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LOGICS AND ALGEBRAS FOR MULTIPLE PLAYERS

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Abstract. We study a generalization of the standard syntax and game-theoretic semantics of logic, which is based on a duality between two players, to a multiplayer setting. We define propositional and modal languages of multiplayer formulas, and provide them with a semantics involving a multiplayer game. Our focus is on the notion of equivalence between two formulas, which is defined by saying that two formulas are equivalent if under each valuation, the set of players with a winning strategy is the same in the two respective associated games. We provide a derivation system which enumerates the pairs of equivalent formulas, both in the propositional case and in the modal case. Our approach is algebraic: We introduce multiplayer algebras as the analogue of Boolean algebras, and show, as the corresponding analog to Stone's theorem, that these abstract multiplayer algebras can be represented as concrete ones which capture the game-theoretic semantics. For the modal case we prove a similar result. We also address the computational complexity of the problem whether two given multiplayer formulas are equivalent. In the propositional case, we show that this problem is co-NP-complete, whereas in the modal case, it is PSPACE-hard.

§1. Introduction. During the second half of the twentieth century, games have become standard tools in many branches of formal logic and theoretical computer science. Manifestations of this development include *dialogue games* in formal argumentation theory, dating back to Lorenzen (1955); *model comparison games* used in model theory, such as the back-and-forth games introduced by Fraïssé (1950) and Ehrenfeucht (1961); *evaluation games*, introduced by Henkin (1959) but very much extended and promoted by Hintikka (1973), which characterize the semantics of formal languages; and finally the *satisfiability games*, used by model theorists to construct models by games (cf. Hodges, 1985), and which can be seen as providing a sophisticated game-theoretic account of semantic tableaux. For a survey of the role of game-theoretic concepts in logic, and conversely, of the increasing importance of logical notions in game theory, we refer to van Benthem (2009). Virtually *all* applications of games in logic or computer science concern the interaction between *two* players, with the game-theoretic opposition naturally corresponding to a logical duality. As examples we mention the dualities between models and proofs in satisfiability games, and between universal and existential quantification in evaluation games.

From a general game-theoretical perspective (Osborne & Rubinstein, 1994) however, it is neither natural nor necessary to restrict attention to two-player games. This has certainly been realized in closely related areas such as multiagent theory, where modal systems

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have been introduced to formalize the power of players (or coalitions of players) to achieve certain outcomes in multiplayer games (see for instance Pauly, 2002, or Alur *et al.*, 2002). While these are examples of logics designed to reason *about* multiplayer games, recently, some first steps have been made to introduce multiplayer games *into* logic. We are aware of three proposals in this direction. The first two of these are partly motivated by the desire to model (in)dependencies in logics related to Hintikka's independence-friendly logic. Abramsky (2007) addresses the question which kind of logic has a natural semantics in multiplayer games. He generalizes the syntax of standard logical languages by introducing connectives and quantifiers that are indexed by the various players. The game-theoretic semantics of this language is based on multiple concurrent strategies, which can be formalized as closure operators on certain domains. Väänänen (2007) introduces *team logic* as a basic logic of functional dependencies, and provides a game-theoretic characterization of its expressive power in terms of certain Ehrenfeucht–Fraïssé games. The third proposal, which turned out to be related to a fairly simple fragment of Abramsky's formalism, was studied in detail by Tulenheimo & Venema (2008).

Since it is the latter formalism that forms the topic of the current paper, we now introduce its basic ideas. Recall that in Hintikka's game-theoretic semantics of propositional logic, a formula φ is true (false) with respect to a valuation V, if there is a winning strategy for Verifier/Éloise (respectively, for Falsifier/Abélard) in the two-player, perfect information game $\mathcal{E}(\varphi, V)$ associated with φ and V. In this game, the connectives $\vee$ and $\wedge$ correspond to choices for Éloise and Abélard, respectively, while negation is interpreted in terms of a role switch between the two players. But from the observation that classical propositional logic involves choice and role switch in a *two*-player setting, it takes a small step to enter the world of *multi*-player logic. More specifically, we will consider a language $PL_{\mathtt{I}}$, indexed by a set $\mathtt{I}$ of players, in which there is a *family* of 'disjunctions',

a disjunction/choice operator $\vee_i$, for each player $i \in \mathtt{I}$,

and another family of 'negations',

a negation/role switch operation $\neg_{ij}$, for each pair i, j of players in $\mathtt{I}$.

Our game-theoretic semantics will generalize the two-player evaluation games for classical propositional logic in a very natural way. This 'multiplication of players' can be applied to any connective that comes in a pair of duals, corresponding to a choice for one of the two opposite players. In this paper we will consider a multiplayer version of modal logic, which we can obtain from multiplayer propositional logic by adding

a modality $\Diamond_i$, for each player $i \in \mathtt{I}$.

In the game-theoretic semantics, the modality $\Diamond_i$ requires player i to pick a successor of the current evaluation point in the Kripke structure.

Two notions arise naturally in this setting: equivalence and i-satisfiability. We say that a formula is *i-satisfiable* if we can find a valuation under which player i has a winning strategy for the given formula in the associated evaluation game. Two formulas are called *equivalent* if under each valuation, the set of players with a winning strategy is the same in the two respective associated games. The main purpose of this paper is to study these two notions; and since i-satisfiability can be expressed in terms of equivalence, we may concentrate on the latter concept. In particular, we shall be interested in methods for finding out whether two given formulas are equivalent.

We will focus on a *mathematical* treatment of this question, since we believe this perspective to be of interest in its own right. As a consequence of this, we leave aside the more philosophical matter as to what (if anything) formulas of this logic *denote*. We confine ourselves to noting that perhaps, multiplayer formulas should be seen as terms of a process-algebra language, witnessing the interaction between various agents, rather than as syntactic entities denoting propositions in a classical sense.

In order to get a grip on the notion of equivalence between multiplayer formulas, we take an *algebraic* perspective. Roughly speaking, we follow the approach of algebraic logic analogous to the algebraization of classical propositional logic in the class of Boolean algebras. Restricting our attention to the propositional case for the moment, we think of multiplayer formulas as terms in the obvious algebraic similarity type, and provide an algebraic encoding of the 'logical' semantics of this language, in terms of a single, small algebra $\mathbb{O}_{\mathtt{I}}$ (corresponding to the Boolean algebra $\mathbb{2}$ of truth values). Thus we have turned the problem of finding a derivation system generating all pairs of equivalent formulas, into the algebraic problem of *axiomatizing* the set of equations that are valid in this algebra $\mathbb{O}_{\mathtt{I}}$. We propose a (fully equational) axiomatization in Definition 3.24, thus introducing a variety $\mathsf{PA}_{\mathtt{I}}$ of *multiplayer algebras*. The main technical result in the paper is Theorem 3.28, a *representation* result for these multiplayer algebras, analogous to Stone's representation theorem for Boolean algebras. More precisely, our result states that every abstract multiplayer algebra can be represented as a concrete one (corresponding to the fields of sets in the Boolean case). Since these concrete algebras are directly obtained from the algebra $\mathbb{O}_{\mathtt{I}}$, it follows immediately that the variety $\mathsf{PA}_{\mathtt{I}}$ is in fact generated by $\mathbb{O}_{\mathtt{I}}$. More or less as an aside, we make some categorical observations, showing that, much in the spirit of Stone duality, the natural constructions between concrete and abstract algebras give rise to a (dual) adjunction between the category of sets with functions and that of multiplayer algebras with homomorphisms, and that this adjunctions provides a categorical duality in the finite case.

These multiplayer algebras are much more than just a convenient tool for addressing the equivalence problem for multiplayer formulas; we believe that they are worth a study in their own right, and this has in fact been a separate motivation for our investigations. The point is that multiplayer algebras provide a nice and natural generalization of Boolean algebras. Whereas the latter structures have a natural internal order duality incarnated by the negation operator, multiplayer algebras are subject to a family of symmetries, indexed by the set of players. In fact, multiplayer algebras come with a collection of (quasi-)orders (one for each player), each of which is intricately related to the algebra structure. Thus multiplayer algebras provide fascinating examples of ordered algebraic structures.

Returning to the problem of understanding the notion of equivalence between multiplayer formulas, we find as a consequence of our representation result that the axiomatization of multiplayer algebras yields a sound and complete derivation system for the notion of equivalence between propositional multiplayer formulas. Similarly, in the modal case we prove a multiplayer analog of the Jónsson–Tarski representation theorem for modal algebras. This result, Theorem 4.43, states that every algebra in the axiomatically defined variety of multiplayer modal algebras can be represented as a concrete algebra associated with some Kripke frame.

Finally, next to the algebraic results on multiplayer equivalence, as a second contribution of this paper we make some observation of a *computational* nature. More precisely, we show that the problem whether two multiplayer formulas are equivalent is $\mathsf{co\text{-}NP}$-complete in the propositional case (Theorem 5.50), and PSPACE-hard in the modal case (Theorem 5.51).

Overview of the paper. In the next section we provide a detailed introduction to both the propositional and the modal variant of multiplayer logic. After introducing the syntax, we provide not only a game-theoretic semantics, but also a compositional, Tarskian one. In Section 3 we develop our algebraic perspective, introducing the algebra $\mathbb{O}_{\mathtt{I}}$ with its induced class of concrete algebras, and the axiomatically defined class $\mathsf{PA}_{\mathtt{I}}$ of *multiplayer algebras*. We make some preliminary observations about these algebras, prove the Representation Theorem 3.28, and then make some category-theoretic observations. Section 4 extends these results to the modal setting. We introduce a class of concrete algebras associated with Kripke frames, and an axiomatically defined class of multiplayer modal algebras. The main result of this section, Theorem 4.43, is the multiplayer version of the Jónsson-Tarski representation theorem. In Section 5 we consider the computational complexity of the problem whether two given multiplayer formulas are equivalent, proving this problem to be $\mathsf{co\text{-}NP}$-complete in the propositional case (Theorem 5.50), and PSPACE-hard in the modal case (Theorem 5.51). Finally, Section 6 lists some open problems concerning the material in this paper, and some ideas for further research.

§2. Multiplayer logics.

In this section we will introduce various versions of multiplayer logic. We will provide the formal syntax and semantics, and make some first observations about the arising formalisms. We start with a detailed discussion of the basic system of *propositional* multiplayer logic, and then move on to some of its variations and special cases. Finally we discuss a *modal* extension of the basic formalism.

CONVENTION 2.1. *Throughout this paper* $\mathtt{I}$ *denotes a finite set of size at least 2. Elements of* $\mathtt{I}$ *will be referred to as* players *or* agents, *and denoted by the letters* $i, j, k, \ldots$. *Sometimes we will use* κ *and* λ *to denote fixed agents; in this case it is always understood that* $\kappa \neq \lambda$.

We will write $2 = \{0, 1\}$ and intuitively think of the elements of this set as outcome values of games, with '1' denoting 'winning', and '0', 'losing'. The set of maps from $\mathtt{I}$ to 2 is denoted as $2^{\mathtt{I}}$; intuitively we think of a map $f \in 2^{\mathtt{I}}$ as identifying the agents i such that $f(i) = 1$ as winners, and the other players as losers.

We also fix a countable set $\mathtt{X} = \{x, y, \ldots\}$ of (propositional) variables.

2.1. Propositional multiplayer logic: syntax and semantics.

Let us start with the *syntax*. There is a large number of propositional connectives (including constants) to be considered in the multiplayer setting. Further on we will discuss which of these can be considered as primitives, and which ones as abbreviated operators. For the time being we include all the connectives that will be considered in this paper.

DEFINITION 2.2. *The language* $PL_{\mathtt{I}}$ *of* multiplayer propositional logic *can be defined using the following induction:*

$$\varphi ::= x \mid \bot \mid \top \mid \bot_i \mid \top_i \mid \varphi \vee_i \varphi \mid \neg_{ij}\varphi,$$

where $x \in \mathtt{X}$ *and* $i, j \in \mathtt{I}$. *Elements of* $PL_{\mathtt{I}}$ *will be called* (propositional, multiplayer) formulas. *The (obviously defined) set of subformulas of a formula* φ *is denoted as* $Sfor(\varphi)$.

The *semantics* of this language can be defined in two ways. We start with an informal description of the multiplayer *evaluation game* $\mathcal{E}(\varphi, V)$ associated with a formula φ and a valuation V. Basically, a match of this game consists of the set $\mathtt{I}$ of players moving a token from one position to another. The key idea of the game is that whenever the game arrives at a position involving a formula $\psi_0 \vee_i \psi_1$, player i determines the next position by picking one of the disjuncts, ψ_0 or ψ_1. The intuitive meaning of the connective $\neg_{ij}$ is that the players i and j *switch roles*, and in order to formalize this properly we follow Tulenheimo & Venema (2008) by making *role distributions* an explicit part of the game.

DEFINITION 2.3. *A* role distribution *is a permutation* $\rho : \mathtt{I} \to \mathtt{I}$ *on the set of players. The set of role distributions on* $\mathtt{I}$ *is denoted as* $RD(\mathtt{I})$. *The identity map is denoted as id; the* transposition *that swaps the agents i and j, while keeping all other players fixed, is denoted as* $[i, j]$. *The composition of two permutations* σ *and* ρ *is denoted as* $\sigma \circ \rho$.

A *position* is a pair (ψ, ρ) consisting of a formula ψ and a role distribution ρ. Intuitively, at position (ψ, ρ), the role distribution indicates which roles the various agents play, with $\rho(i)$ denoting the agent whose role i is taking, and $\rho^{-1}(i)$ referring to the player who takes the role of i. Accordingly, at position $(\psi_0 \vee_i \psi_1, \rho)$ it is player $\rho^{-1}(i)$ who picks a disjunct ψ_q, thus making (ψ_q, ρ) the next position of the match. And at position $(\neg_{ij}\psi, \rho)$ the next position is determined as the pair $(\psi, \rho \circ [i, j])$. Here $\rho \circ [i, j]$ is the role distribution ρ_{ij} given by $\rho_{ij}(i) = \rho(j)$, $\rho_{ij}(j) = \rho(i)$ and $\rho_{ij}(k) = \rho(k)$ for all $k \notin \{i, j\}$.

Clearly, each move of the game reduces the complexity of the formula, so that ultimately, any match will arrive at a position of the form (ψ, ρ) with ψ an *atomic* formula, that is, $\psi \in \mathtt{X} \cup \{\bot, \top, \bot_i, \top_i\}$. At this moment the game comes to an end, and we need to declare the winner(s) and loser(s) of the match. The meaning of the constants is as follows: nobody wins at position $(\bot, \rho)$, everybody wins at position $(\top, \rho)$, everybody but player $\rho^{-1}(i)$ wins at $(\bot_i, \rho)$, and only player $\rho^{-1}(i)$ wins at $(\top_i, \rho)$. For a position of the form (x, ρ) we look at the *valuation* in order to find out who wins and who loses.

DEFINITION 2.4. *A* valuation *is a map* $V : \mathtt{X} \to 2^{\mathtt{I}}$. *Depending on whether* $V(x)(i)$ *is 1 or 0, we say that, according to* V, *player i* wins *or* loses x, *respectively.*

To finish our informal introduction of the evaluation game, when a match arrives at a position of the form (x, ρ), the winners of the match are exactly those players i for which $V(x)(\rho(i)) = 1$.

Summarizing, we define the evaluation game as follows.

DEFINITION 2.5. *Given a* $PL_{\mathtt{I}}$*-formula* φ *and a valuation* V, *we define the* evaluation game $\mathcal{E}(\varphi, V)$ *as the multiplayer graph game played on the set* $Sfor(\varphi) \times RD(\mathtt{I})$, *according to the rules given by Table 1.*

We shall be mainly interested in the following questions:

Table 1. *The evaluation game* $\mathcal{E}(\varphi, V)$ *for* $PL_{\mathtt{I}}$

Position	Player	Admissible moves	Winners
$(\bot, \rho)$	-	-	$\varnothing$
$(\top, \rho)$	-	-	$\mathtt{I}$
$(\bot_i, \rho)$	-	-	$\{j \in \mathtt{I} \mid \rho(j) \neq i\}$
$(\top_i, \rho)$	-	-	$\{j \in \mathtt{I} \mid \rho(j) = i\}$
(x, ρ)	-	-	$\{i \in \mathtt{I} \mid V(x)(\rho(i)) = 1\}$
$(\chi_1 \vee_i \chi_2, \rho)$	$\rho^{-1}(i)$	$\{(\chi_q, \rho) \mid q \in \{1, 2\}\}$	-
$(\neg_{ij}\chi, \rho)$	-	$\{(\chi, \rho \circ [i, j])\}$	-

What is a suitable notion of equivalence between multiplayer formulas,
and how can we determine when two formulas are equivalent?

