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DOI
10.1137/130913687
Publication date
2014
Document Version
Final published version
Published in
SIAM Journal on Computing

SIAM J. COMPUT.
Vol. 43, No. 1, pp. 150–178

POSITION-BASED QUANTUM CRYPTOGRAPHY: IMPOSSIBILITY AND CONSTRUCTIONS*

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Abstract. In this work, we study position-based cryptography in the quantum setting. The aim is to use the geographical position of a party as its only credential. On the negative side, we show that if adversaries are allowed to share an arbitrarily large entangled quantum state, the task of secure position-verification is *impossible*. To this end, we prove the following very general result. Assume that Alice and Bob hold respectively subsystems A and B of a (possibly) unknown quantum state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. Their goal is to calculate and share a new state $|\varphi\rangle = U|\psi\rangle$, where U is a fixed unitary operation. The question that we ask is how many rounds of mutual communication are needed. It is easy to achieve such a task using two rounds of classical communication, whereas, in general, it is impossible with no communication at all. Surprisingly, in case Alice and Bob share enough entanglement to start with and we allow an arbitrarily small failure probability, we show that the same task can be done using a *single* round of classical communication in which Alice and Bob exchange two classical messages. Actually, we prove that a relaxed version of the task can be done with *no* communication at all, where the task is to compute instead a state $|\varphi'\rangle$ that coincides with $|\varphi\rangle = U|\psi\rangle$ up to local operations on A and on B, which are determined by classical information held by Alice and Bob. The one-round scheme for the original task then follows as a simple corollary. We also show that these results generalize to more players. As a consequence, we show a generic attack that breaks any position-verification scheme. On the positive side, we show that if adversaries do not share any entangled quantum state but can compute arbitrary quantum operations, secure position-verification is achievable. Jointly, these results suggest the interesting question whether secure position-verification is possible in case of a bounded amount of entanglement. Our positive result can be interpreted as resolving this question in the simplest case, where the bound is set to zero. In models where secure position-verification is achievable, it has a number of interesting applications. For example, it enables secure communication over an insecure channel without having any preshared key, with the guarantee that only a party at a specific location can learn the content of the conversation. More generally, we show that in settings where secure position-verification is achievable, other position-based cryptographic schemes are possible as well, such as secure position-based authentication and position-based key agreement.

*Received by the editors March 20, 2013; accepted for publication (in revised form) October 29, 2013; published electronically February 4, 2014. A preliminary version of this paper appeared in *Proceedings of the* 31*st Annual International Conference on Cryptology*, 2011 [10].

http://www.siam.org/journals/sicomp/43-1/91368.html

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Key words. position-based cryptography, quantum distributed computation, quantum key distribution

AMS subject classifications. 81P68, 94A60

DOI. 10.1137/130913687

1. Introduction.

1.1. Background. The goal of *position-based cryptography* is to use the geographical position of a party as its only "credential." For example, one would like to send a message to a party at a geographical position *pos* with the guarantee that the party can decrypt the message only if he or she is physically present at *pos*. Such a setting has plenty of real-life applications for wireless security, for instance, granting access to resources only when a user is present at a certain perimeter (e.g., inside a bank branch, or inside a military base). The general concept of position-based cryptography was introduced by Chandran et al. [16]; certain specific related tasks have been considered before under different names (see below and section 1.3).

A central task in position-based cryptography is the problem of *position-verification*. We have a *prover* P at position *pos* wishing to convince a set of *verifiers* $V_0, \ldots, V_k$ (at different points in geographical space) that P is indeed at that position *pos*. The prover can run an interactive protocol with the verifiers in order to convince them. The main technique for such a protocol is known as distance bounding [9]. In this technique, a verifier sends a random nonce to P and measures the time taken for P to reply back with this value. Assuming that the speed of communication is bounded by the speed of light, this technique gives an upper bound on the distance of P from the verifier.

The problem of secure position-verification has been studied before in the field of wireless security, and there have been several proposals for this task [9, 43, 12, 14, 45, 49, 13, 48]. However, Chandran et al. [16] show that there exists no protocol for secure position-verification that offers security in the presence of *multiple colluding* adversaries. In other words, the set of verifiers cannot distinguish between the case when they are interacting with an honest prover at *pos* and the case when they are interacting with multiple colluding dishonest provers, none of which is at position *pos*. Their impossibility result holds even if one makes computational hardness assumptions, and it also rules out most other interesting position-based cryptographic tasks.

In light of the strong impossibility result, Chandran et al. [16] consider a setting that assumes restrictions on the parties' storage capabilities, called the bounded-retrieval model (BRM) in the full version of [16], and construct secure protocols for position-verification and for position-based key exchange (wherein the verifiers, in addition to verifying the position claim of a prover, also exchange a secret key with the prover). While these protocols give us a way to realize position-based cryptography, the underlying setting is relatively hard to justify in practice.

This leaves us with the following question: Are there any other assumptions or settings in which position-based cryptography is realizable?

1.2. Our approach and our results. In this work, we study position-based cryptography in the *quantum* setting. To start with, let us briefly explain why moving to the quantum setting might be useful. The impossibility result of [16] relies heavily on the fact that an adversary can locally store all information she receives *and* at the same time share this information with other colluding adversaries, located elsewhere.

Recall that the positive result of [16] in the BRM circumvents the impossibility result by assuming that an adversary *cannot* store all information he receives. By considering the quantum setting, one may be able to circumvent the impossibility result thanks to the following observation. If some information is encoded into a quantum state, then the above attack fails due to the no-cloning principle: the adversary can either store the quantum state or send it to a colluding adversary (or do something in between, like store part of it) but *not both*.

However, this intuition turns out to be not completely accurate. Once the adversaries preshare entangled states, they can make use of quantum teleportation [6]. Although teleportation on its own does not appear to immediately conflict with the above intuition, we show that, based on techniques by Vaidman [47], adversaries holding a large number of entangled quantum states can perform *instantaneous nonlocal quantum computation*, which in particular implies that they can perform any unitary operation on a state shared between them, using only local operations and *one* round of classical communication (where both can send messages). Based on this technique, we show how a coalition of adversaries can attack and break any position-verification scheme.

Interestingly, sharing entangled quantum systems is vital for attacking the position-verification scheme. We show that there exist schemes that are secure in the information-theoretic sense if the adversary is not allowed to preshare or maintain entanglement. Furthermore, we show how to construct secure protocols for several position-based cryptographic tasks: position-verification, authentication, and key exchange.

This leads to an interesting open question regarding the amount of preshared entanglement required to break the position-verification scheme: the case of a large number of preshared states yields a complete break of any scheme, while having no preshared states leads to information-theoretically secure schemes. The exact number of preshared quantum systems that keeps the system secure is yet unknown.

1.3. Related work. To the best of our knowledge, quantum schemes for position-verification were first considered by Kent in 2002 under the name of "quantum tagging." Together with Munro, Spiller, and Beausoleil, a patent for an (insecure) scheme was filed for HP Labs in 2004 and granted in 2006 [32]. Their results did not appear in the academic literature until 2010 [30, 31]. In that paper, they describe several basic schemes and describe how to break them using teleportation-based attacks. They propose other variations (Schemes IV–VI in [31]) which are not broken by their teleportation attack and leave their security as an open question. Our general attack shows that these schemes are insecure as well.

Concurrent and independent of our work and the work on quantum tagging described above, the approach of using quantum techniques for secure position-verification was proposed by Malaney [36, 37]. However, the proposed scheme is merely claimed secure, and no rigorous security analysis is provided. As pointed out in [31], Malaney's schemes can also be broken by a teleportation-based attack.

In a preliminary version of this paper [15] (by a subset of the current authors), a secure quantum scheme for position-verification was proposed. However, that proof implicitly assumed that the adversaries have no preshared entanglement; as shown by [31], that scheme also becomes insecure without such an assumption.

In a subsequent paper [34], Lau and Lo use ideas similar to those in [31] to show the insecurity of position-verification schemes that are of a certain (yet rather restricted) form, which include the schemes from [36, 37] and [15]. Furthermore, they

propose a modified position-verification scheme that resists their attack. This scheme is shown to be secure against a restricted adversary that shares at most two qubits and, in an earlier version of their paper [33] (published before our impossibility result), is conjectured to be secure also against an unrestricted adversary.

In a recent note Kent [28] considers a different model for position-based cryptography where the prover's position is *not* his only credential, but he is assumed to additionally share with the verifiers a classical key unknown to the adversary. In this case, quantum key distribution (QKD) can be used to expand that key ad infinitum. This classical key stream is then used as an authentication resource.

The idea of performing "instantaneous measurements of nonlocal variables" was initiated by Aharonov and Albert [1, 2], who showed how to measure certain nonlocal properties (namely, the Bell operator). Vaidman [47] provided a scheme that is capable of performing any nonlocal measurement (with overwhelming success probability, which depends on the amount of preshared entanglement). Clark et al. [19] further improved Vaidman's scheme and obtained a scheme that guarantees success while using a finite amount of preshared entanglement. The concept of instantaneous nonlocal quantum computation presented here is an extension of Vaidman's task.

The appearance and circulation of our work has triggered a number of follow-up results. Beigi and König [3] used the technique of port-based teleportation by Ishizaka and Hiroshima [25, 26] to reduce the amount of entanglement required to perform instantaneous nonlocal quantum computation (from our doubly exponential) to exponential in size of the quantum system (i.e., the number of qubits being measured).

In [3], it is also shown that any *one-round* position-verification scheme that is secure with a sufficiently small error probability against an adversary with *no* preshared entanglement is automatically also secure (with a larger error probability, though) against an adversary with some limited amount of entanglement. Unfortunately, this result is not applicable to our positive results since our schemes are multiround schemes. However, it shows that analyzing schemes against an adversary with no entanglement is a good starting point to obtaining security against a limited amount of entanglement. In fact, this path was taken by Tomamichel et al. [46], where a parallel-repetition result for a so-called monogamy game is proven and used to obtain a one-round position-verification scheme, with a sufficiently small error probability so that it gives rise to a scheme that is secure against an adversary with a limited amount of entanglement.

In [11], Buhrman et al. suggest that a certain class of attacks on a particularly simple protocol for position-verification can be analyzed by using techniques from communication complexity. Motivated by this observation, they introduce the new notion of garden-hose (communication) complexity of a function which turns out to be connected to the number of Einstein–Podolsky–Rosen (EPR) pairs required to (perfectly) attack this class of position-verification protocols.

Kent proposes a general framework for quantum tasks in Minkowski space [29]. Position-based cryptographic tasks can be seen as special cases of such a general framework.

Giovannetti, Lloyd, and Maccone [23] show how to measure the distance between two parties by quantum cryptographic means so that only trusted people have access to the result. This is a different kind of problem than what we consider, and the techniques used there are not applicable in our setting.

1.4. Our attack and our schemes in more detail.

Position verification—a simple approach. Let us briefly discuss the 1-dimensional case in which we have two verifiers, V_0 and V_1, and a prover, P, at position *pos* that lies on the straight line between V_0 and V_1. Roughly speaking, to verify P's position, V_0 sends a BB84 qubit $H^\theta|x\rangle$ to P, where $x \in \{0,1\}$ and H is the Hadamard matrix; simultaneously, V_1 sends the corresponding basis $\theta \in \{0,1\}$ to P. The sending of these messages is timed in such a way that $H^\theta|x\rangle$ and θ arrive at position *pos* at the same time. P has to measure the qubit in basis θ to obtain x and immediately send x to both V_0 and V_1, who verify the correctness of x and if it has arrived "in time." See section 6 for a formal definition of the above scheme and its full analysis.

The intuition for this scheme is the following. Consider a dishonest prover $\hat{P}_0$ between V_0 and P and a dishonest prover $\hat{P}_1$ between V_1 and P. (It is not too hard to see that additional dishonest provers do not help.) When $\hat{P}_0$ receives the BB84 qubit, she does not know the corresponding basis θ yet. Thus, if she measures it immediately when she receives it, she is likely to measure it in the wrong basis, and $\hat{P}_0$ and $\hat{P}_1$ will not be able to provide the correct x. However, if she waits until she knows the basis θ, $\hat{P}_0$ and $\hat{P}_1$ will be too late in sending x to V_1 in time. Similarly, if she forwards the BB84 qubit to $\hat{P}_1$, who receives θ before $\hat{P}_0$ does, then $\hat{P}_0$ and $\hat{P}_1$ will be too late in sending x to V_0. It seems that in order to break the scheme, $\hat{P}_0$ needs to store the qubit until she receives the basis θ and at the same time send a copy of it to $\hat{P}_1$. But such actions are excluded by the no-cloning principle.

The attack and instantaneous nonlocal quantum computation. The above intuition turns out to be wrong. Using preshared entanglement yet no communication, $\hat{P}_0$ and $\hat{P}_1$ can in fact jointly perform a measurement of $|\psi\rangle = H^\theta|x\rangle$ in basis θ when one of them holds $|\psi\rangle$ and the other holds θ, and the outcome will be *classically* distributed between them (that is, the measurement outcome, obtained by one party, is correct up to some Pauli-corrections, known by the other party). Combining this fact with the observation by Kent, Munro, and Spiller [31] that the Pauli-corrections resulting from the teleportation commute with the actions of the honest prover in the above protocol shows that colluding adversaries can perfectly break the protocol.

