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Singularities of the matrix exponent of a Markov additive process with one-sided jumps

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Abstract

We analyze the number of zeros of $\det(F(\alpha))$, where $F(\alpha)$ is the matrix exponent of a Markov Additive Process (MAP) with one-sided jumps. The focus is on the number of zeros in the right half of the complex plane, where $F(\alpha)$ is analytic. In addition, we also consider the case of a MAP killed at an independent exponential time. The corresponding zeros can be seen as the roots of a generalized Cramér–Lundberg equation. We argue that our results are particularly useful in fluctuation theory for MAPs, which leads to numerous applications in queueing theory and finance.

Keywords: Markov additive processes; Lévy processes; Queueing theory; Markov modulation; First passage; Roots of Cramér–Lundberg equation; Argument principle

1. Introduction

In this paper we consider Markov Additive Processes (MAPs) with one-sided jumps. For reasons of symmetry, we may restrict ourselves to the case of *no negative* jumps, which we will do throughout this work. Loosely speaking, a MAP is a Lévy process in a Markov environment,

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in the sense that its characteristic exponent is determined by an independent Markovian background process. MAPs thus constitute a natural generalization of a Lévy process, with many analogous properties and characteristics. Any Lévy process (with no negative jumps) is characterized by a Laplace exponent $\phi(\alpha)$; its counterpart for MAPs is the *matrix exponent* $F(\alpha)$, which is essentially a multi-dimensional analogue of $\phi(\alpha)$.

In the theory of Lévy processes the celebrated Cramér–Lundberg (C–L) equation $\phi(\alpha) = q$, where $q \geq 0$, plays an important role. The MAP analogue of this equation has the form $\det(F(\alpha) - q\mathrm{I}) = 0$, where I is the identity matrix. We determine the number of zeros of $\det(F(\alpha) - q\mathrm{I})$ for a fixed $q \geq 0$ in the right half of the complex plane, where $F(\alpha)$ is known to be analytic in the absence of negative jumps. The proofs of our results are entirely analytic and are based on elementary results and techniques.

Similarly to the theory of Lévy processes the roots of the generalized C–L equation play an important role in the fluctuation theory, which leads to numerous application in queueing theory and finance. The analysis of the number of roots is often an important first step when considering a problem in this domain. We provide a number of examples in Section 3: one on a dense class of Lévy processes that allows for positive and negative jumps, one on Markov-modulated M/G/1 queue, one on first passage times, and one on martingale-based calculations. This list of applications is far from complete, and is here to stress the importance of the present problem, which we solve in a very general setting.

A number of special cases of the present problem can be found in the literature, see e.g. [19,25,26]. A common, rather restrictive, assumption in these papers is that the process $X(t)$ evolves *linearly* between the jumps of the underlying Markov chain $J(t)$. In [14] the case of Markov modulated Brownian motion is considered. Finally, the case of Markov modulated compound Poisson process is analyzed under a number of assumptions in [23]. In this respect the findings of our paper considerably generalize results from the existing literature.

This paper is organized as follows. In Section 2 we provide some background on MAPs and state our results. In Section 3 we discuss a number of applications of our results. The rest of the paper is devoted to the proofs. We present two rather general results on the number of zeros of certain functions in Section 4, and some analytic properties of the Laplace exponent of a Lévy process without negative jumps in Section 5. Using this material we prove our main results, i.e., Theorems 1 and 2, in Section 6.

2. Model and results

A right-continuous process $(X(t), J(t))_{t\geq 0}$ is said to be a MAP if for any $T > 0$ it holds that given $\{J(T) = j\}$

$$\begin{aligned}&(X(T+t) - X(T), J(T+t))_{t\geq 0} \text{ is independent of } (X(t), J(t))_{0\leq t\leq T}\\ &\quad\text{and has the same law as } (X(t), J(t))_{t\geq 0} \text{ given } J(0) = j.\end{aligned} \tag{1}$$

Loosely speaking, a MAP is a Lévy process in a Markov environment. Following [2, Ch. XI] we assume that $J(t)$ takes values in some finite set $\{1, \dots, N\}$, hence $J(t)$ is a continuous-time Markov chain. In this case the structure of a MAP is well understood. Namely, while $J(t)$ is in state i, the additive component $X(t)$ evolves like some Lévy process $X_i(t)$. In addition, a jump of $J(t)$ from i to $j \neq i$ triggers a jump of $X(t)$ at the same time, which is distributed as some random variable U_{ij}. It is assumed that the Markov chain $J(t)$ is irreducible, and its transition rate matrix is denoted through Q.

In this paper we assume that $X(t)$ has no negative jumps. Let $\phi_i(\alpha)$ be the Laplace exponent of the Lévy process $X_i(t)$:

$$\begin{aligned}\phi_i(\alpha) &:= \log(\mathbb{E}\mathrm{e}^{-\alpha X_i(1)})\\ &= a_i\alpha + \frac{1}{2}\sigma_i^2\alpha^2 + \int_0^\infty (-1 + \mathrm{e}^{-\alpha x} + \alpha x \mathbb{1}_{\{x<1\}})\nu_i(\mathrm{d}x),\end{aligned} \tag{2}$$

where $(a_i, \sigma_i, \nu_i(\mathrm{d}x))$ is a *Lévy triple*, that is, $a_i \in \mathbb{R}, \sigma_i \geq 0$ and $\nu_i(\mathrm{d}x)$ is a measure on $(0, \infty)$ satisfying $\int_0^\infty (1 \wedge x^2)\nu_i(\mathrm{d}x) < \infty$. Note that restricting the support of measure $\nu_i(\mathrm{d}x)$ to $(0, \infty)$ amounts to forbidding negative jumps. Let $\tilde{G}_{ij}(\alpha) := \mathbb{E}\mathrm{e}^{-\alpha U_{ij}}$ be the Laplace–Stieltjes transform of the distribution of $U_{ij} \geq 0$. Without loss of generality we set $U_{ii} \equiv 0$, and $U_{ij} \equiv 0$ whenever $q_{ij} = 0$.

Letting $\tilde{G}(\alpha) := (\tilde{G}_{ij}(\alpha))$ and $A \circ B := (a_{ij}b_{ij})$, where A and B are two square matrices of the same dimensions, we define

$$F(\alpha) := Q \circ \tilde{G}(\alpha) + \mathrm{diag}(\phi_1(\alpha), \ldots, \phi_N(\alpha)) \tag{3}$$

and note that

$$\mathbb{E}[\mathrm{e}^{-\alpha X(t)}\mathbb{1}_{\{J(t)=j\}} \mid J(0) = i] = \left(\mathrm{e}^{F(\alpha)t}\right)_{ij}, \tag{4}$$

see [2, Ch. XI, Prop. 2.2]. It is not difficult to see from (1) that the matrix exponent $F(\alpha)$ identifies the law of the MAP $(X(t), J(t))$, the reasoning being the same as in the case of a Lévy process. Finally, the absence of negative jumps implies that $F(\alpha)$ is finite for all $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}$ and is analytic in $\mathbb{C}^{\mathrm{Re}>0}$, where $\mathbb{C}^{\mathrm{Re}\geq 0} := \{\alpha \in \mathbb{C} : \mathrm{Re}(\alpha) \geq 0\}$ and $\mathbb{C}^{\mathrm{Re}>0} := \{\alpha \in \mathbb{C} : \mathrm{Re}(\alpha) > 0\}$.

An important concept in fluctuation theory is the concept of 'killing', see e.g. [7,16]. Let e_q be an exponential random variable of rate $q > 0$ independent of the process $(X(t), J(t))$ then

$$\mathbb{E}[\mathrm{e}^{-\alpha X(t)}\mathbb{1}_{\{J(t)=j, t<e_q\}} \mid J(0) = i] = (\mathrm{e}^{(F(\alpha)-q\mathrm{I})t})_{ij}. \tag{5}$$

Hence $F(\alpha) - q\mathrm{I}$ can be seen as the matrix exponent of the MAP $(X(t), J(t))$, which is only considered up to the time e_q (at this random time the MAP is 'killed'). The matrix $F(\alpha) - q\mathrm{I}$ frequently appears in the fluctuation theory for MAPs (or, equivalently, in the theory of storage systems with MAP input), see for instance [10, Thm. 3.1], which motivates the importance of structural properties of the zeros of $\det(F(\alpha) - q\mathrm{I})$.

