Chapter 1

The gauge/gravity dualities

The various gravity/gauge theory dualities are among the most far-reaching recent developments in theoretical high-energy physics. These dualities claim that certain quantum theories of gravity in \( (d + 1) \)-dimensional backgrounds are equivalent, or dual, to certain quantum field theories in \( d \) dimensions. (In many cases this quantum field theory is a gauge theory which led to the name ‘gauge/gravity duality’.) The dualities offer a completely new perspective on gravitational physics and in particular realize the idea of holography put forward in [7, 8]. Since typically the quantum field theory is strongly coupled when the dual gravity theory becomes weakly coupled and classical, these dualities also open up a window onto the strong-coupling dynamics of gauge theories. In recent times they have been applied to a variety of physical systems that range from high-energy phenomenology to condensed matter physics.

In this chapter we give a brief introduction to these dualities. A reasonable overview of all past and current developments would require a textbook on its own, so we shall have to omit a great number of topics and details. Our aim is to present enough material to convey a general picture and focus on those details that we need in the subsequent chapters of this thesis.

The outline of this chapter is as follows. We begin with a concrete example of a gauge/gravity duality where the gravity theory is type IIB string theory on \( \text{AdS}_5 \times S^5 \) and the dual gauge theory is the \( \mathcal{N} = 4 \) supersymmetric Yang-Mills theory. This example is intended to clarify the general ideas presented in later sections. In section 1.2 we discuss the basic structure common to all the gauge/gravity dualities. We then consider the Ward identities of the gauge theory in some detail in section 1.3 and discuss the necessary steps towards their realization in the

1
1. The gauge/gravity dualities

In section 1.4 we present an example computation of a field theory correlation function using the gravity theory. In the final section 1.6 we consider the realization in the gravity theory of the scale and diffeomorphism Ward identities.

1.1 The AdS$_5$/CFT$_4$ correspondence

In this section we discuss a concrete example of a gauge/gravity duality [9]. As we mentioned above, this section is intended to support the general ideas that we will subsequently present. We will necessarily be rather brief and refer the reader to the extensive literature, in particular the reviews [10, 11], for more details on this realization of the duality.

Our example concerns a duality between the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and type IIB string theory on AdS$_5 \times S^5$. The former turns out to be a conformal field theory (CFT) and our example is therefore called an AdS/CFT correspondence. It is in fact but one example of a more general class of AdS/CFT correspondences which all involve gravity theories in AdS$_{d+1}$ spacetimes and $d$-dimensional CFT’s.

1.1.1 Open versus closed strings

Consider a system of $N$ coincident D3-branes in $\mathbb{R}^{1,9}$ in type IIB string theory. Let us take the branes to be extended along the $x^i$ directions, with $i \in \{0, 1, 2, 3\}$ and with $x^0$ playing the role of time. On the six transverse dimensions we will pick spherical coordinates so they are spanned by an $S^5$ plus a radial direction denoted $y$. The tension of the branes is given by

$$T = \frac{N}{g_s (2\pi)^3 \alpha'^2},$$ (1.1)

where $g_s$ is the string coupling constant. The IIB background geometry created by the branes is uniquely fixed by the translational and rotational symmetries, the energy density and five-form charge of the branes. It takes the following form:

$$ds^2 = f(y)^{-1/2} \eta_{ij} dx^i dx^j + f(y)^{1/2} (dy^2 + y^2 d\Omega_5^2) \quad f(y) = 1 + \frac{4\pi g_s N \alpha'^2}{y^4},$$ (1.2)

with a constant dilaton $g_s = e^\Phi$ and axion $C_0$ and with a five-form flux:

$$F_5 = (1 + *_{10}) dx^0 \ldots dx^3 df^{-1}.$$ (1.3)
1.1. The AdS$_5$/CFT$_4$ correspondence

In the metric (1.2) the branes are at $y = 0$ but they are completely redshifted away and have disappeared from view.

Let us now consider the low-energy limit of this system. A convenient way to obtain this limit is by sending $\alpha' \to 0$ while keeping the energies $E$ of physical processes fixed. This way the dimensionless energies

$$\epsilon = \sqrt{\alpha'} E$$

indeed go to zero. In the metric (1.2) this limit implies that $f(y) \to 1$ for all nonzero $y$. Away from the branes we therefore recover the ten-dimensional flat space metric on which we have to consider low-energy excitations, which are described by IIB supergravity.

On the other hand we shall see that the behavior of the branes themselves can be described in two different ways, namely either in terms of open strings or in terms of closed strings. The fact that these are two descriptions of the same physical system will then lead to the AdS/CFT correspondence.