Clearly, in order to answer these questions we need to study not so much single matches of evaluation games, but rather the connections between different matches of the same game, and the relation between different games. More specifically, we need to understand the power of players to achieve certain outcomes when playing evaluation games, and for this purpose we will employ the notion of a *(winning) strategy*. For the sake of completeness we will provide a formal definition of this notion. In general however, we advise the reader to think of a winning strategy for player i simply as a way of playing which guarantees her to win the match, no matter what her opponents do.

DEFINITION 2.6. *A* graph game *is a structure* $\mathbb{G} = \langle G, F, G_i, E, W\rangle_{i\in\mathtt{I}}$ *such that* G *is a set of objects called* positions, *the collection* $F \cup \{G_i \mid i \in \mathtt{I}\}$ *is a partition of* G, $\langle G, E\rangle$ *is a directed acyclic graph such that* $E[d] = \varnothing$ *for all* $d \in F$, *and* $W : F \to 2^{\mathtt{I}}$ *is a map associating with each position* $d \in F$ *a set of winners. Given a graph game* $\mathbb{G}$ *and a position* p *in* $\mathbb{G}$, *we let* $\mathbb{G}@p$ *denote the game* $\mathbb{G}$ initialized at p.

Positions $p \in F$ *are called* final. *A* $\mathbb{G}$-match *is a finite path through* $\langle G, E\rangle$, *that is, a sequence* $\pi = p_0 p_1 \dots p_k$ *such that* $p_{l-1} E p_l$ *for each* $l < k$. *By definition, a match* π *of the initialized game* $\mathbb{G}@p$ *starts at position* p, *that is,* $\mathit{first}(\pi) = p$. *A match* π *is* full *if its last position* $\mathit{last}(\pi)$ *has no successor, that is, if* $E[\mathit{last}(\pi)] = \varnothing$, *and* partial *otherwise. The set of (partial) matches* π *such that* $\mathit{last}(\pi) \in G_i$ *is denoted as* $PM_i(\mathbb{G})$.

The set of winners *of a full match* π *is given as* $W(\mathit{last}(\pi))$ *in case* $\mathit{last}(\pi) \in F$, *and as* $\mathtt{I} \setminus \{i\}$ *if* $\mathit{last}(\pi) \in G_i$—*in the latter case we say that player* i got stuck. *A* strategy for player i, *or briefly, an* i-strategy, *is a map* $f : PM_i \to G$. *A match* $\pi = p_0 p_1 \dots p_k$ *is* consistent *with an* i*-strategy* f, *or* f-conform, *if* $p_{l+1} = f(p_0 \dots p_l)$ *whenever* $p_l \in G_i$. *An* i*-strategy* f *is* legitimate *(from position* p*) if* $\mathit{last}(\pi) E f(\pi)$ *for every* f*-conform match* $\pi \in PM_i$ *(of* $\mathbb{G}@p$*).*

A i*-strategy* f *is* winning for i at position p *if it is legitimate, and every* f*-conform full match* π *of* $\mathbb{G}@p$ *has* i *as one of its winners.*

We assume that the reader will have no difficulties in applying the previous definition to the evaluation games of Definition 2.5.

DEFINITION 2.7. *We say that player* i has a winning strategy *for an evaluation game* $\mathcal{E}(\varphi, V)$ *if she has a winning strategy, in the sense of Definition 2.6, at the initial position* (φ, id) *of* $\mathcal{E}(\varphi, V)$. *A formula* φ *is* i-satisfied *by a valuation* V *if player* i *has a winning strategy in* $\mathcal{E}(\varphi, V)$. *A formula* φ *is* i-satisfiable *if it is* i*-satisfied by some valuation, and* i-valid *if it is* i*-satisfied by every valuation.*

Most of all, we will be interested in two notions of equivalence between formulas.

DEFINITION 2.8. *Two formulas* φ *and* ψ *are* i-equivalent, *notation:* $\varphi \equiv_i \psi$, *if for every valuation* V, φ *is* i*-satisfied by* V *iff* ψ *is* i*-satisfied by* V, *and* equivalent, *notation:* $\varphi \equiv \psi$, *if they are* i*-equivalent for every* $i \in \mathtt{I}$.

EXAMPLE 2.9. *Let* V *be an arbitrary valuation. We will have a look at some examples of formulas* φ *of* $PL_{\mathtt{I}}$.

(i) Let $\varphi = x$. Exactly the players i such that $V(x)(i) = 1$ have a winning strategy.
(ii) Let $\varphi = \bot_\kappa$. Every player except for κ has a winning strategy. Note that $\bot_\kappa$ is i-valid for all $i \neq \kappa$, but not κ-satisfiable.

(iii) Let $\varphi = \neg_{\kappa\lambda}\psi$. Player κ has a winning strategy for $\mathcal{E}(\varphi, V)$ iff λ has a winning strategy for $\mathcal{E}(\psi, V)$, and vice versa. Each of the other players has a winning strategy for $\mathcal{E}(\varphi, V)$ iff he has a winning strategy for $\mathcal{E}(\psi, V)$.

(iv) Let $\varphi = \psi_1 \vee_\lambda \psi_2$. Since player κ is not in the position to make the first move she has a winning strategy for $\mathcal{E}(\varphi, V)$ iff she has a winning strategy both for $\mathcal{E}(\psi_1, V)$ and for $\mathcal{E}(\psi_2, V)$. Player λ, on the other hand, can choose one of the ψ_q in his first move and therefore has a winning strategy for $\mathcal{E}(\varphi, V)$ iff he has a winning strategy for either $\mathcal{E}(\psi_1, V)$ or $\mathcal{E}(\psi_2, V)$.

(v) Let $\varphi = \bot \vee_\kappa \top$. As a special case of (iv), it follows that player κ is the *only* player with a winning strategy of this formula, which is therefore equivalent to the formula $\top_\kappa$.

(vi) Let $\mathtt{I} = \{0, 1, 2, 3\}$ and take $\varphi = (((\bot \vee_1 \top) \vee_2 \top) \vee_3 \top)$. It is not difficult to see that each of the Players 1, 2, and 3 has a winning strategy (which consists in choosing $\top$ when their turn comes), whereas Player 0 does not: if each of her opponents would *not* pick $\top$ when their turn comes, she will lose the match (and each of her opponents would lose with her). Generalizing this to the setting where $\mathtt{I} = \{0, 1, \ldots, N\}$, one may show that the formula $\bot_0$ can be expressed as $(\ldots((\bot \vee_1 \top) \vee_2 \top)\ldots) \vee_N \top$.

(vii) Let $\varphi = x \vee_\kappa (x \vee_\lambda y)$. It is not difficult to see that player κ has a winning strategy for φ iff she has one for the formula x, and that the same applies to player λ. Nevertheless, the two formulas are only equivalent in case there are no *other* players, that is, if $\mathtt{I} = \{\kappa, \lambda\}$. If there is an additional player $i \notin \{\kappa, \lambda\}$, this player may win x according to V without having a winning strategy for the formula φ: The players κ and λ might team up to arrive at the formula y which may not be a win for player i.

(viii) Let $\varphi = \bigvee_\kappa\{\top_j \mid j \neq \kappa\}$. In this case no player has a winning strategy. Clearly, κ cannot win because for any of the $\top_j$ with $j \neq \kappa$ only j will win. As for the other players, they don't have a guarantee that κ will play 'their' $\top_j$. Hence they do not have a winning strategy. In other words, we have $\bigvee_\kappa\{\top_j \mid j \neq \kappa\} \equiv \bot$.

The notions of i-equivalence and full equivalence between formulas are interdefinable; the less obvious reduction is given as follows:

$$\varphi \equiv_i \psi \text{ iff } \varphi \vee_i \bot \equiv \psi \vee_i \bot,$$

as a straightforward proof will reveal.

2.2. Basic observations. Now that we have defined the syntax and semantics of propositional multiplayer logic, we make some preliminary observations.

First of all, the question naturally arises whether our language admits a natural *compositional* semantics which is equivalent to the game-theoretic one we have just given. Recall that in classical logic, the game-theoretical semantics can be seen as an operational *alternative* for the compositional (Tarskian) semantics. Fortunately, we have an affirmative answer.

DEFINITION 2.10. *We define a ternary forcing relation $\Vdash$, relating valuations, players, and formulas, by the following induction on the complexity of $PL_{\mathtt{I}}$-formulas:*

$$V \nVdash_i \bot$$

$$V \Vdash_i \top$$

$$V \Vdash_i \bot_j \quad \textit{if} \quad j \neq i$$
$$V \Vdash_i \top_j \quad \textit{if} \quad j = i$$
$$V \Vdash_i x \quad \textit{if} \quad V(x)(i) = 1$$
$$V \Vdash_i \varphi \vee_j \psi \quad \textit{if} \quad \begin{cases} V \Vdash_i \varphi \textit{ or } V \Vdash_i \psi & (i = j) \\ V \Vdash_i \varphi \textit{ and } V \Vdash_i \psi & (i \neq j) \end{cases}$$
$$V \Vdash_i \neg_{jk}\varphi \quad \textit{if} \quad V \Vdash_{[j,k](i)} \varphi$$

In case $V \Vdash_i \varphi$ *we say that player* i *forces* φ *under* V.

It is easy to prove that the two approaches are in fact equivalent:

PROPOSITION 2.11. *For any valuation* $V : \mathtt{X} \to 2^{\mathtt{I}}$*, any player* $i \in \mathtt{I}$*, and any formula* $\varphi \in PL_{\mathtt{I}}$ *we have*

$$V \Vdash_i \varphi \textit{ iff player } i \textit{ has a winning strategy in } \mathcal{E}(\varphi, V). \tag{1}$$

Proof. By a straightforward formula induction it can be proved that for any subformula ψ of φ, and any role distribution ρ,

$$V \Vdash_{\rho(i)} \psi \text{ iff player } i \text{ has a winning strategy for the game } \mathcal{E}(\varphi, V)@(\psi, \rho).$$

From this the proposition is immediate (take $\varphi = \psi$ and $\rho = id$). □

Second, comparing our framework to classical propositional logic, we observe that even in the two-player setting our semantics is more liberal than the standard one, in which valuations declare a *unique* winner for a proposition letter: such an atomic formula is either true ($\exists$ winning) or false ($\forall$ winning). Our setup allows valuations that declare *both* or *neither* of the players as the winners of a match associated with a proposition letter. As a consequence, our two-player formalism is closer to multivalued logic than to classical propositional logic.

In fact, if we restrict attention to the disjunction operators, our system corresponds exactly to the *four-valued logic* introduced by Belnap (1977). This formalism was proposed as the underlying logic of reasoning systems that need to deal with incomplete and inconsistent information. In Belnap's logic, each atomic piece of information is marked with one of four truth values: T ('true'), F ('false'), N ('no information available'), or B ('both positive and negative information available'). Since the database is supposed to deal with compound formulas as well, Belnap suggested truth tables for the conjunction and disjunction connectives:

$\wedge$	N	F	T	B
N	N	F	N	F
F	F	F	F	F
T	N	F	T	B
B	F	F	B	B

$\vee$	N	F	T	B
N	N	N	T	T
F	N	F	T	B
T	T	T	T	T
B	T	B	T	B

The resulting logic relates to our formalism as follows. With each of Belnap's truth values W we associate a set $W_s \subseteq \{\forall, \exists\}$: $T \sim \{\forall\}$, $F \sim \{\exists\}$, $N \sim \varnothing$, and $B \sim \{\forall, \exists\}$. Intuitively, think of W_s as the set of players with a winning strategy for the formula having W as its meaning. Then the truth tables for $\wedge$ and $\vee$ exactly capture the semantics of our game connectives $\vee_\forall$ and $\vee_\exists$. In other words: Belnap's logic of $\{\wedge, \vee\}$ is the same as ours of $\{\vee_\forall, \vee_\exists\}$.

Unfortunately, this correspondence does not extend to Belnap's full system, since his negation $\sim$, given by the following table:

$$\begin{array}{c|cccc} \sim & N & F & T & B \\ \hline & B & T & F & N \end{array} \tag{2}$$

does not behave the same as our role switch operation $\neg_{\forall\exists}$. Thus we see that in our setting the classical negation splits into two distinct operators, $\neg_{01}$ and $\sim$.

If we want to regain classical propositional logic, we need to drop the constants $\top$ and $\bot$ from the language, and *restrict* valuations to those maps assigning to each proposition letter *exactly one winner*. Observe that in the two-player case, this property is propagated by the semantics: if V is a restricted valuation in the sense above, then for every formula φ in the resulting fragment of $PL_{\{\forall,\exists\}}$ there is exactly one player $i \in \{\forall, \exists\}$ with a winning strategy for φ, and it is easy to see that the resulting logic is exactly classical propositional logic. Obviously, these restricted valuations have analogs in the multiplayer setting, which would give rise to alternative versions of our logic.

DEFINITION 2.12. *A valuation $V : \mathtt{X} \to 2^{\mathtt{I}}$ is called a* single-winner *valuation if for each $x \in \mathtt{X}$ there is exactly one player i such that $V(x)(i) = 1$.*

Note, however, that the restriction to single-winner valuations does not rule out the existence of formulas for which no player has a winning strategy, see for instance the formula $\bigvee_\kappa\{\top_j \mid j \neq \kappa\}$ of Example 2.9(viii). In other words, the property of having exactly one winner does not propagate in the multiplayer case. It might be of interest to investigate the logic of these single-winner valuations, but we will not go into the details here.

The third and final issue we address in this subsection concerns the interdefinability of various connectives and the relative expressive strength of various sets of connectives.

Let us first consider the constants. In Example 2.9(v,vi) we already saw in a specific setting how the constants $\top_i$ and $\bot_i$ can be defined using $\bot$, $\top$, and the disjunctions. In the general setting, where $\mathtt{I} = \{i_0, i_1, \ldots, i_N\}$, we obtain the definitions as in Table 2.

Conversely, it is not hard to see that we can express the constant $\top_i$ as the formula $\bot_{i_0} \vee_i \bot_{i_1} \vee_i \cdots \vee_i \bot_{i_N}$, and $\bot$ as the formula $\top_{i_1} \vee_{i_0} \top_{i_2} \vee_{i_0} \cdots \vee_{i_0} \top_{i_N}$.

This shows that, in the presence of the full set of disjunctions, both sets $\{\top, \bot\}$ and $\{\top\} \cup \{\bot_i \mid i \in \mathtt{I}\}$ are equally expressive as the set $\{\top, \bot\} \cup \{\bot_i, \top_i \mid i \in \mathtt{I}\}$ of all constants. (We leave it as an exercise for the reader to verify that we cannot do without the symbol $\top$.) In the sequel we will make use of this by restricting the *primitive* constants to the set $\{\bot, \top\}$, using $\top_i$ and $\bot_i$ as abbreviations of the formulas given in Table 2.

When it comes to the disjunction operators, it should be obvious that in the presence of the role switch operators, one single disjunction, say $\vee_\kappa$, suffices to define all others: $\varphi \vee_i \varphi'$ can be abbreviated as $\neg_{\kappa i}(\neg_{\kappa i}\varphi \vee_\kappa \neg_{\kappa i}\varphi')$. Similarly, using some well-known,

Table 2. $\top_i$ *and* $\bot_i$ *as abbreviations*

Symbol	Abbreviation for	
$\top_i$	$\bot \vee_i \top$	
$\bot_i$	$(\ldots((\bot \vee_{i_1} \top) \vee_{i_2} \top)\ldots) \vee_{i_N} \top$	(where $\mathtt{I} = \{i_1, \ldots, i_N\} \uplus \{i\}$)

basic results on permutation groups, we may restrict the role switch operators to the set $\{\neg_{\kappa i} \mid i \in \mathtt{I}\}$ without compromising the expressive power of the language.

It might be of some interest to investigate in more detail the relative expressive power of various combinations of multiplayer connectives, but we will not pursue this matter now. However, a question that we will address here is whether the set of connectives introduced thus far is *functionally complete*, that is, whether *any* 'reasonable' multiplayer connective can be expressed using the repertoire of $PL_{\mathtt{I}}$. In analogy to the Boolean case, we will interpret 'reasonable' here as 'definable using a truth table over the set $2^{\mathtt{I}}$', that is, the collection of potential meanings of formulas.