Before we can state our main results, we introduce a number of useful notions. Firstly, Lévy processes whose paths are non-decreasing are called *subordinators*. The number of processes $X_i(t), i \in \{1, \ldots, N\}$ which are *not* subordinators plays a crucial role in our work. We denote this number by N^*. Secondly, Perron–Frobenius theory entails that there exists a unique eigenvalue $k(\alpha), \alpha \geq 0$ of $F(\alpha)$ with maximal real part. This eigenvalue is real and simple. Moreover, it is well known that $k(0) = 0$ and

$$\lim_{t\to\infty} \frac{X(t)}{t} = -k'(0^+) \quad \text{a.s. for any } J(0), \tag{6}$$

where $k'(0^+)$ is the right-sided derivative of $k(\alpha)$ at 0. In this sense $k'(0^+)$ can be interpreted as the asymptotic drift of $-X(t)$. These results can be found in [2, Ch. XI].

We are now ready to state the main theorems.

Theorem 1. *If* $q > 0$, *then* $\det(F(\alpha) - q\mathrm{I})$ *has no zeros on the imaginary axis and has exactly* N^* *zeros (counting multiplicities) in* $\mathbb{C}^{\mathrm{Re}>0}$.

Remark 2.1. The above result still holds if one considers $\det(F(\alpha) - \mathrm{diag}(\mathbf{q}))$, where $\mathbf{q} \neq \mathbf{0}$ is a vector with non-negative elements. We note that $F(\alpha) - \mathrm{diag}(\mathbf{q})$ can be seen as a matrix exponent of a 'killed' MAP, where the killing rates depend on the state of $J(t)$.

Interestingly, it turns out that in the situation *without* killing the statement becomes slightly less clean, as we see in Theorem 2. Note the important role played by the asymptotic drift: the result depends on whether the process eventually tends to $-\infty$ or $+\infty$.

Theorem 2. *If* $N^* \neq 0$ *and* $k'(0^+)$ *is finite and non-zero, then* $\det(F(\alpha))$ *has a unique zero on the imaginary axis at* $\alpha = 0$ *and* $N^* - \mathbb{1}_{\{k'(0^+)\geq 0\}}$ *zeros (counting multiplicities) in* $\mathbb{C}^{\mathrm{Re}>0}$.

We believe that the case when all the underlying Lévy processes $X_i(t)$ are subordinators, in other words, $N^* = 0$, is not of much interest. For completeness we make the following remark.

Remark 2.2. If $N^* = 0$, then either $X(t) \equiv 0$ or $\det(F(\alpha))$ has no zeros in $\mathbb{C}^{\mathrm{Re}>0}$. In the latter case $\det(F(\alpha))$ has either a unique zero (at 0) or infinitely many distinct zeros on the imaginary axis.

Earlier we mentioned that, when considering MAPs with one-sided jumps, we can without loss of generality assume that there are no *negative* jumps. This claim is made precise in the following remark.

Remark 2.3. Clearly, if $(X(t), J(t))$ is a MAP without positive jumps then $(-X(t), J(t))$ is a MAP without negative jumps. Let $F(\alpha)$ be the matrix exponent of the latter MAP. In the case of no positive jumps it is common to use $\phi_i(\alpha) := \log(\mathbb{E}e^{\alpha X_i(1)})$ and $\tilde{G}_{ij}(\alpha) := \mathbb{E}e^{\alpha U_{ij}}$ in the definition (3) of the matrix exponent. As a consequence $(X(t), J(t))$ has the same matrix exponent $F(\alpha)$. It is easy to see now that Theorems 1 and 2 also hold in the case of no positive jumps, but now N^* is defined as the number of processes which are not *downward* subordinators.

Finally, we make a note about a special class of MAPs, viz. *time-reversible* MAPs $(X(t), J(t))$. This means that $J(t)$ is time-reversible, and U_{ij} has the same distribution as U_{ji}. It is shown in [13] that in this case $\det(F(\alpha))$ has $N^* - \mathbb{1}_{\{k'(0^+)\geq 0\}}$ positive zeros (counting multiplicities). Moreover, if $\alpha_0 > 0$ is a zero of $\det(F(\alpha))$ of multiplicity m then the null space of $F(\alpha_0)$ has rank m (such a zero is called semi-simple). Now we can use Theorem 2 to see that all the zeros of $\det(F(\alpha))$ in $\mathbb{C}^{\mathrm{Re}>0}$ are real and semi-simple. As argued in detail in [13], these properties greatly simplify the analysis of the stationary distribution of the corresponding Markov modulated storage system. They, however, do not hold in general (i.e., when dropping the time-reversibility requirement), as demonstrated by the following example.

Example 2.1. We specify the jump-free MAP $(X(t), J(t))$ as follows. Let

$$U_{ij} \equiv 0, \qquad \phi_1(\alpha) = \alpha, \qquad \phi_2(\alpha) = \alpha + \alpha^2, \qquad \phi_3(\alpha) = \frac{2}{5}\alpha, \quad \text{and}$$

$$Q = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

Note that the asymptotic drift of the MAP is nagative. Moreover, none of the Lévy processes $X_i(t)$ is a subordinator. Hence $F(\alpha)$ should have 2 zeros in $\mathbb{C}^{\mathrm{Re}>0}$ (counting multiplicities). Clearly, $\det(F(\alpha))$ is a fourth order polynomial. The zeros are $-3/2, 0, 2, 2$. Thus the only zero in $\mathbb{C}^{\mathrm{Re}>0}$ is $\alpha = 2$, which has multiplicity 2. The null space of $F(2)$, however, has rank 1.

3. Motivation and applications of the results

In this section we motivate our interest in the roots of $\det(F(\alpha) - q\mathrm{I}) = 0$. This equation can be seen as a matrix analogue of the famous C–L equation $\phi(\alpha) = q$ for a Lévy process with Laplace exponent $\phi(\alpha)$. One would therefore expect that the roots are important in a wide range of applications, and through the following examples we show that this is indeed the case.

3.1. A dense class of Lévy processes and the Cramér–Lundberg equation

We start by recalling that a distribution is called phase-type if it is the distribution of an absorption time in a finite-state continuous-time Markov chain. Such a distribution is usually characterized by the triplet $(m, \mathbf{a}, T)$, where m is the number of phases, $\mathbf{a}$ is the initial distribution and T is the transient transition rate matrix. Moreover, the vector $\mathbf{t} = -T\mathbf{1}$ is interpreted as a vector of exit intensities. The reader is advised to consult [2, Ch. III.4] for a further discussion of phase-type distributions.

We consider a dense class of Lévy processes of the form $X(t) = X^+(t) - X^{PH}(t)$, where $X^+(t)$ is a Lévy process without negative jumps and $X^{PH}(t)$ is an independent compound Poisson process with intensity λ and jumps of phase-type with parameters $(m, \mathbf{a}, T)$. This class of processes received a lot of attention in the literature [4,21]. It is rich enough as it allows both negative and positive jumps, but at the same time the analysis of various fluctuation identities remains tractable. Importantly, the joint Laplace transform of the first passage over a negative level and the corresponding overshoot can be expressed through the roots of the C–L equation with positive real parts [4]. This leads to numerous applications in, e.g., finance and queueing theory.

The above C–L equation takes the following form:

$$\phi^+(\alpha) - \lambda(\mathbf{a}(T + \alpha\mathrm{I})^{-1}\mathbf{t} + 1) = q,$$

where $\phi^+(\alpha)$ is the Laplace exponent of $X^+(t)$. Often it is convenient to associate a certain MAP $(X_0(t), J_0(t))$ with the above Lévy process. More concretely, we consider a MAP without negative jumps whose matrix exponent is

$$F(\alpha) = \begin{pmatrix} \phi^+(\alpha) - \lambda & \lambda\mathbf{a} \\ \mathbf{t} & T + \alpha\mathrm{I} \end{pmatrix};$$

that is, $X_0(t)$ evolves as $X^+(t)$ while $J_0(t) = 1$ and as $-t$ while $J_0(t) \neq 1$. This corresponds to 'levelling out' the phase-type jumps into linear parts of gradient -1. Note also that exponential killing of $X(t)$ with rate q corresponds to the killing of $(X_0(t), J_0(t))$ with the same rate but only when $J_0(t) = 1$. Hence the MAP analogue of the C–L equation reads

$$d(\alpha) := \det \begin{pmatrix} \phi^+(\alpha) - \lambda - q & \lambda\mathbf{a} \\ \mathbf{t} & T + \alpha\mathrm{I} \end{pmatrix} = 0.$$

Assume that $T + \alpha\mathrm{I}$ is invertible otherwise $\phi(\alpha)$ has a pole. Then by a well-known formula for block matrices we obtain

$$d(\alpha) = \det(T + \alpha\mathrm{I}) \det\left((\phi^+(\alpha) - \lambda - q) - \lambda\mathbf{a}(T + \alpha\mathrm{I})^{-1}\mathbf{t}\right)$$

and hence we recover the roots of the original C–L equation. One can show that $\det(T + \alpha\mathrm{I})$ and $d(\alpha)$ cannot be simultaneously 0 if the phase-type representation is minimal (m is the least possible number of phases leading to the same distribution). Thus the C–L equation and its MAP analogue have the same roots.