Much more generally, we will show how to break *any* position-verification scheme, possibly consisting of multiple (and interleaved) rounds. To this end, we will show how to perform *instantaneous nonlocal quantum computation*.[1] That is, we prove that any unitary operation U acting on a composite system shared between players can be performed using only a single round of mutual classical communication. We obtain our computation scheme by using ideas by Vaidman [47]. Informally speaking, the players "teleport" quantum states back and forth many times in a clever way, *without* awaiting the classical measurement outcomes from the other party's teleportations. After each such "teleportation," the state on the receiving end has a constant probability of being identical to the original state, or otherwise it is the same up to some (tensor product of) Pauli-corrections; furthermore, the sender knows whether or not Pauli-corrections are needed. The receiving side performs U on the joint state, assuming no Pauli-corrections are needed, and "teleports" the result back to sender. In the case that no Pauli-corrections are needed, the computation ends. Otherwise, the scheme is repeated; however, since now the parties hold an altered version of the input state,

[1] Although in physics it is common to say that an operator U acts on a system $|\psi\rangle$, we adopt a more computer-science oriented terminology and use the word "computation" to indicate the joint execution of a unitary U on $|\psi\rangle$. The output of this distributed computation is the state $|\varphi\rangle = U|\psi\rangle$, possibly distributed between the parties.

the unitary U' to be computed this time should first undo the computation of the first round and only then perform U. See full details in section 4.

Position verification in the no-preshared entanglement (No-PE) model. On the other hand, the above intuition about the security of the protocol is correct in the *no-preshared entanglement* (No-PE) model, where the adversaries are not allowed to have preshared entangled quantum states prior to the execution of the protocol, or, more generally, prior to the execution of each round of the protocol in case of multiround schemes. Even though this model may be somewhat unrealistic and artificial, analyzing protocols in this setting serves as stepping stone to obtaining protocols which tolerate adversaries who preshare and maintain some *limited* amount of entanglement [46]. But also, rigorously proving security in the restrictive (for the adversary) No-PE model is already nontrivial and requires heavy machinery. Our proof uses the *strong complementary information trade-off* (CIT) due to Renes and Boileau [40], and it guarantees that for any strategy against the simple scheme outlined above, the success probability of $\hat{P}_0$ and $\hat{P}_1$ is bounded by approximately 0.89. By repeating the above simple scheme sequentially, we get a secure multiround position-verification scheme with exponentially small soundness error. We note that when performing sequential repetitions in the No-PE model, the adversaries must enter each round with no entanglement; thus, they are not allowed to generate entanglement in one round, store it, and use it in the next round(s).

Position-based authentication and key-exchange in the No-PE model. In the task of position-based authentication, the prover P sends a message m to the verifiers. The requirement is that the verifiers acknowledge m as "authenticated" only if its originator P is located at *pos*; see formal definition in section 7. Our position-based authentication scheme is based on our position-verification scheme. The idea is to start with a "weak" authentication scheme for a 1-bit message m: the verifiers and P execute the secure position-verification scheme; if P wishes to authenticate $m = 1$, then P correctly finishes the scheme by sending x back, but if P wishes to authenticate $m = 0$, P sends back an "erasure" symbol $\perp$ instead of the correct reply x with some probability q (which needs to be carefully chosen). This authentication scheme is weak in the sense that turning 1 into 0 is easy for the adversary, but turning a 0 into a 1 fails with constant probability.

The scheme is now augmented by performing multiple sequential repetitions of the weak authentication scheme. Based on ideas from [42, 27, 17], we use a suitable *balanced* encoding of the actual message to be authenticated, so that for any two messages, the adversary needs to turn many 0's into 1's. Unfortunately, an arbitrary balanced encoding is not good enough. The reason is that we do not assume the verifiers and the honest P to be synchronized. This asynchrony allows the adversary to use an (out of sync) honest P in order to authenticate a different message. Nevertheless, we show that the above approach does work for carefully chosen codes. We show that, for instance, the bitwise encoding which maps 0 into $00\ldots0\,11\ldots1$ and 1 into $11\ldots1\,00\ldots0$ is such a code.

Given a position-based authentication scheme, one can immediately obtain a position-based key exchange scheme simply by (essentially) executing an arbitrary QKD scheme (e.g., [5]), which assumes an authenticated classical communication channel, and authenticate the classical communication by means of the position-based authentication scheme.

1.5. Organization of the paper. In section 2, we begin by introducing notation and presenting the relevant background from quantum information theory.

In section 3, we describe the problem of position-verification and define our standard quantum model as well as the No-PE model in more detail. A protocol for computing any unitary operation using local operations and one round of classical communication is provided and analyzed in section 4, and in section 5 we conclude that there does not exist any protocol for position-verification (and hence for position-based authentication or key exchange) in the standard quantum model. We present our position-verification protocol in the No-PE model in section 6. Section 7 is devoted to our position-based authentication protocol and showing how to combine the above tools to obtain position-based key exchange.

2. Preliminaries.

2.1. Notation and terminology. We assume the reader to be familiar with the basic concepts of quantum information theory and refer the reader to [39] for an excellent introduction; we merely fix some notation.

Qubits. A *qubit* is a quantum system A with a 2-dimensional state space $\mathcal{H}_A = \mathbb{C}^2$. The *computational basis* $\{|0\rangle, |1\rangle\}$ (for a qubit) is given by $|0\rangle = \binom{1}{0}$ and $|1\rangle = \binom{0}{1}$, and the *Hadamard basis* by $H\{|0\rangle, |1\rangle\} = \{H|0\rangle, H|1\rangle\}$, where H denotes the 2-dimensional *Hadamard matrix*, which maps $|0\rangle$ to $(|0\rangle + |1\rangle)/\sqrt{2}$ and $|1\rangle$ to $(|0\rangle - |1\rangle)/\sqrt{2}$. The state space of an n-qubit system $A = A_1 \cdots A_n$ is given by the 2^n-dimensional space $\mathcal{H}_A = (\mathbb{C}^2)^{\otimes n} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$.

Since we mainly use the above two bases, we can simplify terminology and notation by identifying the computational basis $\{|0\rangle, |1\rangle\}$ with the bit 0 and the Hadamard basis $H\{|0\rangle, |1\rangle\}$ with the bit 1. Hence, when we say that an n-qubit state $|\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$ is measured in basis $\theta \in \{0,1\}^n$, we mean that the state is measured qubitwise where basis $H^{\theta_i}\{|0\rangle, |1\rangle\}$ is used for the ith qubit. As a result of the measurement, the string $x \in \{0,1\}^n$ is observed with probability $|\langle\psi|H^\theta|x\rangle|^2$, where $H^\theta = H^{\theta_1} \otimes \cdots \otimes H^{\theta_n}$ and $|x\rangle = |x_1\rangle \otimes \cdots \otimes |x_n\rangle$.

An important example of a 2-qubit state is the *EPR pair*, which is given by $|\Phi_{AB}\rangle = (|0\rangle|0\rangle + |1\rangle|1\rangle)/\sqrt{2} \in \mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^2 \otimes \mathbb{C}^2$ and has the following properties: if qubit A is measured in the computational basis, a uniformly random bit $x \in \{0,1\}$ is observed, and qubit B collapses to $|x\rangle$. Similarly, if qubit A is measured in the Hadamard basis, a uniformly random bit $x \in \{0,1\}$ is observed, and qubit B collapses to $H|x\rangle$.

Density matrices and trace distance. For any complex Hilbert space $\mathcal{H}$, we write $\mathcal{D}(\mathcal{H})$ for the set of all *density matrices* acting on $\mathcal{H}$. We measure closeness of two density matrices ρ and σ in $\mathcal{D}(\mathcal{H})$ by their *trace distance*: $\delta(\rho,\sigma) := \frac{1}{2}\mathrm{tr}|\rho - \sigma|$. One can show that for any physical processing of two quantum states described by ρ and σ, respectively, the two states behave in an indistinguishable way except with probability at most $\delta(\rho,\sigma)$. Thus, informally, if $\delta(\rho,\sigma)$ is very small, then without making a significant error, the two quantum states can be considered equal.

Classical and hybrid systems (and states). Subsystem X of a bipartite quantum system XE is called *classical* if the state of XE is given by a density matrix of the form $\rho_{XE} = \sum_{x\in\mathcal{X}} P_X(x)|x\rangle\langle x| \otimes \rho_E^x$, where $\mathcal{X}$ is a finite set of cardinality $|\mathcal{X}| = \dim(\mathcal{H}_X)$, $P_X : \mathcal{X} \to [0,1]$ is a probability distribution, $\{|x\rangle\}_{x\in\mathcal{X}}$ is some fixed orthonormal basis of $\mathcal{H}_X$, and ρ_E^x is a density matrix on $\mathcal{H}_E$ for every $x \in \mathcal{X}$. Such a state, called the *hybrid* state (also known as the *cq*-state, for *c*lassical and *q*uantum), can equivalently be understood as consisting of a *random variable* X with distribution P_X and range $\mathcal{X}$, and a system E that is in state ρ_E^x exactly when X takes on the value x. This formalism naturally extends to two (or more) classical systems X, Y, etc. as well as to two (or more) quantum systems.

Teleportation. The goal of teleportation is to transfer a quantum state from one location to another by only communicating classical information. Teleportation requires preshared entanglement among the two locations. Specifically, to teleport a qubit Q in an arbitrary (and typically unknown) state $|\psi\rangle$ from Alice to Bob, Alice performs a Bell measurement on Q and her half of an EPR pair, yielding a classical measurement outcome $k \in \{0,1,2,3\}$. Instantaneously, the other half of the corresponding EPR pair, which is held by Bob, turns into the state $\sigma_k^\dagger|\psi\rangle$, where $\sigma_0 = \mathbb{I}$, $\sigma_1, \sigma_2, \sigma_3$ denote the four Pauli matrices, and $\sigma_k^\dagger$ denotes the complex conjugate of the transpose of P_k. The classical information k is then communicated to Bob, who can recover the state $|\psi\rangle$ by performing σ_k on his EPR half. Note that the operator σ_k is Hermitian and unitary; thus $\sigma_k^\dagger = \sigma_k$ and $\sigma_k\sigma_k^\dagger = \mathbb{I}$.

2.2. Some quantum information theory. The *von Neumann entropy* of a quantum state $\rho \in \mathcal{D}(\mathcal{H})$ is given by $\mathrm{H}(\rho) := -\mathrm{tr}\big(\rho \log(\rho)\big)$, where here and throughout the article, log denotes the binary logarithm. $\mathrm{H}(\rho)$ is nonnegative and upper bounded by $\log(\dim(\mathcal{H}))$. For a bipartite quantum state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, the *conditional* von Neumann entropy of A given B is defined as $\mathrm{H}(\rho_{AB}|B) := \mathrm{H}(\rho_{AB}) - \mathrm{H}(\rho_B)$. In cases where the state ρ_{AB} is clear from the context, we may write $\mathrm{H}(A|B)$ instead of $\mathrm{H}(\rho_{AB}|B)$. If X and Y are both classical, $\mathrm{H}(X|Y)$ coincides with the classical conditional Shannon entropy. Furthermore, in case of conditioning (partly) on a classical state, the following holds.

LEMMA 2.1. *For any tripartite state ρ_{ABY} with classical Y,*

$$\mathrm{H}(A|BY) = \sum_y P_Y(y)\, \mathrm{H}(\rho_{AB}^y|B).$$

Lemma 2.1 along with the concavity of H and Jensen's inequality implies that for classical Y, $\mathrm{H}(A) \geq \mathrm{H}(A|Y) \geq 0$. The proof of Lemma 2.1 is given in Appendix A.

The following theorem is a generalization of the well-known Holevo bound [24] (see also [39]) and follows from the *monotonicity of mutual information*. Informally, it says that measuring only reduces your information. Formally, and tailored to the notation used here, it ensures the following.

THEOREM 2.2. *Let $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ be an arbitrary bipartite state, and let ρ_{AY} be obtained by measuring B in some basis to observe (classical) Y. Then $\mathrm{H}(A|Y) \geq \mathrm{H}(A|B)$.*

For classical X and Y, the Fano inequality [22] (see also [20]) allows us to bound the probability of correctly guessing X when having access to Y. In the statement below and throughout the paper, $\mathrm{h} : [0,1] \to [0,1]$ denotes the *binary entropy function* defined as $\mathrm{h}(p) = -p\log(p) - (1-p)\log(1-p)$ for $0 < p < 1$ and as $\mathrm{h}(p) = 0$ for $p = 0$ or 1, and $\mathrm{h}^{-1} : [0,1] \to [0, \frac{1}{2}]$ denotes its inverse on the branch $0 \leq p \leq \frac{1}{2}$.

THEOREM 2.3 (Fano inequality). *Let X and Y be random variables with ranges $\mathcal{X}$ and $\mathcal{Y}$, respectively, and let $\hat{X}$ be a guess for X computed solely from Y. Then $q := P[\hat{X} \neq X]$ satisfies*

$$\mathrm{h}(q) + q\log(|\mathcal{X}| - 1) \geq \mathrm{H}(X|Y).$$

In particular, for binary X, $q \geq \mathrm{h}^{-1}(\mathrm{H}(X|Y))$.