Open string description

The first way to describe the branes is in terms of open strings. At low energies the effective theory for the branes becomes a field theory, namely the $\mathcal{N} = 4$ super Yang-Mills (or SYM) theory with gauge group $U(N)$. This is a four-dimensional gauge theory with four chiral fermions and six real scalars, all transforming in the adjoint representation of the gauge group. The $\mathcal{N} = 4$ supersymmetry implies an $SU(4) \simeq \text{Spin}(6)$ group of global R-symmetries, for which the fermions transform in the fundamental and the scalars in the fundamental of $SO(6)$. The Lagrangian schematically has the form:

$$\mathcal{L} = \frac{1}{g^2} \text{Tr} \left( \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} D_\mu \Phi D^\mu \Phi - i \bar{\psi} \sigma^\mu D_\mu \psi \right) + \psi [\Phi, \psi] + \bar{\psi} [\Phi, \bar{\psi}] - \frac{1}{4} (|\Phi, \Phi|)^2 + \frac{\theta}{2\pi} \text{Tr} (F \wedge F),$$

where $D_\mu$ is the gauge covariant derivative and we suppressed spinor, gauge and $SU(4)$ indices. The only coupling constant $g$ in the theory is given in terms of the string coupling constant $g_s$ via $g^2 = g_s$. Furthermore, its $\theta$ angle is given in terms of the axion background value, $\theta = 2\pi C_0$. It turns out that the beta function for $g$ vanishes to all orders in perturbation theory and it is generally believed that it vanishes non-perturbatively as well [12]. This implies that $g$ is a physical parameter that is not transmuted into a scale. The resulting scale invariance of the theory can in fact be extended to the full conformal group and this makes the $\mathcal{N} = 4$ SYM theory a conformal field theory or CFT.
1. The gauge/gravity dualities

Since all fields transform in the adjoint representation of the $U(N) = SU(N) \times U(1)$ gauge group the overall $U(1)$ becomes free. In the brane picture this $U(1)$ multiplet essentially describes the motion of the center of mass of the brane. We may safely consider the brane system at rest and ignore it in what follows. One is then left with an $SU(N)$ theory.

Finally, there are in principle nonzero interactions between the ambient IIB supergravity modes and the $\mathcal{N} = 4$ theory, but these vanish as $\alpha' \to 0$ since the ten-dimensional Newton’s constant $G_N \sim \alpha'^4$.

Closed string description

The second way to describe the brane system is in terms of closed strings. If we choose to use this description we have to take into account the gravitational redshift of the excitations in the geometry (1.2). The energy $E$ of an excitation at a certain fixed $y$ is redshifted at infinity to the value:

$$E_\infty = Ef(y)^{-1/4}. \quad (1.6)$$

To take a low-energy limit in the closed-string description we again send the dimensionless energies $E_\infty \sqrt{\alpha'}$ to zero, which we may again implement by sending $\alpha' \to 0$. However from (1.6) we see that this does not necessarily imply that we have to send $E\sqrt{\alpha'}$ to zero since we may also send $f(y) \to \infty$. This in turn can be done by sending $y \to 0$ at the same time as $\alpha' \to 0$ while keeping the new coordinate

$$z = \frac{\alpha'}{y} \quad (1.7)$$

fixed. In this limit we find:

$$f(y) \to \frac{4\pi g_s N z^4}{\alpha'^2}. \quad (1.8)$$

After taking the limit, in terms of the coordinate $z$, we find the following metric:

$$ds^2 = \alpha' \left( \frac{1}{4\pi g_s N z^2} \eta_{ij} dx^i dx^j + \sqrt{4\pi g_s N} \frac{dz^2}{z^2} + \sqrt{4\pi g_s N} d\Omega_5^2 \right). \quad (1.9)$$

This is precisely the $AdS_5 \times S^5$ product geometry where both factors have a radius of curvature

$$R^4 = 4\pi g_s N \alpha'^2. \quad (1.10)$$

(Notice that the Ricci scalar $\text{Ric} \sim 1/R^2$.) This curvature is supported by the five-form flux, which in the above limit takes the form:

$$F_5 \to -(1 + *_{10}) \frac{\alpha'^2}{\pi g_s N z^5} dx^0 \ldots dx^3 dz. \quad (1.11)$$
The factor $\alpha'$ in front of the metric (1.9) makes the entire metric small. However this factor cancels in the worldsheet sigma model, for example for the bosonic part of the worldsheet action of a string in the above background we find:

$$S = \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{-h(h^{\alpha\beta}G_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu + \ldots)}$$

$$= \frac{1}{2\pi} \int d^2\sigma \sqrt{-h(h^{\alpha\beta}\tilde{G}_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu + \ldots)},$$

where $\tilde{G}_{\mu\nu} = G_{\mu\nu}/\alpha'$. We therefore obtain a finite theory in the low-energy limit. Indeed, we described above that all the excitations in the $\text{AdS}_5 \times S^5$ region have a low energy from the viewpoint of asymptotic observers. We may therefore forget about the smallness of $\alpha'$ altogether and consider excitations with arbitrary finite dimensionless energy $E\sqrt{\alpha'}$ on this background. In this description the ‘sum over histories’ becomes a sum over arbitrary backgrounds which are asymptotically of the $\text{AdS}_5 \times S^5$ form. This concept will be made more precise below.