Recall that Theorem 1 still holds for state-dependent killing, see Remark 2.1. Hence Theorem 1 implies that if $q > 0$ then the number of roots of the C–L equation with positive real parts is m or $m + 1$ according to whether $X^+(t)$ is a subordinator or not. This result was proven in [4] using the very special structure of this problem and the Wiener–Hopf factorisation.

3.2. Markov modulated M/G/1 queue

Following [23] we consider a Markov modulated M/G/1 queue. The so-called free process of such a queue is a Markov modulated compound Poisson process minus linear drift, where the jumps correspond to the service times; hence it is a MAP without negative jumps. The workload process is obtained by reflecting the free process at 0. In the powerful paper [23], the transforms are determined for both the waiting time and the number of customers (at arrival epochs as well as at an arbitrary time). The basis of this result is a Wiener–Hopf type equation and the analysis of the number of zeros of $\det(F(\alpha) - q\mathrm{I})$ in the right half-plane, see [23, Lemma 3.1] for the latter. It is noted that this crucial Lemma is a special case of our Theorems 1 and 2. In [23] it is assumed that all $X_i(t) + t$ are compound Poisson processes, there are no jumps at the switching epochs, and the service times are light-tailed, that is $F(\alpha)$ is analytic in the neighborhood of 0. We anticipate that our results may open the possibility of relaxing some of these assumptions.

3.3. First passage process

The first passage time over level $-x$ is defined through

$$\tau_x = \inf\{t \geq 0 : X(t) < -x\}, \tag{7}$$

where $x \geq 0$. It is known that the property (1) still holds when choosing $T = \tau_x$. In the absence of negative jumps this property implies that the first passage process $(\tau_x, J(\tau_x))$ is a MAP itself. Hence there exists a matrix-valued function $\Lambda(q)$ characterizing the law of the first passage process:

$$\mathbb{E}[\mathrm{e}^{-q\tau_x} \mathbb{1}_{\{J(\tau_x)=j\}} \mid J(\tau_0) = i] = \left(\mathrm{e}^{\Lambda(q)x}\right)_{ij}.$$

Compare this to the setting of Lévy processes, where τ_x is known to be a Lévy process itself. It should be noted that $J(\tau_x)$ never jumps to a state j if $X_j(t)$ is a subordinator, because in these states a new minimum cannot be attained, so that $\Lambda(q)$ is a $N^* \times N^*$-dimensional matrix. It was observed before that the knowledge of the matrix exponent $\Lambda(q)$ of the first passage process is of crucial importance in fluctuation theory for MAPs [17]; as a consequence, it also plays a pivotal role when analyzing storage systems with MAP input [22,24], and financial models in a Markov modulated environment [4].

Using our main results we can determine $\Lambda(q)$ under minor assumptions. It is noted that the well-known Wald martingale can be generalized to the MAP setting. Namely, it takes the following form: $\mathrm{e}^{-\lambda X(t)} v_{J(t)}$, where $(\lambda, \mathbf{v})$ are such that $F(\lambda)\mathbf{v} = \mathbf{0}$, see for example [2, Prop. XI.2.4]. Applying the optional stopping theorem to the killed version of the Wald martingale with $\tau = \tau_x \wedge t$ yields

$$\mathbb{E}_i[\mathrm{e}^{-\lambda X(\tau)} \mathbb{1}_{\{\tau < e_q\}} v_{J(\tau)}] = v_i,$$

where λ and $\mathbf{v}$ are such that $\mathrm{Re}(\lambda) > 0$ and $(F(\lambda) - q\mathrm{I})\mathbf{v} = \mathbf{0}$. The dominated convergence theorem applies since $X(\tau) \geq -x$ for all $t \geq 0$. We thus obtain

$$\mathrm{e}^{\lambda x} \sum_j \mathbb{P}_i(\tau_x < e_q, J(\tau_x) = j) v_j = v_i.$$

By noting that $\mathbb{P}_i(\tau_0 = 0)$ if i corresponds to a non-subordinator, we get

$$\mathrm{e}^{\Lambda(q)x}\mathbf{v}^* = \mathrm{e}^{-\lambda x}\mathbf{v}^*,$$

where $\mathbf{v}^*$ is a vector of N^* elements of $\mathbf{v}$ corresponding to non-subordinators. Differentiate the above equation at $x = 0$ to find that $\Lambda(q)\mathbf{v}^* = -\lambda\mathbf{v}^*$. But Theorem 1 states that there are exactly N^* possible choices of $(\lambda, \mathbf{v})$ assuming that all the zeros of $\det(F(\alpha) - q\mathrm{I})$ in $\mathbb{C}^{\mathrm{Re}>0}$ are distinct. Thus under this assumption we can find all the eigenvalues and eigenvectors of $\Lambda(q)$, which is enough to determine $\Lambda(q)$ through its Jordan normal form. The case of $q = 0$ is slightly more subtle. In this case a similar conclusion can be reached using Theorem 2 under the additional assumption that $k'(0^+)$ is finite and non-zero.

The assumption of zeros being distinct is often made in the literature, see e.g., [3,23]. Actually, it is enough to assume that the zeros are semi-simple: multiplicities of the zeros coincide with the ranks of corresponding null spaces. Even this weaker assumption does not always hold as demonstrated in Example 2.1. If, however, one considers time-reversible MAPs, then the latter assumption does hold and, moreover, the zeros are all real as shown in [13]. One can see that the above method of determining $\Lambda(q)$ will certainly fail if there exists an eigenvalue of $\Lambda(q)$ whose algebraic multiplicity is strictly larger than its geometric multiplicity. What are the objects associated to $F(\alpha) - q\mathrm{I}$ leading to such eigenvalues and the corresponding Jordan chains? It turns out that one has to resort to the theory of analytic matrix functions and generalized Jordan chains. In [9] we present this general theory and solve a number of open questions. It is noted that Theorem 1 plays a crucial role in the underlying argumentation.

3.4. Martingale calculations

Various problems related to MAPs can be approached by the use of the Wald martingale, see [2, Section XI.4a] and the above Section 3.3. It is essential for this approach that the number of zeros of $\det(F(\alpha) - q\mathrm{I})$ is sufficiently large. Moreover, the celebrated Kella–Whitt martingale can be generalized to the MAP setting too, see [5], where also many applications of this martingale are discussed. Most of these results require identification of some unknown constants, which can be done by solving a system of linear equations. There is an open problem however: one needs to show that the number of equations is sufficient, where this number is closely related to the number of zeros of $\det(F(\alpha) - q\mathrm{I})$ in $\mathbb{C}^{\mathrm{Re}>0}$. Our main result can be seen as an important step towards an answer to these problems.

Let us provide an example. We consider an arbitrary MAP without negative jumps and its reflection at 0 as in [5, Section 4]. Let (W, J) be a random vector distributed as the stationary version of the reflected process. The pair (W, J) characterizes the steady-state buffer content of a queue driven by the underlying MAP. For stability one has to require that $(X(t), J(t))$ has a negative asymptotic drift, which is the same as $k'(0^+) > 0$, see (6). In [5] it is shown that the Laplace–Stieltjes transform of (W, J) can be expressed in terms of $F(\alpha)$ and a generally unknown row vector $\boldsymbol{\ell}$. More precisely,

$$(\mathbb{E}\mathrm{e}^{-\alpha W}\mathbb{1}_{\{J=1\}}, \ldots, \mathbb{E}\mathrm{e}^{-\alpha W}\mathbb{1}_{\{J=N\}}) = \alpha\boldsymbol{\ell}F(\alpha)^{-1}, \tag{8}$$

where $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}$. The authors observed that the computation of the unknown vector $\boldsymbol{\ell}$ in general is a difficult problem, and proposed to set up a system of linear equations of the type $\boldsymbol{\ell}\mathbf{v}_i = 0$, where $F(\lambda_i)\mathbf{v}_i = \mathbf{0}$ and $\lambda_i \in \mathbb{C}^{\mathrm{Re}>0}$. In addition, they noted that firstly $\ell_i = 0$ if $X_i(t)$ is a subordinator, and secondly $\boldsymbol{\ell}\mathbf{1} = k'(0^+)$, where $\mathbf{1}$ is a vector of 1-s. Hence the unknown constants can be identified if there exist $N^* - 1$ vectors $\mathbf{v}_i$, and moreover their restrictions $\mathbf{v}_i^*$ and in addition $\mathbf{1}^*$ are linearly independent. But our Theorem 2 shows that there are exactly $N^* - 1$ zeros of $\det(F(\alpha))$ in $\mathbb{C}^{\mathrm{Re}>0}$. Hence if these zeros are semi-simple then the number of equations is sufficient. Finally, the linear independence follows from the fact that $\mathbf{v}_i^*$-s and $\mathbf{1}^*$ are the eigenvectors of $\Lambda(0)$ as hinted in Section 3.3.