2.3. Strong complementary information trade-off. At the heart of our security proofs is the following entropic uncertainty principle due to [40], called *strong complementary information trade-off* (CIT), and which was later generalized by [7]

to bases other than computational and Hadamard bases. Loosely speaking, it relates the uncertainty of the measurement outcome of a system A using some basis θ with the uncertainty of the measurement outcome when the complementary basis $\bar{\theta}$ is used instead, and it guarantees that no two systems E and F coexist such that E has full information on the outcome in basis θ and F has full information on the outcome in basis $\bar{\theta}$. Note that by the *complementary* basis $\bar{\theta}$ of a basis $\theta = (\theta_1, \ldots, \theta_n) \in \{0,1\}^n$, we mean the n-bit string $\bar{\theta} = (\bar{\theta}_1, \ldots, \bar{\theta}_n) \in \{0,1\}^n$ with $\bar{\theta}_i \neq \theta_i$ for all i.

THEOREM 2.4 (CIT). *Let $|\psi_{AEF}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_E \otimes \mathcal{H}_F$ be an arbitrary tripartite state where $\mathcal{H}_A = (\mathbb{C}^2)^{\otimes n}$. Let the hybrid state ρ_{XEF} be obtained by measuring A in basis $\theta \in \{0,1\}^n$, and let the hybrid state σ_{XEF} be obtained by measuring A (of the original state $|\psi_{AEF}\rangle$) in the complementary basis $\bar{\theta}$. Then*

$$\mathrm{H}(\rho_{XE}|E) + \mathrm{H}(\sigma_{XF}|F) \geq n.$$

CIT in particular implies the following.

COROLLARY 2.5. *Let $|\psi_{AEF}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_E \otimes \mathcal{H}_F$ be an arbitrary tripartite state where $\mathcal{H}_A = (\mathbb{C}^2)^{\otimes n}$. Let Θ be uniformly distributed in $\{0,1\}^n$, and let X be the result of measuring A in basis Θ. Then*

$$\mathrm{H}(X|\Theta E) + \mathrm{H}(X|\Theta F) \geq n.$$

Note that by convexity of the entropy, the claims also hold for mixed states.

Proof. By Lemma 2.1, we can write

$$\begin{aligned}\mathrm{H}(X|\Theta E) + \mathrm{H}(X|\Theta F) &= \frac{1}{2^n}\sum_{\theta} \mathrm{H}(\rho^{\theta}_{XE}|E) + \frac{1}{2^n}\sum_{\theta} \mathrm{H}(\rho^{\theta}_{XF}|F) \\ &= \frac{1}{2^n}\sum_{\theta} \big(\mathrm{H}(\rho^{\theta}_{XE}|E) + \mathrm{H}(\rho^{\bar{\theta}}_{XF}|F)\big).\end{aligned}$$

Note that ρ^{θ}_{XE} is obtained by measuring A of $|\psi_{AEF}\rangle$ in basis θ (and ignoring F), and $\rho^{\bar{\theta}}_{XF}$ is obtained by measuring A of $|\psi_{AEF}\rangle$ in the complementary basis $\bar{\theta}$ (and ignoring E). Hence, Theorem 2.4 applies, and we can conclude that $\mathrm{H}(\rho^{\theta}_{XE}|E)+\mathrm{H}(\rho^{\bar{\theta}}_{XF}|F) \geq n$ and thus $\mathrm{H}(X|\Theta E) + \mathrm{H}(X|\Theta F) \geq n$. ∎

3. Position verification.

3.1. Setting, space and time, and communication model. We consider entities $V_0, \ldots, V_k$, called *verifiers*, and an entity P, the (honest) *prover*. Additionally, we consider a coalition $\hat{P}$ of *dishonest provers* (or *adversaries*) $\hat{P}_0, \ldots, \hat{P}_\ell$. All entities are restricted by the laws of quantum mechanics; they can perform arbitrary quantum (and classical) operations and can communicate quantum (and classical) messages among them. We assume that quantum operations and communication are noise-free and that local computations take no time.

Each entity is assigned an arbitrary fixed position *pos* in the d-dimensional space $\mathbb{R}^d$. Throughout the paper, we require that the honest prover P is *enclosed* by the verifiers $V_0, \ldots, V_k$ in that the prover's position $pos \in \mathbb{R}^d$ lies within the polyhedron, i.e., convex hull, $\mathrm{Hull}(pos_0, \ldots, pos_k) \subset \mathbb{R}^d$ formed by the respective positions of the verifiers.

We assume that messages to be communicated travel at fixed velocity v (e.g., with the speed of light in a vacuum), and hence the time needed for a message to travel from one entity to another equals the Euclidean distance between the two (assuming

that v is normalized to 1). This timing assumption holds for honest and dishonest entities.

We also assume that the verifiers have precise and synchronized clocks, so that they can coordinate exact times for sending messages and can measure the exact time of a message arrival. We do not require P's clock to be precise or in sync with the verifiers. Actually, we will allow the adversary to fully control the frequency of P's clock.

With the above model, we can reason as follows. Consider a verifier V_0 at position pos_0 who sends a challenge ch_0 to the (supposedly honest) prover claiming to be at position pos. If V_0 receives a reply within time $2d(pos_0, pos)$, where $d(\cdot,\cdot)$ is the Euclidean distance measure in $\mathbb{R}^d$ and thus also measures the time a message takes from one point to the other, then V_0 can conclude that he is communicating with a prover that is within distance $d(pos_0, pos)$.

Remark. Our model above relies on several idealized assumptions—mainly, flat space(time), fixed velocity of information travel, instantaneous local computations, point-shaped locations, error-free quantum communication, and information processing. This is necessary in order to obtain a clean model that is simple enough to prove rigorous results. Relaxing these assumptions is beyond the scope of this work. As such, our work should be appreciated from a theoretical perspective, as trying to understand the possibilities and the limitations of the *theory* of quantum mechanics to position-based cryptography; we do not make any real-life practicality claims.

We stress, however, that relaxing the assumptions (e.g., introducing noise) typically makes the life of the honest parties harder and simplifies the life of the dishonest parties.[2] As such, our negative result for position-verification does not strongly rely on these assumptions and is likely to carry over also to more realistic settings. See further discussion about relaxing the assumptions in section 8.

3.2. The security model. In this work we consider only *stand-alone security*; i.e., we analyze only a single execution with a single honest prover, and we do not guarantee any kind of composable or concurrent security. We note that an impossibility result for stand-alone security implies impossibility in any composable setting; on the other hand, our positive results are restricted to the stand-alone case. Furthermore, we assume that P cannot be *reset*; that is, the adversary cannot restart P's program and make P run the protocol again from its starting point.

We require that the verifiers have private and authenticated channels among themselves, which allow them to coordinate their actions by communicating before, during, or after protocol execution. We stress, however, that this assumption does not hold for the communication between the verifiers and P: $\hat{P}$ has full control over messages communicated between the verifiers and P (both ways). In particular, the verifiers do not know per se if they are communicating with the honest or a dishonest prover (or a coalition of dishonest provers).

We stress that in our model, the honest prover P has no advantage over the dishonest provers beyond being at position pos. In particular, P does not share any secret information with the verifiers, nor can he per se authenticate his messages by any other means.

[2] For instance, in the case of noise, one would have to introduce some error-correction mechanism for the honest parties, whereas one would still want to prove security against dishonest parties that are not affected by noise. This is because even if there is noise in practice, if possible we do not want to base the security on that.

For our positive results, we consider a restricted model, which prohibits entanglement between the dishonest provers. Specifically, the *No-PE model* is such that the dishonest provers enter every new round of communication, initiated by the verifiers, with no preshared entanglement. That is, in every round, a dishonest prover can distribute an entangled quantum state only *after* it receives the verifier's message, and the dishonest provers cannot maintain such an entangled state in order to use it in the next round. As mentioned in the introduction, considering this simple (but possibly unrealistic) model may help in obtaining protocols that are secure against adversaries with *limited* entanglement [46].

3.3. Secure position verification. A position-verification scheme should allow a prover P at position $pos \in \mathbb{R}^d$ (in d-dimensional space) to convince a set of $k+1$ verifiers $V_0, \ldots, V_k$, who are located at respective positions $pos_0, \ldots, pos_k \in \mathbb{R}^d$, that he is indeed at position pos. We assume that P is enclosed by $V_0, \ldots, V_k$. We require that the verifiers jointly accept if an honest prover P is at position pos, and we require that the verifiers reject with "high" probability in case of a dishonest prover that is not at position pos. The latter should hold even if the dishonest prover consists of a *coalition* of collaborating dishonest provers $\hat{P}_0, \ldots, \hat{P}_\ell$ at arbitrary positions $apos_0, \ldots, apos_\ell \in \mathbb{R}^d$ with $apos_i \neq pos$ for all i. We refer the reader to [16] for the general formal definition of the completeness and security of a position-verification scheme. In this paper, we mainly focus on position-verification schemes of the following form.

DEFINITION 3.1. *A one-round* position-verification *scheme* $\mathsf{PV} = (\mathsf{Chlg}, \mathsf{Resp}, \mathsf{Ver})$ *consists of the following three parts: A challenge generator* Chlg*, which outputs a list of challenges* $(ch_0, \ldots, ch_k)$ *and auxiliary information* x*; a response algorithm* Resp*, which on input a list of challenges outputs a list of responses* $(x'_0, \ldots, x'_k)$*; and a verification algorithm* Ver *with* $\mathsf{Ver}(x'_0, \ldots, x'_k, x) \in \{0, 1\}$.

PV *is said to have* perfect completeness *if* $\mathsf{Ver}(x'_0, \ldots, x'_k, x) = 1$ *with probability* 1 *for* $(ch_0, \ldots, ch_k)$ *and* x *generated by* Chlg *and* $(x'_0, \ldots, x'_k)$ *by* Resp *on input* $(ch_0, \ldots, ch_k)$.

The algorithms Chlg, Resp, and Ver are used as described in Figure 3.1 to verify the claimed position of a prover P. We clarify that in order to have all the challenges arrive at P's (claimed) location pos *at the same time*, the verifiers agree on a time T and each V_i sends off his challenge ch_i at time $T - d(pos_i, pos)$. As a result, all ch_i's arrive at P's position pos at time T. In step 3, V_i receives x'_i *in time* if x'_i arrives at V_i's position pos_i at time $T + d(pos_i, pos)$. Throughout the paper, we use this simplified terminology. Furthermore, we are sometimes a bit sloppy in distinguishing a party, like P, from its location pos.

We stress that we allow Chlg, Resp, and Ver to be *quantum* algorithms and ch_i, x, and x'_i to be quantum information. In our constructions, only ch_0 will actually be quantum; thus, we will only require quantum communication from V_0 to P; all other communication is classical. Also, in our constructions, $x'_0 = \cdots = x'_k$, and $\mathsf{Ver}(x'_0, \ldots, x'_k, x) = 1$ exactly if $x'_i = x$ for all i.

DEFINITION 3.2. *A one-round position-verification scheme* $\mathsf{PV} = (\mathsf{Chlg}, \mathsf{Resp}, \mathsf{Ver})$ *is called* ε-sound *if for any position* $pos \in \mathrm{Hull}(pos_0, \ldots, pos_k)$*, and any coalition of dishonest provers* $\hat{P}_0, \ldots, \hat{P}_\ell$ *at arbitrary positions* $apos_0, \ldots, apos_\ell$*,* $all \neq pos$*, when executing the scheme from Figure* 3.1 *the verifiers accept with probability at most* ε*. We write* PV^ε *for such a protocol.*

A position-verification scheme can also be understood as a (position-based) *identification* scheme, where the identification is not done by means of a cryptographic key or a password, but by means of the geographical location.

Common input to the verifiers: Their respective positions $pos_0, \ldots, pos_k$ and P's (claimed) position pos.

0. V_0 generates a list of challenges $(ch_0, \ldots, ch_k)$ and auxiliary information x using Chlg and privately sends ch_i to V_i for $i = 1, \ldots, k$.
1. Every V_i sends ch_i to P in such a way that all ch_i's arrive at the same time at P's position pos.
2. P computes $(x'_0, \ldots, x'_k) := \mathsf{Resp}(ch_0, \ldots, ch_k)$ as soon as all the ch_i's arrive, and he sends x'_i to V_i for every i.
3. The V_i's jointly accept if and only if all V_i's receive x'_i in time and $\mathsf{Ver}(x'_0, \ldots, x'_k, x) = 1$.