Our answer to the functional completeness question is negative. Consider the 'strong negation' connective $\sim$ with the following semantics:

$$V \Vdash_i \sim\varphi \text{ iff } V \nVdash_i \varphi.$$

In words, $\sim$ turns winners into losers and vice versa—this is in fact the multiple player version of Belnap's negation, compare (2). Not only do we fail to see how to incorporate this connective into the framework of our multiplayer evaluation game, in fact we can prove that this connective is not definable in terms of the $PL_{\mathtt{I}}$-operations. For a proof of this fact, simply observe that every formula in our language is *monotone* in the sense that bigger valuations mean more winners, whereas this property obviously fails for a simple formula like $\sim x$.

2.3. Modal multiplayer logic. The key ideas underlying the extension of propositional multiplayer logic to the modal setting are to extend the language with a modal operator $\Diamond_i$ for each player $i \in \mathtt{I}$, and to provide these modalities with a Kripke-style semantics which make each $\Diamond_i$ behave like a diamond for i and like a box for every player $j \neq i$.

Starting with syntax, in the propositional part of the language we restrict the language so that $\bot$ and $\top$ are the only constants; that is, we think of $\bot_i$ and $\top_i$ as abbreviations, as in Table 2. To this propositional language we now add modalities.

DEFINITION 2.13. *The language $ML_{\mathtt{I}}$ of* multiplayer modal logic *is defined using the following induction:*

$$\varphi ::= x \mid \bot \mid \top \mid \varphi \vee_i \varphi \mid \neg_{ij}\varphi \mid \Diamond_i\varphi.$$

Elements of $ML_{\mathtt{I}}$ will be called modal multiplayer formulas.

Formulas of this language will be interpreted in multiplayer versions of Kripke models.

DEFINITION 2.14. *A* Kripke frame *is a structure $\langle S, R\rangle$ such that S is a set of objects called* states *and R is a binary relation on S, often thought of as a map $R[\cdot] : s \mapsto \{t \mid Rst\}$. A* (multiplayer) Kripke model *is a triple $\langle S, R, V\rangle$ such that $\langle S, R\rangle$ is a Kripke frame, and V is a* (modal) valuation*, that is, a map assigning to each state s in S a propositional valuation $V_s : \mathtt{X} \to 2^{\mathtt{I}}$.*

As in the propositional case, the semantics of multiplayer modal logic can be defined using evaluation games, or via a direct compositional definition. Here we restrict ourselves to the latter.

DEFINITION 2.15. *Given a Kripke model $\mathbb{S} = \langle S, R, V\rangle$, we define the* forcing relation *$\Vdash \subseteq S \times \mathtt{I} \times ML_{\mathtt{I}}$ by the following formula induction:*

$$\mathbb{S}, s \nVdash_i \bot$$

$$\mathbb{S}, s \Vdash_i \top$$

$$\mathbb{S}, s \Vdash_i x \quad \textit{if} \quad V_s(x)(i) = 1$$

$$\mathbb{S}, s \Vdash_i \varphi \vee_j \psi \quad \textit{if} \quad \begin{cases} \mathbb{S}, s \Vdash_i \varphi \textit{ or } \mathbb{S}, s \Vdash_i \psi & (i = j) \\ \mathbb{S}, s \Vdash_i \varphi \textit{ and } \mathbb{S}, s \Vdash_i \psi & (i \neq j) \end{cases}$$

$$\mathbb{S}, s \Vdash_i \neg_{jk}\varphi \quad \textit{if} \quad \mathbb{S}, s \Vdash_{[j,k](i)} \varphi$$

$$\mathbb{S}, s \Vdash_i \Diamond_j \varphi \quad \textit{if} \quad \begin{cases} \mathbb{S}, t \Vdash_i \varphi \textit{ for some } t \in R[s] & (i = j) \\ \mathbb{S}, t \Vdash_i \varphi \textit{ for all } t \in R[s] & (i \neq j) \end{cases}$$

In case $\mathbb{S}, s \Vdash_i \varphi$ *we say that* player i forces φ at s in $\mathbb{S}$.

The notions of i-satisfiability, i-validity, and equivalence for modal multiplayer formulas are defined in the obvious way.

EXAMPLE 2.16. *Let* $\mathbb{S} = \langle S, R, V \rangle$ *be some Kripke model. We will have a look at some examples of modal multiplayer formulas* $\varphi \in ML_{\mathtt{I}}$.

(i) Let $\varphi = \Diamond_\kappa \bot_\lambda$. In this case $\mathbb{S}, s \Vdash_\kappa \varphi$ iff s has a successor: If s has a successor s' then player κ can win by picking this successor at her first move; if s has no successor then player κ loses immediately. For player λ, $\mathbb{S}, s \Vdash_\lambda \varphi$ iff s has *no* successor: If s has a successor then player λ will lose no matter which successor is picked by player κ; if s has no successor then player λ wins since player κ gets stuck right away. Any other player i is guaranteed to win the game, implying that $\mathbb{S}, s \Vdash_i \Diamond_\kappa \bot_\lambda$: In case s has no successors player i wins immediately because player κ gets stuck; but if s has successors, then no matter which one is picked by κ, player i will win since the resulting formula is $\bot_\lambda$.

(ii) The formulas $\Diamond_\kappa(x \vee_\kappa y)$ and $\Diamond_\kappa x \vee_\kappa \Diamond_\kappa y$ are equivalent: note that for player κ, the formulas read like $\Diamond(x \vee y)$ and $\Diamond x \vee \Diamond y$, respectively, while for any player $i \neq \kappa$, the formulas read like $\Box(x \wedge y)$ and $\Box x \wedge \Box y$, respectively.

REMARK 2.17. *It might be of interest to develop a multiplayer version of modal* model theory. *A key notion in this theory would be a multiplayer version of the notion of bisimilarity, defined as follows.*

A *multiplayer bisimulation* between two multiplayer Kripke models $\mathbb{S}$ and $\mathbb{S}'$ is a binary relation $Z \subseteq S \times S'$ which satisfies, in addition to the back-and-forth clauses of standard bisimulations, the condition that for all $(s, s') \in Z$, all variables x and all agents i, $V_s(x)(i) = V'_{s'}(x)(i)$. For such relations a straightforward formula induction reveals that if s and s' are bisimilar (i.e., linked by some multiplayer bisimulation), then each player i forces exactly the same formulas in s as in s'.

In addition, it might be interesting to study *frame definability* for modal multiplayer logic. We leave this as a topic for future work.

REMARK 2.18. *Recent years have witnessed some fruitful and interesting interaction between the areas of modal logic and* coalgebra*, see Venema (2006) for an overview. On the one hand, tools and techniques from modal logic, suitably adapted, have found applications in a far wider settings than Kripke structures. On the other hand, the abstract, general* coalgebraic perspective *has offered some new insights on the position of modal logic in its wider mathematical landscape.*

From this perspective, it is interesting to note that multiplayer Kripke models can be seen as coalgebras for the functor $\mathsf{K}_{\mathtt{I},\mathtt{X}}$ given by $\mathsf{K}_{\mathtt{I},\mathtt{X}}(-) = 2^{\mathtt{I} \times \mathtt{X}} \times \mathcal{P}(-)$. Thus coalgebraic methods can be applied in the study of multiplayer modal logic—to mention just one example, the notion of bisimilarity mentioned in Remark 2.17 is exactly the one that arises naturally from this coalgebraic perspective.

§3. Multiplayer algebras: the propositional case. In the previous section we got interested in the equivalence relations $\equiv$ and $\equiv_i$ between propositional formulas. In this section we develop our *algebraic* approach toward multiplayer propositional logics. Throughout the paper we assume some familiarity with basic notions from universal algebra; in particular, we let **S** denote the operation on classes of algebras of taking all isomorphic copies of subalgebras. Our main aim will be to regard the equivalence relation $\equiv$ between propositional formulas in an equational way. For this purpose, we first make explicit what kind of algebras we will be working with.

DEFINITION 3.19. *Let $Bool_{\mathtt{I}}$ be the algebraic similarity type having constants $\bot$ and $\top$, a binary symbol $\vee_i$ for each $i \in \mathtt{I}$, and a unary function symbol $\neg_{ij}$ for every pair $i, j \in \mathtt{I}$.*

Equations of terms of this similarity type will be denoted as $\varphi \approx \psi$ (where φ and ψ are $Bool_{\mathtt{I}}$-terms). We will use the symbol $\approx_i$ to abbreviate the formula $\bot \vee_i \varphi \approx \bot \vee_i \psi$ as $\varphi \approx_i \psi$.

Intuitively, think of $\varphi \approx \psi$ and $\varphi \approx_i \psi$ as the statements, respectively, that ψ and ψ are equivalent/i-equivalent.

3.1. Concrete multiplayer algebras. It is well known how to give an algebraic version of the semantics of classical propositional logic in the Boolean algebra $\mathbb{2}$ of truth values. Similarly, we can encode the semantics of multiplayer logic in a fixed small algebra of type $Bool_{\mathtt{I}}$. Elements of this algebra can be seen either as subsets of the set $\mathtt{I}$ of players, or, equivalently, as maps from $\mathtt{I}$ to the set $2 = \{0, 1\}$. Concretely, think of $f : \mathtt{I} \to 2$ as characterizing the members of the set $f^{-1}(1) = \{i \in \mathtt{I} \mid f(i) = 1\}$ as the winners associated with f.

DEFINITION 3.20. *The algebra $\mathbb{O}_{\mathtt{I}}$ is given as the $Bool_{\mathtt{I}}$-algebra*

$$\mathbb{O}_{\mathtt{I}} := \langle 2^{\mathtt{I}}, \overline{0}, \overline{1}, +_i, -_{ij} \rangle_{i,j \in \mathtt{I}},$$

with as carrier the set $2^{\mathtt{I}}$ of maps from $\mathtt{I}$ to 2. The designated elements $\overline{0}, \overline{1} : \mathtt{I} \to 2$ are given by

$$\begin{aligned} \overline{0}(k) &:= 0, \\ \overline{1}(k) &:= 1. \end{aligned}$$

The operation $+_i$ is given by putting, for $f, g \in 2^{\mathtt{I}}$:

$$(f +_i g)(k) := \begin{cases} \max(f(k), g(k)) & \text{if } i = k, \\ \min(f(k), g(k)) & \text{if } i \neq k, \end{cases}$$

and the operation $-_{ij}$ is given by putting, for $f \in 2^{\mathtt{I}}$:

$$-_{ij} f := f \circ [i, j].$$

It is not difficult to see how this algebra encodes the game semantics defined in the previous section. The key observation here is that the valuations of the previous sections can be identified with algebraic *assignments* on $\mathbb{O}_{\mathtt{I}}$. Given a valuation/assignment V and a multiplayer formula/term φ, let $\widetilde{V}(\varphi)$ denote the (uniquely induced) meaning of φ under V in the algebra $\mathbb{O}_{\mathtt{I}}$. Now a straightforward verification will reveal that for any multiplayer formula/term φ, and any player i:

$$\varphi \text{ is } i\text{-satisfied by } V \text{ iff } \widetilde{V}(\varphi)(i) = 1. \tag{3}$$

From this it follows directly that two formulas φ and ψ are equivalent iff the equation $\varphi \approx \psi$ holds in the algebra $\mathbb{O}_{\mathtt{I}}$, see Theorem 3.23 below.

Analogous to the case of Boolean algebras, there is also a natural set-theoretic interpretation for this algebraic language.

DEFINITION 3.21. *Given a set S, we define the* full multiplayer set algebra over *S, as the following structure:*

$$\mathbb{C}_{\mathtt{I}}(S) := \langle \mathcal{P}(S)^{\mathtt{I}}, \overline{\emptyset}, \overline{S}, \emptyset_i, \cup_i, \sim_{ij} \rangle_{i,j \in \mathtt{I}},$$

based on the carrier $\mathcal{P}(S)^{\mathtt{I}}$ of $\mathtt{I}$-indexed sequences of subsets of S. The designated elements $\overline{\emptyset}, \overline{S} : \mathtt{I} \to \mathcal{P}(S)$ are given by

$$\begin{aligned} \overline{\emptyset}(k) &:= \varnothing, \\ \overline{S}(k) &:= S. \end{aligned}$$

Given two sequences $X = \{X(i) \mid i \in \mathtt{I}\}$ and $Y = \{Y(i) \mid i \in \mathtt{I}\}$, we define $X \cup_i Y$ by putting

$$(X \cup_i Y)(k) := \begin{cases} X(k) \cup Y(k) & \text{if } k = i, \\ X(k) \cap Y(k) & \text{if } k \neq i, \end{cases}$$

and we define $\sim_{ij} X$ by

$$\sim_{ij} X := X \circ [i, j].$$

The class of full multiplayer set algebras indexed by $\mathtt{I}$ is denoted as $\mathsf{CPA}_{\mathtt{I}}$.

Also analogous to the case of Boolean algebra is the following natural relation between these set algebras and the small algebra $\mathbb{O}_{\mathtt{I}}$:

PROPOSITION 3.22. *For any set S, the map $h_S : \mathcal{P}(S)^{\mathtt{I}} \to (2^{\mathtt{I}})^S$ given by*

$$h_S(X)(s)(i) := \begin{cases} 1 & \text{if } s \in X(i), \\ 0 & \text{if } s \notin X(i). \end{cases}$$

is an isomorphism between the algebra $\mathbb{C}_{\mathtt{I}}(S)$ and the S-fold power $\mathbb{O}_{\mathtt{I}}^S$ of $\mathbb{O}_{\mathtt{I}}$.

Proof. The proof of this proposition is routine and left as an exercise for the reader. □

The following theorem bears witness to our claim that the multiplayer logic introduced in the previous section has an elegant algebraization.

THEOREM 3.23. *The following are equivalent, for all formulas φ, ψ:*

(1) $\varphi \equiv \psi$;
(2) $\mathbb{O}_{\mathtt{I}} \models \varphi \approx \psi$;
(3) $\mathsf{CPA}_{\mathtt{I}} \models \varphi \approx \psi$.

Proof. As we mentioned already, the equivalence 1 ⇔ 2 is a straightforward consequence of (3). The equivalence 2 ⇔ 3 is immediate by Proposition 3.22. □

3.2. Abstract multiplayer algebras.

DEFINITION 3.24. *A $Bool_{\mathtt{I}}$-type algebra $\mathbb{A} = \langle A, \bot, \top, \vee_i, \neg_{ij}\rangle_{i,j\in\mathtt{I}}$ is called a* multiplayer *algebra if it satisfies the equations (P1–8) and (N1–11) of Table 3. The class of multiplayer algebras associated with the set $\mathtt{I}$ of players will be denoted as $\mathsf{PA}_{\mathtt{I}}$.*

Most of the axioms speak for themselves. For instance, the axioms (P1–5) express that for each $i \in \mathtt{I}$, the algebra $\mathbb{A}_i := \langle A, \vee_i, \bot_i, \top_i\rangle$ is a semilattice with bottom $\bot_i$ and top $\top_i$. Any pair $(\vee_i, \vee_j)$ of such semilattice operations is linked by two additional axioms: the *distributive law* (P6) and the *absorption law* (P7). This does *not* mean that the structure $\langle A, \vee_i, \vee_j\rangle$ is a distributive lattice however, compare Example 2.9(vii). The role of the axiom (P8), which will become clearer when we prove Proposition 3.26 below, is to express that $i_0, \dots, i_n$ are *all* the players in the set $\mathtt{I}$.

The axioms involving the *role switch operations* are completely straightforward. In particular, (N1–7) are simply the manifestation in this logical setting of the familiar laws governing the behavior of transpositions. The axioms (N8) and (N9) encode that the constants $\bot$ and $\top$ represent fixpoints of every role switch operation. Finally, the axioms (N10) and (N11) can be seen as a multiplayer version of the de Morgan laws—note that we could have combined the two axioms into one: $\neg_{ij}(x \vee_k y) \approx \neg_{ij}x \vee_{[i,j](k)} \neg_{ij}y$.

Finally, note that we did not aim for the shortest presentation of our axioms, or even for irredundancy. For example, it is not hard to see that the axiom (P1) can be derived from (P3), (P4), and (P6).