The above list of applications, where the number of zeros of $\det(F(\alpha) - q\mathrm{I})$ is required, is far from complete. Knowledge of the number of zeros is essential in the analysis of a Markov-modulated risk model, see e.g., equation (10) in [18]. Another example is a Markov-modulated feedforward network in [15], especially if one is interested in replacing fluid input by a more general Lévy process. We conclude by mentioning [6], see in particular Thm. 5.2, where loss rates for MAPs with two reflecting barriers are computed assuming that the number of zeros is sufficient.

4. On the number of zeros of certain functions

This section presents two general results on the number of zeros of certain functions (that is, functions satisfying a given set of assumptions) in a bounded domain. We would like to stress that we rely in this section on techniques that were developed earlier. To enhance the paper's transparency, we have isolated these results from the rest of the paper; for the sake of completeness their proofs are given in Appendix A.

In the following we assume that

$$\begin{aligned} &D \subset \mathbb{C} \text{ is a bounded domain with boundary } \gamma,\\ &\qquad \text{where } \gamma \text{ is a piecewise smooth simple loop.}\end{aligned} \tag{9}$$

One can find the basic notions of complex analysis in, e.g., [12]. We use $B(z, r)$ to denote an open ball of radius $r > 0$ centered at a point $z \in \mathbb{C}$.

The first theorem concerns the number of zeros of the determinant of a matrix-valued function in a bounded domain.

Theorem 3. *Let $M(z) = (m_{ij}(z))$ be a $n \times n$-matrix-valued function and $f(z) := \det(M(z))$. If*

A1 *$m_{ij}(z)$ are analytic on D and continuous on $D \cup \gamma$,*
A2 *$\forall i \in \{1, \ldots, n\}, z \in \gamma : \sum_{j\neq i} |m_{ij}(z)| \leq |m_{ii}(z)| \neq 0$,*
A3 *$f(z) \neq 0$ for $z \in \gamma$,*

then $f(z)$ and $\prod_{i=1}^{n} m_{ii}(z)$ have the same number of zeros in D.

Proof. See Appendix A. □

The main idea of the proof of Theorem 3 is taken from [11], where the authors use the following procedure. First they introduce an additional parameter t; the original function is retrieved by taking $t = 1$. For $t = 0$, however, the function has a nice form (that is, it nicely factorizes) making the analysis of the number of zeros easy. Then essentially continuity arguments are used to conclude that the number of zeros, as a function of the new parameter t, is constant. This basic idea used in a related context can be also found in [8,25].

It is noted that Theorem 3 does not allow $f(z)$ to be zero on the boundary of the domain. The analysis of the number of zeros becomes substantially harder if this assumption does not hold. In case of a simple zero on the boundary the following powerful result may be used.

Theorem 4 shows that if a function of interest and a given sequence of 'approximating' functions satisfy certain assumptions, then the functions in the tail of the sequence have the same number of zeros as the original function. This turns out to be useful in situations where the approximating functions have particular crucial properties (such as being analytic) which the original function does not necessarily have.

Theorem 4. *Let complex functions $f(z), f_n(z), n \in \mathbb{N}$ satisfy the following assumptions for some $z_0 \in \gamma$:*

A1 $f(z), f_n(z), n \in \mathbb{N}$ *are analytic on D and continuous on $D \cup \gamma$,*
A2 $f_n(z) \to f(z)$ *and* $f_n'(z) \to f'(z)$ *as $n \to \infty$ uniformly in $z \in D$,*
A3 $f(z_0) = f_1(z_0) = f_2(z_0) = \cdots = 0$ *and* $f(z) \neq 0, z \in \gamma \setminus \{z_0\}$,
A4 $\exists \epsilon > 0$, *such that $f_n(z), n \in \mathbb{N}$ are analytic on $B(z_0, \epsilon)$,*

$$f'(z_0) := \lim_{z \to z_0, z \in D} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, is non-zero and coincides with $\lim_{n\to\infty} f_n'(z_0)$.

Then for large enough n, the functions $f_n(z)$ are non-zero on $\gamma \setminus \{z_0\}$ and have the same number of zeros in D as the function $f(z)$.

Proof. The proof of this result relies on a technical argument borrowed from [1] and is given in Appendix A. □

5. Analytic properties of the Laplace exponent

In this section we discuss some analytic properties of the Laplace exponent of a Lévy process without negative jumps. These properties will turn out to be crucial in the analysis of the zeros of $\det(F(\alpha))$. Throughout this section we assume that $X(t)$ is a Lévy process without negative jumps, $(\alpha, \sigma, \nu(\mathrm{d}x))$ is the associated Lévy triple, and $\phi(\alpha)$ is the Laplace exponent of $X(t)$, cf. (2).

We start by recalling a number of well-known facts about Lévy processes, see [7] or [16] for a general reference. It is well known that $\phi(\alpha)$ is finite on $\mathbb{C}^{\mathrm{Re}\geq 0}$. Due to dominated convergence, the derivative of $\phi(\alpha), \alpha \in \mathbb{C}^{\mathrm{Re}>0}$ can be computed by interchanging the differentiation and integration operators when using representation (2). It then follows easily that $\phi(\alpha)$ is analytic on $\mathbb{C}^{\mathrm{Re}>0}$. If it is additionally assumed that the jumps of $X(t)$ are bounded by a constant, then similar arguments show that $\phi(\alpha)$ is analytic on $\mathbb{C}$. These facts can be found in [16].

The following two lemmas play an important role in the analysis of the zeros of $\det(F(\alpha))$. Their proofs do not provide much intuition and hence are given in Appendix B.

Lemma 5. *At least one of the following holds:*

(i) $\mathrm{Re}(\phi(\alpha)) \leq 0$ *for all* $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}$,
(ii) $\lim_{|\alpha|\to\infty, \alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}} |\phi(\alpha)| = \infty$.

Lemma 6. *For $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0} \setminus \mathbb{R}$ it holds that $\phi(\alpha) \notin (0, \infty)$ and, in addition, $\phi(\alpha) \neq 0$ if $X(t)$ is not a compound Poisson process.*

We finish this section with a simple lemma.

Lemma 7. *For any $c > 0$ function $\phi(\alpha) - c$ has no zeros in $\mathbb{C}^{\mathrm{Re}\geq 0}$ if $X(t)$ is a subordinator, and has a unique simple zero otherwise.*

Proof. Lemma 6 shows that $\phi(\alpha) \neq c$ for $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0} \setminus \mathbb{R}$. It remains to analyze the case when $\alpha \geq 0$. It is well known that $\phi(0) = 0$ and $\phi(\alpha)$, $\alpha \geq 0$ is convex. The claim then follows from another well-known fact, viz. that $\phi(\alpha) \leq 0$ if $X(t)$ is a subordinator and $\lim_{\alpha\to\infty} \phi(\alpha) = \infty$ otherwise. $\square$

From Lemma 7 we see that a special role is played by *subordinators*, which was to be expected in view of Theorem 2.

6. Proofs of the main results

The primary goal of this section is to prove our main results, viz. Theorems 1 and 2. Throughout this section we assume that $(X(t), J(t))$ is a MAP without negative jumps, and $F(\alpha)$ is the associated matrix exponent as defined in (3). In the following we extensively use a bounded domain D_R, defined through

$$D_R := \{\alpha \in \mathbb{C} : \mathrm{Re}(\alpha) > 0, |\alpha| < R\}. \tag{10}$$

Note that this domain satisfies (9). Furthermore, recall that a square $n \times n$ matrix $M = (m_{ij})$ is called *non-strictly diagonally dominant* if $\forall i : |m_{ii}| \geq \sum_{j\neq i} |m_{ij}|$. If, moreover, M is irreducible and at least one of the above inequalities is strict then M is called *irreducibly diagonally dominant*. It is well known that an irreducibly diagonally dominant matrix is non-singular, see for instance p. 226 of [20].

The following lemma is a key result on the way to prove the main theorems. It allows us to restrict our attention to a bounded domain D_R instead of considering the whole $\mathbb{C}^{\mathrm{Re}\geq 0}$. Note that this is an essential prerequisite required by Theorems 3 and 4.