Just as in the open string picture, the low-energy limit is again a decoupling limit in which the interactions with the low-energy excitations of the ambient flat geometry vanish. In the closed string description this follows from the fact that the redshift diverges as $y \to 0$ so only modes that have infinite energy in the AdS region can escape to infinity. Furthermore the closed strings in the bulk cannot probe the AdS region since it becomes of zero size, which for example can be concretely seen from the fact that the absorption cross section of the branes vanishes in the low-energy limit [9].

**The AdS/CFT correspondence**

We have given two different description of the low-energy dynamics of the brane system in its rest frame. In one description we recovered the $\mathcal{N} = 4$ SU($N$) SYM theory (with coupling constants $g = g_s$ and $\theta = 2\pi C_0$) and in the other we found type IIB closed string theory on the $\text{AdS}_5 \times S^5$ background (with radius of curvature $R^4 = 4\pi g_s N\alpha'$/2 and dilaton $g_s = e^\Phi$ and axion $C_0$). One is therefore led to conjecture that these two theories are equivalent. This is precisely the content of the AdS/CFT correspondence we set out to derive.

**The supergravity limit**

In this work we will be exclusively working in the classical limit of the closed string theory, which for the case at hand implies that it reduces to classical IIB supergravity on $\text{AdS}_5 \times S^5$. This can only be a reliable approximation when the curvature radius of the geometry (1.10) is large in units of $\alpha'$, which is when

$$g_s N \gg 1.$$  

(1.13)
For example, the low-energy effective action for the gravitons schematically takes
the form:

\[ S = \frac{1}{2\kappa^2} \int d^{10}x (R + \alpha' R^4 + \ldots) \]  

(1.14)

and substituting the background values of the curvature we find that the correction
term \( \alpha'^3 R^4 \) is of order \((g_s N)^{-3/2}\). Furthermore, we should also suppress string
loops so we require \( g_s \ll 1 \) (and therefore \( N \gg 1 \)). Indeed, the ten-dimensional
Newton’s constant is given by

\[ 2\kappa^2 = (2\pi)^7 g_s^2 \alpha'^4, \]  

(1.15)

and for small \( g_s \) we find that \( \kappa^2 \) is small in units of \( \alpha' \) as well. In what follows we
will often set \( R^4 = 1 \), which means that \( \alpha'^2 = (4\pi g_s N)^{-1} \). In that case

\[ \kappa^2 = \frac{4\pi^5}{N^2}. \]  

(1.16)

Notice that the \( S^5 \) has now unit radius, so when we reduce over the sphere (as
we will do in the next section) we see that the effective five-dimensional Newton’s
constant is again of order \( N^{-2} \).

In the dual field theory the large \( N \) limit is actually the well-known ’t Hooft limit
[13] in which the so-called planar Feynman diagrams are dominant. In this limit
the effective loop counting parameter for an \( SU(N) \) gauge theory is \( \lambda = g_s N \). Since
according to (1.13) it is actually very big in the supergravity limit, we cannot at
the same time use any perturbative approaches to the quantum field theory. This
makes an explicit verification of the correspondence very hard, but on the other
hand it does offer a unique opportunity to obtain results in a strongly coupled
field theory from relatively straightforward computations in supergravity.

### 1.1.2 Bulk fields and boundary operators

The correspondence in particular implies that we can compute in two different
ways the response of the brane system to probes like external closed strings. In
the gauge theory these correspond to the insertion of certain local gauge-invariant
operators, whereas they correspond to nontrivial boundary conditions for the fields
in the AdS\(_5 \times S^5 \) geometry. One therefore expects a relation between the fields in
the closed string theory and certain gauge-invariant operators in the gauge theory.
Here we will present some details of this map in the aforementioned supergravity
limit.
1.1. The AdS$_5$/CFT$_4$ correspondence

Symmetries and boundary operators

In this subsection we discuss the relevant class of gauge-invariant operators in the $\mathcal{N} = 4$ SYM theory. We begin with a discussion of the global symmetries in the theory, since it is natural to organize the operators in representations of these symmetries.

The symmetry algebra of $\mathcal{N} = 4$ SYM is built up as follows. The fact that the theory is conformal means that the usual Lorentz algebra $so(3,1)$ is extended to the full conformal algebra $so(4,2)$. There are four independent supersymmetries which rotate into each other under an $su(4) \simeq so(6)$ algebra of R-symmetries. We shall denote the corresponding charges as $Q_{\alpha a}$ with $\alpha \in \{1,2\}$ a chiral spinor index and $a \in \{1,\ldots,4\}$ a fundamental $SU(4)$ index. The combination of the $so(4,2) \simeq psu(2,2)$ conformal symmetry and the $\mathcal{N} = 4$ supersymmetry generates a so-called superconformal algebra of symmetries. In this case we find that the full superconformal algebra is $psu(2,2|4)$. Notice that we can act with the special conformal transformations on the $Q_{\alpha a}$ to get additional fermionic symmetries. Their charges are again chiral spinors and we shall denote them by $S_{\alpha a}$.