DEFINITION 3.25. *Let $\mathbb{A} = \langle A, \bot, \top, \vee_i, \neg_{ij}\rangle_{i,j\in\mathtt{I}}$ be a multiplayer algebra. On the carrier A of this algebra we define the relations $\leq_i^{\mathbb{A}}$, $\sqsubseteq_i$ and $\equiv_i$ as follows:*

$$\begin{array}{lll} a \leq_i^{\mathbb{A}} b & \text{iff} & a \vee_i b = b, \\ a \sqsubseteq_i^{\mathbb{A}} b & \text{iff} & \bot \vee_i a \leq_i^{\mathbb{A}} \bot \vee_i b \quad (\text{iff } \bot \vee_i a \vee_i b = \bot \vee_i b), \\ a \equiv_i^{\mathbb{A}} b & \text{iff} & a \sqsubseteq_i^{\mathbb{A}} b \text{ and } b \sqsubseteq_i^{\mathbb{A}} a \quad (\text{iff } \bot \vee_i a = \bot \vee_i b). \end{array}$$

Table 3. *Axioms for multiplayer algebras*

(P1)	$x \vee_i x \approx x$	
(P2)	$x \vee_i y \approx y \vee_i x$	
(P3)	$x \vee_i (y \vee_i z) \approx (x \vee_i y) \vee_i z$	
(P4)	$x \vee_i \bot_i \approx x$	
(P5)	$x \vee_i \top_i \approx \top_i$	
(P6)	$x \vee_i (y \vee_j z) \approx (x \vee_i y) \vee_j (x \vee_i z)$	
(P7)	$x \vee_i (x \vee_j y) \approx_i x$	$(i \neq j)$
(P8)	$x \approx (\cdots((\bot \vee_{i_0} x) \vee_{i_1} x) \cdots \vee_{i_N} x)$	$(I = \{i_0, \dots, i_N\})$
(N1)	$\neg_{ii}x \approx x$	
(N2)	$\neg_{ij}x \approx_k x$	$(k \notin \{i, j\})$
(N3)	$\neg_{ij}x \approx \neg_{ji}x$	
(N4)	$\neg_{ij}\neg_{ij}x \approx x$	
(N5)	$\neg_{ij}\neg_{kl}x \approx \neg_{lk}\neg_{ij}x$	$(\{i, j\} \cap \{k, l\} = \varnothing)$
(N6)	$\neg_{ij}\neg_{jk}x \approx_i \neg_{ik}x$	
(N7)	$\neg_{ij}\neg_{ik}x \approx_i \neg_{ij}x$	$(j \notin \{i, k\})$
(N8)	$\neg_{ij}\bot \approx \bot$	
(N9)	$\neg_{ij}\top \approx \top$	
(N10)	$\neg_{ij}(x \vee_i y) \approx \neg_{ij}x \vee_j \neg_{ij}y$	
(N11)	$\neg_{ij}(x \vee_k y) \approx \neg_{ij}x \vee_k \neg_{ij}y$	$(k \notin \{i, j\})$

When no confusion is likely we will drop superscripts and write $\leq_i$ instead of $\leq_i^{\mathbb{A}}$, and so forth.

Intuitively, $\vee_i$ is like a join for player i, and like a meet for all the other players. Therefore, one may read $a \sqsubseteq_i b$ as stating that player i prefers a over b, and $a \equiv_i b$, that i is indifferent between them. The other relationship, $a \leq_i b$ indicates that player i prefers a over b, and that, in addition, all the other players have the opposite preference. Formally we can prove the following.

PROPOSITION 3.26. *Let $\mathbb{A}$ be a multiplayer algebra. For each $i \in \mathtt{I}$:*

(1) *the relation $\leq_i$ is a partial order with bottom $\bot_i$ and top $\top_i$;*
(2) *the relation $\sqsubseteq_i$ is a quasi-order;*
(3) *for any $a, b \in A$ we have $a = b$ iff $a \equiv_i b$ for all i;*
(4) *for any $a, b \in A$ we have $a \leq_i b$ iff $a \sqsubseteq_i b$ and $b \sqsubseteq_j a$ for all $j \neq i$.*

Proof. The first item of the proposition is completely standard, given the axioms (P1–5). For the second item we need to prove that each $\sqsubseteq_i$ is reflexive and transitive; these proofs are routine, and details are left to the reader.

In order to prove the third item, we introduce, for an arbitrary subset $J \subseteq \mathtt{I}$, an operation $\star_J : A \times A \to A$. This operation is defined by the following induction on the size of J:

$$\begin{array}{rcl} a \star_\varnothing b & := & a, \\ a \star_{J \cup \{i\}} b & := & (a \star_J b) \vee_i b. \end{array}$$

The correctness of this definition follows by the distribution axiom (P6) which guarantees that $(a \vee_j b) \vee_i b = (a \vee_i b) \vee_j b$.

Define the relation $\equiv_J$ on A by putting

$$a \equiv_J b \text{ iff } \bot \star_J a = \bot \star_J b.$$

We claim that for all $J \subseteq \mathtt{I}$:

$$\Big(\bigcap_{i \in \mathtt{I}} \equiv_i\Big) \subseteq \equiv_J . \tag{4}$$

In order to see why Part 3 follows from this, consider the case where $J = \mathtt{I}$. Take $a, b \in A$ with $a \equiv_i b$ for all $i \in \mathtt{I}$, then by (4) and axiom (P8) we obtain that $a = \bot \star_{\mathtt{I}} a = \bot \star_{\mathtt{I}} b = b$. From this the direction from right to left in Part 3 is immediate; the other direction is trivial.

We will *prove* (4) by induction on the size of J. For $J = \varnothing$ we find by definition of $\star_\varnothing$ that $\equiv_J = A \times A$; from this, (4) is straightforward. If J is a singleton, say, $J = \{i\}$, then we have that $\equiv_J = \equiv_i$, so again (4) is immediate.

For the inductive case assume that $|J| \geq 2$, and write $J = K \cup \{i, j\}$ where $|K| = |J| - 2$. Consider two elements a, b in $\mathbb{A}$ such that $a \equiv_i b$ for all $i \in \mathtt{I}$. Then inductively we may assume that $\bot \star_L a = \bot \star_L b$ for all proper subsets L of J. In order to prove that $a \equiv_J b$, we compute

$$\begin{aligned} \bot \star_J a &= ((\bot \star_K a) \vee_i a) \vee_j a && \text{(definition of } \star) \\ &= ((\bot \star_K b) \vee_i b) \vee_j a && \text{(inductive hypothesis on } K \cup \{i\}) \\ &= ((\bot \star_K b) \vee_j a) \vee_i (b \vee_j a) && \text{(distributivity—P6)} \\ &= ((\bot \star_K a) \vee_j a) \vee_i (b \vee_j a) && \text{(inductive hypothesis on } K) \end{aligned}$$

Likewise we find that

$$\bot \star_J b = ((\bot \star_K b) \vee_j b) \vee_i (a \vee_j b).$$

But then by (P2) and the inductive hypothesis on $K \cup \{j\}$, we see that $\bot \star_J a = \bot \star_J b$, as required to show that $a \equiv_J b$. This proves (4) and thus finishes the proof of Part 3 of the proposition.

Finally, we turn to Item 4 of the proposition. The direction from left to right is not hard to show: if $a \leq_i b$ then it is immediate that $a \sqsubseteq_i b$. We also have, for $j \neq i$, that

$$\begin{aligned} \bot \vee_j a \vee_j b &= \bot \vee_j a \vee_j (a \vee_i b) && \text{(assumption)} \\ &= \bot \vee_j a, && \text{(P7)} \end{aligned}$$

which shows that $b \sqsubseteq_j a$.

For the other direction of Item 4, assume that $a, b \in A$ are such that $a \sqsubseteq_i b$ and $b \sqsubseteq_j a$ for all $j \neq i$. In order to prove that $a \leq_i b$, we prove that

$$a \vee_i b \equiv_k b, \text{ for all } k \in \mathtt{I}. \tag{5}$$

Turning to the proof of (5), the case where $k = i$ is immediate by the assumption that $a \sqsubseteq_i b$. In case $k \neq i$ we make the following calculation:

$$\begin{aligned} \bot \vee_k (a \vee_i b) &= (\bot \vee_k a) \vee_i (\bot \vee_k b) && \text{(distributivity—P6)} \\ &= (\bot \vee_k a \vee_k b) \vee_i (\bot \vee_k b) && \text{(assumption } b \sqsubseteq_k a) \\ &= (a \vee_k (\bot \vee_k b)) \vee_i (b \vee_k (\bot \vee_k b)) && \text{(semilattice axioms)} \\ &= (a \vee_i b) \vee_k (\bot \vee_k b) && \text{(distributivity—P6)} \\ &= \bot \vee_k b \vee_k (b \vee_i a) && \text{(semilattice axioms)} \\ &= \bot \vee_k b && \text{(absorption—P7)} \end{aligned}$$

This means that $a \vee_i b \equiv_k b$ for $k \neq i$, and finishes the proof of (5). But then we are done, since by the previous item it follows from (5) that $a \vee_i b = b$, that is, $a \sqsubseteq_i b$. □

In the proofs below we will need some additional facts on multiplayer algebras; we list the following observations but leave their (straightforward) verification as an exercise.

PROPOSITION 3.27. *Let $\mathbb{A}$ be a multiplayer algebra.*

(1) $\neg_{ij} \bot_i = \bot_j$, *and* $\neg_{ij} \bot_k = \bot_k$ *(provided* $k \notin \{i, j\}$*);*
(2) $\neg_{ij} \top_i = \top_j$, *and* $\neg_{ij} \top_k = \top_k$ *(provided* $k \notin \{i, j\}$*);*
(3) $\top_j \equiv_i \bot_i \equiv_i \bot$ *and* $\bot_j \equiv_i \top_i \equiv_i \top$ *(provided* $i \neq j$*);*
(4) $a = b$ *iff* $\neg_{ij} a = \neg_{ij} b$
(5) $a \sqsubseteq_i b$ *iff* $\neg_{ij} a \sqsubseteq_j \neg_{ij} b$*;*
(6) $a \vee_j b \sqsubseteq_i a$ *if* $i \neq j$*;*
(7) $a \vee_i b \equiv_k a \vee_j b$ *if* $k \notin \{i, j\}$*;*
(8) *if* $a \sqsubseteq_i b$ *and* $a' \sqsubseteq_i b'$ *then* $a \vee_j a' \sqsubseteq_i b \vee_j b'$.

3.3. A representation theorem. The main purpose of this subsection is to prove that every abstract multiplayer algebra can be represented as a concrete, set-based one.

THEOREM 3.28. *For any set* $\mathtt{I}$ *of players with* $|\mathtt{I}| \geq 2$*:*

$$\mathsf{PA}_{\mathtt{I}} = \mathbf{S}(\mathsf{CPA}_{\mathtt{I}}). \tag{6}$$

The right-to-left inclusion of (6) is easy to prove: It suffices to show that the concrete multiplayer algebras satisfy the axioms defining (P1-8) and (N1-11). These are routine checks that we leave as an exercise to the reader.

For the opposite inclusion ($\subseteq$) of (6) more work is needed. Given an abstract multiplayer algebra $\mathbb{A}$, the first problem is over which set to represent it.

DEFINITION 3.29. *A* representation *of a multiplayer algebra* $\mathbb{A}$ *is an embedding* $\rho : \mathbb{A} \to \mathbb{C}_{\mathtt{I}}(S)$ *for some set* S*. If such an embedding exists we say that* $\mathbb{A}$ *is* represented over S*, and algebras that can be represented over some set are called* representable.

In the two-player setting, by Stone's celebrated theorem, every distributive lattice can be represented over either the set of its *prime filters*, or the set of its *prime ideals*. As an analogous notion for multiplayer algebras we propose the following.

DEFINITION 3.30. *Let* $\mathbb{A} = \langle A, \bot, \top, \vee_i, \neg_{ij}\rangle_{i,j\in\mathtt{I}}$ *be a multiplayer algebra. An* i-filter *of* $\mathbb{A}$ *is a nonempty subset* $F \subseteq A$ *such that (F1)* $\top_i \in F$*; (F2) if* $a, b \in F$ *then* $a \vee_j b \in F$*, for all* $j \neq i$*; (F3) if* $a \in F$ *and* $a \sqsubseteq_i b$ *then* $b \in F$*. An* i*-filter* F *is* proper *if* $\bot_i \notin F$*, and* prime *when it satisfies (F4) if* $a \vee_i b \in F$ *then* $a \in F$ *or* $b \in F$*.*

An i-ideal *of* $\mathbb{A}$ *is a nonempty subset* $D \subseteq A$ *such that (I1)* $\bot_i \in D$*; (I2) if* $a, b \in D$ *then* $a \vee_i b \in D$*; (I3) if* $a \in D$ *and* $b \sqsubseteq_i a$ *then* $b \in D$*. An* i*-ideal* D *is* proper *if* $\bot_i \notin D$*, and* prime *when it satisfies (I4) if* $a \vee_i b \in D$ *then* $a \in D$ *or* $b \in D$*.*

The collections of i*-filters, prime* i*-filters,* i*-ideals, and prime* i*-ideals of* $\mathbb{A}$ *are denoted as* $Fil_i(\mathbb{A})$, $Fil_i^p(\mathbb{A})$, $Idl_i(\mathbb{A})$*, and* $Idl_i^p(\mathbb{A})$*, respectively.*

Clearly, i-ideals are defined almost exactly like ideals of distributive lattices: they are up-directed downsets with respect to the order $\sqsubseteq_i$. For i-filters the situation is marginally more involved: they are $\sqsubseteq_i$-downsets that are 'closed under arbitrary i-conjunctions', where the latter expression would be an i-based way to state condition (F2). In any case, observe that there is no direct order duality between the notions of filter and ideal in our setup.

Nevertheless there are some interesting observations to be made about filters and ideals. The following proposition collects some basic facts; the proof, being routine, is omitted.

PROPOSITION 3.31. *Let* $\mathbb{A} = \langle A, \bot, \top, \vee_i, \neg_{ij}\rangle_{i,j\in\mathtt{I}}$ *be a multiplayer algebra. Then*

1. $X \subseteq A$ *is a prime* i*-filter iff* $A \setminus X$ *is a prime* i*-ideal;*
2. $X \subseteq A$ *is a prime* i*-filter iff* $\{\neg_{ij} a \mid a \in X\}$ *is a prime* j*-filter;*
3. *both* $\langle Fil_i(\mathbb{A}), \subseteq\rangle$ *and* $\langle Idl_i(\mathbb{A}), \subseteq\rangle$ *are complete lattices, with meet given by set intersection.*

In passing we note that it follows from Part 3 of the above proposition that for any multiplayer algebra $\mathbb{A}$ and subset X of the carrier of $\mathbb{A}$ there is a *smallest* i-filter (i-ideal, respectively) containing X.

Other notions that can be generalized from the lattice case to the multiplayer settings are those of *upsets* and *downsets* generated by a subset of the algebra.

DEFINITION 3.32. *Let* $\mathbb{A} = \langle A, \bot, \top, \vee_i, \neg_{ij}\rangle_{i,j\in\mathtt{I}}$ *be a multiplayer algebra. Given a subset* $X \subseteq A$*, we define*

$$\begin{aligned} \uparrow_i X &:= \{a \in A \mid x \sqsubseteq_i a \text{ for some } x \in X\}, \\ \downarrow_i X &:= \{a \in A \mid a \sqsubseteq_i x \text{ for some } x \in X\}. \end{aligned}$$

In case X *is a singleton* $\{x\}$*, we write* $\uparrow_i x$ *and* $\downarrow_i x$ *rather than* $\uparrow_i\{x\}$ *and* $\downarrow_i\{x\}$*; sets of these forms are called* principal i*-upsets and* i*-downsets, respectively.*

These notions are used to gather some additional facts on i-filters and i-ideals.