Lemma 8. *Let $C > 0$ then there exists $R > 0$ such that for any $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0} \setminus D_R$ and any $c_1, \ldots, c_N \in \mathbb{C}^{\mathrm{Re}>0} \cup \{0\}$, such that $c_i < C$ for all i, the following holds true. If $\mathbf{c} \neq \mathbf{0}$, or $N^* > 0$ and $\alpha \neq 0$, then the matrix*

$$Q + \mathrm{diag}(\phi_1(\alpha), \ldots, \phi_N(\alpha)) - \mathrm{diag}(\mathbf{c})$$

is irreducibly diagonally dominant.

Proof. Choose $i \in \{1, \ldots, N\}$ and note that

$$\mathrm{e}^{\mathrm{Re}(\phi_i(\mathrm{i}r))} = |\mathrm{e}^{\phi_i(\mathrm{i}r)}| = |\mathbb{E}\mathrm{e}^{-\mathrm{i}rX_i(1)}| \leq 1, \quad r \in \mathbb{R}.$$

Therefore, $\mathrm{Re}(\phi_i(\alpha)) \leq 0$ for all α on the imaginary axis. This statement and Lemma 5 imply that there exists $R_i > 0$, such that, for all $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0} \setminus D_{R_i}$ and $c_i < C$ it holds that

$$|q_{ii} + \phi_i(\alpha) - c_i| > -q_{ii} \quad \text{or} \quad \mathrm{Re}(\phi_i(\alpha)) \leq 0, \tag{11}$$

where the $q_{ii} = -\sum_{j\neq i} q_{ij} < 0$ $(i = 1, \ldots, N)$ are the diagonal elements of the transition rate matrix Q. But $\mathrm{Re}(\phi_i(\alpha)) \leq 0$ implies $|q_{ii} + \phi_i(\alpha) - c_i| \geq -q_{ii}$, because $\mathrm{Re}(c_i) \geq 0$. Note that this inequality is strict unless $c_i = \phi_i(\alpha) = 0$, because c_i takes values in $\mathbb{C}^{\mathrm{Re}>0} \cup \{0\}$. We see that for $R = \max\{R_1, \ldots, R_N\}$ our matrix is non-strictly diagonally dominant.

Assume for a moment that $\forall i : |q_{ii} + \phi_i(\alpha) - c_i| = -q_{ii}$, then $c_i = 0$ and $\phi_i(\alpha) = 0$ for all i. To finish the proof, it is enough to show that $N^* > 0$ and $\alpha \neq 0$ imply that $\phi_i(\alpha) \neq 0$ for some i. Take i, such that $X_i(t)$ is not a subordinator, which is possible due to $N^* > 0$. Then $\phi_i(\alpha) \neq 0$ for all $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0} \setminus \mathbb{R}$ by Lemma 6 (ii). Considering $\phi_i(r), r \in \mathbb{R}^+$, we note that $\lim_{r\to\infty} \phi_i(r) = \infty$, thus $\phi_i(r)$ has no zeros larger than some constant C_i. Clearly, we were initially able to choose $R > C_i$. Hence $\phi_i(\alpha) \neq 0$ for $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0} \setminus (D_R \cup \{0\})$, which concludes the proof. $\square$

Note that if matrix $Q + \mathrm{diag}(\phi_1(\alpha), \ldots, \phi_N(\alpha)) - \mathrm{diag}(\mathbf{c})$ is irreducibly diagonally dominant, then so is

$$Q \circ \tilde{G}(\alpha) + \mathrm{diag}(\phi_1(\alpha), \ldots, \phi_N(\alpha)) - \mathrm{diag}(\mathbf{c}),$$

because $0 < |\tilde{G}_{ij}(\alpha)| \leq 1$ and $\tilde{G}_{ii}(\alpha) = 1$ for $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}$. Moreover, it is easy to see from the above proof that $\det(F(\alpha)) \equiv 0$ on $\mathbb{C}^{\mathrm{Re}>0}$ if and only if $\forall i, j : \phi_i(\alpha) \equiv 0$ and $\tilde{G}_{ij}(\alpha) \equiv 1$, which is the same as $X(t) \equiv 0$. Furthermore, it is a trivial consequence of the above lemma that $F(\alpha)$ is non-singular for all α on the imaginary axis except $\alpha = 0$, whenever $N^* > 0$. On the other hand, a simple non-degenerate example of $X(t)$ can be constructed with $N^* = 0$, such that $F(\alpha)$ is singular at infinitely many points on the imaginary axis (let $X_i(t), i \in \{1, \ldots, N\}$ be Poisson processes and set $U_{ij} \equiv 0$).

In the remainder of this section we distinguish between two cases: killing is present (Section 6.1, containing the proof of Theorem 1) and no killing (Section 6.2, containing the proof of Theorem 2).

6.1. Killing is present

We are ready to prove our first main result, Theorem 1. The statement of the theorem is an immediate consequence of Lemma 8, Theorem 3 and Lemma 7.

Proof of Theorem 1. Apply Lemma 8 to see that there exists $R > 0$, such that $F(\alpha) - q\mathrm{I}$ is irreducibly diagonally dominant (and thus non-singular) for $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0} \setminus D_R$, because $q > 0$. Now we can apply Theorem 3 to show that $\det(F(\alpha) - q\mathrm{I})$ and $\prod_{i=1}^N (q_{ii} + \phi_i(\alpha) - q)$ have the same number of zeros in D_R, and no zeros in $\mathbb{C}^{\mathrm{Re}\geq 0} \setminus D_R$, because of diagonal dominance. Conclude by noting that the latter function has exactly N^* zeros in D_R according to the statement of Lemma 7. $\square$

Next we study the limiting behavior of the zeros of $\det(F(\alpha) - q\mathrm{I})$ in $\mathbb{C}^{\mathrm{Re}>0}$ as the killing rate $q > 0$ converges to 0. This is an important step in the analysis of the case of no killing.

Theorem 9. *If $N^* > 0$ then the zeros of $\det(F(\alpha) - q_n\mathrm{I})$ in $\mathbb{C}^{\mathrm{Re}>0}$ converge as $q_n \downarrow 0$ to some limit points $z_1, \ldots, z_{N^*} \in \mathbb{C}^{\mathrm{Re}>0} \cup \{0\}$ (not necessarily distinct). The set*

$$Z := \bigcup_{i=1}^{N^*} \{z_i\} \cup \{0\}$$

is the set of all the distinct zeros of $\det(F(\alpha))$ in $\mathbb{C}^{\mathrm{Re}\geq 0}$, and the multiplicity of a zero $z \in Z, z \neq 0$ is given by the number of zeros of $\det(F(\alpha) - q_n\mathrm{I})$ converging to z.

Proof. Let Z_0 be the set of all the distinct zeros of $\det(F(\alpha))$ in $\mathbb{C}^{\mathrm{Re}\geq 0}$. Recall that $\det(F(\alpha))$ is *not* identically zero, because $N^* > 0$. Now Hurwitz's theorem [12, p. 173] shows that every zero of $\det(F(\alpha))$ in $\mathbb{C}^{\mathrm{Re}>0}$ (analyticity region) of multiplicity m is a limit point of exactly m zeros of $\det(F(\alpha)-q_n\mathrm{I})$. Recall also that $\det(F(\alpha))$ has a unique zero on the imaginary axis, which is at 0. Clearly, Z_0 can have at most finitely many elements, so it remains to show that for sufficiently large n_0 the zeros of $\det(F(\alpha)-q_n\mathrm{I})$, for $n > n_0$, are arbitrarily close to the elements of Z_0.

Suppose the latter claim is not true. So we can pick a sequence of the zeros which are at least ϵ away from the elements of Z_0. Apply Lemma 8 to see that we can choose $R > 0$, such that the zeros of $\det(F(\alpha)-q_n\mathrm{I})$, $n \in \mathbb{N}$ in $\mathbb{C}^{\mathrm{Re}>0}$ are all in D_R. Thus, given the above sequence of zeros, we can choose a converging subsequence (D_R is bounded) with some limit z_0. Clearly, $\det(F(z_0)) = 0$ and $z_0 \in \mathbb{C}^{\mathrm{Re}\geq 0}$ which means that $z_0 \in Z_0$, and, thus, the above sequence cannot exist. $\square$

It is easy to see that the above proof also shows that if $N^* = 0$ then either $X(t)$ is degenerate or $F(\alpha), \alpha \in \mathbb{C}^{\mathrm{Re}>0}$ is non-singular. Considering the question about the number of zeros of $\det(F(\alpha))$, we note that there is essentially one thing left unknown: the number of zeros which converge to 0 as the killing rate goes to 0. We address this seemingly simple question in what follows.

6.2. *No killing*

We now concentrate on the proof of Theorem 2. The statement of Theorem 2 shows that a critical role is played by the sign of the asymptotic drift. The next lemma presents a relation between the sign of the asymptotic drift and the sign of $\det(F(0^+))'$, the right-sided derivative of $\det(F(r)), r \geq 0$ at 0.