The field theory operators we consider are composite, which means that they are built up out of the fundamental fields:

$$D_\mu, \Phi, \psi, F_{\mu\nu}. \quad (1.17)$$

Notice that we use the gauge covariant derivative $D_\mu$ as an independent field rather than the gauge field $A_\mu$ itself since the former transforms homogeneously under gauge transformations. We obtain gauge-invariant combinations by taking a product of these fields, all evaluated at the same spacetime point, and taking a trace:

$$O_I = \text{Tr} \left( \ldots D_\mu \ldots \Phi \ldots \psi \ldots F_{\rho\sigma} \ldots \right). \quad (1.18)$$

A composite operator is constructed from either a single such trace or a product of multiple traces. Such an operator still carries an arbitrary number of Lorentz and $SU(4)$ indices, which we summarized with a single index $I$.

The operators dual to the supergravity fields form a subclass of this set, namely the so-called single-trace superconformal primary operators and their descendants. The superconformal primary operators are the lowest dimension operators in a multiplet which implies that they must be annihilated by all the $S_{\alpha a}$. The single-trace ones take the form

$$\text{Tr} \left( \Phi^{\{I_1 I_2 I_3 I_4 \ldots I_n\}} \right), \quad (1.19)$$

where the $I_i$ are $SO(6)$ indices and $\{I_1 \ldots I_n\}$ denotes the symmetrized traceless part. The fact that these operators take the above form follows essentially from
the fact that they cannot be the supersymmetry variation of other operators. The
descendants in these representations are found by acting with the supercharges
$Q_{\alpha a}$.

The superconformal primary operators are Lorentz scalars so all operators with
nonzero spin in these representations have to be descendants obtained by acting
with the supercharges $Q_{\alpha a}$ and $\bar{Q}_{\dot{\alpha} a}$. However the operators (1.19) are in fact 1/2
BPS and therefore annihilated by half of these supercharges. So out of the 16
independent real supercharges we can act with at most 8 on the above operators,
or at most 4 with the same helicity. Therefore the helicities in this multiplet range
between 0 and 2.

The final quantum number that labels their $\text{psu}(2,2|4)$ representation is their
dilatation weight or scaling dimension which is denoted as $\Delta$. The scaling dimen-
sion of composite operators may in principle renormalize because of short-distance
singularities. However the dilatation weight for the above operators is tied to their
$su(4)$ representation labels and therefore does not renormalize. It is then simply
the sum of the dimensions of the elementary fields out of which the operator is
composed.

**Symmetries and bulk fields**

In this section we shall briefly discuss the relation between the above operators
and supergravity fields on AdS$_5$.

The bosonic field content of type IIB supergravity consists of the metric $G$, the
dilaton $\Phi$, the axion $C_0$, two real antisymmetric two-forms $C_2$ and $B_2$ and a real
antisymmetric four-form $C_4$. In terms of the field strengths:

\[ H_3 = dB_2 \]
\[ F_1 = dC_0 \]
\[ \tilde{F}_3 = dC_2 - C_0 \wedge dB_2 \]
\[ \tilde{F}_5 = dC_4 - \frac{1}{2} C_2 \wedge dB_2 + \frac{1}{2} B_2 \wedge dC_2, \]

the string frame action takes the form:

\[ S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-G} e^{-2\Phi} \left( R + 4|d\phi|^2 - \frac{1}{2} |H_3|^2 \right) \]
\[ - \frac{1}{4\kappa^2} \int d^{10}x \left( \sqrt{-G} (|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2} |\tilde{F}_5|^2) + C_4 \wedge H_3 \wedge dC_2 \right). \]

The five-form field strength is self-dual, $F_5 = *_{10} F_5$, which is an additional con-
straint that should be imposed separately from the field equation obtained from
(1.21).
Let us now consider the AdS$_5 \times S^5$ solution whose metric was given in (1.9) and which solves the equations of motion derived from (1.21). (We use conventions where $\Phi = 0$ for the above solutions.) Its isometries form the group $SO(4,2) \times SO(6)$ and we also find 32 independent Killing spinors. Their combined algebra is precisely the $psu(2,2|4)$ we already discussed for the dual gauge theory. (To be precise the symmetries of the AdS$_5 \times S^5$ geometry are dual to the symmetries respected by the superconformal vacuum of the field theory.)

In the supergravity limit we may obtain a map between the fields introduced above (plus their fermionic superpartners) and the aforementioned class of 1/2 BPS superconformal multiplets in the dual field theory. To find this relation we have to put the fields in the appropriate representations, in particular we have to make manifest their $so(6) \simeq su(4)$ representation properties. This we can do by decomposing the fields into spherical harmonics on the $S^5$ part of the geometry. For example, for the dilaton $\Phi$ we may write a general field configuration as:

$$\Phi(x,y) = \sum_I \phi_I(x)Y^I(y).$$

Here $x$ denotes the collective coordinates on the AdS$_5$ part of the geometry, $y$ the coordinates on the $S^5$ part and the $Y^I$ are a complete set of spherical harmonics on the $S^5$, $\Box_{S^5} Y^I = -k(I)Y^I$. From the decomposition (1.22) we obtain an infinite number of effective fields $\phi_I(x)$ in the AdS$_5$ geometry, one for each independent solution of the Laplacian on the $S^5$. The fields $\phi_I(x)$ are by construction organized in $SO(6)$ representations. Let us now impose the lowest-order equations of motion, which would be the massless Klein-Gordon equation for $\Phi$. We find:

$$0 = \Box_{10} \Phi = (\Box_{\text{AdS}_5} + \Box_{S^5}) \Phi = \sum_I [\Box_{\text{AdS}_5} \phi_I(x) - k(I)\phi_I(x)]Y^I(y).$$