PROPOSITION 3.33. *Let* $\mathbb{A} = \langle A, \bot, \top, \vee_i, \neg_{ij} \rangle_{i,j \in \mathrm{I}}$ *be a multiplayer algebra. Then*

(1) *for* $a \in A$*, the set* $\uparrow_i a$ *is the smallest* i*-filter containing* a*;*
(2) *for* $a \in A$*, the set* $\downarrow_i a$ *is the smallest* i*-ideal containing* a*;*
(3) *for* $a \in A$*,* $F \in Fil_i(\mathbb{A})$ *and* $j \neq i$*, the set* $\uparrow_i \{a \vee_j b \mid b \in F\}$ *is an* i*-filter;*
(4) *for* $a \in A$ *and* $D \in Idl_i(\mathbb{A})$*, the set* $\downarrow_i \{a \vee_i c \mid c \in D\}$ *is the smallest* i*-ideal extending* $\{a\} \cup D$*;*
(5) *if* $\mathbb{A}$ *is finite then every prime* i*-filter is of the form* $\uparrow_i a$ *for some* $a \in A$*.*

Proof. As a sample case of the first four items, we sketch the proof of Item 3. Let a and F be as stated, and define $G := \uparrow_i \{a \vee_j c \mid c \in F\}$. In order to prove that G is an i-filter, we show that it satisfies the conditions (F1–3). Starting with (F1), note that $\top_i \in F$ by assumption on F. Since by (P5) we have $a \vee_j \top_i \sqsubseteq_i \top_i$, this gives $\top_i \in G$, as required. For (F2), assume that e and e' belong to G; we will show that $e \vee_k e' \in G$ for an arbitrary $k \neq i$. By definition of G, there are $b, b' \in F$ such that $a \vee_j b \sqsubseteq_i e$, and $a \vee_j b' \sqsubseteq_i e'$, respectively. Then the element $a \vee_j (b \vee_k b')$, which is identical to $(a \vee_j b) \vee_k (a \vee_j b')$ by distributivity (P6), satisfies $a \vee_j (b \vee_k b') \sqsubseteq_i e \vee_k e'$ by Proposition 3.27(8). But since F is an i-filter, it contains the element $b \vee_k b'$, and so by definition G contains the element $e \vee_k e'$, which suffices to prove that G satisfies (F2). Finally, condition (F3) is immediate by the transitivity of $\sqsubseteq_i$.

Concerning the proof of Item 5, assume that $\mathbb{A}$ is finite, and let $F = \{a_1, \ldots, a_n\}$ be one of its prime i-filters. Fix a player j distinct from i, and define

$$
\begin{aligned}
c_0 &:= \top_i, \\
c_{q+1} &:= \begin{cases} c_q & \text{if } c_q \sqsubseteq_i a_{q+1}, \\ c_q \vee_j a_{q+1} & \text{otherwise.} \end{cases}
\end{aligned}
$$

It is straightforward to verify that each c_q belongs to F, and (using Proposition 3.27(6)) that $c_q \sqsubseteq_i a_p$ for all $p \leq q$. From this it is immediate that $F = \uparrow_i c_n$. □

In the representation theorem for multiplayer algebras we will apply the following analog of the Prime Filter Theorem. The particular instance of this result that we will need states that given an i-filter G and an element $a \notin G$ we can always extend G to a prime i-filter F not containing a.

THEOREM 3.34. *Let* $\mathbb{A} = \langle A, \bot, \top, \vee_i, \neg_{ij} \rangle_{i,j \in \mathrm{I}}$ *be a multiplayer algebra, and let* D *and* G *be respectively an* i*-ideal and an* i*-filter of* $\mathbb{A}$*. If* $D \cap G = \varnothing$*, then there is a prime* i*-filter* F *and a prime* i*-ideal* C *such that* $G \subseteq F$*,* $D \subseteq C$*, and* $C = A \setminus F$*.*

Proof. A straightforward application of Zorn's lemma provides us with a maximal i-ideal C that is disjoint from G. We claim that C is a *prime* i-ideal. This suffices to prove the theorem by Proposition 3.31(1).

To see that C is a prime ideal, it suffices to prove condition (I4) since C, being disjoint from G, is a proper ideal. Suppose for contradiction that for some $j \neq i$ we have $a \vee_j b \in C$ but $a, b \notin C$. Then the set $C_a := \downarrow_i \{a \vee_i c \mid c \in C\}$ properly extends C, while it is an ideal by Proposition 3.33(4). So by our assumption on C, C_a must intersect with G. In other words, there is an element $c_a \in C$ with $g \sqsubseteq_i a \vee_i c_a$ for some $g \in G$. Likewise, we find a $c_b \in C$ with $g' \sqsubseteq_i b \vee_i c_b$. Using the closure properties of i-filters we can prove that subsequently the elements $a \vee_i c_a$, $b \vee_i c_b$, $a \vee_i c_a \vee_i c_b$, $b \vee_i c_a \vee_i c_b$, and

$$(a \vee_i c_a \vee_i c_b) \;\vee_j\; (b \vee_i c_a \vee_i c_b)$$

belong to G. On the other hand, since both c_a and c_b, and by assumption, in addition $a \vee_j b$ belong to C, by properties of ideals we find that the object

$$(a \vee_j b) \vee_i (c_a \vee_i c_b)$$

belongs to C. This provides the desired contradiction since it is not hard to prove that the two displayed elements are identical, while C and G are disjoint. □

We are now ready to prove the main technical result of this section, which states that any multiplayer algebra can be represented over the set of its i-filters (for any $i \in \mathtt{I}$). Before we formulate and prove this representation theorem we first give an explicit definition of the representation map.

DEFINITION 3.35. *Let* $\mathbb{A} = \langle A, \bot, \top, \vee_i, \neg_{ij}\rangle_{i,j\in\mathtt{I}}$ *be a multiplayer algebra. Given a player* $i \in \mathtt{I}$*, let*

$$\rho_i : A \to \Big(\mathtt{I} \to \mathcal{P}(Fil_i^p(\mathbb{A}))\Big)$$

be the map given by

$$\rho_i(a) : j \mapsto \{F \in Fil_i^p(\mathbb{A}) \mid \neg_{ij} a \in F\}. \tag{7}$$

PROPOSITION 3.36. *For any multiplayer algebra* $\mathbb{A}$ *and any* $i \in \mathtt{I}$*, the map* ρ_i *is a representation of* $\mathbb{A}$ *over the set* $Fil_i^p(\mathbb{A})$*.*

Proof. For simplicity of notation we fix $i = \kappa$. In order to show that the map ρ_κ is a representation of $\mathbb{A}$ over the set of prime κ-filters of $\mathbb{A}$, we need to prove that it is an injective homomorphism from $\mathbb{A}$ into the complex algebra $\mathbb{C}_{\mathtt{I}}(Fil_\kappa^p(\mathbb{A}))$. Abbreviate $S := Fil_\kappa^p(\mathbb{A})$ and $\rho := \rho_\kappa$.

We start by proving that ρ is a homomorphism, first considering the constants. For $\bot$, we need to show that

$$\rho(\bot) = \overline{\emptyset}. \tag{8}$$

In order to prove this, first note that, by (N8), $\bot = \neg_{\kappa i}\bot$, and that by Proposition 3.27(3), $\bot \equiv_\kappa \bot_\kappa$. From this it follows that there is no prime κ-filter F such that $\neg_{\kappa i}\bot \in F$, and thus by (7), we obtain that $\rho(\bot)(i) = \varnothing$ for each $i \in \mathtt{I}$. From this (8) is immediate.

Similarly, if considering $\top$, using axiom (N9) and Proposition 3.27(3), the reader may verify that

$$\rho(\top) = \overline{S}, \tag{9}$$

Turning to the disjunction operations, in order to show that $\rho(a \vee_i b) = \rho(a) \cup_i \rho(b)$ for all $i \in \mathtt{I}$, we will prove that

$$\rho(a \vee_i b)(j) = (\rho(a) \cup_i \rho(b))(j), \text{ for all } i, j \in \mathtt{I}. \tag{10}$$

We will distinguish four cases, depending on whether $i = \kappa$ and $j = i$.

If $i = j = \kappa$, we have

$$\begin{aligned}
\rho(a \vee_i b)(j) &= \{F \in S \mid \neg_{\kappa\kappa}(a \vee_\kappa b) \in F\} && \text{(definition of } \rho) \\
&= \{F \in S \mid a \vee_\kappa b \in F\} && \text{(N1)} \\
&= \{F \in S \mid a \in F\} \cup \{F \in S \mid b \in F\} && (F \text{ is } \kappa\text{-prime}) \\
&= \{F \in S \mid \neg_{\kappa\kappa} a \in F\} \cup \{F \in S \mid \neg_{\kappa\kappa} b \in F\} && \text{(N1)} \\
&= \rho(a)(j) \cup \rho(b)(j) && \text{(definition of } \rho) \\
&= (\rho(a) \cup_i \rho(b))(j) && (j = i)
\end{aligned}$$

If $i = \kappa$ but $j \neq i$, we have

$$\begin{aligned}
\rho(a \vee_i b)(j) &= \{F \in S \mid \neg_{\kappa j}(a \vee_\kappa b) \in F\} && \text{(definition of } \rho) \\
&= \{F \in S \mid \neg_{\kappa j} a \vee_j \neg_{\kappa j} b \in F\} && \text{(N10)} \\
&= \{F \in S \mid \neg_{\kappa j} a \in F\} \cap \{F \in S \mid \neg_{\kappa j} b \in F\} && \text{(F2,F3)} \\
&= \rho(a)(j) \cap \rho(b)(j) && \text{(definition of } \rho) \\
&= (\rho(a) \cup_i \rho(b))(j) && (j \neq i)
\end{aligned}$$

If $i = j \neq \kappa$, we have

$$\begin{aligned}
\rho(a \vee_i b)(j) &= \{F \in S \mid \neg_{\kappa j}(a \vee_j b) \in F\} && \text{(definition of } \rho) \\
&= \{F \in S \mid \neg_{\kappa j} a \vee_\kappa \neg_{\kappa j} b \in F\} && \text{(N10)} \\
&= \{F \in S \mid \neg_{\kappa j} a \in F\} \cup \{F \in S \mid \neg_{\kappa j} b \in F\} && (F \text{ is } \kappa\text{-prime}) \\
&= \rho(a)(j) \cup \rho(b)(j) && \text{(definition of } \rho) \\
&= (\rho(a) \cup_i \rho(b))(j) && (j = i)
\end{aligned}$$

Finally, if $i \neq \kappa$ and $j \neq i$ (possibly $j = \kappa$), we have

$$\begin{aligned}
\rho(a \vee_i b)(j) &= \{F \in S \mid \neg_{\kappa j}(a \vee_i b) \in F\} && \text{(definition of } \rho) \\
&= \{F \in S \mid \neg_{\kappa j} a \vee_i \neg_{\kappa j} b \in F\} && \text{(N11)} \\
&= \{F \in S \mid \neg_{\kappa j} a \in F\} \cap \{F \in S \mid \neg_{\kappa j} b \in F\} && \text{(F2,F3)} \\
&= \rho(a)(j) \cap \rho(b)(j) && \text{(definition of } \rho) \\
&= (\rho(a) \cup_i \rho(b))(j) && (j \neq \kappa)
\end{aligned}$$

Finally, we turn to the role switch operations: in order to prove that $\rho(\neg_{ij}(a)) = \sim_{ij} \rho(a)$ (for all $i, j \in \mathtt{I}$) we will show that

$$\rho(\neg_{ij}(a))(k) = \sim_{ij} \rho(a)(k) \text{ for all } i, j, k \in \mathtt{I}. \tag{11}$$

Also in this case there are a number of distinct cases to consider.

If $\kappa \in \{i, j\}$, by axiom (N3) we may without loss of generality assume that $\kappa = i$. We distinguish subcases as to whether $k \in \{\kappa, j\}$ or not.

For $k = \kappa$, we have

$$\begin{aligned}
\rho(\neg_{ij}(a))(k) &= \{F \in S \mid \neg_{\kappa\kappa}\neg_{\kappa j} a \in F\} && \text{(definition of } \rho)\\
&= \{F \in S \mid \neg_{\kappa j} a \in F\} && \text{(N1)}\\
&= \rho(a)(j) && \text{(definition of } \rho)\\
&= \sim_{\kappa j} \rho(a)(\kappa) && \text{(definition of } \sim_{\kappa j})\\
&= \sim_{ij} \rho(a)(k) && (\kappa = i = k)
\end{aligned}$$

For $k = j$, we obtain

$$\begin{aligned}
\rho(\neg_{ij}(a))(k) &= \{F \in S \mid \neg_{\kappa j}\neg_{\kappa j} a \in F\} && \text{(definition of } \rho)\\
&= \{F \in S \mid \neg_{\kappa\kappa} a \in F\} && \text{(N6 \& F3)}\\
&= \rho(a)(\kappa) && \text{(definition of } \rho)\\
&= \sim_{\kappa j} \rho(a)(j) && \text{(definition of } \sim_{\kappa j})\\
&= \sim_{ij} \rho(a)(k) && (\kappa = i, k = j)
\end{aligned}$$

For $k \notin \{\kappa, j\}$, we find

$$\begin{aligned}
\rho(\neg_{ij}(a))(k) &= \{F \in S \mid \neg_{\kappa k}\neg_{\kappa j} a \in F\} && \text{(definition of } \rho)\\
&= \{F \in S \mid \neg_{\kappa k} a \in F\} && \text{(N7 \& F3)}\\
&= \rho(a)(k) && \text{(definition of } \rho)\\
&= \sim_{\kappa j} \rho(a)(k) && \text{(definition of } \sim_{\kappa j})\\
&= \sim_{ij} \rho(a)(k) && (\kappa = i)
\end{aligned}$$

If, on the other hand, $\kappa \notin \{i, j\}$, we distinguish two subcases.

For $k \in \{i, j\}$, we may (again because of (N3)) without loss of generality assume that $k = i$. Then we find that

$$\begin{aligned}
\rho(\neg_{ij}(a))(k) &= \{F \in S \mid \neg_{\kappa k}\neg_{kj} a \in F\} && \text{(definition of } \rho)\\
&= \{F \in S \mid \neg_{\kappa j} a \in F\} && \text{(N6 \& F3)}\\
&= \rho(a)(j) && \text{(definition of } \rho)\\
&= \sim_{kj} \rho(a)(k) && \text{(definition of } \sim_{kj})\\
&= \sim_{ij} \rho(a)(k) && (k = i)
\end{aligned}$$

Finally, for $k \notin \{i, j\}$ we obtain that

$$\begin{aligned}
\rho(\neg_{ij}(a))(k) &= \{F \in S \mid \neg_{\kappa k}\neg_{ij} a \in F\} && \text{(definition of } \rho)\\
&= \{F \in S \mid \neg_{ij}\neg_{\kappa k} a \in F\} && \text{(N5)}\\
&= \{F \in S \mid \neg_{\kappa k} a \in F\} && \text{(N2 \& F3)}\\
&= \rho(a)(k) && \text{(definition of } \rho)\\
&= \sim_{ij} \rho(a)(k) && (k \notin \{i, j\})
\end{aligned}$$

This finishes the proof that ρ is a homomorphism.

It is left to show that ρ is injective. Consider two distinct elements $a, b \in A$. By antisymmetry of $\leq_\kappa$ we may assume without loss of generality that $a \not\leq_\kappa b$. Then by

Proposition 3.26 it follows that either $a \not\sqsubseteq_\kappa b$ or $b \not\sqsubseteq_j a$ for some $j \neq \kappa$. In the first case we find that $b \notin \uparrow_\kappa a$, and so by Proposition 3.33 and Theorem 3.34 there is a prime κ-filter $F \supseteq \uparrow_\kappa a$ (and thus containing a), but not containing b. From this it follows that $F \in \rho(a)(\kappa) \setminus \rho(b)(\kappa)$. In the second case, it follows by Proposition 3.27(5) that $\neg_{\kappa j} a \not\sqsubseteq_\kappa \neg_{\kappa j} b$. Reasoning as above we find a prime κ-filter F containing $\neg_{\kappa j} a$ but not $\neg_{\kappa j} b$, from which it follows that $F \in \rho(b)(j) \setminus \rho(a)(j)$. In both cases we have shown that $\rho(a) \neq \rho(b)$, as required. $\square$

From Proposition 3.36 the proof of the (hard direction of) Theorem 3.28 is immediate.

3.4. Some categorical observations. Representation theorems frequently have the additional benefit of opening doors between different mathematical worlds. In many cases, representation results like our Theorem 3.28, can be strengthened to *duality theorems* between on the one hand a category of algebras, and on the other hand a category of spaces, possibly endowed with relational and/or topological structure. Such duality theorems may come in pairs consisting of a *discrete* duality linking arbitrary objects on the space side to rather special structures on the algebraic side, and a *topological* duality connecting arbitrary algebras to structures on the spatial side that are endowed with nontrivial topological structure. These two dualities often coincide when one restricts attention to *finite* objects. For instance, in the case of Boolean algebras, the well-known duality between finite Boolean algebras and finite sets splits into a discrete duality between arbitrary sets and complete, atomic, Boolean algebras, and Stone's topological duality between arbitrary Boolean algebras and zero-dimensional, compact Hausdorff spaces. For more information on this topic, and more examples, see Venema (2006).

In this subsection we study the constructions we defined earlier on in this section from such a duality-theoretic perspective. We assume familiarity with the basic notions from category theory, see for instance Lane (1998).