Lemma 10. *It holds that*

$$\mathrm{sign}(k'(0^+)) = (-1)^{N-1}\mathrm{sign}(\det(F(0^+))'). \tag{12}$$

Proof. Let $\lambda_1(\alpha), \ldots, \lambda_{N-1}(\alpha), \lambda_N(\alpha) = k(\alpha)$ be the eigenvalues of $F(\alpha)$, then $\det(F(\alpha)) = \prod_{i=1}^N \lambda_i(\alpha)$. So we have

$$\det(F(0^+))' = k'(0^+)\prod_{i=1}^{N-1}\lambda_i(0),$$

because $k(0) = 0$. Hence it is enough to show that $\prod_{i=1}^{N-1}(-\lambda_i(0)) > 0$.

Take any $i < N$ and set $\lambda = \lambda_i(0)$. If λ is real then it is negative, since $k(0) = 0$ is a simple eigenvalue with the maximal real part. If, however, λ has a non-zero imaginary part and is of multiplicity m, then there is an eigenvalue $\bar{\lambda}$ (complex conjugate of λ) of multiplicity m. The product of these $2m$ eigenvalues is a positive number. $\square$

The next lemma specifies the number of zeros of $\det(F(\alpha))$ in $\mathbb{C}^{\mathrm{Re}>0}$ under the additional assumption of analyticity.

Lemma 11. *Let $N^* > 0$ and $k'(0^+) \neq 0$. If the function $\det(F(\alpha))$ is analytic in some open neighborhood of 0, then it has $N^* - \mathbb{1}_{\{k'(0^+)\geq 0\}}$ zeros in $\mathbb{C}^{\mathrm{Re}>0}$.*

Proof. Consider the setting of Theorem 9. In view of this result, we only need to show the following: (A) if $k'(0^+) > 0$ then exactly one zero out of the N^* zeros of $\det(F(\alpha)-q_n\mathrm{I})$ in $\mathbb{C}^{\mathrm{Re}>0}$ converges to 0, and (B) if $k'(0^+) < 0$ then none of these zeros converges to 0. Using Lemma 10 we note that $k'(0^+) \neq 0$ implies $\det(F(0))' \neq 0$, so the multiplicity of the zero of $\det(F(\alpha))$ at 0 is 1. The assumption of analyticity in the neighborhood of 0 allows us to apply

Hurwitz's theorem to show that there is exactly one zero of $\det(F(\alpha) - q_n\mathrm{I})$ converging to 0. Note that this zero either converges from $\mathbb{C}^{\mathrm{Re}>0}$ or from $\mathbb{C}^{\mathrm{Re}<0}$. So it remains to show that the first case corresponds to $k'(0^+) > 0$ and the second to $k'(0^+) < 0$. Before we proceed we note that $(-1)^N \det(F(0) - q_n\mathrm{I}) > 0$, which follows by an argument similar to the one appearing in the proof of Lemma 10.

We restrict ourselves to the domain of reals and assume without loss of generality that $k'(0^+) > 0$. So $(-1)^{N-1}\det(F(0))' > 0$ by Lemma 10. Now for any small $\delta > 0$ we can pick $x \in (0, \delta)$, such that $(-1)^{N-1}\det(F(x)) > 0$. Hence for large enough n the inequality $(-1)^{N-1}\det(F(x) - q_n\mathrm{I}) > 0$ holds. This means that $\det(F(x) - q_n\mathrm{I})$ and $\det(F(0) - q_n\mathrm{I})$ have opposite signs, thus by continuity there exists $x_n \in (0, x)$, such that, $\det(F(x_n) - q_n\mathrm{I}) = 0$. This concludes the proof. $\square$

It is noted that the second paragraph of the above proof uses an idea from Prop. 9 of [11].

Now we outline the proof of Theorem 2. We start by constructing a sequence of functions, which approximates $\det(F(\alpha))$. Then Lemma 8 is applied to bound the region of zeros of the above functions. Next, using Theorem 4, we relate the number of zeros of $\det(F(\alpha))$ to the number of zeros of an approximating function from the tail of the sequence. Finally, due to the enlarged region of analyticity of the approximating functions, Lemma 11 can be applied to obtain the latter number.

In order to implement the above ideas, we introduce a sequence of 'truncations' of $(X(t), J(t))$. For every $n \in \mathbb{N}$ define a MAP $(X^{[n]}(t), J(t))$ through

$$\nu_i^{[n]}(\mathrm{d}x) := \mathbb{1}_{\{x \le n\}}\nu_i(\mathrm{d}x) \quad \text{and} \quad U_{ij}^{[n]} := U_{ij}\mathbb{1}_{\{U_{ij} \le n\}}, \tag{13}$$

where the other characteristics are kept unchanged. Using self-evident notation, we note that $\tilde{G}_{ij}^{[n]}(\alpha)$, $\phi_i^{[n]}(\alpha)$, and thus $\det(F^{[n]}(\alpha))$ are analytic on $\mathbb{C}$ (see the introduction to Section 5). Next we consider a sequence of functions $\det(F^{[n]}(\alpha))$ and prove some convergence results required by Theorem 4. In the following lemma we implicitly assume that the derivative of any function $f(\alpha)$ at a point α_0 on the imaginary axis is understood in the following sense: $\lim_{h\to 0, h\in\mathbb{C}^{\mathrm{Re}\ge 0}}(f(\alpha_0 + h) - f(\alpha_0))/h$. It is noted that $f(\alpha)$ may be infinite for all $\alpha \in \mathbb{C}^{\mathrm{Re}<0}$, and yet $f'(\alpha_0)$ is well-defined and finite.

Lemma 12. *If $\mathbb{E}X_i(1)$ and $\mathbb{E}U_{ij}$ exist for all i and j, then for any $R > 0$ it holds that*

$$\det(F^{[n]}(\alpha)) \to \det(F(\alpha)) \quad \text{and} \quad \det(F^{[n]}(\alpha))' \to \det(F(\alpha))' \tag{14}$$

as $n \to \infty$ uniformly in $\alpha \in D_R$. Moreover,

$$\lim_{n\to\infty} \det(F^{[n]}(0))' = \det(F(0))' \in (-\infty, \infty). \tag{15}$$

Proof. The statements of the lemma follow immediately from the following two observations: (A) $\phi_i^{[n]}(\alpha)$, $\tilde{G}_{ij}^{[n]}(\alpha)$ as well as their derivatives converge to the corresponding 'non-truncated' functions as $n \to \infty$ uniformly in $\alpha \in \mathbb{C}^{\mathrm{Re}\ge 0}$, and (B) $|\phi_i(\alpha)|$, $|\phi_i'(\alpha)|$, $|\tilde{G}_{ij}(\alpha)|$ and $|\tilde{G}_{ij}'(\alpha)|$ are bounded on D_R. Statement (B) follows from (A) and the fact that the corresponding truncated functions are bounded for every n, which is true, because D_R is bounded and functions $\phi_i^{[n]}(\alpha)$ and $\tilde{G}_{ij}^{[n]}(\alpha)$ are analytic on $\mathbb{C}$.

With regard to statement (A) we only show uniform convergence of the derivatives of $\phi_i^{[n]}(\alpha)$, because the other results are either trivial or follow by a similar argument. That is we show that

$\Delta_n(\alpha) := |\partial\phi_i^{[n]}(\alpha)/\partial\alpha - \partial\phi_i(\alpha)/\partial\alpha| \to 0$ as $n \to \infty$ uniformly in $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}$. Recall that $\mathbb{E}X_i(1) < \infty$ implies $\int_1^\infty x\nu_i(\mathrm{d}x) < \infty$. Now use dominated convergence to see that

$$\Delta_n(\alpha) = \left|\frac{\partial \int_n^\infty (-1+\mathrm{e}^{-\alpha x})\nu_i(\mathrm{d}x)}{\partial\alpha}\right| = \left|\int_n^\infty x\mathrm{e}^{-\alpha x}\nu_i(\mathrm{d}x)\right| \\ \leq \int_n^\infty x\nu_i(\mathrm{d}x),$$

which goes to 0 as $n \to \infty$. □

It is not difficult to show using (6) that $k'(0^+) \in [-\infty, \infty)$ and, moreover,

$$k'(0^+) \text{ is finite if and only if } \forall i, j : \mathbb{E}X_i(1) \text{ and } \mathbb{E}U_{ij} \text{ exist.} \tag{16}$$

Hence the above lemma can be applied whenever $k'(0^+) \neq -\infty$.