FIG. 3.1. *Generic one-round position-verification scheme.*

4. Instantaneous nonlocal quantum computation. In order to analyze the (in)security of position-verification schemes, we first address a more general task, which is interesting in its own right: *instantaneous nonlocal quantum computation*.[3] Consider the following problem, involving two parties Alice and Bob. Alice holds A and Bob holds B of a tripartite system ABE that is in some unknown state $|\psi\rangle$. The goal is to apply a known unitary transformation U to AB, but *without* using any communication—just by local operations. In general, such a task is clearly impossible, as it violates the nonsignaling principle [19]. The goal of instantaneous nonlocal quantum computation is to achieve almost the above but without violating nonsignaling. Specifically, the goal is for Alice and Bob to compute, without communication, a state $|\varphi'\rangle$ that coincides with $|\varphi\rangle = (U \otimes \mathbb{I}_E)|\psi\rangle$ up to *local* and *qubitwise* operations on A and B (that is, tensor products of operators on qubits), where $\mathbb{I}_E$ denotes the identity on E.[4] Furthermore, these local and qubitwise operations are determined by *classical* information that Alice and Bob obtain as part of their actions. In particular, if Alice and Bob share their classical information, which can be done with *one* round of communication (where a "round" means a simultaneous mutual exchange of messages), then they can transform $|\varphi'\rangle$ into $|\varphi\rangle = (U \otimes \mathbb{I}_E)|\psi\rangle$ by local qubitwise operations. Following ideas by Vaidman [47], we show below that instantaneous nonlocal quantum computation, as described above, is possible if Alice and Bob share sufficiently many EPR pairs.

In the following, let $\mathcal{H}_A$, $\mathcal{H}_B$, and $\mathcal{H}_E$ be Hilbert spaces where the former two consist of n_A and n_B qubits, respectively, i.e., $\mathcal{H}_A = (\mathbb{C}^2)^{\otimes n_A}$ and $\mathcal{H}_B = (\mathbb{C}^2)^{\otimes n_B}$. Furthermore, let U be a unitary matrix acting on $\mathcal{H}_A \otimes \mathcal{H}_B$. Alice holds system A, and Bob holds system B of an arbitrary and unknown state $|\psi\rangle \in \mathcal{H}_{ABE} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E$. Additionally, Alice and Bob share an arbitrary but finite number of EPR pairs.

THEOREM 4.1. *For every unitary U and for every $\varepsilon > 0$, given sufficiently many shared EPR pairs, there exist local operations $\mathcal{A}$ and $\mathcal{B}$, acting on Alice's and Bob's respective sides, with the following property. For any initial state $|\psi\rangle \in \mathcal{H}_{ABE}$, the joint execution $\mathcal{A} \otimes \mathcal{B}$ transforms $|\psi\rangle$ into $|\varphi'\rangle$ and provides classical outputs k to Alice and ℓ to Bob, such that the following holds except with probability ε. The state $|\varphi'\rangle$*

[3] This is an extension of the task of "instantaneous measurement of nonlocal variables" introduced by Vaidman [47].

[4] The requirement on the local operations to be *qubit-wise* is not essential, but it simplifies matters.

coincides with $|\varphi\rangle = (U \otimes \mathbb{I}_E)|\psi\rangle$ *up to local qubitwise operations on* A *and* B *that are determined by* k *and* ℓ.

We stress that $\mathcal{A}$ acts on A as well as on Alice's shares of the EPR pairs, and the corresponding holds for $\mathcal{B}$. Furthermore, being equal up to local qubitwise operations on A and B means that $|\varphi\rangle = (V^A_{k,\ell} \otimes V^B_{k,\ell} \otimes \mathbb{I}_E)|\varphi'\rangle$, where $\{V^A_{k,\ell}\}_{k,\ell}$ and $\{V^B_{k,\ell}\}_{k,\ell}$ are fixed families of unitaries which act qubitwise on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. In our construction, the $V^A_{k,\ell}$ and $V^B_{k,\ell}$'s will actually be tensor products of one-qubit Pauli operators.

As an immediate consequence of Theorem 4.1, we get the following.

COROLLARY 4.2. *For every unitary* U *and for every* $\varepsilon > 0$, *given sufficiently many shared EPR pairs, there exists a nonlocal operation* $\mathcal{AB}$ *for Alice and Bob which consists of local operations and* one *round of mutual communication, such that for any initial state* $|\psi\rangle \in \mathcal{H}_{ABE}$ *of the tripartite system* ABE, *the joint execution of* $\mathcal{AB}$ *transforms* $|\psi\rangle$ *into* $|\varphi\rangle = (U \otimes \mathbb{I}_E)|\psi\rangle$, *except with probability* ε.

For technical reasons, we will actually prove the following extension of Theorem 4.1, which is easily seen to be equivalent. The difference from Theorem 4.1 is that Alice and Bob are additionally given classical inputs: x to Alice and y to Bob, and the unitary U that is to be applied to the quantum input depends on x and y. In the statement below, x ranges over some arbitrary but fixed finite set $\mathcal{X}$, and y ranges over some arbitrary but fixed finite set $\mathcal{Y}$.

THEOREM 4.3. *For every family* $\{U_{x,y}\}$ *of unitaries and for every* $\varepsilon > 0$, *given sufficiently many shared EPR pairs, there exist families* $\{\mathcal{A}_x\}$ *and* $\{\mathcal{B}_y\}$ *of local operations, acting on Alice's and Bob's respective sides, with the following property. For any initial state* $|\psi\rangle \in \mathcal{H}_{ABE}$ *and for every* $x \in \mathcal{X}$ *and* $y \in \mathcal{Y}$, *the joint execution* $\mathcal{A}_x \otimes \mathcal{B}_y$ *transforms the state* $|\psi\rangle$ *into* $|\varphi'\rangle$ *and provides classical outputs* k *to Alice and* ℓ *to Bob, such that the following holds except with probability* ε. *The state* $|\varphi'\rangle$ *coincides with* $|\varphi\rangle = (U_{x,y} \otimes \mathbb{I}_E)|\psi\rangle$ *up to local qubitwise operations on* A *and* B *that are determined by* k *and* ℓ.

The solution works by "teleporting" states back and forth in a clever way [47], but *without* communicating the respective classical outcomes of the Bell measurements, so that only local operations are performed. More precisely, the respective sender makes a Bell measurement, resulting in some classical information, but he does *not* (yet) communicate the outcome to the receiver, and the receiver takes his share of the EPR pair as the received state, but does not/cannot (yet) correct it. To emphasize this difference from standard teleportation, we refer to it as *teleportation** (with an asterisk) in the proof below.

There is a small probability that the classical outcomes of the Bell measurement indicate that the receiver holds exactly the original state (no correction is needed). In this case he can perform the unitary on the joint state, and the parties are practically done (although they do not know that they are done). Otherwise, the parties still need to perform the unitary and in addition to correct the effect caused by the Bell measurement. This can be seen as performing a new unitary on their joint state, such that the new unitary takes into account both the original unitary and the corrections needed due to the Bell measurement. The parties now repeat the process with this new unitary.

Note that the sender knows when the teleported* state needs no correction, but the receiver does not. Thus, the receiver always performs the unitary and then teleports* the result back to the sender, who can decide whether to continue the scheme with an updated unitary, or stop (the receiver side always continues, yet if the sender has stopped, any further actions are meaningless). At the end (after a fixed

number of rounds performed by the receiver) the parties communicate all the classical information gathered along the scheme. This allows them to perform the correction of the final teleportation* of the scheme.

Another crucial trick, due to Vaidman [47], that is used in the above approach is as follows. Before Alice actually teleports* her state to Bob, they prepare a *list* of "teleportation* channels," one for each possible choice of x, and Alice then teleports* her state using the channel that is labeled by her actual input x, whereas Bob applies $U_{x',y}$ to the channel labeled by x' for *every* possible x'. This way, Bob applies the right unitary (to the right state) without Alice having to communicate x to him.

Proof of Theorem 4.3. To simplify notation, we assume that the joint state of A and B is pure, and thus we may ignore system E. However, all our arguments also hold in case the state of A and B is entangled with E.

Next, we observe that it is sufficient to prove Theorem 4.3 for the case where B is "empty," i.e., $\dim \mathcal{H}_B = 1$ and thus $n_B = 0$. Indeed, if this is not the case, Alice and Bob can do the following. Bob first teleports* B to Alice. We stress that the asterisk means that Bob does not communicate the outcome of the Bell measurements. Now, Alice holds $A' = AB$ with $n_{A'} = n_A + n_B$, and Bob's system has collapsed, and thus Bob holds no quantum state anymore but only classical information. Then, they do the nonlocal computation, and in the end Alice teleports* B back to Bob. The modification to the state of B introduced by teleporting* it to Alice can be taken care of by modifying the set of unitaries $\{U_{x,y}\}$ accordingly (and making it dependent on Bob's measurement outcome, thereby extending the set $\mathcal{Y}$). Also, the modification to the state of B introduced by teleporting* it back to Bob does not harm the requirement of the joint state being equal to $|\varphi\rangle = U_{x,y}|\psi\rangle$ up to local qubitwise operations.

Hence, from now on, we may assume that B is "empty," and we write n for n_A. Next, we describe the core of how the local operations $\mathcal{A}_x$ and $\mathcal{B}_y$ work. To simplify notation, we assume that $\mathcal{X} = \{1, \dots, m\}$. Recall that Alice and Bob share (many) EPR pairs. We may assume that the EPR pairs are grouped into groups of size n; each such group we call a *teleportation channel*. Furthermore, we may assume that m of these teleportation channels are labeled by the numbers 1 up to m and that another m of these teleportation channels are labeled by the numbers $m+1$ up to $2m$.

1. Alice teleports* $|\psi\rangle$ to Bob, using the teleportation channel that is labeled by her input x. Let us denote her measurement outcome by $k_\circ \in \{0,1,2,3\}^n$.
2. For every $i \in \{1, \dots, m\}$, Bob does the following. He applies the unitary $U_{i,y}$ to the n qubits that make up his share of the EPR pairs given by the teleportation channel labeled i. Then, he teleports* the resulting state to Alice using the teleportation channel labeled $m+i$. We denote the corresponding measurement outcome by $\ell_{\circ,i}$.
3. Alice specifies the n qubits that make up her share of the EPR pairs given by the teleportation channel labeled $m+x$ to be the state $|\varphi'\rangle$.

Let us analyze the above. With probability $1/4^n$, namely, if $k_\circ = 0 \cdots 0$, teleporting* $|\psi\rangle$ to Bob leaves the state unchanged. In this case, it is easy to see that the resulting state $|\varphi'\rangle$ satisfies the required property of being identical to $|\varphi\rangle = U_{x,y}|\psi\rangle$ up to local qubitwise operations determined by $\ell_{\circ,x}$ and thus determined by x and $\ell_\circ = (\ell_{\circ,1}, \dots, \ell_{\circ,m})$. This proves the claim for the case where $\varepsilon \geq 1 - 1/4^n$.

We show how to reduce ε. The crucial observation is that if in the above procedure $k_\circ \neq 0 \cdots 0$, and thus $|\varphi'\rangle$ is not necessarily identical to $|\varphi\rangle$ up to local qubitwise operations, then

$$|\varphi'\rangle = (V_{\ell_{\circ,x}} U_{x,y} V_{k_\circ} \otimes \mathbb{I}_E)|\psi\rangle = (V_{\ell_{\circ,x}} U_{x,y} V_{k_\circ} U_{x,y}^\dagger \otimes \mathbb{I}_E)|\varphi\rangle,$$

where $V_{\ell_\circ,x}$ and $V_{k_\circ}$ are tensor products of Pauli matrices acting on $\mathcal{H}_A \otimes \mathcal{H}_B$. Thus, setting $|\psi'\rangle := |\varphi'\rangle$, $x' := (x, k_\circ)$, $y' := (y, \ell_\circ)$, and $U'_{x',y'} := U_{x,y} V_{k_\circ} U^\dagger_{x,y} V_{\ell_\circ,x}$, the state $|\varphi\rangle$ can be written as $|\varphi\rangle = (U'_{x',y'} \otimes \mathbb{I}_E)|\psi'\rangle$. This means we are back to the original problem of applying a unitary, $U'_{x',y'}$, to a state, $|\psi'\rangle$, held by Alice, where the unitary depends on classical information x' and y', known by Alice and Bob, respectively. Thus, we can reapply the above procedure to the new problem instance. Note that in the new problem instance, the classical inputs x' and y' come from larger sets than the original inputs x and y, but the new quantum input, $|\psi'\rangle$, has the same number of qubits, n. Therefore, reapplying the procedure will succeed with the same probability $1/4^n$.

As there is a constant probability of success in each round, reapplying the above procedure sufficiently many times to the resulting new problem instances guarantees that, except with arbitrarily small probability, the state $|\varphi'\rangle$ will be of the required form at some point (when Alice gets $k_\circ = 0 \cdots 0$). Say this is the case at the end of the jth iteration. Then, Alice stops with her part of the procedure at this point, keeps the state $|\varphi'\rangle$, and specifies k to consist of j and of her classical input into the jth iteration (which consists of x and of the $k_\circ$'s from the prior $j-1$ iterations). Since, Bob does not learn whether an iteration is successful or not, he has to keep on reiterating up to some bound, and in the end he specifies ℓ to consist of the $\ell_\circ$'s collected over all the iterations. The state $|\varphi'\rangle$ equals $|\varphi\rangle = (U_{x,y} \otimes \mathbb{I}_E)|\psi\rangle$ up to local qubitwise operations that are determined by k and ℓ. □

The number of EPR pairs needed by Alice and Bob in the scheme described in the proof is doubly exponential in $n_A + n_B$, the number of qubits that make the joint quantum system.[5] In recent subsequent work, Beigi and König [3] used a different kind of quantum teleportation by Ishizaka and Hiroshima [25, 26] to reduce the amount of entanglement needed to perform instantaneous nonlocal quantum computation to exponential in the number of qubits of the joint quantum system. It remains an interesting open question whether such an exponentially large amount of entanglement is necessary.