First of all we extend the constructions $\mathbb{C}_{\mathtt{I}}(\cdot)$ and $Fil_i^p(\cdot)$ to (contravariant) *functors* from the category Set (of sets with functions), to the category $\mathsf{PA}_{\mathtt{I}}$ (of multiplayer algebras with homomorphisms), and vice versa. Starting with $\mathbb{C}_{\mathtt{I}}(\cdot)$, given a function $f : S \to S'$, define the map $\mathbb{C}_{\mathtt{I}}(f) : \mathcal{P}(S')^{\mathtt{I}} \to \mathcal{P}(S)^{\mathtt{I}}$ by putting, for $X' \in \mathcal{P}(S')^{\mathtt{I}}$:

$$\mathbb{C}_{\mathtt{I}}(f) : X' \mapsto \lambda i.f^{-1}(X'(i)).$$

We leave it for the reader to verify that $\mathbb{C}_{\mathtt{I}}(f)$ is a homomorphism from $\mathbb{C}_{\mathtt{I}}(S')$ to $\mathbb{C}_{\mathtt{I}}(S)$, which suffices to show indeed we have obtained a functor: $\mathbb{C}_{\mathtt{I}} : \mathsf{Set} \to \mathsf{PA}_{\mathtt{I}}^{op}$.

Conversely, given multiplayer algebras $\mathbb{A}$ and $\mathbb{A}'$, assume that κ is a player in $\mathtt{I}$, and define $\mathsf{F}\mathbb{A} := Fil_\kappa^p(\mathbb{A})$. For a homomorphism $\alpha : \mathbb{A} \to \mathbb{A}'$, define the map $\mathsf{F}\alpha$ by putting, for a prime κ-filter $G' \in \mathsf{F}\mathbb{A}'$:

$$\mathsf{F}\alpha(G') := \{a \in A \mid \alpha a \in G'\}.$$

It is straightforward to verify that $\mathsf{F}\alpha$ maps prime κ-filters to prime κ-filters, so that we have $\mathsf{F}\alpha : \mathsf{F}\mathbb{A}' \to \mathsf{F}\mathbb{A}$. Thus F is a functor $\mathsf{F} : \mathsf{PA}_{\mathtt{I}}^{op} \to \mathsf{Set}$.

As our next step we show that the functors $\mathbb{C}_{\mathtt{I}}$ and F form an *adjoint pair*. Here the main task is to find, for a set S and a multiplayer algebra $\mathbb{A}$, a bijection between the arrows in Set from S to $\mathsf{F}\mathbb{A}$, and the ones in $\mathsf{PA}_{\mathtt{I}}$ from $\mathbb{A}$ to $\mathbb{C}_{\mathtt{I}} S$:

$$\mathsf{Set}(S, \mathsf{F}\mathbb{A}) \cong \mathsf{PA}_{\mathtt{I}}(\mathbb{A}, \mathbb{C}_{\mathtt{I}} S).$$

Given a map $f : S \to \mathsf{F}\mathbb{A}$, define $\varphi_f : A \to \mathcal{P}(S)^{\mathtt{I}}$ by putting

$$\varphi_f : a \mapsto \lambda i.\{s \in S \mid \neg_{\kappa i} a \in fs\}.$$

It is easy to see that indeed φ_f is a homomorphism from $\mathbb{A}$ to $\mathbb{C}_{\mathtt{I}}S$. Conversely, given a homomorphism $\alpha : \mathbb{A} \to \mathbb{C}_{\mathtt{I}}S$, define the map $q_\alpha : S \to \mathsf{F}\mathbb{A}$ by putting

$$q_\alpha : s \mapsto \{a \in A \mid s \in (\alpha a)(\kappa)\}.$$

Some straightforward verifications reveal that $\varphi : \mathsf{Set}(S, \mathsf{F}\mathbb{A}) \to \mathsf{PA}_{\mathtt{I}}(\mathbb{A}, \mathbb{C}_{\mathtt{I}}S)$ and $q : \mathsf{PA}_{\mathtt{I}}(\mathbb{A}, \mathbb{C}_{\mathtt{I}}S) \to \mathsf{Set}(S, \mathsf{F}\mathbb{A})$ are natural in S and $\mathbb{A}$, and that they are each other's converse. For instance, for $\alpha : \mathbb{A} \to \mathbb{C}_{\mathtt{I}}S$, to see that $\varphi_{q_\alpha} = \alpha$, we compute, for $a \in A$ and $i \in \mathtt{I}$:

$$\begin{aligned}
\varphi_{q_\alpha}(a)(i) &= \{s \in S \mid \neg_{\kappa i} a \in q_\alpha(s)\} && \text{(definition of } \varphi) \\
&= \{s \in S \mid s \in (\alpha(\neg_{\kappa i} a))(\kappa)\} && \text{(definition of } q) \\
&= \{s \in S \mid s \in (\sim_{\kappa i}(\alpha a))(\kappa)\} && (\alpha \text{ is a homomorphism}) \\
&= \{s \in S \mid s \in (\alpha a))(i)\} && \text{(definition of } \sim_{\kappa i}) \\
&= \alpha(a)(i) && \text{(immediate)}
\end{aligned}$$

From this it follows that $\mathbb{C}_{\mathtt{I}} : \mathsf{Set} \to \mathsf{PA}_{\mathtt{I}}^{op}$ is left adjoint to $\mathsf{F} : \mathsf{PA}_{\mathtt{I}}^{op} \to \mathsf{Set}$, with φ/q as the witnessing natural isomorphism.

As expected, in the case we restrict on both sides (i.e., in Set and in $\mathsf{PA}_{\mathtt{I}}$) to *finite* objects, more can be said. We will see that in fact, our adjunction restricts to a *duality* between the categories FinSet of *finite* sets, and $\mathsf{FinPA}_{\mathtt{I}}$ of *finite* multiplayer algebras.

In order to show this, we need to prove that the unit and counit of the adjunction are natural *isomorphisms*. The key claim in both cases is Proposition 3.33(5), stating that in finite algebras, all κ-prime filters are principal. As an immediate consequence of this we see that in the case of finite algebras, the representation map of Definition 3.35 is not just an embedding, but in fact an isomorphism. Since these representation maps provide the *counit* of the adjunction (as a straightforward verification will reveal), we have established our claim for the counit of the adjunction.

Turning to the *unit*, our key claim is that if S is a finite set, then every prime κ-filter of $\mathbb{C}_{\mathtt{I}}S$ is a principal filter of the form $\uparrow_\kappa U_s$ for some unique $s \in S$, where $U_s \in \mathcal{P}(S)^{\mathtt{I}}$ is given by

$$U_s(i) = \begin{cases} \{i\} & \text{if } i = \kappa, \\ \varnothing & \text{if } i \neq \kappa. \end{cases}$$

In order to prove this claim, let G be an arbitrary prime κ-filter of $\mathbb{A}$. By Proposition 3.33(5), $G = \uparrow_\kappa X$ for some $X \in \mathcal{P}(S)^{\mathtt{I}}$. The point is that $X(i)$ must be a singleton: for otherwise, partitioning $X(i) := X'(i) \uplus X''(i)$, we could write $X = X' \cup_\kappa X''$ without either X' or X'' belonging to G, and this would contradict the primeness of G. Hence, we may assume that there is some $s \in S$ such that $G = \uparrow_\kappa X$ for some X with $X(\kappa) = \{s\}$. It is not hard to see that $X \equiv_\kappa Y$ for *any* Y with $Y(\kappa) = X(\kappa)$, and so $G = \uparrow_\kappa Y$ for any such Y. In particular, we find that $G = \uparrow_\kappa U_s$. Finally, uniqueness of s is given by the fact that if G could be written as $\uparrow_\kappa U_t$ for some $t \neq s$ as well, then by property (F2) we would have $U_s \cup_\lambda U_t = \overline{\emptyset} \in G$, which would contradict the fact that G is proper.

As a consequence of the above claim, given a finite set S, the function

$$e_S : S \to \mathsf{F}\mathbb{C}_{\mathtt{I}}S$$

mapping $s \in S$ to the κ-prime filter $\uparrow_\kappa U_s$, is surjective. It is routine to verify that the collection of maps e_S provide the unit $e : \mathsf{Id}_{\mathsf{Set}} \to \mathsf{F}\mathbb{C}_{\mathtt{I}}$ of the adjunction between

$\mathbb{C}_{\mathtt{I}}$ and F, and since each e_S is easily proven to be injective, it follows that this unit is a natural isomorphism indeed.

We summarize our findings in the following theorem.

THEOREM 3.37.

1. *The functors* $\mathbb{C}_{\mathtt{I}} : \mathsf{Set} \to \mathsf{PA}_{\mathtt{I}}^{op}$ *and* $\mathsf{F} : \mathsf{PA}_{\mathtt{I}}^{op} \to \mathsf{Set}$ *form, with the natural bijections* $\varphi : \mathsf{Set}(S, \mathsf{F}\mathbb{A}) \cong \mathsf{PA}_{\mathtt{I}}(\mathbb{A}, \mathbb{C}_{\mathtt{I}} S) : q$, *an adjunction from* Set *to* $\mathsf{PA}_{\mathtt{I}}^{op}$.
2. *This adjunction restricts to a dual equivalence between the categories* FinSet *and* $\mathsf{FinPA}_{\mathtt{I}}^{op}$.

It would certainly be interesting to develop this category-theoretic perspective further. It should not be difficult to extend Theorem 3.37(2) to a discrete duality between the category Set and a certain subcategory of $\mathsf{PA}_{\mathtt{I}}$ consisting of 'perfect' multiplayer algebras with complete homomorphisms. To obtain a duality for the category $\mathsf{PA}_{\mathtt{I}}$ of *all* multiplayer algebras one needs to put appropriate topological structure on the dual side. We conjecture that some multiplayer version of the recent *bitopological* approach toward Stone duality taken by Bezhanishvili *et al.* (2010) will provide the right framework for addressing this problem, but leave this as a topic for further research.

§4. Multiplayer algebras: the modal case. In this section we extend the results of the previous section to the modal setting. That is, we will introduce a class of concrete algebras associated with the semantics of multiplayer modal logic in Kripke frames, and an axiomatically defined class of multiplayer modal algebras. The main result of this section will be a representation theorem linking these two classes.

All algebras that we consider in this section will be of the following similarity type.

DEFINITION 4.38. *Let* $MBool_{\mathtt{I}}$ *be the algebraic similarity type extending that of* $Bool_{\mathtt{I}}$ *with a unary function symbol* $\Diamond_i$ *for each* $i \in \mathtt{I}$.

4.1. Concrete modal multiplayer algebras. Analogous to the case for standard modal logic, we can encode the Kripke semantics of multiplayer modal logic in a class of concrete algebras. These algebras arise as so-called complex algebras of Kripke frames: given a Kripke frame $\mathbb{S} = \langle S, R \rangle$, we obtain its full complex algebra by expanding the full multiplayer set algebra $\mathbb{C}_{\mathtt{I}}(S)$ with a *modality* $\langle R \rangle_i : \mathcal{P}(S)^{\mathtt{I}} \to \mathcal{P}(S)^{\mathtt{I}}$ for each player i. These operations are defined coordinatewise, with $\langle R \rangle_i$ acting as a classical modal diamond for coordinate i, and as a box for coordinates $j \neq i$.

DEFINITION 4.39. *The* full complex multiplayer algebra over $\mathbb{S}$ *of a Kripke frame* $\mathbb{S} = \langle S, R \rangle$ *is the following* $MBool_{\mathtt{I}}$*-type algebra:*

$$\mathbb{C}_{\mathtt{I}}(\mathbb{S}) := \langle \mathcal{P}(S)^{\mathtt{I}}, \overline{\emptyset}, \overline{S}, \cup_i, \sim_{ij}, \langle R \rangle_i \rangle_{i,j \in \mathtt{I}}.$$

Here the operation $\langle R \rangle_i : \mathcal{P}(S)^{\mathtt{I}} \to \mathcal{P}(S)^{\mathtt{I}}$ *is given by putting, for* $X \in \mathcal{P}(S)^{\mathtt{I}}$*:*

$$(\langle R \rangle_i X)(k) := \begin{cases} \{s \in S \mid Rst \text{ for some } t \in X(k)\} & \text{if } k = i, \\ \{s \in S \mid Rst \text{ implies } t \in X(k), \text{ for all } t\} & \text{if } k \neq i. \end{cases}$$

The class of all full complex multiplayer algebras is denoted as $\mathsf{CMA}_{\mathtt{I}}$.

As in the propositional case, it is straightforward to verify that these concrete algebras encode the multiplayer modal Kripke semantics.

PROPOSITION 4.40. *Let φ, ψ be modal multiplayer formulas. Then $\varphi \equiv \psi$ iff* $\mathsf{CMA}_{\mathtt{I}} \models \varphi \approx \psi$.

Proof. Given a Kripke frame $\mathbb{S} = \langle S, R\rangle$, we may obviously identify (logical) valuations on $\mathbb{S}$ with (algebraic) assignments on $\mathbb{C}_{\mathtt{I}}(\mathbb{S})$. Then, given such a valuation/assignment V, let $\widetilde{V}(\varphi)$ denote the (uniquely defined) meaning of φ under V in the algebra $\mathbb{C}_{\mathtt{I}}(\mathbb{S})$. It is a routine exercise to verify that

$$S, R, V, s \Vdash_i \varphi \text{ iff } s \in \widetilde{V}(\varphi)(i).$$

for every formula φ and for every player i.

But from this it easily follows that $\varphi \not\equiv \psi$ iff for some Kripke frame $\mathbb{S}$ we have $\mathbb{C}_{\mathtt{I}}(\mathbb{S}) \not\models \varphi \approx \psi$. This suffices to prove the proposition. $\square$

As a consequence, we may obtain all game-semantical equivalences of formulas by axiomatizing the equational theory of the class of complex multiplayer algebras.

4.2. Abstract modal multiplayer algebras. The axiomatization that we propose for modal multiplayer equivalence of formulas, is given in Table 4. The corresponding algebraic variety is defined as follows.

DEFINITION 4.41. *An MBool$_{\mathtt{I}}$-type algebra $\mathbb{A} = \langle A, \bot, \top, \vee_i, \neg_{ij}, \Diamond_i\rangle_{i,j\in\mathtt{I}}$ is called a* multiplayer modal algebra *if it satisfies, in addition to the equations (P1–8) and (N1–11) of Table 3, the equations (M1–11) of Table 4. The class of modal multiplayer algebras will be denoted as* $\mathsf{MA}_{\mathtt{I}}$.

The axioms (M1–4) concern the interaction between the modalities and the constants $\bot$, $\top$, and $\bot_i$. (M5) is the equational way to express the monotonicity of $\Diamond_j$ with respect to the ordering $\leq_i$, while (M6) states that $\Diamond_i$ distributes over $\vee_i$. The next axiom, (M7) expresses some interaction between distinct modalities $\Diamond_i$ and $\Diamond_j$, and is like a multiplayer version of the axiom $\Diamond x \wedge \Box y \approx \Diamond(x \wedge y)$ which is well known from positive modal logic. (M8) states that a player k distinct from i and j will have no preference between $\Diamond_i x$ and $\Diamond_j x$. The axioms (M9) and (M10) state the obvious interaction between the modalities and the role switch operations, and could be combined into $\Diamond_i \neg_{jk} x \approx \neg_{jk} \Diamond_{[j,k](i)} x$. Finally, axiom (M11) is needed to ensure that $\Diamond_j$ is monotone with respect to the relation $\sqsubseteq_i$.

Based on these axioms we can prove that in any multiplayer modal algebra, each modality is monotone (order preserving) with respect to each of the orders $\leq_i$ and $\sqsubseteq_i$.