We are now ready to prove Theorem 2. In this proof we use $X^{[\infty]}(t)$ to denote the process $X(t)$.

Proof of Theorem 2. Note that for all $n \in \mathbb{N} \cup \{\infty\}$ it holds that

$$\phi_i^{[n]}(\alpha) = \phi_i^{[1]}(\alpha) - \left(-\int_1^n (-1+\mathrm{e}^{-\alpha x})\nu_i(\mathrm{d}x)\right) = \phi_i^{[1]}(\alpha) - g_i^n(\alpha),$$

where $g_i^n(\alpha) := -\int_1^n(-1+\mathrm{e}^{-\alpha x})\nu_i(\mathrm{d}x)$. It is an easy exercise to show that for all $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}$ functions $g_i^n(\alpha)$ take values in $\mathbb{C}^{\mathrm{Re}>0} \cup \{0\}$ and are bounded in absolute value by a common constant $C = 2\max_i\{\nu_i(1,\infty)\}$. So we can apply Lemma 8 to the MAP $(X^{[1]}(t), J(t))$ to show that there exists $R > 0$, such that the matrices $Q + \mathrm{diag}(\phi_1^{[n]}(\alpha), \ldots, \phi_N^{[n]}(\alpha))$ are irreducibly diagonally dominant for all $n \in \mathbb{N} \cup \{\infty\}$ and all $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0} \setminus (D_R \cup \{0\})$. Hence the zeros of $\det(F^{[n]}(\alpha))$, $n \in \mathbb{N} \cup \{\infty\}$ in $\mathbb{C}^{\mathrm{Re}\geq 0}$ are all in $D_R \cup \{0\}$. Now use (16) and Lemma 12 to see that Theorem 4 applies. So it remains to analyze the number of zeros of $\det(F^{[n]}(\alpha))$ in $\mathbb{C}^{\mathrm{Re}\geq 0}$ for a large n.

First note that $X_i(t)$ is a subordinator if and only if $X_i^{[n]}(t)$ is a subordinator. Thus the number of non-subordinators corresponding to any truncated MAP is N^*. Secondly, Lemmas 12 and 10 show that $k^{[n]\prime}(0^+)$ has the same sign as $k'(0^+)$ for n large enough. Now Lemma 11 completes the proof. □

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Appendix A. Proofs of the results from Section 4

Proof of Theorem 3. Define $f(z,t) := \det(M_t(z))$ for $t \in [0,1]$, where $M_t(z)$ is a $n \times n$ matrix obtained from $M(z)$ by multiplying the off-diagonal elements by t. Note that $f(z,0) = \prod_{i=1}^n m_{ii}(z)$ and $f(z,1) = f(z)$. Moreover, $f(z,t) \neq 0$ for all $z \in \gamma$. To see this use assumption A3 when $t = 1$ and A2 when $t < 1$. In the second case $M_t(z), z \in \gamma$ is strictly diagonally dominant and thus non-singular, see p. 226 of [20]. Since $f(z,t)$ is a continuous function on $\overline{D} \times [0,1]$, one can choose $\delta > 0$, such that $f(z,t) \neq 0$ on $[0,1] \times E_\delta$, where

$E_\delta := \{z \in D : y \in \gamma, |z-y| < \delta\}$ is a boundary strip of D. This is true, because otherwise there exists a converging sequence of the zeros with a limit (z^*, t^*), such that $z^* \in \gamma$ and $f(z^*, t^*) = 0$.

Let n_t denote the number of zeros (counting multiplicities) of the function $f_t(z) := f(z,t)$ in D. Take some piecewise smooth simple loop $\gamma' \subset E_\delta$ (which is possible) and write using the argument principle

$$n_t = \frac{1}{2\pi \mathrm{i}} \oint_{\gamma'} \frac{f_t'(z)}{f_t(z)} \mathrm{d}z.$$

Note that n_t is integer-valued and continuous, because $f_t'(z)/f_t(z)$ is continuous in t uniformly in $z \in \gamma'$. This means that n_t is constant. $\square$

Proof of Theorem 4. We start by noting that there exists $\delta > 0$, such that $f(z) \neq 0$ on $E_\delta := \{z \in D : y \in \gamma, |z-y| \leq \delta\}$, because otherwise there would exist a converging sequence (z_n) in D with a limiting point $z^* \in \gamma$, such that $f(z_n) = 0$ for all n. But then $f(z^*) = 0$ and $\lim_{n\to\infty}(f(z^*) - f(z_n))/(z^* - z_n) = 0$, which contradicts the assumptions. Now take a piecewise smooth simple loop $\gamma' \subset E_\delta$ and write using the argument principle and the fact that $f_n{}'(z)/f_n(z)$ converges uniformly to $f'(z)/f(z)$ on γ':

$$k = \frac{1}{2\pi \mathrm{i}} \oint_{\gamma'} \frac{f'(z)}{f(z)} \mathrm{d}z = \lim_{n\to\infty} \frac{1}{2\pi \mathrm{i}} \oint_{\gamma'} \frac{f_n{}'(z)}{f_n(z)} \mathrm{d}z,$$

where k is the number of zeros of $f(z)$ inside γ'. Thus for a sufficiently large n the numbers of zeros of $f(z)$ and $f_n(z)$ inside γ' are the same.

It remains to show that $f_n(z)$ has neither zeros in E_δ, nor in $\gamma \setminus \{z_0\}$, for sufficiently large n. Uniform convergence $f_n'(z) \to f'(z), z \in D$ and continuity of $f_n'(z)$ on $D \cup \{z_0\}$ imply that $f'(z)$ is continuous on $D \cup \{z_0\}$, where $f'(z_0)$ is defined in the statement of the theorem. Now it is easy to see that one can pick $\eta > 0$, such that for a sufficiently small $\epsilon > 0$ and large n the following holds:

$$|f_n'(z) - f'(z_0)| < \frac{1}{2}\eta < \eta < |f'(z_0)|, \quad z \in D \cap B(z_0, \epsilon),$$

which implies

$$|f_n'(z) - f'(z_0)| \leq \frac{1}{2}\eta < \eta < |f'(z_0)|, \quad z \in \overline{D} \cap B(z_0, \epsilon).$$

Here we assume that ϵ is taken small enough, so that the $f_n(z)$ are analytic on $B(z_0, \epsilon)$. Note that for a sufficiently small $\epsilon > 0$ one can connect the points z_0 and $z \in \overline{D} \cap B(z_0, \epsilon)$ by a piecewise smooth path $\tilde{\gamma}$, so that $\tilde{\gamma} \subset \overline{D} \cap B(z_0, \epsilon)$ and the length of $\tilde{\gamma}$ is less than $2|z - z_0|$, because the contour of D is assumed to be piecewise smooth. Now

$$\begin{aligned} |f_n(z) - f'(z_0)(z - z_0)| &= \left| \int_{\tilde{\gamma}} (f_n{}'(s) - f'(z_0)) \mathrm{d}s \right| \\ &\leq 2|z - z_0| \max_{s \in \tilde{\gamma}} |f_n{}'(s) - f'(z_0)| \leq \eta |z - z_0| \end{aligned}$$

and

$$\begin{aligned} |f_n(z)| &\geq |f'(z_0)(z - z_0)| - |f_n(z) - f'(z_0)(z - z_0)| \\ &\geq (|f'(z_0)| - \eta)|z - z_0| > 0 \end{aligned}$$

for $z \in \overline{D} \cap B(z_0, \epsilon)$, $z \neq z_0$ and sufficiently large n.

Finally, consider the set $E' := (\gamma \cup E_\delta) \setminus B(z_0, \epsilon)$. The set E' is compact and $f(z) \neq 0$ on E', thus $f_n(z) \neq 0$ on E' for sufficiently large n, which completes the proof. $\square$

Appendix B. Proofs of the results from Section 5

In order to prove the two lemmas from Section 5 we need to discuss some additional properties of Lévy processes and to prove a technical lemma. It is known that

$$\begin{aligned} &X(t) \text{ has paths of bounded variation iff} \\ &\sigma = 0 \text{ and } \int_0^1 x\nu(\mathrm{d}x) < \infty. \end{aligned} \tag{B.1}$$

The Laplace exponent of such a process has a unique representation of the form

$$\phi(\alpha) = a'\alpha + \int_0^\infty (-1 + \mathrm{e}^{-\alpha x})\nu(\mathrm{d}x). \tag{B.2}$$

Note that any subordinator has paths of bounded variation, so it can be written in the form given in (B.2). If, in addition, $a' = 0$ then such a subordinator is called *pure jump subordinator*.

The first lemma is similar to Prop. 2 on p. 16 of [7], and will only be used to prove Lemma 5.