We see that each of the $\phi_I$ satisfies an independent Klein-Gordon equation with increasing masses given by the eigenvalues $k(I)$ of the $S^5$ Laplacian.

The details of the $S^5$ reduction of the IIB supergravity fields were presented in [14]. We will not repeat them here, but suffice it to say that we find precisely one AdS$_5$ field for every one of the local single-trace 1/2 BPS gauge-invariant operators we mentioned above. In particular the fields again transform in short representations of $psu(2,2|4)$ with helicities at most equal to two. As we shall see in more detail below, the scaling dimension $\Delta$ of these operators is determined directly in terms of the $k(I)$, so in terms of their effective mass on the AdS$_5$ part of the geometry. It precisely matches the spectrum on the gauge theory side.

Notice that in principle there is an infinite number of fields originating from the Kaluza-Klein decomposition of a single field. However we cannot trust the supergravity approximation if the fields fluctuate on the $S^5$ on distances smaller than
the Planck scale. This puts an upper bound on the number of fields we can reliably consider within supergravity. In the dual gauge theory there is a similar upper bound on the number of single-trace superconformal primaries since the fields are $su(N)$ matrices and if their length $n > N$ then one may show that the operators are no longer linearly independent. Within the regime of supergravity, then, the results are mutually consistent.

It is rather remarkable that the supergravity only ‘sees’ a fraction of the full set of gauge-invariant operators. It means in particular that, if the AdS/CFT correspondence is correct, the other operators should correspond to excitations with masses of order the string scale and therefore they should gain large anomalous dimensions at strong coupling. This is certainly possible: unlike the 1/2 BPS operators their dimensions are in general unprotected against quantum corrections.

1.1.3 Summary and generalizations

By describing the low-energy excitations of a system of D-branes in two different ways we have discovered a duality between a theory of gravity and a gauge theory. We have discussed the relation between the parameters of the two theories, found a matching of their global symmetry groups and discussed how the supergravity limit was dual to the planar, strong-coupling limit in the field theory. We also argued that a class of 1/2 BPS gauge-invariant operators is in one-to-one correspondence with supergravity fields. Notice that we have not yet presented a method for actually computing the field theory observables, i.e. the correlation functions of these operators, from the gravity side. Doing so will be the subject of the next section.

As we mentioned in the introduction to this chapter, this correspondence belongs to a much larger class of AdS/CFT correspondences. One may for example study the near-horizon limit of a variety of brane systems to obtain AdS$_{d+1}$/CFT$_d$ dualities for other dimensions $d$ as well. For each of these there is a compact manifold, equivalent to the $S^5$ for $d = 4$, which makes the total closed string background ten- or eleven-dimensional. One may also obtain generalizations by putting the branes in different background geometries which then result in different compact mani- folds when one takes the decoupling limit. These correspond to different dual field theories with generally a different internal symmetry group. Another generalization concerns deformations of the theory by switching on nonzero sources, which would result in theories with nontrivial scale dependence. One may also consider nonzero vacuum expectation values of certain operators which can partially break the gauge symmetry. Finally, there are also non-conformal examples where the dual field theory is no longer conformal even in the UV. Although these dualities
are sometimes still called ‘AdS/CFT correspondences’, it is more appropriate to refer to these more general instances as ‘gauge-gravity dualities’.

It turns out that all these dualities have a very similar structure. In particular this is the case for the basic ‘dictionary’ that ‘translates’ observable quantities between the gauge theory and the gravity side and which we will present in the next section.

1.2 General prescription

In this section we shall be concerned with the main ingredient of the gauge/gravity duality, namely the general prescription to compute observables in the two theories. Since this prescription is to a large extent universal, i.e. independent of the specific theory under consideration, we shall from now on consider a general gauge/gravity duality between a $d$-dimensional field theory and a quantum theory of gravity for which $d+1$ dimensions are noncompact.

More precisely our assumptions will be the following. First of all we will be working in a low-energy limit in which we suppose that we may reliably use classical gravity and which is dual to a strong-coupling (and planar) limit of some sorts in the dual field theory. Second, we suppose that a Kaluza-Klein reduction has been performed so we no longer have to explicitly consider the compact part of the closed string background geometry. We then work with an effective supergravity theory in $d+1$ dimensions which we will assume to be a consistent truncation of the full supergravity theory. Third, we will suppose that there is an operator in the dual field theory corresponding to every bulk field we consider in this $(d+1)$-dimensional effective theory. Last, we will restrict ourselves to the cases where the closed string theory is defined on manifolds which are roughly speaking of the ‘asymptotically AdS’ form discussed above. As we shall see below this implies that the dual field theory is a CFT (and the dimensions of the operators we consider are protected) at least at asymptotically high energies.