Table 4. *Modal axioms for multiplayer algebras*

(M1)	$\Diamond_i \bot \approx_i \bot$	
(M2)	$\Diamond_i \top \approx_j \top$	
(M3)	$\Diamond_i \bot_i \approx \bot_i$	
(M4)	$\Diamond_i \bot_j \approx_j \bot_j$	
(M5)	$\Diamond_j x \vee_i \Diamond_j (x \vee_i y) \approx \Diamond_j (x \vee_i y)$	
(M6)	$\Diamond_i x \vee_i \Diamond_i y \approx \Diamond_i (x \vee_i y)$	
(M7)	$\Diamond_i x \vee_j \Diamond_j y \preceq_i \Diamond_i (x \vee_j y)$	
(M8)	$\Diamond_i x \approx_k \Diamond_j x$	$(k \notin \{i, j\})$
(M9)	$\Diamond_i \neg_{ij} x \approx \neg_{ij} \Diamond_j x$	
(M10)	$\Diamond_i \neg_{jk} x \approx \neg_{jk} \Diamond_i x$	$(i \notin \{j, k\})$
(M11)	$\Diamond_j (x \vee_i \bot) \approx_i \Diamond_j x$	

PROPOSITION 4.42. *Let* $\mathbb{A} = \langle A, \bot, \top, \vee_i, \neg_{ij}, \Diamond_i \rangle_{i,j\in \mathtt{I}}$ *be a multiplayer modal algebra. Then for any* $a, b \in A$, *and any* $i, j \in \mathtt{I}$,

(1) $a \leq_i b$ *implies* $\Diamond_j a \leq_i \Diamond_j b$;
(2) $a \sqsubseteq_i b$ *implies* $\Diamond_j a \sqsubseteq_i \Diamond_j b$.

Proof. In order to prove Item 1, assume that $a \leq_i b$, that is, $a \vee_i b = b$. By (M6) it follows from this that $\Diamond_i a \vee_i \Diamond_i b = \Diamond_i (a \vee_i b) = \Diamond_i b$, which means that $\Diamond_i a \leq_i \Diamond_i b$ indeed. Similarly, by (M5) it follows that $\Diamond_j a \vee_i \Diamond_j b = \Diamond_j a \vee_i \Diamond_j (a \vee_i b) = \Diamond_j (a \vee_i b) = \Diamond_j b$, which by definition means $\Diamond_j a \leq_i \Diamond_j b$.

For Item 2, assume $a \sqsubseteq_i b$. In order to prove that $\Diamond_i a \sqsubseteq_i \Diamond_i b$ we first observe that by definition we have $a \vee_i \bot \leq_i b \vee_i \bot$. From this we obtain by Part 1 and Proposition 3.26(4) that

$$\Diamond_i (a \vee_i \bot) \sqsubseteq_i \Diamond_i (b \vee_i \bot). \tag{12}$$

But for any $c \in A$ we find by, respectively, reasoning in propositional multiplayer algebras, axiom (M1), and axiom (M6), that

$$\Diamond_i c \equiv_i \Diamond_i c \vee_i \bot \equiv_i \Diamond_i c \vee_i \Diamond_i \bot = \Diamond_i (c \vee_i \bot). \tag{13}$$

Thus we obtain that

$$\Diamond_i a \equiv_i \Diamond_i (a \vee_i \bot) \sqsubseteq_i \Diamond_i (b \vee_i \bot) \equiv_i \Diamond_i b,$$

from which it is immediate that $\Diamond_i a \sqsubseteq_i \Diamond_i b$.

To prove that $\Diamond_j a \sqsubseteq_i \Diamond_j b$, we reason as follows. From $a \vee_i \bot \leq_i b \vee_i \bot$ we may infer, using $a \leq_i a \vee_i \bot$, that $a \leq_i b \vee_i \bot$. Using Item 1, we obtain that $\Diamond_j a \leq_i \Diamond_j (b \vee_i \bot)$, whereas by axiom (M11) we have that $\Diamond_j (b \vee_i \bot) \equiv_i \Diamond_j b$. Combining these facts with the observations on $\leq_i$ and $\sqsubseteq_i$ made in Proposition 3.26, we find that $\Diamond_j a \sqsubseteq_i \Diamond_j b$, as required. □

4.3. A modal representation theorem. In this subsection we shall prove the modal analog of Theorem 3.28, stating that every abstract multi-modal algebra can be represented as a concrete one based on some Kripke frame.

THEOREM 4.43. *For any set* $\mathtt{I}$ *of players with* $|\mathtt{I}| \geq 2$:

$$\mathsf{MA}_{\mathtt{I}} = \mathbf{S}(\mathsf{CMA}_{\mathtt{I}}). \tag{14}$$

Again, we leave the proof of the right-to-left inclusion of (14), which comes down to proving the validity of the (M) axioms in any complex multiplayer algebra, as an exercise to the reader. The proof of the opposite inclusion builds on the propositional case. Given an abstract multiplayer modal algebra, we already know how to represent its propositional reduct, namely, over the set of its prime i-filters (for some player i). What is needed on top of this, is a binary *relation* turning the set of prime i-filters into an appropriate Kripke frame.

DEFINITION 4.44. *Let* $\mathbb{A} = \langle A, \bot, \top, \vee_i, \neg_{ij}, \Diamond_i \rangle_{i,j\in \mathtt{I}}$ *be a multiplayer modal algebra, and let* $i \in \mathtt{I}$ *be a player. The* i-dual frame $\mathbb{K}_i(\mathbb{A})$ *is defined as the Kripke frame* $\langle Fil_i^p(\mathbb{A}), Q_i \rangle$, *where* $Fil_i^p(\mathbb{A})$ *is the set of prime* i*-filters over* $\mathbb{A}$, *and* $Q_i \subseteq Fil_i^p(\mathbb{A}) \times Fil_i^p(\mathbb{A})$ *is the relation given by*

$$Q_i FG \ :\Longleftrightarrow \ \Diamond_i G \subseteq F \textit{ and, for all } j \neq i, \Diamond_j^{-1} F \subseteq G.$$

Here we use the following notation, for $X \subseteq A$, and $i \in \mathtt{I}$:

$$\begin{array}{rcl} \Diamond_i X & := & \{\Diamond_i a \mid a \in X\}, \\ \Diamond_i^{-1} X & := & \{a \mid \Diamond_i a \in X\}. \end{array}$$

REMARK 4.45. *The dependence of the definition of the dual frame on the player $i \in \mathtt{I}$, is only superficial. In fact, we can prove that for any multiplayer modal algebra $\mathbb{A}$, for any pair of agents $i, j \in \mathtt{I}$, the frames $\mathbb{K}_i(\mathbb{A})$ to $\mathbb{K}_j(\mathbb{A})$ are* isomorphic *via the map*

$$F \in Fil_i^p(\mathbb{A}) \mapsto \{\neg_{ij} a \mid a \in F\} \in Fil_j^p(\mathbb{A}).$$

The proof of this fact is routine, and left as an exercise to the reader.

The following is one of the key propositions in the proof of the Representation Theorem for modal multiplayer algebras.

PROPOSITION 4.46. *Let $\mathbb{A} = \langle A, \bot, \top, \vee_i, \neg_{ij}, \Diamond_i \rangle_{i,j \in \mathtt{I}}$ be a multiplayer modal algebra. Furthermore, let $i \neq j$ be distinct players in $\mathtt{I}$, and let F be a prime i-filter of $\mathbb{A}$.*

(1) *$\Diamond_i a \in F$ iff there is a $G \in Fil_i^p(\mathbb{A})$ such that $Q_i F G$ and $a \in G$;*
(2) *$\Diamond_j a \notin F$ iff there is a $G \in Fil_i^p(\mathbb{A})$ such that $Q_j F G$ and $a \notin G$.*

Proof. Given an i-filter F of the multiplayer modal algebra $\mathbb{A}$, we first prove that

$$X := \Diamond_i^{-1}(A \setminus F) \text{ is an } i\text{-ideal,} \tag{15}$$

and that

$$Y := \bigcup_{j \neq i} \Diamond_j^{-1} F \text{ is an } i\text{-filter.} \tag{16}$$

In order to prove (15), we check that X satisfies the conditions (I1-3). (I1) Since F is proper, we have $\bot_i \notin F$. Then by (M3) and (F3), we also have $\Diamond_i \bot_i \notin F$, and so $\bot_i \in X$. (I2) Assume $b, b' \in X$, then neither $\Diamond_i b$ nor $\Diamond_i b'$ belongs to F, and so by (F2) and (M6), we have $\Diamond_i (b \vee_i b') \notin F$. Thus we find $b \vee_i b' \in X$, by definition of X. (I3) Assume $b \in X$ and $b' \sqsubseteq_i b$. By Proposition 4.42(2) we have that $\Diamond_i b' \sqsubseteq_i \Diamond_i b$. Since $\Diamond_i b \notin F$, it follows from (F3) that $\Diamond_i b' \notin F$, and thus $b' \in X$.

For (16), we verify the properties (F1-3) for Y. (F1) By Proposition 3.27(3) we have $\top_i \equiv_i \top$, which implies $\Diamond_j \top_i \equiv_i \Diamond_j \top$ by Proposition 4.42, and since $\Diamond_j \top \equiv_i \top$ by (M2), we obtain that $\Diamond_j \top_i \equiv_i \top_i$. From this it follows that $\Diamond_j \top_i$ belongs to F, and so we find $\top_i \in Y$. (F2) Assume $b, b' \in Y$, and let j be distinct from i. By assumption there are k and l, both distinct from i, such that $\Diamond_k b$ and $\Diamond_l b'$ belong to F. But then by (M8) and (F3) we have both $\Diamond_j b$ and $\Diamond_j b'$ in F, and so by (M6) and (F2) we find $\Diamond_j (b \vee_j b') \in F$. This immediately shows that $b \vee_j b' \in Y$. (F3) Assume $b \in Y$ and $b \sqsubseteq_i b'$. By definition of Y, we have $\Diamond_j b \in F$ for some $j \neq i$, and by Proposition 4.42(2), we have $\Diamond_j b \sqsubseteq_i \Diamond_j b'$. So by (F3) we find $\Diamond_j b' \in F$, and thus, $b' \in Y$.

We turn to the proof of the Item 1. Let $a \in A$ be such that $\Diamond_i a \in F$. It follows from (15) and Proposition 3.33 that

$$H := \uparrow_i \{a \vee_j y \mid y \in Y\} \text{ is an } i\text{-filter,} \tag{17}$$

Furthermore, we claim that

$$X \cap H = \varnothing. \tag{18}$$

To see this, suppose for contradiction that $x \in X \cap H$; then $a \vee_j y \sqsubseteq_i x$, for some $y \in Y$. By definition of Y, we have $\Diamond_k y \in F$ for some $k \neq i$, and so $\Diamond_j y \in F$ by (F3) and (M8).

Also, by assumption, we have $\Diamond_i a \in F$. Thus by (F2) we find that $\Diamond_i a \vee_j \Diamond_j y \in F$, and using (M7) and (F3), we may infer that $\Diamond_i (a \vee_j y) \in F$. However, since X is an i-ideal, it follows from $a \vee_j y \sqsubseteq_i x$ that $a \vee_j y \in X$, implying $\Diamond_i (a \vee_j y) \notin F$. This gives the desired contradiction, finishing the proof of (18).

By (15), (17), and (18), the Prime i-Filter Theorem yields a prime i-filter G such that $H \subseteq G$ and $X \cap G = \varnothing$. We claim that G has the desired properties. First of all, it is easy to see that a belongs to H, and hence, to G. Second, in order to prove that $Q_i FG$, first take an arbitrary $b \in G$. It follows that $b \notin X$, and so $\Diamond_i b \in F$ by definition of X. On the other hand, if $\Diamond_j b \in F$ for $j \neq i$, we obtain $b \in Y$. Then, by $a \vee_j b \sqsubseteq_i b$, b belongs to H, and so by $H \subseteq G$ we find $b \in G$, as required.

Item 2 of the proposition is proved in a similar way—we confine ourselves to a sketch. It follows from (15) and Proposition 3.33 that

$$J := {\downarrow_i}\{a \vee_i x \mid x \in X\} \text{ is an } i\text{-ideal,} \tag{19}$$

while we also have

$$Y \cap J = \varnothing. \tag{20}$$

As in the proof of Item 1, the Prime i-Filter Theorem guarantees the existence of a prime i-filter G such that $Y \subseteq G$ and $J \cap G = \varnothing$. We leave it as an exercise for the reader to verify that $Q_i FG$ and $a \notin G$. □

We are now well equipped for the proof of the representation theorem. Given a multiplayer modal $\mathbb{A}$ and a player i, recall that the map ρ_i of Definition 3.35:

$$\rho_i : A \to \Big(\mathtt{I} \to \mathcal{P}(Fil_i^p(\mathbb{A}))\Big)$$

provides a representation of $\mathbb{A}$ over its set of prime i-filters. As we will see now, these maps will also provide representations of a multiplayer *modal* algebra $\mathbb{A}$ over its dual Kripke frame.

PROPOSITION 4.47. *For any multiplayer modal algebra $\mathbb{A}$ and any $i \in \mathtt{I}$, the map ρ_i is a representation of $\mathbb{A}$ over the Kripke frame $\mathbb{K}_i(\mathbb{A})$.*

Proof. As in the proof of Proposition 3.36, we fix $i = \kappa$, and abbreviate $\rho = \rho_\kappa$. We will also write Q rather than Q_κ.

Since we already know ρ to be injective, and a homomorphism with respect to the propositional operation symbols, we only need to prove that it is a homomorphism for the modal operators as well. In other words, we need to establish that $\rho(\Diamond_i a) = \langle Q \rangle_i \rho(a)$, for each $i \in \mathtt{I}$, and for all $a \in A$. For this purpose it suffices to show, for a fixed $a \in A$, and a fixed prime κ-filter F, that

$$F \in \rho(\Diamond_i a)(j) \text{ iff } F \in \Big(\langle Q \rangle_i \rho(a)\Big)(j), \text{ for all } i, j \in \mathtt{I}. \tag{21}$$

We will distinguish four cases, depending on whether $i = \kappa$ and $j = i$.

If $\kappa = i = j$, we have

$$\begin{aligned}
F \in \rho(\Diamond_i a)(j) &\text{ iff } \neg_{\kappa\kappa} \Diamond_\kappa a \in F && \text{(definition of } \rho)\\
&\text{ iff } \Diamond_\kappa a \in F && \text{(N1)}\\
&\text{ iff } a \in G \text{ for some } G \text{ such that } QFG && \text{(Proposition 4.46(1))}\\
&\text{ iff } G \in \rho(a)(\kappa) \text{ for some } G \text{ such that } QFG && \text{(N1, definition of } \rho)
\end{aligned}$$

$$
\begin{aligned}
&\text{iff } F \in \Big(\langle Q\rangle_\kappa \rho(a)\Big)(\kappa) && (\text{definition of } \langle Q\rangle_\kappa)\\
&\text{iff } F \in \Big(\langle Q\rangle_i \rho(a)\Big)(j) && (\kappa = i = j)
\end{aligned}
$$

If $\kappa = i$, but $j \neq i$, we have

$$
\begin{aligned}
F \in \rho(\Diamond_i a)(j) &\text{ iff } \neg_{\kappa j}\Diamond_\kappa a \in F && (\text{definition of } \rho)\\
&\text{ iff } \Diamond_j \neg_{\kappa j} a \in F && (\text{M9})\\
&\text{ iff } \neg_{\kappa j} a \in G \text{ for all } G \text{ such that } QFG && (\text{Proposition 4.46(2)})\\
&\text{ iff } G \in \rho(a)(j) \text{ for all } G \text{ such that } QFG && (\text{definition of } \rho)\\
&\text{ iff } F \in \Big(\langle Q\rangle_\kappa \rho(a)\Big)(j) && (\text{definition of } \langle Q\rangle_\kappa,\ \kappa \neq j)\\
&\text{ iff } F \in \Big(\langle Q\rangle_i \rho(a)\Big)(j) && (\kappa = i)
\end{aligned}
$$

In case $\kappa \neq i = j$, we obtain

$$
\begin{aligned}
F \in \rho(\Diamond_i a)(j) &\text{ iff } \neg_{\kappa i}\Diamond_i a \in F && (\text{definition of } \rho)\\
&\text{ iff } \Diamond_\kappa \neg_{\kappa i} a \in F && (\text{M9})\\
&\text{ iff } \neg_{\kappa i} a \in G \text{ for some } G \text{ such that } QFG && (\text{Proposition 4.46(1)})\\
&\text{ iff } G \in \rho(a)(i) \text{ for some } G \text{ such that } QFG && (\text{definition of } \rho)\\
&\text{ iff } F \in \Big(\langle Q\rangle_i \rho(a)\Big)(i) && (\text{definition of } \langle Q\rangle_i)\\
&\text{ iff } F \in \Big(\langle Q\rangle_i \rho(a)\Big)(j) && (i = j)
\end{aligned}
$$

Finally, if $\kappa \neq i$ and $i \neq j$ (possibly $j = \kappa$), we have

$$
\begin{aligned}
F \in \rho(\Diamond_i a)(j) &\text{ iff } \neg_{\kappa j}\Diamond_i a \in F && (\text{definition of } \rho)\\
&\text{ iff } \Diamond_i \neg_{\kappa j} a \in F && (\text{M10})\\
&\text{ iff } \neg_{\kappa j} a \in G \text{ for all } G \text{ such that } QFG && (\text{Proposition 4.46(2)})\\
&\text{ iff } G \in \rho(a)(j) \text{ for all } G \text{ such that } QFG && (\text{definition of } \rho)\\
&\text{ iff } F \in \Big(\langle Q\rangle_i \rho(a)\Big)(j) && (\text{definition of } \langle Q\rangle_i,\ i \neq j)
\end{aligned}
$$

□

As in the propositional case, given the soundness of the axiomatization, Theorem 4.43 is an immediate consequence of Proposition 4.47.