Lemma 13. *It holds that*

$$\lim_{|\alpha|\to\infty, \alpha\in\mathbb{C}^{\mathrm{Re}\geq 0}} \alpha^{-2}\phi(\alpha) = \sigma^2/2. \tag{B.3}$$

Moreover, if $\phi(\alpha)$ has representation (B.2) then

$$\lim_{|\alpha|\to\infty, \alpha\in\mathbb{C}^{\mathrm{Re}\geq 0}} \alpha^{-1}\phi(\alpha) = a'. \tag{B.4}$$

Proof. First note that

$$|-1 + \mathrm{e}^{-y} + y| \leq 3|y|^2 \quad \text{for } y \in \mathbb{C}^{\mathrm{Re}\geq 0}.$$

This inequality holds, because if $|y| \geq 1$ then $|-1 + \mathrm{e}^{-y} + y| \leq 2 + |y| \leq 3|y| \leq 3|y|^2$. On the other hand if $|y| < 1$ then using a power series expansion we have $|-1 + \mathrm{e}^{-y} + y| = |y^2/2! - y^3/3! + \cdots| \leq |y|^2(1/2! + |y|/3! + \cdots) \leq 3|y|^2$.

Now we see that $|\alpha|^{-2}|-1 + \mathrm{e}^{-\alpha x} + \alpha x| \leq 3x^2$ when $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}$, $\alpha \neq 0$ and $x > 0$. Since $\int_0^1 x^2\nu(\mathrm{d}x) < \infty$, dominated convergence gives

$$\lim_{|\alpha|\to\infty, \alpha\in\mathbb{C}^{\mathrm{Re}\geq 0}} \alpha^{-2} \int_0^1 (-1 + \mathrm{e}^{-\alpha x} + \alpha x)\nu(\mathrm{d}x) = 0$$

and then (B.3) follows from (2). The second part can be proven in the same way by noting that $|-1 + \mathrm{e}^{-y}| \leq 2|y|$ for $y \in \mathbb{C}^{\mathrm{Re}\geq 0}$. $\square$

Proof of Lemma 5. If the Gaussian component σ^2 (see (2)) is non-zero or $\phi(\alpha)$ can be written as in (B.2) with $a' \neq 0$, then the result follows trivially from Lemma 13. If, on the other hand, $\phi(\alpha) = \int_0^\infty (-1 + \mathrm{e}^{-\alpha x})\nu(\mathrm{d}x)$, then it is easy to see that $\mathrm{Re}(\phi(\alpha)) \leq 0$ for $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}$. It follows from (B.1) that the only case left is the following:

$$\phi(\alpha) = a\alpha + \int_0^\infty (-1 + \mathrm{e}^{-\alpha x} + \alpha x \mathbb{1}_{\{x<1\}})\nu(\mathrm{d}x),$$

where $\int_0^1 x\nu(\mathrm{d}x) = \infty$. We now show that in this case statement (i) holds. Note that $|\int_1^\infty(-1 + \mathrm{e}^{-\alpha x})\nu(\mathrm{d}x)|$ is bounded for all $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}$, so we can truncate Lévy measure $\nu(\mathrm{d}x)$ to the interval $(0, 1)$.

Step 1. We show that $\mathrm{Im}(\phi(u + \mathrm{i}v))/v \to \infty$ as $|v| \to \infty$ uniformly in $u \geq 0$. Note that

$$\mathrm{Im}(\phi(u + \mathrm{i}v)) = av + \int_0^1 (vx - \mathrm{e}^{-ux}\sin(vx))\nu(\mathrm{d}x)$$

is an odd function in v, thus it is enough to consider the case when $v > 0$. Note also that $vx - \mathrm{e}^{-ux}\sin(vx) \geq 0$ when $x > 0$. Thus we have for any $\epsilon > 0$

$$\begin{aligned}\frac{\mathrm{Im}(\phi(u + \mathrm{i}v))}{v} &\geq a + \int_\epsilon^1 \left(x - \frac{\mathrm{e}^{-ux}\sin(vx)}{v}\right)\nu(\mathrm{d}x)\\ &\geq a + \int_\epsilon^1 x\nu(\mathrm{d}x) - \int_\epsilon^1 \frac{1}{v}\nu(\mathrm{d}x) \to a + \int_\epsilon^1 x\nu(\mathrm{d}x) \quad \text{as } v \to \infty.\end{aligned}$$

Send ϵ to 0 and use $\int_0^1 x\nu(\mathrm{d}x) = \infty$ to complete the proof of the first step.

Step 2. We show that given any constants $M > 0$ and $V > 0$ one can choose a large $U > 0$, so that $\mathrm{Re}(\phi(u + \mathrm{i}v)) > M$ for all u and v such that $|v| \leq V$ and $u > U$. First recall that the process we consider has paths of unbounded variation and thus is not a subordinator. It is well known that in this case $\phi(u) \to \infty$ as $u \to \infty$. Next note that

$$\frac{\partial \mathrm{Re}(\phi(u + \mathrm{i}v))}{\partial v} = -\int_0^1 x\mathrm{e}^{-ux}\sin(vx)\nu(\mathrm{d}x)$$

and

$$\left|\int_0^1 x\mathrm{e}^{-ux}\sin(vx)\nu(\mathrm{d}x)\right| \leq V\int_0^1 x^2\nu(\mathrm{d}x) < \infty,$$

when $|v| \leq V$. So it is enough to choose U such that $\phi(u) > M + V^2\int_0^1 x^2\nu(\mathrm{d}x)$ for all $u > U$.

Now pick any $M > 0$. The result of Step 1 implies that there exists a large enough $V > 0$, so that $|\mathrm{Im}(\phi(u + \mathrm{i}v))| > M$ for all $u \geq 0$ and all v satisfying $|v| > V$. Combining this with the result of Step 2, we see that there exists $U > 0$, such that $|\phi(\alpha)| > M$ when $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}$ and $|\alpha| > U + V$, which implies (i). □

The above proof provides more information than stated in the lemma. Namely, we can add that the first statement is true at least for those $X(t)$ which are not pure jump subordinators. If $X(t)$ is a compound Poisson process then $|\phi(r)|$ is bounded for all $r \in [0, \infty)$, and thus the first statement of the above lemma does not hold.

Proof of Lemma 6. Let $u \geq 0$, $v \neq 0$ and assume that $\phi(u + \mathrm{i}v) \geq 0$, then

$$\begin{aligned}&au + \frac{1}{2}\sigma^2(u^2 - v^2) + \int_0^\infty (-1 + \mathrm{e}^{-ux}\cos(vx) + ux\mathbb{1}_{\{x<1\}})\nu(\mathrm{d}x) \geq 0,\\ &av + \sigma^2 uv + \int_0^\infty (-\mathrm{e}^{-ux}\sin(vx) + vx\mathbb{1}_{\{x<1\}})\nu(\mathrm{d}x) = 0.\end{aligned}$$

Divide the second equation by v, multiply it by u and subtract it from the first inequality to obtain:

$$\frac{1}{2}\sigma^2(-u^2 - v^2) + \int_0^\infty \left(\frac{u}{v}\mathrm{e}^{-ux}\sin(vx) - 1 + \mathrm{e}^{-ux}\cos(vx)\right)\nu(\mathrm{d}x) \geq 0.$$

Now note that

$$\cos r + \frac{q}{r}\sin r \le \mathrm{e}^{q} \quad \text{when } q \ge 0, r \ne 0$$

with equality when $q = 0$ and $\cos r = 1$. This shows that the integrand is non-positive, which proves (i).

Finally, from the above we conclude that $\phi(u + \mathrm{i}v) = 0$ if and only if either (A) $X(t) \equiv 0$, or (B) $\sigma^2 = 0$, $u = 0$, and

$$\int_0^\infty (1 - \cos(vx))\nu(\mathrm{d}x) = 0, \qquad av + \int_0^\infty (-\sin(vx) + vx\mathbb{1}_{\{x<1\}})\nu(\mathrm{d}x) = 0.$$

It can be further deduced that in the latter case $a = -\int_0^1 x\nu(\mathrm{d}x)$. Therefore we have that

$$\phi(\alpha) = \int_0^\infty (-1 + \mathrm{e}^{-\alpha x})\nu(\mathrm{d}x)$$

with $\int_0^1 x\nu(\mathrm{d}x) < \infty$, which means that $X(t)$ is a compound Poisson process. $\square$

Note that if $X(t)$ is not identically zero and for some $\alpha_0 \in \mathbb{C}^{\mathrm{Re}\ge 0} \setminus \mathbb{R}$ it holds that $\phi(\alpha_0) = 0$, then Lemma 6 implies that $X(t)$ is a compound Poisson process. Moreover, the above proof shows that α_0 lies on the imaginary axis.

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