In Appendix B, we explain how to perform instantaneous nonlocal quantum computation among more than two parties.

5. Impossibility of unconditional position verification. In this section we show that no position-verification scheme is secure against a coalition of quantum adversaries in the standard model specified in section 3.2. For simplicity, we consider the 1-dimensional case, with two verifiers V_0 and V_1, but the attack can be generalized to higher dimensions and more verifiers.

We consider an arbitrary position-verification scheme in our model (as specified in section 3). We recall that in this model, the verifiers must base their decision solely on *what* the prover replies and *how long* it takes him to reply, and the honest prover has no advantage over a coalition of dishonest provers beyond being at the claimed position.[6] Such a position-verification scheme may be of the form specified in Figure 3.1 but may also be made up of several, possibly interleaved, rounds of interaction between the prover and the verifiers.

[5] The probability that teleporting* needs no correction is $4^{-(n_A+n_B)}$; hence, the basic scheme needs to be executed about $4^{n_A+n_B}$ times. Furthermore, in every execution, the number of required "teleportation channels" grows by a factor $4^{n_A+n_B}$. See [3] for a detailed calculation.

[6] In particular, the prover does not share any secret information with the verifiers, differentiating our setting from models as described, for example, in [28].

For the honest prover P, such a general scheme consists of steps that look as follows. P holds a local quantum register R, which is set to some default value at the beginning of the scheme. In each step, P obtains a system A from V_0 along with some classical information x; simultaneously, he gets a system B from V_1 and some classical information y. The separation between classical and quantum inputs is convenient for technical reasons similar to those in section 4. In addition, V_0 and V_1 jointly keep some system E. Let $|\psi\rangle$ be the state of the four-partite system $ABRE$; it is determined by the scheme and by the step within the scheme we are focusing on. P has to apply a unitary transformation $U_{x,y}$ that depends on x and y to ABR and send the (transformed) systems A and B back to V_0 and V_1 (and keep R). Note that, after the transformation, the state of $ABRE$ is given by $|\varphi\rangle = (U_{x,y} \otimes \mathbb{I}_E)|\psi\rangle$.

We show that a coalition of two dishonest provers $\hat{P}_0$ and $\hat{P}_1$, where $\hat{P}_0$ is located in between V_0 and P and $\hat{P}_1$ is located in between V_1 and P, can perfectly simulate the actions of the honest prover P, and therefore it is impossible for the verifiers to distinguish between an honest prover at position *pos* and a coalition of dishonest provers at positions different from *pos*. The simulation of the dishonest provers perfectly imitates the *computation* as well as the *timing* of an honest P. Since in our model this information is what the verifiers have to base their decision on, the general impossibility of position-verification in our model follows.

Consider a step in the scheme as described above, but now from the point of view of $\hat{P}_0$ and $\hat{P}_1$. Since $\hat{P}_0$ is closer to V_0, she will first receive A and x; similarly, $\hat{P}_1$ will first receive B and y. We specify that $\hat{P}_1$ takes care of and maintains the local register R. If the step we consider is the *first* step in the scheme, the state of $ABRE$ equals $|\psi\rangle$, as in the case of an honest P. In order to have an invariant that holds for all the steps, we actually relax this statement and merely observe that the state of $ABRE$, say, $|\psi'\rangle$, equals $|\psi\rangle$ up to local and qubitwise operations on the subsystem R, determined by classical information $x_\circ$ and $y_\circ$, where $\hat{P}_0$ holds $x_\circ$ and $\hat{P}_1$ holds $y_\circ$. This invariant clearly holds for the first step in the scheme, when R is in some default state, and we will show that it also holds for the other steps.

By Theorem 4.3, it follows that without communication (just by instantaneous local operations), $\hat{P}_0$ and $\hat{P}_1$ can transform the state $|\psi'\rangle$ into a state $|\varphi'\rangle$ that coincides with $|\varphi\rangle = (U_{x,y} \otimes \mathbb{I}_E)|\psi\rangle$ up to local and qubitwise transformations on A, B, and R, determined by classical information k (known to $\hat{P}_0$) and ℓ (known to $\hat{P}_1$). Note that the initial state is not $|\psi\rangle$ but rather a state of the form $|\psi'\rangle = (V_{x_\circ,y_\circ} \otimes \mathbb{I}_E)|\psi\rangle$, where $x_\circ$ is known to $\hat{P}_0$ and $y_\circ$ to $\hat{P}_1$. Thus, Theorem 4.3 is actually applied to the unitary $U'_{x',y'} = U_{x,y} V^\dagger_{x_\circ,y_\circ}$, where $x' = (x_\circ, x)$ and $y' = (y_\circ, y)$. Given $|\varphi'\rangle$, k, and ℓ, the parties $\hat{P}_0$ and $\hat{P}_1$ can exchange k and ℓ using *one* mutual round of communication and transform $|\varphi'\rangle$ into $|\varphi''\rangle$ which coincides with $|\varphi\rangle$ up to qubitwise operations only on R and send A to V_0 and B to V_1. It follows that the state of ABE and the time it took $\hat{P}_0$ and $\hat{P}_1$ for the computation and communication are identical to those of an honest P; i.e., $\hat{P}_0$ and $\hat{P}_1$ have perfectly simulated this step of the scheme.

Finally, we see that the invariant is satisfied, when moving on to the next step in the scheme, where $\hat{P}_0$ and $\hat{P}_1$ receive new A and B (along with new classical x and y) from V_0 and V_1, respectively. Even if this new round interleaves with the previous round in that the new A and B, etc., arrive *before* $\hat{P}_0$ and $\hat{P}_1$ have finished exchanging (the old) k and ℓ, it still holds that the state of $ABRE$ is as in the case of honest P up to qubitwise operations on the subsystem R. It follows that the above procedure works for all the steps and thus that $\hat{P}_0$ and $\hat{P}_1$ can indeed perfectly simulate honest P's actions throughout the whole scheme.

6. Secure position-verification in the No-PE model.

6.1. Basic scheme and its analysis. In this section we show the possibility of secure position-verification in the No-PE model. We consider the following basic one-round position-verification scheme, given in Figure 6.1. It is based on the BB84 encoding.

0. V_0 chooses two random bits $x, \theta \in \{0,1\}$ and privately sends them to V_1.
1. V_0 prepares the qubit $H^\theta|x\rangle$ and sends it to P, and V_1 sends the bit θ to P, so that $H^\theta|x\rangle$ and θ arrive at the same time at P.
2. When $H^\theta|x\rangle$ and θ arrive, P measures $H^\theta|x\rangle$ in basis θ to observe $x' \in \{0,1\}$ and sends x' to V_0 and V_1.
3. V_0 and V_1 accept if on both sides x' arrives in time and $x' = x$.

FIG. 6.1. *Position-verification scheme* $\mathsf{PV}_{\mathrm{BB84}}$ *based on the BB*84 *encoding.*

We implicitly specify that parties abort if they receive any message that is inconsistent with the protocol, for instance, (classical) messages with a wrong length, or different number of received qubits than expected.

THEOREM 6.1. *The one-round position-verification scheme* $\mathsf{PV}_{\mathrm{BB84}}$ *from Figure* 6.1 *is* ε*-sound with* $\varepsilon = 1 - \mathrm{h}^{-1}(\frac{1}{2})$ *in the No-PE model.*

Recall that h denotes the binary entropy function and h^{-1} its inverse on the branch $0 \le p \le \frac{1}{2}$. A numerical calculation shows that $\mathrm{h}^{-1}(\frac{1}{2}) \ge 0.11$ and thus $\varepsilon \le 0.89$. A particular attack for a dishonest prover $\hat{P}$, sitting in between V_0 and P, is to measure the qubit $H^\theta|x\rangle$ in the *Breidbart* basis, resulting in an acceptance probability of $\cos(\pi/8)^2 \approx 0.85$. This shows that our analysis is pretty tight.

Proof. In order to analyze the position-verification scheme it is convenient to consider an equivalent *purified* version, given in Figure 6.2. The only difference between the original and the purified scheme is the preparation of the bit $H^\theta|x\rangle$. In the purified version, it is done by preparing $|\Phi_{AB}\rangle = (|0\rangle|0\rangle + |1\rangle|1\rangle)/\sqrt{2}$ and measuring A in basis θ. This way of preparation changes the point in time when V_0 measures A and the point in time when V_1 learns x. This, however, has no influence on the view of the (dishonest or honest) prover, nor on the joint distribution of θ, x, and x', and thus nor on the probability that V_0 and V_1 accept. It therefore suffices to analyze the purified version.

0. V_0 and V_1 privately agree on a random bit $\theta \in \{0,1\}$.
1. V_0 prepares an EPR pair $|\Phi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, keeps qubit A, and sends B to P, and V_1 sends the bit θ to P, so that B and θ arrive at the same time at P.
2. When B and θ arrive, P measures B in basis θ to observe $x' \in \{0,1\}$ and sends x' to V_0 and V_1.
3. Only now, when x' arrives, V_0 measures A in basis θ to observe x and privately sends x to V_1. V_0 and V_1 accept if on both sides x' arrives in time and $x' = x$.

FIG. 6.2. *EPR version of* $\mathsf{PV}_{\mathrm{BB84}}$.

We first consider security against two dishonest provers $\hat{P}_0$ and $\hat{P}_1$, where $\hat{P}_0$ is between V_0 and P and $\hat{P}_1$ is between V_1 and P. In the end we will argue that a similar argument holds for multiple dishonest provers on either side.

Since V_0 and V_1 do not accept if x' does not arrive in time and dishonest provers do not use preshared entanglement in the No-PE-model, any potentially successful strategy of $\hat{P}_0$ and $\hat{P}_1$ must look as follows. As soon as $\hat{P}_1$ receives the bit θ from V_1, she forwards (a copy of) it to $\hat{P}_0$. Also, as soon as $\hat{P}_0$ receives the qubit A, she applies an arbitrary quantum operation to the received qubit A (and maybe some ancillary system she possesses) that maps it into a bipartite state E_0E_1 (with arbitrary state space $\mathcal{H}_{E_0}\otimes\mathcal{H}_{E_1}$), and $\hat{P}_0$ keeps E_0 and sends E_1 to $\hat{P}_1$. Then, as soon as $\hat{P}_0$ receives θ, she applies some measurement (which may depend on θ) to E_0 to obtain $\hat{x}_0$, and as soon as $\hat{P}_1$ receives E_1, she applies some measurement (which may depend on θ) to E_1 to obtain $\hat{x}_1$, and both send $\hat{x}_0$ and $\hat{x}_1$ immediately to V_0 and V_1, respectively. We will argue that the probability that $\hat{x}_0 = x$ *and* $\hat{x}_1 = x$ is upper bounded by ε as claimed.

Let $|\psi_{A\,E_0E_1}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_{E_0} \otimes \mathcal{H}_{E_1}$ be the state of the tripartite system $A\,E_0E_1$ after $\hat{P}_0$ has applied the quantum operation to the qubit B. Note that in the No-PE model, the quantum operation does not depend on θ. Therefore, the global state $|\psi_{A\,E_0E_1}\rangle$ does not depend on θ.[7] Recall that x is obtained by measuring A in either the computational (if $\theta = 0$) or the Hadamard (if $\theta = 1$) basis. Writing x, θ, etc. as random variables X, Θ, etc., it follows from CIT (specifically Corollary 2.5) that $\mathrm{H}(X|\Theta E_0) + \mathrm{H}(X|\Theta E_1) \geq 1$. Let Y_0 and Y_1 denote the classical information obtained by $\hat{P}_0$ and $\hat{P}_1$ as a result of measuring E_0 and E_1, respectively, with bases that may depend on Θ. By the (generalized) Holevo bound Theorem 2.2, it follows from the above that

$$\mathrm{H}(X|\Theta Y_0) + \mathrm{H}(X|\Theta Y_1) \geq 1;$$

therefore, $\mathrm{H}(X|\Theta Y_i) \geq \frac{1}{2}$ for at least one $i \in \{0,1\}$. By Fano's inequality (Theorem 2.3), we can conclude that the corresponding error probability $q_i = P[\hat{X}_i \neq X]$ satisfies $\mathrm{h}(q_i) \geq \frac{1}{2}$. It thus follows that the failure probability

$$q = P[\hat{X}_0 \neq X \vee \hat{X}_1 \neq X] \geq \max\{q_0, q_1\} \geq \mathrm{h}^{-1}\left(\frac{1}{2}\right),$$

and the probability of V_0 and V_1 accepting, $P[\hat{X}_0 = X \wedge \hat{X}_1 = X] = 1 - q$, is indeed upper bounded by ε, as claimed.

It remains to argue that more than two dishonest provers in the No-PE model cannot do any better. The reasoning is the same as above. Namely, in order to respond in time, the dishonest provers that are closer to V_0 than P must map the qubit A—possibly jointly—into a bipartite state E_0E_1 *without knowing* θ and jointly keep E_0 and send E_1 to the dishonest provers that are "on the other side" of P (i.e., closer to V_1). Then, the reply for V_0 needs to be computed from E_0 and θ (possibly jointly by the dishonest provers that are closer to V_0), and the response for V_1 from E_1 and θ. Thus, it can be argued as above that the success probability is bounded by ε as claimed. ∎

6.2. Reducing the soundness error. In order to obtain a position-verification scheme with a negligible soundness error, we can simply repeat the one-round scheme

[7] We stress that this independence breaks down if $\hat{P}_0$ and $\hat{P}_1$ start off with an entangled state, because then $\hat{P}_1$ can act on his part of the entangled state in a θ-dependent way, which makes the overall state dependent on θ.

