arXiv:1507.00047] [INSPIRE].

[68] D. Nandan, J. Plefka and W. Wormsbecher, *Collinear limits beyond the leading order from the scattering equations*, arXiv:1608.04730 [INSPIRE].

JHEP12(2016)121