1.2.1 Quantum field theory partition function

The field theory observables we will consider are the correlation functions of the local gauge-invariant operators $O_I$. In the present context $I$ is a general index labelling the different operators and it may include Lorentz indices as well. The correlation functions take the form:

$$\langle O_{I_1}(x_1)O_{I_2}(x_2)\ldots O_{I_n}(x_n)\rangle_{X,g}. \tag{1.24}$$
The subscripts $X, g$ mean that the theory lives on a manifold $X$ with a general metric $g_{ij}$ which we take to be positive definite for now; in the next chapter we discuss the Lorentzian theories. Notice that the symbol $\langle \ldots \rangle$ defines a certain state in the theory and the correlation functions depend implicitly on that state. For example, this state may include nonzero vacuum expectation values or operator insertions at infinity.

A convenient way to collect these correlation functions is to introduce sources $\phi^I_{(0)}(x)$ for every $O_I$ and write down the partition function:

$$Z_{\text{CFT}}[\phi_{(0)}, g_{ij}] = \langle \exp \left( - \int_{\mathcal{M}} d^d x \sqrt{g} \phi^I_{(0)}(x) O_I(x) \right) \rangle_{X, g}.$$  \hspace{1cm} (1.25)

Notice that when the sources are irrelevant the partition function makes sense only as a formal power series in these sources.

Upon functional differentiation of $Z$ with respect to the sources and setting the sources to zero we obtain again the field theory correlation functions. To find the connected correlation functions we introduce:

$$W = \ln(Z),$$  \hspace{1cm} (1.26)

from which we may obtain:

$$\langle O_I(x) O_J(y) \ldots \rangle_{X, g} = \left. \frac{-1}{\sqrt{g(x)} \delta \phi^I_{(0)}(x)} \frac{-1}{\sqrt{g(y)} \delta \phi^J_{(0)}(y)} \ldots W \right|_{\phi_{(0)}=0}. \hspace{1cm} (1.27)$$

Functional derivatives with respect to the inverse metric by definition generate insertions of the energy-momentum tensor $T_{ij}$ of the theory,

$$\langle T_{ij}(x) \rangle_{X, g, \phi} = \frac{-2}{\sqrt{g(x)} \delta g^{ij}(x)} \delta W.$$  \hspace{1cm} (1.28)

Notice that this time we did not set the sources in $W$ to zero which we indicated with the subscript $\phi$ on the left-hand side. We shall continue to use this subscript to indicate nonzero sources, but we will omit the subscripts $X, g$ henceforth.

### 1.2.2 Relation to gravity

According to the gauge/gravity duality the operators $O_I$ should be in one-to-one correspondence with string theory fields $\Phi^I$ which are defined on a $(d+1)$-dimensional manifold $(M, G)$. Since the corresponding sources $\phi^I_{(0)}$ represent external deformations of the system, it is natural that they correspond to the boundary conditions for the fields $\Phi^I$. In particular the boundary conditions for the bulk
metric $G$ should be defined in terms of the field theory metric $g$. Of course this can only be sensible when $\partial M = X$ which we therefore shall assume to be the case.

Once the boundary conditions for the gravity fields are fixed we can in principle perform a ‘stringy path integral’ on the closed string side. This would result in a partition function which depends on these boundary conditions,

$$Z_{\text{string}}[\phi(0), g_{ij}].$$

The main statement of the gauge/gravity duality is that this partition function is equal to the field theory partition function:

$$Z_{\text{CFT}}[\phi(0), g_{ij}] = Z_{\text{string}}[\phi(0), g_{ij}].$$

Upon functional differentiation with respect to the sources, this formula allows one to in principle compute arbitrary field theory correlation functions from the gravity theory.

Unfortunately, computing $Z_{\text{string}}$ seems a formidable task as it presumably contains some path integral over arbitrary bulk field configurations $\Phi^I$ and bulk manifolds $(M, G)$ that satisfy the boundary conditions. However, in the supergravity limit we can restrict ourselves to a saddle-point approximation on the gravity side. The string theory partition function is then expressed in terms of the on-shell supergravity action $I[\phi^I(0), g_{ij}]$, evaluated on the solution that is determined by these boundary data. We then find:

$$W[\phi^I(0), g_{ij}] = -I[\phi^I(0), g_{ij}]$$

in the supergravity limit. (In the case where multiple such solutions exist the dominant contribution comes from the solution with the smallest on-shell action.) The formula (1.31) is the main formula of this thesis. It was first presented in [15, 16]. Notice that upon functional differentiation of both sides with respect to the sources we find that we can compute correlation functions in a strongly coupled field theory using classical gravity!