REMARK 4.48. *Extending the categorical observations of Subsection 3.4 we can easily establish a duality between the category of finite Kripke frames with bounded morphisms on the one hand, and that of finite multiplayer modal algebras with homomorphisms on the other. As in the propositional case, we leave it as a matter for further research how to obtain a (multi-)topological duality for the category of all multiplayer modal algebras.*

§5. Computational issues. In this section we look at some computational issues related to the questions that we introduced in Section 2. More in particular, we shall be interested in the computational complexity of the following two problems (with i denoting a fixed player in $\mathtt{I}$):

i-SAT(L) Given a multiplayer formula $\varphi \in L$, is φ i-satisfiable?
EQ(L) Given two multiplayer formulas φ and ψ in L, are φ and ψ equivalent?

We will consider these questions both for the propositional case and for the modal case.

5.1. The propositional case. Starting with the propositional case, we first observe that since we allow valuations in which every player wins, it is not so hard to satisfy a given multiplayer formula. This makes it very easy to determine whether, for a fixed player i, a given formula is i-satisfiable or not.

THEOREM 5.49. *i-SAT($PL_{\mathtt{I}}$), the problem whether a given propositional multiplayer formula is i-satisfiable, can be solved in linear time.*

Proof. It is straightforward to verify that a formula φ is i-satisfiable iff it is i-satisfiable under the *maximal* valuation V^s which makes all players winners of each propositional variable. But solving the latter problem is merely a matter of calculating the meaning of the term φ in the small algebra $\mathbb{O}_{\mathtt{I}}$ under the valuation/assignment V^s. This can clearly be executed in time linear in the length of φ. □

Clearly the low complexity of this problem is caused by our admitting valuations declaring *all* players as the winners associated with a proposition letter. In fact, when we restrict attention to, for example, the single-winner valuations of Definition 2.12, the complexity of the satisfiability problem immediately becomes NP-complete.

A more interesting issue concerns the complexity of the problem whether two given propositional multiplayer formulas are equivalent.

THEOREM 5.50. *EQ($PL_{\mathtt{I}}$), the problem whether two given propositional multiplayer formulas are equivalent, is co-NP-complete.*

Proof. Given Theorem 3.23, it is not hard to come up with a nondeterministic polynomial algorithm which determines whether two propositional multiplayer formulas are *not* equivalent: For two propositional multiplayer formulas φ and ψ, simply guess a valuation/assignment on $\mathbb{O}_{\mathtt{I}}$, and compute whether φ and ψ obtain a *different* meaning under this valuation. Clearly φ and ψ are not equivalent iff this algorithm returns the answer 'YES'. This shows that EQ belongs to co-NP.

In order to prove that the problem is in fact co-NP-hard, consider the word problem for bounded distributive lattices, a problem well known to be co-NP-hard (see Bloniarz *et al.*, 1984). Clearly then it suffices to give a polytime reduction of this problem to EQ.

Consider the following translation $(\cdot)^t$ from bounded distributive lattice terms to $PL_{\mathtt{I}}$-formulas (where i and j are two fixed, distinct members of $\mathtt{I}$):

$$\begin{array}{rcl} p^t & := & p \\ \top^t & := & \top_i \\ \bot^t & := & \bot_i \\ (\varphi \vee \psi)^t & := & \varphi^t \vee_i \psi^t \\ (\varphi \wedge \psi)^t & := & \varphi^t \vee_j \psi^t \end{array} \tag{22}$$

It is straightforward to verify that the equation $\varphi \approx \psi$ is valid in the two-element distributive lattice $\mathbb{2}$ iff the equation $\varphi^t \approx_i \psi^t$ holds in the algebra $\mathbb{O}_{\mathtt{I}}$. But the algebra $\mathbb{2}$ generates the variety of distributive lattices, and by Theorem 3.23, and two $PL_{\mathtt{I}}$-terms are equivalent iff the corresponding equation holds in the algebra $\mathbb{O}_{\mathtt{I}}$. Combining the above observations we have indeed reduced the word problem for distributive lattices to the problem EQ. $\square$

5.2. The modal case. We first consider the complexity of the equivalence problem.

THEOREM 5.51. *$\mathsf{EQ}(ML_{\mathtt{I}})$, the problem whether two given modal multiplayer formulas are equivalent, is PSPACE-complete.*

Proof. It is straightforward to adapt the standard *Witness* algorithm, determining the satisfiability of ordinary modal logic formulas Blackburn *et al.* (2001), to an algorithm that in a multiplayer setting determines whether two formulas are equivalent or not. Since *Witness* operates in PSPACE, this shows that EQ can be solved in polynomial space as well.

PSPACE-*hardness* of $\mathsf{EQ}(ML_{\mathtt{I}})$ can be proved similarly to the $\mathsf{co\text{-}NP}$-hardness of EQ in the propositional case (see the proof of Theorem 5.50), using the fact that the problem, whether two given positive modal formulas are equivalent, is PSPACE-hard. Here the set *PML* of *positive modal formulas* is defined by the following grammar:

$$\varphi ::= p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \Box\varphi.$$

CLAIM $\mathsf{EQ}(PML)$ is PSPACE-hard.

The following *proof* of this claim is due to Carsten Lutz (personal communication); we include it here with his kind permission. It is in fact not very hard to reduce the validity problem for basic modal logic, a problem proven to be PSPACE-hard by Ladner (1977), to $\mathsf{EQ}(PML)$. The key observation underlying the reduction is that for any proposition letter p, and any natural number m, the formula

$$\bigwedge_{0\leq i\leq m} \Box^i(p \vee p') \wedge \neg \bigvee_{0\leq i\leq m} \Diamond^i(p \wedge p')$$

holds at some state s in a Kripke model $\mathbb{M}$ iff at every state t that can be reached in at most m steps from s, the formula p' is equivalent to the formula $\neg p$. This observation provides the intuitions for handling negations when reducing modal formulas to equations between positive modal formulas.

Turning to the details, given a modal logic formula φ, let $\mathtt{X}_\varphi$ denote the set of proposition letters occurring in φ, and let n be the modal depth of φ. First rewrite φ into negation normal form, that is, with negations only allowed directly in front of proposition letters. Let φ' be the formula obtained by replacing each occurrence of a negated propositional variable $\neg p$ with a fresh variable p'. Define the auxiliary formulas ψ and ψ':

$$\begin{aligned}\psi &:= \textstyle\bigwedge_{p\in\mathtt{X}_\varphi} \bigwedge_{0\leq i\leq n} \Box^i(p \vee p')\\ \psi' &:= \textstyle\bigvee_{p\in\mathtt{X}_\varphi} \bigvee_{0\leq i\leq n} \Diamond^i(p \wedge p')\end{aligned}$$

It follows from the above key observation that

$$\varphi \text{ is satisfiable} \iff \varphi' \wedge \psi \wedge \neg\psi' \text{ is satisfiable.}$$

But from this it is immediate that

$$\neg\varphi \text{ is valid} \iff (\varphi' \wedge \psi) \to \psi' \text{ is valid} \iff \varphi' \wedge \psi \text{ is equivalent to } \varphi' \wedge \psi \wedge \psi'.$$

Since each of the formulas φ', ψ, and ψ' is of size polynomial in the size of φ, we have indeed found the required reduction, and as a consequence, $\mathsf{EQ}(\mathit{PML})$ is PSPACE-hard. This finishes the proof of the claim.

In order to prove the theorem, it suffices to reduce $\mathsf{EQ}(\mathit{PML})$ to $\mathsf{EQ}(\mathit{ML}_{\mathtt{I}})$. For that purpose, extend the translation $(\cdot)^t$ of (22) with clauses for the modalities:

$$\begin{array}{lcl} (\Diamond\varphi)^t & := & \Diamond_i \varphi^t, \\ (\Box\varphi)^t & := & \Diamond_j \varphi^t. \end{array} \tag{23}$$

It is now straightforward to prove that two given positive modal formulas φ and ψ are equivalent iff their translations φ^t and ψ^t are i-equivalent. Then from the above claim the PSPACE-hardness of $\mathsf{EQ}(\mathit{ML}_{\mathtt{I}})$ is immediate. □

Concerning the *satisfiability* problem, it is fairly obvious that i-$\mathsf{SAT}(\mathit{ML}_{\mathtt{I}})$ can be solved in PSPACE (again using the *Witness* algorithm Blackburn *et al.*, 2001), but we have not been able to find sharper complexity bounds. As in the propositional case, it should be relatively easy to i-satisfy multiplayer formulas, since we may use valuations in which every player wins. Putting it differently: an $\mathit{ML}_{\mathtt{I}}$-formula φ is i-satisfiable iff the closed formula $\varphi^{\top}$, obtained from φ by substituting $\top$ for each variable p, is i-satisfiable. We believe that this observation could lead to a good complexity but in the modal case there is not a small algebra like $\mathbb{O}_{\mathtt{I}}$ that can be used to obtain a nicer upper bound for the complexity of i-$\mathsf{SAT}(\mathit{ML}_{\mathtt{I}})$. We leave this as a topic for further research.

§6. Further research. In this section we list some questions that were left open in this paper, and we indicate some interesting directions for further research.

Open problems. First of all, there are some specific open problems related to the formalisms introduced in this paper. Some rather *concrete problems* include the question whether the role switch operations are determined by the other operations (in the sense that any Boolean algebra is completely determined by its bounded lattice reduct), the issue whether the axiom (M11) is really needed in the axiomatization of modal multiplayer algebras, and the matter of determining the complexity of the i-satisfiability problem of modal multiplayer logic.

More generally, we would like to see the development of a useful *duality theory* for (modal) multiplayer algebras, if possible in terms of multi-topological spaces. Related to this, one might consider the theory of complex multiplayer modal algebras stemming from Kripke frames satisfying certain *frame conditions*. And finally, we feel it might be interesting to develop the *general* algebraic theory of multiplayer algebras.

Apart from these concrete open problems, we believe there are some interesting research directions to take.

Multiplayer fixpoint logics. The basic idea underlying this paper has been to generalize the duality characterizing the semantics of many logical systems to a family of symmetries by generalizing logical evaluation games between two players to a multiagent setting. We elaborated this idea for two dual pairs of ‘choice connectives’: disjunction/conjunction and diamond/box, and our approach is easily extended to the existential and universal quantifier in first-order logic.

But what about *fixpoint operators*? These also come in dual pairs: μ/ν, corresponding to the least and greatest fixpoint of a monotone function, respectively. While the semantics of fixpoint operators lends itself very well to a game-theoretic treatment, namely, that of infinite *parity games* (Grädel *et al.*, 2002), the μ/ν duality is not directly related to a partition of the two players' positions in evaluation games. Nevertheless, it seems straightforward to generalize for instance the modal μ-calculus to a multiplayer setting, simply by 'multiplying' the two operators μ and ν to a *family* $\{\mu_i \mid i \in \mathtt{I}\}$ of fixpoint operators. Thus the language of the multiplayer m-calculus would be given as follows:

$$\varphi ::= x \in \mathtt{X} \mid \bot \mid \top \mid \varphi \vee_i \varphi \mid \neg_{ij}\varphi \mid \Diamond_i \varphi \mid \mu_i x.\varphi,$$

with a proviso that $\mu_i x.\varphi$ is only admitted as a formula if all occurrences of x in φ are 'i-positive', for a suitably defined player-related notion of positivity. The semantics of this language could be defined in two ways, either via a multiplayer version of parity games (in which the single loser of an infinite match is determined by the nature of the highest fixpoint variable that is unfolded infinitely often during the match), or algebraically, using the fact that on full complex algebras, each relation $\leq_i$ is a complete partial order.

Note however, that by associating a single player with each fixpoint variable, we restrict ourselves to a setting where infinite matches always have a unique loser. If we want to stay closer to the spirit of this paper, we should not exclude the option that infinite matches can be won by an arbitrary set of players. This would suggest to index every fixpoint operator with a *set* of agents, that is, extend the language with a family $\{\mu_{\mathtt{J}} \mid \mathtt{J} \subseteq \mathtt{I}\}$ of fixpoint operators.

In the future we hope to have a more detailed look at such multiplayer fixpoint logics.

Game theory in multiplayer logic. A second interesting line of research would be to study the logic from a more sophisticated game-theoretical perspective. We could, for instance, incorporate concepts like rationality, uncertainty, and coalitions into our framework.

To see where *rationality* may come in, consider the formula $\varphi = \bot_0 \vee_1 \top_0$, with $\mathtt{I} = \{0, 1, 2\}$. Regardless of the valuation V, the only player with a winning strategy for the game $\mathcal{E}(\varphi, V)$ is Player 1. In particular, Player 2 does not have a winning strategy since 1 might choose to play the disjunct $\top_0$. However, this move is clearly irrational, because it would make 1 lose the match, while the alternative move $\top_0$ would guarantee him to win. Thus, under the assumption of rationality, it would seem reasonable to assume that Player 2 also has a guaranteed win in the game $\mathcal{E}(\varphi, V)$. More generally, one may introduce a more sophisticated semantics for multiplayer logics, by incorporating not only the assumption of rationality of players, but also that of common knowledge of rationality, so that players may base their actions on their belief that the other players will behave rationally later on. One way to incorporate this idea is by modifying (only) the semantic definition of the $\vee_j$-connective for players i *different* from j:

$$V \Vdash_i \varphi \vee_j \psi \quad \text{if} \quad \begin{cases} V \Vdash_i \varphi & V \Vdash_j \varphi \text{ and } V \nVdash_j \psi \\ V \Vdash_i \psi & V \Vdash_j \psi \text{ and } V \nVdash_j \varphi \\ V \Vdash_i \varphi \text{ and } V \Vdash_i \psi & \text{otherwise} \end{cases}$$

Thus, if player j strictly prefers one of the disjuncts, say φ, we assume that he plays φ, and other players will have a winning strategy for the whole formula only if they have a winning strategy for φ. If, on the other hand, player j is indifferent between φ and ψ—he either has a winning strategy for both or for neither of the subformulas—we assume nothing about his move, and the semantics of $\vee_j$ remains as in the original framework. With this modification we guarantee that $V \Vdash_i \varphi$ if and only if player i has a winning strategy for the game $\mathcal{E}(\varphi, V)$, given the assumption of common knowledge of rationality.

Second, we would like to mention the possibility of making knowledge of the game, or the lack thereof, explicit in our framework. This can for example be done by introducing *imperfect information*, uncertainty about players' actual moves, or *incomplete information*, uncertainty about the rules of the game, like players' payoffs, into the logic. Clearly, such modifications will change the game, and hence the semantics of the logic, and we leave investigation of these concepts for future work.

Assuming (common knowledge of) rationality and incorporating uncertainty into the framework of multiplayer logic will allow us to relate our semantics to existing solution concepts from game theory, like backward induction and various notions of equilibria. In future work, the first author hopes to elaborate on these connections.

Finally, a new feature that is enabled by the introduction of multiple players, concerns the agents' potential to form *coalitions* in order to enhance their prospects. Again, let $\mathtt{I} = \{0, 1, 2\}$ and consider the formula $\varphi = \top_0 \vee_1 (\top_0 \vee_2 \bot_0)$. According to our formulation of the game, none of the players has a winning strategy for this formula. However, if Players 1 and 2 would team up, as a coalition they would have the power to force the game to end in the basic position $\bot_0$, and doing so they would both win the match. Thus, allowing for cooperation or coalition forming could change the semantics of the logic. The formation of coalitions could be incorporated into our logic by allowing operators to be indexed by groups of agents. Another way would be to define negations as coalition-forming operators, such that $\neg_{ij}$ denotes that players i and j form a coalition. Alternatively, we could have negations represent arbitrary permutation functions that are not necessarily injective. That is, two or more agents could be mapped to the same agent and assume the same role. We think it would be interesting to incorporate coalitions into our framework, but leave the investigations of the resulting systems as a topic for further research.

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