$\mathsf{PV}_{\text{BB84}}$ from Figure 6.1. Repeating the scheme n times *in sequence*, where the verifiers launch the next execution only after the previous one is finished, reduces the soundness error to ε^n. Recall that in the No-PE model defined in section 3.2, the adversaries must start every round without preshared entanglement. Therefore, the security of the sequentially repeated scheme follows immediately from the security of the one-round scheme.

COROLLARY 6.2. *In the No-PE model, the n-fold sequential repetition of $\mathsf{PV}_{\text{BB84}}$ from Figure 6.1 is ε^n-sound with $\varepsilon = 1 - \mathrm{h}^{-1}(\frac{1}{2})$.*

In terms of round complexity, a more efficient way of repeating $\mathsf{PV}_{\text{BB84}}$ is by repeating it *in parallel*: V_0 sends n BB84 qubits $H^{\theta_1}|x_1\rangle, \ldots, H^{\theta_n}|x_n\rangle$, and V_1 sends the corresponding bases $\theta_1, \ldots, \theta_n$ to P so that they all arrive at the same time at P's position, and P needs to reply with the correct list $x_1, \ldots, x_n$ in time. This protocol is obviously more efficient in terms of round complexity and appears to be the preferred solution. However, we do not have a proof for the security of the parallel repetition of $\mathsf{PV}_{\text{BB84}}$.

6.3. Position-verification in higher dimensions. The scheme $\mathsf{PV}_{\text{BB84}}$ can easily be extended into higher spatial dimensions. The scheme for d-dimensional space is a generalization of the scheme $\mathsf{PV}_{\text{BB84}}$ in Figure 6.1, where the challenges of the verifiers $V_1, V_2, \ldots, V_d$ form a *sum sharing* of the basis θ, i.e., are random $\theta_1, \theta_2, \ldots, \theta_d \in \{0,1\}$ such that their modulo-2 sum equals θ. As specified in Figure 3.1, the state $H^\theta|x\rangle$ and the shares θ_i are sent by the verifiers to P such that they arrive at P's (claimed) position at the same time. P can reconstruct θ and measure $H^\theta|x\rangle$ in the correct basis to obtain $x' = x$, which he sends to all the verifiers who check if x' arrives in time and equals x.

We can argue security by a reduction to the scheme in one dimension. For the sake of concreteness, we consider three dimensions. For three dimensions, we need a set of (at least) four noncoplanar verifiers $V_0, \ldots, V_3$, and the prover P needs to be located inside the tetrahedron defined by the positions of the four verifiers. We consider a coalition of dishonest provers $\hat{P}_0, \ldots, \hat{P}_\ell$ at arbitrary positions but different from P. We may assume that $\hat{P}_0$ is closest to V_0. It is easy to see that there exists a verifier V_j such that $d(\hat{P}_0, V_j) > d(P, V_j)$. Furthermore, we may assume that V_j is not V_0, and thus we assume for concreteness that it is V_1. We strengthen the dishonest provers by giving them θ_2 and θ_3 for free from the beginning. Since, when θ_2 and θ_3 are given, θ can be computed from θ_1 and vice versa, we may assume that V_1 actually sends θ as a challenge rather than θ_1. But now, θ_2 and θ_3 are just two random bits, independent of θ and x, and are thus of no help to the dishonest provers, and we can safely ignore them.

As $\hat{P}_0$ is further away from V_1 than P is, $\hat{P}_0$ cannot afford to store $H^\theta|x\rangle$ until she has learned θ. Indeed, otherwise V_1 will not get a reply in time. Therefore, before she learns θ, $\hat{P}_0$ needs to apply a quantum transformation to $H^\theta|x\rangle$ with a bipartite output and keep one part of the output, E_0, and send the other part, E_1, to $\hat{P}_1$. Note that this quantum transformation is independent of θ as long as $\hat{P}_0$ does not share an entangled state with the other dishonest provers (who might know θ by now). Then, $\hat{x}_0$ and $\hat{x}_1$, the replies that are sent to V_0 and V_1, respectively, need to be computed from θ and E_0 alone and from θ and E_1 alone. It follows from the analysis of the scheme in one dimension that the probability that both $\hat{x}_0$ and $\hat{x}_1$ coincide with x is at most $\varepsilon = 1 - \mathrm{h}^{-1}(\frac{1}{2})$.

COROLLARY 6.3. *The above generalization of $\mathsf{PV}_{\text{BB84}}$ to d dimensions is ε-sound in the No-PE model with $\varepsilon = 1 - \mathrm{h}^{-1}(\frac{1}{2})$.*

7. Position-based authentication and key exchange. In this section we consider a new primitive: position-based authentication. In contrast to position-verification, where the goal of the verifiers is to make sure that entity P is at the claimed location *pos*, the verifiers want to make sure that a given message m originates from an entity P that is at the claimed location *pos*. We stress that it is not sufficient to first execute a position-verification scheme with P to ensure that P is at position *pos* and then have P send or confirm m, because a coalition of dishonest provers may do a *man-in-the-middle* attack and stay passive during the execution of the position-verification scheme but modify the communicated message m.

Formally, in a position-based authentication scheme the prover takes as input a message m (that can be empty), and the verifiers $V_0, \ldots, V_k$ take as input a message m' and the claimed position *pos* of P, and we require the following security properties.

ε_c-*completeness*. If $m = m'$, P is honest and at the claimed position *pos*, and if there is no (coalition of) dishonest prover(s), then the verifiers jointly accept except with probability ε_c.

ε_s-*soundness*. For any $pos \in \mathrm{Hull}(pos_0, \ldots, pos_k)$ and for any coalition of dishonest provers $\hat{P}_0, \ldots, \hat{P}_\ell$ at locations all different to *pos*, if $m \neq m'$, the verifiers jointly reject except with probability ε_s.

We stress that soundness should hold even when an honest prover is located at *pos* (but also when there is no honest prover at *pos*). Furthermore, in the case of a man-in-the-middle attack P, recall that the adversary fully controls P's clock; thus the adversary can temporarily stop P from executing its program or "fast forward" P's program to a later point (but the adversary cannot reset P and make P rerun its program from its starting point).

We build a position-based authentication scheme based on our position-verification scheme. The idea is to incorporate the message to be authenticated into the replies of the position-verification scheme. Our construction is very generic and may also be useful for turning other kinds of identification schemes (not necessarily position-based schemes) into corresponding authentication schemes. Our aim is merely to show the existence of such a scheme; we do not strive for optimization. We begin by proposing a weak position-based authentication scheme for a 1-bit message m.

7.1. Weak 1-bit authentication scheme. Let PV^ε be a one-round position-verification scheme between $k+1$ verifiers $V_0, \ldots, V_k$ and a prover P. For simplicity we assume, like in the scheme $\mathsf{PV}_{\mathrm{BB84}}$ of section 6, that x and $x'_0, \ldots, x'_k$ are classical (but ch_i are quantum), and Ver accepts if $x'_i = x$ for all i, and thus we understand the output of $\mathsf{Resp}(ch_0, \ldots, ch_k)$ as a single element x' (supposed to be x). We require PV^ε to have perfect completeness and soundness $\varepsilon < 1$. We let $\perp$ be some special symbol. We consider the weak authentication scheme given in Figure 7.1 for a 1-bit message $m \in \{0,1\}$. We assume that m has already been communicated to the verifiers and thus there is agreement among the verifiers on the message to be authenticated. The weak authentication scheme works by executing the one-round position-verification scheme PV^ε, but letting P replace his response x' by $\perp$ with probability q, to be specified later.

We analyze the success probability of an adversary authenticating a bit $m' \in \{0,1\}$. We consider both the case where there is no honest prover present (we call this an *impersonation attack*), and the case where an honest prover is active and authenticates the bit $m \neq m'$ (we call this a *substitution attack*).

The following properties are easy to verify and follow from the security property of PV^ε.

Let $\mathsf{PV}^\varepsilon = (\mathsf{Chlg}, \mathsf{Resp}, \mathsf{Ver})$ be a perfect-complete and ε-sound 1-round position-verification scheme.

0. V_0 generates $(ch_0, \ldots, ch_k)$ and x using Chlg and sends ch_i and x to V_i for $i = 1, \ldots, k$.
1. Every verifier V_i sends ch_i to P in such a way that all ch_i's arrive at the same time at P.
2. When the ch_i's arrive, P computes the authentication tag t as follows and sends it back to all the verifiers.
   If $m = 1$, then $t := \mathsf{Resp}(ch_0, \ldots, ch_k)$. If $m = 0$, then $t := \perp$ with probability q and $t := \mathsf{Resp}(ch_0, \ldots, ch_k)$ otherwise.
3. If different verifiers have received different values for t, or the replies did not arrive in time, the verifiers abort. Otherwise, they jointly accept if $t = x$ or both $m = 0$ and $t = \perp$.

FIG. 7.1. *Generic position-based weak authentication scheme* $\mathsf{wAUTH}^{\varepsilon,q}$ *for* 1*-bit message* m.

LEMMA 7.1. *Let* $\hat{P}$ *be a coalition of dishonest provers not at the claimed position and trying to authenticate message* $m' = 1$. *In case of an impersonation attack, the verifiers accept with probability at most* ε, *and in case of a substitution attack (with* $m = 0$*), the verifiers accept with probability at most* $\delta = (1-q)+q\varepsilon = 1-q(1-\varepsilon) < 1$.

On the other hand, $\hat{P}$ can obviously authenticate $m' = 0$ by means of a substitution attack with success probability 1; however, informally, $\hat{P}$ has bounded success probability in authenticating message $m' = 0$ by means of an impersonation attack unless she uses the tag $\perp$. (This fact is used later to obtain a strong authentication scheme.)

Let us try to extend the above in order to get a strong authentication scheme. Based on the observation that by performing a substitution attack on $\mathsf{wAUTH}^{\varepsilon,q}$, it is easy to substitute the message bit $m = 1$ by $m' = 0$ but nontrivial to substitute $m = 0$ by $m' = 1$, a first approach to obtain an authentication scheme with good security might be to apply $\mathsf{wAUTH}^{\varepsilon,q}$ bitwise to a *balanced encoding* of the message. Such an encoding should ensure that for any distinct messages m and m', there are many positions in which the encoding of m' is 1 but the encoding of m is 0. Unfortunately, this is not good enough. The reason is that P and the verifiers are not necessarily synchronized. For instance, assume we encode $m = 0$ into $c = 010101\ldots01$ and $m' = 1$ into $c' = 101010\ldots10$, and authentication works by doing $\mathsf{wAUTH}^{\varepsilon,q}$ bitwise on all the bits of the encoded message. If $\hat{P}$ wants to substitute $m = 0$ by $m' = 1$, then she can simply do the following. She tries to authenticate the first bit 1 of c' toward the verifiers by means of an impersonation attack. If she succeeds, which she can with constant probability, she simply authenticates the remaining bits $01010\ldots10$ of c' by using P, who is happy to authenticate all of the bits of $c = 010101\ldots01$. Because of this issue of $\hat{P}$ bringing P and the verifiers out of sync, we need to be more careful about the exact encoding we use.

7.2. Secure position-based authentication scheme. We specify a special class of codes, which is strong enough for our purpose.

DEFINITION 7.2. *Let* $c \in \{0,1\}^N$. *A vector* $e \in \{-1,0,1\}^{2N}$ *is called an* embedding *of* c *if by removing all the* -1 *entries in* e *we obtain* c. *Furthermore, for two strings* $c, c' \in \{0,1\}^N$ *we say that* c' λ-dominates c *if for all embeddings* e *and* e' *of* c *and* c' *(at least) one of the following holds:* (a) *the number of positions* $i \in \{1, \ldots, 2N\}$ *for which* $e'_i = 1$ and $e_i < 1$ *is at least* λ, *or* (b) *there exists a*

consecutive sequence of indices I such that the set $J = \{i \in I : e'_i > -1\}$ has size $|J| \geq 4\lambda$ and it contains at least λ indices $i \in J$ with $e_i = -1$.

For instance, let $c = 00\ldots0\,11\ldots1$ and $c' = 11\ldots1\,00\ldots0$, where the blocks of 0's and 1's are of length $N/2$. It is not hard to see that the two codewords $N/4$-dominate each other. However, $\tilde{c}' = 0101\ldots01$ does not dominate $\tilde{c} = 1010\ldots10$, since $\tilde{c}'$ can be embedded into $\ddagger0101\ldots01\ddagger\ddagger\ldots\ddagger$ and $\tilde{c}$ into $1010\ldots10\ddagger\ddagger\ldots\ddagger$, where here and later we use $\ddagger$ to represent -1.

DEFINITION 7.3. *A code $C \subseteq \{0,1\}^N$ is* λ-dominating *if any two codewords in C λ-dominate each other.*

We note that the requirement for λ-dominating codes can be relaxed in various ways to allow a greater range of codes.