In the remainder of this chapter we shall work out some of the details of equation (1.31). We begin by discussing the constraints that $W$ must satisfy as a consequence of the symmetries of the field theory. We shall then consider the precise boundary behavior of the bulk fields and the relation to the field theory sources. We will see that the on-shell action $I$ is actually always infinite, essentially because of the infinite volume of the AdS$_{d+1}$ spacetime. These infinities should be removed with an appropriate renormalization procedure, the so-called holographic renormalization. We shall explain the details of this procedure in two examples.
Finally, we demonstrate how the resulting renormalized on-shell action satisfies the same constraints as \( W \) and can be used to compute field theory correlation functions.

Given the fact that all the field theory data is defined at the boundary of the \((d + 1)\)-dimensional spacetime, it is often said that the field theory ‘lives’ on the boundary of the spacetime. It is then called the boundary theory and the gravity theory is correspondingly called the bulk theory.

### 1.3 Field theory symmetries and Ward identities

If a field theory has a classical symmetry then we can check whether this symmetry also leaves the quantum partition function \( W \) invariant, provided we transform the sources appropriately. If so then this invariance leads to constraints on the form of \( W \) in the form of functional differential equations. In this section we work out some of these constraints.

#### 1.3.1 Global internal symmetries

Let us first consider purely internal global symmetries of the theory that are not related to spacetime invariances. For example, consider a symmetry generated by a constant parameter \( \xi^I \) under which the infinitesimal transformation of the fundamental fields is \( \Psi_I \rightarrow \Psi_I + \xi^J \delta_J \Psi_I \) and the corresponding transformation of the operators \( \mathcal{O}_I \) is given by \( \xi^K M_{KI} J \mathcal{O}_J \). When we make \( \xi^I \) position-dependent, the action changes by:

\[
\delta_\xi S = \int d^d x \sqrt{g} (\nabla_i \xi^I) j_I^i, \tag{1.32}
\]

where \( j_I^i \) is by definition the corresponding Noether current and \( \nabla_i \) is the covariant derivative for the various spacetime indices on \( \xi^I \). Since this transformation of the fields is merely a change of integration variable, \( W \) should be invariant under such a transformation (provided there are no anomalies). This translates into a functional differential equation for \( W \). Namely, when we single out the source for \( j_I^i \) and call it \( A_I^I \) then we find that the transformation is equivalent to:

\[
A_I^I \rightarrow A_I^I + \nabla_i \xi^I \\
\phi_{(0)}^I \rightarrow \phi_{(0)}^I + \phi_{(0)}^I \xi^K M_{KI} J .
\tag{1.33}
\]

The invariance of \( W \) is then seen to lead to the following identity:

\[
0 = \delta_\xi W [A_I^I, \phi_{(0)}^I, g_{ij}] = \int d^d x \left( \nabla_i \xi^I \frac{\delta W}{\delta A_I^I} + \phi_{(0)}^I \xi^K M_{KI} J \frac{\delta W}{\delta \phi_{(0)}^J} \right) . \tag{1.34}
\]
1.3. Field theory symmetries and Ward identities

Since this should hold for any local $\xi^I$ we obtain a constraint for $W$, namely

$$\nabla_i \delta W \frac{\delta W}{\delta A^K_i} - \phi^I_{(0)} M_{KI} J^J \frac{\delta W}{\delta \phi^J_{(0)}} = 0. \quad (1.35)$$

We may alternatively phrase it in terms of correlation functions:

$$\nabla_i \langle j^K_i \rangle - \phi^I_{(0)} M_{KI} J^J \langle O_J \rangle = 0. \quad (1.36)$$

This is the Ward identity corresponding to the symmetry.

1.3.2 Diffeomorphism invariance

We will now discuss diffeomorphism invariance and the conservation of the energy-momentum tensor. To this end we consider an arbitrary infinitesimal diffeomorphism along a vector field $\zeta^i$. The fundamental fields and the composite operators in the theory change according to their Lie derivative, and so do the sources and the metric: $\delta \phi^I_{(0)} = \mathcal{L}_\zeta \phi^I_{(0)}$ and $\delta g^{ij} = -\nabla^i \zeta^j - \nabla^j \zeta^i$. The change of $W$ is then given by:

$$\delta_\zeta W = -\int d^d x 2(\nabla^i \zeta^j) \frac{\delta W}{\delta g^{ij}} + \int d^d x (\mathcal{L}_\zeta \phi^I_{(0)}) \frac{\delta W}{\delta \phi^I_{(0)}}. \quad (1.37)$$

In principle one expects physical quantities to be independent of the coordinate system in which they are described and consequently one would always expect $\delta_\zeta W = 0$. However we will see in chapter 7 that in some cases $W$ is not invariant under general coordinate transformations and we find:

$$\delta_\zeta W = \int d^d x \zeta^i A_i. \quad (1.38)$$

Here $A_i$ is called the diffeomorphism anomaly of the theory.