Let $\mathsf{wAUTH}^{\varepsilon,q}$ be the above weak authentication scheme satisfying Lemma 7.1. In order to authenticate a message $m \in \{0,1\}^\mu$ in a strong way (with λ a security parameter), an encoding c of m using a λ-dominating code C is bitwise authenticated by means of $\mathsf{wAUTH}^{\varepsilon,q}$, and the verifiers perform statistics over the number of $\perp$'s received. The resulting authentication scheme is given in Figure 7.2; as for the weak scheme, we assume that the message m has already been communicated.

0. P and the verifiers encode m into a codeword $c = (c_1, \ldots, c_N) \in C$ for a λ-dominating code C.
1. For $j = 1, \ldots, N$, the following is repeated in sequence.
   1.1 P authenticates c_j by means of $\mathsf{wAUTH}^{\varepsilon,q}$. Let t_i be the corresponding tag received.
   1.2 If $j > 4\lambda$, the verifiers compute
   $$n_\perp(j) = |\{i \in \{j - 4\lambda, \ldots, j\} : c_i = 0 \wedge t_i = \perp\}|.$$
2. If any of the $\mathsf{wAUTH}^{\varepsilon,q}$ executions fail, or if $n_\perp(j) > 8q\lambda$ for some round $j > 4\lambda$, the verifiers jointly reject. Otherwise, m is accepted.

FIG. 7.2. *A generic position-based authentication scheme* $\mathsf{AUTH}^{\varepsilon,q,\lambda}$.

THEOREM 7.4. *The generic position-based authentication scheme* $\mathsf{AUTH}^{\varepsilon,q,\lambda}$ *(Figure 7.2) is $Ne^{-2q\lambda}$-complete.*

Proof. An honest prover which follows the above scheme can fail only if for some round r, $n_\perp > 8q\lambda$. Using the Chernoff bound [18], the probability of having $n_\perp > 8q\lambda$ at a specific round r is upper bounded by $e^{-2q\lambda}$. Using the union bound for every possible round j, we can bound the failure probability with $Ne^{-2q\lambda}$. ∎

Before we analyze the security of the authentication scheme, let us discuss the possible attacks on it. We treat $\hat{P}$ as a single identity; however, $\hat{P}$ represents a collaboration of adversaries. Similarly, we refer the $k+1$ verifiers as a single entity, V. We point out that we do not assume that honest P and V have synchronized clocks. Therefore, we allow $\hat{P}$ to arbitrarily schedule and interleave the N executions of $\mathsf{wAUTH}^{\varepsilon,q}$ that V performs with the N executions that P performs. The only restriction on the scheduling is that P and V perform their executions of $\mathsf{wAUTH}^{\varepsilon,q}$ in the specified order.

This means that at any point in time during the attack when P has executed $\mathsf{wAUTH}^{\varepsilon,q}$ for the bits $c_1, \ldots, c_{j-1}$ and V has executed $\mathsf{wAUTH}^{\varepsilon,q}$ for the bits $c'_1, \ldots, c'_{j'-1}$ and both are momentarily inactive (at the beginning of the attack $j = j' = 1$), $\hat{P}$ can perform one of the following three actions. (1) Activate V to run

$\mathsf{wAUTH}^{\varepsilon,q}$ on $c'_{j'}$ but not activate P; this corresponds to an impersonation attack. (2) Activate V to run $\mathsf{wAUTH}^{\varepsilon,q}$ on $c'_{j'}$ and activate P to run $\mathsf{wAUTH}^{\varepsilon,q}$ on c_j; this corresponds to a substitution attack if $c_j \neq c'_{j'}$. (3) Activate P to run $\mathsf{wAUTH}^{\varepsilon,q}$ on c_j but not activate V; this corresponds to "fast-forwarding" P. We note that $\hat{P}$'s choice on which action to perform may be adaptive and depend on what she has seen so far. However, since V and P execute $\mathsf{wAUTH}^{\varepsilon,q}$ for each position within c independently, information gathered from previous executions of $\mathsf{wAUTH}^{\varepsilon,q}$ does not improve $\hat{P}$'s success probability to break the next execution.

It is easy to see that any attack with its (adaptive) choices of (1), (2), or (3) leads to embeddings e and e' of c and c', respectively. Indeed, start with empty strings $e = e' = \emptyset$ and update them as follows. For each of $\hat{P}$'s rounds, update e by $e\ddagger$ and e' by $e'c'_{j'}$ if $\hat{P}$ chooses (1), update e by ec_j and e' by $e'c'_{j'}$ if she chooses (2), and update e by ec_j and e' by $e'\ddagger$ if she chooses (3). In the end, complete e and e' by padding them with sufficiently many $\ddagger$'s to have them of length $2N$. It is clear that the obtained e and e' are indeed valid embeddings of c and c', respectively.

THEOREM 7.5. *For any $\varepsilon > 0$ and $0 < q < (1-\varepsilon)/8$, the generic position-based authentication scheme $\mathsf{AUTH}^{\varepsilon,q,\lambda}$ (Figure 7.2) is $2^{-\Omega(\lambda)}$-sound in the No-PE model.*

Proof. Let m and $m' \neq m$ be the messages input by P and the verifiers, respectively, and let c and c' be their encodings. Furthermore, let e and e' be their embeddings, determined (as explained above) by $\hat{P}$'s attack. By the condition on the λ-dominating code C we know that one of the two properties (a) or (b) of Definition 7.2 holds. If (a) holds, the number of positions $i \in \{1,\dots,2N\}$ for which $e'_i = 1$ and $e_i \in \{-1, 0\}$ is λ. In this case, by construction of the embeddings, in his attack $\hat{P}$ needs to authenticate (using $\mathsf{wAUTH}^{\varepsilon,q}$) the bit 1 at least λ times (by means of an impersonation or a substitution attack). By Lemma 2, the success probability of $\hat{P}$ is thus at most δ^λ, which is $2^{-\Omega(\lambda)}$. In the case where property (b) holds, there exists a consecutive sequence of indices I such that the set $J = \{i \in I : e'_i > -1\}$ has size $|J| \geq 4\lambda$ and contains at least λ indices $i \in J$ with $e_i = -1$. For any such index $i \in J$ with $e_i = -1$, $\hat{P}$ needs to authenticate (using $\mathsf{wAUTH}^{\varepsilon,q}$) the bit e'_i by means of an impersonation attack, while he may use $\perp$ for (at most) a $8q$-fraction of those i's.

However, by the ε-soundness of PV^ε, if we require $\varepsilon < 1 - 8q$, the probability of $\hat{P}$ succeeding in this attack is exponentially small in λ. ∎

A possible choice for a dominating code for μ-bit messages is the *balanced repetition code* $C^\mu_{\ell\text{-BR}}$, obtained by applying the code $C_{\ell\text{-BR}} = \{00\dots011\dots1, 11\dots100\dots0\} \subset \{0,1\}^{2\ell}$ bitwise.

LEMMA 7.6. *For any ℓ and μ, the balanced repetition code $C^\mu_{\ell\text{-BR}}$ is $\ell/4$-dominating.*

Proof. Let $c, c' \in \{0,1\}^{2\ell\mu}$ be two distinct code words from $C^\mu_{\ell\text{-BR}}$, and let e and e' be their respective embeddings. Note that c is made up of blocks of 0's and 1's of length ℓ. Correspondingly, e is made up of blocks of 0's and 1's of length ℓ, with $\ddagger$'s inserted at various positions. Let $I_1,\dots,I_{2\mu}$ be the index sets that describe these 0 and 1-blocks of e. In other words, they satisfy $I_j < I_{j+1}$ elementwise, $|I_j| = \ell$, and $\{e_i : i \in I_j\}$ equals $\{0\}$ or $\{1\}$. Furthermore, the sequence of e_i's with $i \in I_1 \cup \dots \cup I_{2\mu}$ equals c, and as such, for any odd j, one of I_j and I_{j+1} is a 0-block, and one is a 1-block. Let $\phi : \{1,\dots,\mu\} \to \{1,\dots,2\mu\}$ be the function such that $I_{\phi(k)}$ is the kth 1-block in $I_1,\dots,I_{2\mu}$. We do the same with c' and e', resulting in blocks $I'_1,\dots,I'_{2\mu}$ and function ϕ'. For any j, we define $cl(I'_j)$ to be the smallest "interval" in $\{1,\dots,4\mu\ell\}$ that contains I'_j.

For 1-blocks I_j and $I'_{j'}$, we say that I_j *overlaps* with $I'_{j'}$ if $|I_j \cap cl(I'_{j'})| \geq 3\ell/4$. We make the following case distinction.

Case 1: $I_{\phi(k')}$ *does not overlap with* $I'_{\phi'(k')}$ *for some* k'. If all the indices in $I_{\phi(k')} \setminus cl(I'_{\phi'(k')})$ are larger than those in $cl(I'_{\phi'(k')})$, then $e'_i = 1$ for all $i \in I'_{\phi'(1)} \cup \cdots \cup I'_{\phi'(k')}$, but $e_i < 1$ for at least $\ell/4$ of these i's. A similar argument can be used when all these indices are smaller than those in $cl(I'_{\phi'(k')})$. If neither of the above holds, then $e'_i = 1$ for all $i \in I'_{\phi'(k')}$, but $e_i < 1$ for at least $\ell/4$ of these i's. Hence, property (a) of Definition 7.2 is satisfied (with parameter $\ell/4$).

Case 2: $I_{\phi(k)}$ *overlaps with* $I'_{\phi'(k)}$ *for every* k. Since c and c' are distinct, and by the structure of the code, there must exist two subsequent 1-blocks $I_{\phi(k)}$ and $I_{\phi(k+1)}$ such that the number of 0-blocks between $I_{\phi(k)}$ and $I_{\phi(k+1)}$ is strictly smaller than the number of 0-blocks between the corresponding 1-blocks $I'_{\phi'(k)}$ and $I'_{\phi'(k+1)}$. If there is no 0-block between $I_{\phi(k)}$ and $I_{\phi(k+1)}$ and (at least) one 0-block between $I'_{\phi'(k)}$ and $I'_{\phi'(k+1)}$, then by the assumption on the overlap, at least half of the indices i in the 0-block $I'_{\phi'(k)+1}$ satisfy $e_i = \ddagger$. If there is one 0-block between $I_{\phi(k)}$ and $I_{\phi(k+1)}$ and two 0-blocks between $I'_{\phi'(k)}$ and $I'_{\phi'(k+1)}$, then at least a quarter of the indices $i \in I'_{\phi'(k)+1} \cup I'_{\phi'(k)+2}$ satisfy $e_i = \ddagger$. In both (sub)cases, property (b) of Definition 7.2 is satisfied (with $\lambda = \ell/4$). □

Plugging in the concrete secure position-verification scheme from section 6.3, we obtain a secure realization of a position-based authentication scheme in $\mathbb{R}^d$, in the No-PE model.

7.3. Position-based key exchange. The goal of a position-based key exchange scheme is to have the verifiers agree with honest prover P at location *pos* on a key $K \in \{0,1\}^L$ in such a way that no dishonest prover has any (nonnegligible amount of) information on K beyond its bit-length L, as long as she is not located at *pos*.[8] Formally, we require the following security properties.

ε_c*-completeness.* If P is honest and at the claimed position *pos*, and if there is no (coalition of) dishonest prover(s), then P and $V_0, \ldots, V_k$ output the same key K of positive length, except with probability ε_c.

ε_s*-security.* For any position $pos \in \text{Hull}(pos_0, \ldots, pos_k)$ and for any coalition $\hat{P}$ of dishonest provers at locations all different to *pos*, the hybrid state ρ_{KE}, consisting of the key K output by the verifiers and the collective quantum system of $\hat{P}$ at the end of the scheme, satisfies $\delta(\rho_{KE}, \rho_{\tilde{K}} \otimes \rho_E) \leq \varepsilon_s$, where $\tilde{K}$ is chosen independently and at random of the same bit-length as K.

Note that the security properties only ensure that the *verifiers* can be convinced that $\hat{P}$ has no information on the key they obtain; no such security is guaranteed for P. Indeed, $\hat{P}$ can always honestly execute the scheme with P, acting as verifiers. Also note that the security properties do not provide any guarantee to the verifiers that P has obtained the *same* key that was output by the verifiers, in case of an active attack by $\hat{P}$, but this feature can always be achieved, e.g., with the help of a position-based authentication scheme by having P send an authenticated hash of his key.

A position-based key exchange scheme can easily be obtained by taking any quantum key distribution (QKD) scheme that requires authenticated communication, and do the authentication by means of a position-based authentication scheme, like the scheme from the previous section. One subtlety to take care of is that QKD schemes usually require *two-way* authentication, whereas position-based authentication only provides authentication from the prover to the verifiers. However, this problem can

[8] The length L of the key may depend on the course of the scheme. In particular, an adversary may enforce it to be 0.

easily be resolved as follows. Whenever the QKD scheme instructs V_0 (acting as Alice in the QKD scheme) to send a message m in an authenticated way to P (acting as Bob), V_0 sends m without authentication to P, but in the next step P authenticates the message m' he has received (supposedly $m' = m$) toward the verifiers, who abort and output an empty key K in case the authentication fails.

Using standard BB84 QKD, we obtain a concrete position-based key exchange scheme. The security of that scheme follows from the security of the BB84 protocol [35, 44, 38, 4, 41, 8] and of the position-based authentication scheme.

8. Conclusion, discussion, and open questions. Continuing a very recent line of research [36, 37, 15, 34, 28, 31], we have given a general proof that information-theoretic position-verification quantum schemes are impossible, thereby answering an open question about the security of schemes proposed by [31] to the negative. On the positive side, we have provided schemes secure under the assumption that dishonest provers do not use preshared entanglement.

Regarding our positive results, we now briefly discuss some of our assumptions and their impact when trying to relax them.

- *Noise-free quantum information processing:* Manipulating quantum states is an extremely challenging task and far from noise-free with current technology. To this end, we would like to point out that for all our positive results, the honest parties do not need to perform sophisticated quantum computations; communicating and measuring-upon-arrival BB84 qubits is sufficient. Furthermore, it is not too hard to see that our schemes can be made robust against a certain amount of noise. Working out the details is tedious but in the end straightforward. A related issue is that of *losses* in the communication of quantum states: if there are too many slots where no qubit arrives (because it got lost or because none was emitted), our scheme cannot work. A natural approach to dealing with a too large number of losses is to consider schemes with more than two bases.
- *Fixed communication velocity, and instantaneous local computations:* Relaxing the assumption on the fixed communication velocity by allowing the dishonest parties to communicate (slightly) faster, and/or taking into account some (small amount of) time for P's local computations, has the effect that P's position cannot be verified *exactly* but only up to some radius. Again, working out the details is beyond the scope of this work.
- *Flat space(time):* We do not consider *curved* spaces, like the surface of a sphere, or *curved spacetime* in the presence of matter (or energy). In these cases, the (im)possibility of position-based cryptography is likely to depend on the geometry of the space(time).

Our results naturally lead to the following question: How much entanglement is needed in order to break position-verification protocols? Can we show security in the bounded-quantum-storage model [21] where adversaries are limited to store, say, a linear fraction of the communicated qubits, thus restricting the amount of available entanglement? These questions do remain open in general but have already triggered quite a bit of follow-up work [3, 11, 46] providing partial answers (see section 1.3 for details).

Appendix A. Proof of Lemma 2.1. In this section we prove the following lemma (Lemma 2.1): *For any tri-partite state ρ_{ABY} with classical Y,*

$$\mathrm{H}(A|BY) = \sum_y P_Y(y)\, \mathrm{H}(\rho_{AB}^y|B).$$

Proof. We first consider the case of an "empty" B. Y being classical means that ρ_{AY} is of the form $\rho_{AY} = \sum_y P_Y(y)\rho_A^y \otimes |y\rangle\langle y|$. Let us write $\lambda_1^y, \ldots, \lambda_n^y$ for the eigenvalues of ρ_A^y. Note that the eigenvalues of ρ_{AY} are given by $P_Y(y)\lambda_i^y$ with $y \in \mathcal{Y}$ and $i \in \{1, \ldots, n\}$. It follows that

$$\begin{aligned}
\mathrm{H}(\rho_{AY}|Y) &= \mathrm{H}(\rho_{AY}) - \mathrm{H}(\rho_Y) \\
&= -\mathrm{tr}\big(\rho_{AY}\log(\rho_{AY})\big) + \mathrm{tr}\big(\rho_Y \log(\rho_Y)\big) \\
&= -\Big(\sum_{y,i} P_Y(y)\lambda_i^y \log\big(P_Y(y)\lambda_i^y\big) - \sum_y P_Y(y)\log\big(P_Y(y)\big)\Big) \\
&= -\sum_y P_Y(y)\sum_i \lambda_i^y \log\big(\lambda_i^y\big) = \sum_y P_Y(y)\,\mathrm{H}(\rho_A^y)\,.
\end{aligned}$$

In general, we can conclude that

$$\begin{aligned}
\mathrm{H}(\rho_{ABY}|BY) &= \mathrm{H}(\rho_{ABY}) - \mathrm{H}(\rho_{BY}) \\
&= \sum_y P_Y(y)\,\mathrm{H}(\rho_{AB}^y) - \sum_y P_Y(y)\,\mathrm{H}(\rho_B^y) \\
&= \sum_y P_Y(y)\big(\mathrm{H}(\rho_{AB}^y) - \mathrm{H}(\rho_B^y)\big) \\
&= \sum_y P_Y(y)\,\mathrm{H}(\rho_{AB}^y|B)\,,
\end{aligned}$$

which proves the claim. ∎

Appendix B. Instantaneous nonlocal quantum computation with N parties. We generalize the above result to any N-party distributed computation by generalizing Theorem 4.3 to the case of N parties. We assume that some distinguished user holds the system A and the information $x \in \mathcal{X}$, while for the rest, each user $p = 1 \ldots N-1$ holds the system B_p and the classical input $y_p \in \mathcal{Y}_p$. Let us call the user who holds $\mathcal{H}_A$ Alice, and the rest of the users $\mathcal{U}_p$ with $p = 1 \ldots N-1$. Denote $\mathcal{H}_{all} \triangleq \mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_{N-1}}$. The parties share an arbitrary and unknown state $|\psi\rangle \in \mathcal{H}_{all} \otimes \mathcal{H}_E$ and a unitary operation U defined on $\mathcal{H}_{all}$. The unitary U is determined by x and $\{y_p\}$ out of some fixed family of unitaries.

THEOREM B.1. *For every family $\{U_{x,y_1,\ldots,y_{N-1}}\}$ of unitaries defined on $\mathcal{H}_{all}$ and for every $\varepsilon > 0$, given sufficiently many pairwise shared EPR pairs, there exist families $\{\mathcal{A}_x\}, \{\mathcal{B}_{y_1}^1\}, \ldots, \{\mathcal{B}_{y_{N-1}}^{N-1}\}$ of local operations, acting on Alice's and $\mathcal{U}_p$'s respective sides, with the following property. For any initial state $|\psi\rangle \in \mathcal{H}_{all} \otimes \mathcal{H}_E$ and for every $x \in \mathcal{X}$ and $y_1, \ldots, y_{N-1} \in \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_{N-1}$, the joint execution $\mathcal{A}_x \otimes \mathcal{B}_{y_1}^1 \otimes \cdots \otimes \mathcal{B}_{y_{N-1}}^{N-1}$ transforms the state $|\psi\rangle$ into $|\varphi'\rangle$ and provides classical outputs k to Alice and ℓ_p to $\mathcal{U}_p$, such that the following holds except with probability ε. The state $|\varphi'\rangle$ coincides with $|\varphi\rangle = (U_{x,y_1,\ldots,y_{N-1}} \otimes \mathbb{I}_E)|\psi\rangle$ up to local qubitwise operations on systems A and B_p for $p = 1 \ldots N-1$ that are determined by k and $\{\ell_p\}$.*

Proof. As in the two-party case, we may assume that Alice holds $|\psi\rangle$ and that for each player $\mathcal{U}_p$, $\dim \mathcal{H}_{B_p} = 1$. We prove the theorem by induction on the number of parties. As we have already proven the above for $N = 2$ (and the case of $N = 1$ is trivial), let us assume that the proposition holds for $N = c$ and show that it also holds for $N = c + 1$.

1. Alice begins by teleporting* the state $|\psi\rangle$ to $\mathcal{U}_1$ through teleportation channel number x she shares with $\mathcal{U}_1$. Let $k_\circ \in \{0,1,2,3\}^n$ be the outcome of her measurement performed during the teleportation*.

2. For every $i = 1 \ldots |\mathcal{X}|$, denote with $|\varphi_i\rangle$ the state at $\mathcal{U}_1$'s end of the ith teleportation channel. Next, for $i = 1, \ldots, |\mathcal{X}|$, users $\mathcal{U}_1$ to $\mathcal{U}_c$ perform the scheme given by the induction assumption[9] on the input state $|\varphi_i\rangle$ with respective classical information $((i, y_1), y_2, y_3, \ldots, y_c)$, and with $\{U^i_{y_1,\ldots,y_c} := U_{x=i,y_1,\ldots,y_c}\}$ being the family of unitaries. At the end of the induction step $\mathcal{U}_1$ holds the state $|\varphi'_i\rangle$ and each of $\mathcal{U}_p$ obtains a classical output ℓ^i_p,[10] such that for every i the state $|\varphi'_i\rangle$ coincides with $(U_{x=i,y_1,\ldots,y_{N-1}} \otimes \mathbb{I}_E)|\varphi_i\rangle$ up to local qubitwise operations determined by $\{\ell^i_p\}$.
3. For every i, $\mathcal{U}_1$ teleports* $|\varphi'_i\rangle$ back to Alice, using teleportation channel number $|\mathcal{X}| + i$. Let $\ell_{\circ,i} \in \{0, 1, 2, 3\}^n$ be the outcome of his measurement performed during the teleportation*.
4. Alice specifies the state at her end of teleportation channel number $|\mathcal{X}| + x$ to be the state $|\varphi'\rangle$.

Clearly, if $k_\circ = 0 \cdots 0$, then the parties $\mathcal{U}_1, \ldots, \mathcal{U}_c$ on teleportation channel $i = x$ perform instantaneous quantum computation of the unitary $(U_{x,y_1,\ldots,y_c} \otimes \mathbb{I}_E)$ on the state $|\psi\rangle$, obtaining the state $|\varphi'_x\rangle$ which coincides with $(U_{x,y_1,\ldots,y_c} \otimes \mathbb{I}_E)|\psi\rangle$ up to some local qubitwise operations determined by their classical outputs $\ell^x_1, \ldots, \ell^x_c$, that is, $|\varphi'_x\rangle = (W_{\ell^x_1,\ldots,\ell^x_c} U_{x,y_1,\ldots,y_c} \otimes \mathbb{I}_E)|\psi\rangle$, where W is a tensor product of Pauli matrices determined by their classical input. The state $|\varphi'\rangle$ obtained by Alice at the $|\mathcal{X}| + x$ teleportation channel coincides with $|\varphi'_x\rangle$ up to local qubitwise operations determined by $\ell_{\circ,x}$, which proves the theorem for this case.

On the other hand, assume $k_\circ \neq 0 \cdots 0$; then by the induction assumption

$$\begin{aligned} |\varphi'\rangle &= (V_{\ell_{\circ,x}} W_{\ell^x_1,\ldots,\ell^x_c} U_{x,y_1,\ldots,y_c} V_{k_\circ} \otimes \mathbb{I}_E)|\psi\rangle \\ &= (V_{\ell_{\circ,x}} W_{\ell^x_1,\ldots,\ell^x_c} U_{x,y_1,\ldots,y_c} V_{k_\circ} U^\dagger_{x,y_1,\ldots,y_c} \otimes \mathbb{I}_E)|\varphi\rangle, \end{aligned}$$

where $V_{\ell_{\circ,x}}$ and $V_{k_\circ}$ are tensor products of Pauli matrices, and $W_{\ell^x_1,\ldots,\ell^x_c}$ is the local qubitwise (Pauli) operations asserted by the induction assumption. Thus, setting $|\psi'\rangle := |\varphi'\rangle$, $x' := (x, k_\circ)$, $y'_1 := (y_1, \ell_\circ, \ell^x_1)$, and $y'_p := (y_p, \ell^x_p)$ for $p = 2 \ldots c$, and letting

$$U'_{x',y'_1,\ldots,y'_c} := U_{x,y_1,\ldots,y_c} V_{k_\circ} U^\dagger_{x,y_1,\ldots,y_c} W_{\ell^x_1,\ldots,\ell^x_c} V_{\ell_{\circ,x}},$$

the state $|\varphi\rangle$ can be written as $|\varphi\rangle = (U'_{x',y'_1,\ldots,y'_c} \otimes \mathbb{I}_E)|\psi'\rangle$. Again, we are back to the original problem of applying a unitary, $U'_{x',y'_1,\ldots,y'_c}$, to a state, $|\psi'\rangle$, held by Alice, where the unitary depends on classical information x' and $\{y'_p\}$, known by Alice and the users $\mathcal{U}_p$, respectively. We complete the proof by recalling that the success probability per round is a constant which depends only on $\dim \mathcal{H}_{all}$. Assuming a sufficient number of pairwise shared EPR pairs, reapplying the above procedure sufficiently many times to the resulting new problem instances guarantees that, except with arbitrarily small probability, the state $|\varphi'\rangle$ will be of the required form at some point. □

Acknowledgments. We thank Charles Bennett, Frédéric Dupuis, and Louis Salvail for interesting discussions. H.B. would like to thank Sandu Popescu for explaining Vaidman's scheme and pointing [19] out to him.

[9] To be more precise, the scheme is performed with the given instance $\mathbf{U}$, reduced to the case of c classical inputs, by "merging" the first two inputs, i.e., $\{U_{z_1,z_2,\ldots,z_c}\}_{z_1 \in (\mathcal{X} \times \mathcal{Y}_1), z_2 \in \mathcal{Y}_2, \ldots, z_c \in \mathcal{Y}_c}$.

[10] To simplify notation, we denote by ℓ^i_1 the classical information k^i that $\mathcal{U}_1$ obtains when acting as the distinguished user in the scheme given by the induction assumption.

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