In this chapter we will be concerned with theories for which the diffeomorphism anomaly vanishes. For concreteness let us restrict ourselves to the case where the nonzero $\phi^I_{(0)}$ are all scalars so that

$$\mathcal{L}_\zeta \phi^I_{(0)} = \zeta^i \partial_i \phi^I_{(0)}. \quad (1.39)$$

Substitution in (1.37) and demanding that $\delta_\zeta W = 0$ then leads to the following constraint:

$$2 \nabla^i \frac{\delta W}{\delta g^{ij}} - (\partial_j \phi^I_{(0)}) \frac{\delta W}{\delta \phi^I_{(0)}} = 0, \quad (1.40)$$

which is equal to a conservation equation for $T_{ij}$,

$$\nabla^i \langle T_{ij} \rangle - \langle \partial_j \phi^I_{(0)} \rangle \langle O_I \rangle = 0. \quad (1.41)$$

It is not hard to extend the above result for sources that carry vector or spinor indices.
1. The gauge/gravity dualities

Example

In the specific case where $\zeta^i$ is a Killing vector field we obtain $\delta g_{ij} = 0$ and in the absence of an anomaly (1.37) becomes:

$$0 = \int d^d x \sqrt{|g|} \phi_0 \zeta^i \partial_i \langle O_1 \rangle_\phi .$$

Taking two further functional derivatives with respect to the source and setting all sources to zero afterwards results in:

$$0 = \left( \zeta^i(x) \frac{\partial}{\partial x^i} + \zeta^j(y) \frac{\partial}{\partial y^j} \right) \langle O_I(x) O_J(y) \rangle ,$$

reflecting the translational invariance of the two-point function along the Killing vector field $\zeta^i$.

1.3.3 Scale invariance

The final symmetry we discuss is scale invariance. In flat space classical scale invariance [17] is phrased as the invariance of the action under a combined coordinate rescaling:

$$x^i \rightarrow x^i + \lambda x^i ,$$

plus a transformation of the fundamental fields $\Psi_I$ as given by the dilatation operator:

$$\delta \Psi_I = i \lambda [D, \Psi_I] = -\lambda (x^i \partial_i \Psi_I + \Delta \Psi_I) .$$

The transformation of the composite operators $O_I$ takes the same form, with $\Delta$ replaced by their corresponding scaling dimension.

We can alternatively describe the coordinate transformation (1.44) in terms of its action on the metric and the fundamental fields in the theory. The resulting change is again expressed in terms of a Lie derivative, this time along the specific vector field $\zeta^i = x^i$. We then obtain:

$$\delta_{\lambda} g_{ij} = \lambda (\partial_i x_j + \partial_j x_i) = 2 \lambda g_{ij} .$$

and for the fundamental fields the combined transformation becomes:

$$\delta_{\lambda} \Psi_I = i \lambda [D, \Psi_I] + \lambda \mathcal{L}_\zeta \Psi_I .$$

When one writes out the Lie derivative for a generic tensor (or spinor) one finds that the first factor in the transformation (1.45) vanishes against a corresponding factor in the Lie derivative. Furthermore, the subsequent terms in the Lie derivative are of the form $\pm \nabla_j x^i = \pm \delta_j^i$ for each vector index on $\Psi_I$, with the positive
1.3. Field theory symmetries and Ward identities

sign for lower vector indices and the negative sign for upper vector indices. The terms in the Lie derivative corresponding to the spinor indices on the fields vanish identically for this particular transformation. We thus obtain:

\[ \delta \lambda \Psi_I = -\lambda (\Delta - \tilde{s}) \Psi_I , \]  

(1.48)

where \( \tilde{s} \) is the number of lower minus the number of upper vector indices on the field \( \Psi_I \). The composite operators again \( O_I \) transform again in a similar way and the source-operator coupling therefore changes according to:

\[ \delta \lambda \left( \int d^d x \sqrt{g} \phi_I(0) O_I \right) = \lambda (\Delta + \tilde{s} + d) \left( \int d^d x \sqrt{g} \phi_I(0) O_I \right) , \]  

(1.49)

where the extra \( d \) comes from the metric determinant. The action is by assumption invariant under this transformation, so to completely offset it we only have to shift

\[ \delta \phi_I(0) = \lambda (\Delta - \tilde{s} - d) . \]  

(1.50)

The combined variation of \( W \) is then given by:

\[ \delta \lambda W = - \int d^d x 2 \lambda g^{ij} \frac{\delta W}{\delta g^{ij}} + \int d^d x \lambda (\Delta + \tilde{s} + d) \phi_I(0) \frac{\delta W}{\delta \phi_I(0)} , \]  

(1.51)

and the above reasoning shows that it vanishes if there are no anomalies.

Let us make a distinction between global and local scale invariance, at the classical level for now. From (1.51) one observes that global scale invariance (in the absence of sources) requires \( T^i_i = \nabla_i J^i \) for some \( J^i \). However it may also happen that \( T^i_i = 0 \). In that case (1.51) shows that the invariance is extended to include local rescalings where \( \lambda \) can be an arbitrary function of \( x \). This is called Weyl invariance. It turns out that the theories we shall consider are indeed Weyl invariant up to anomalies. We shall therefore from now on assume that \( \lambda(x) \) is position-dependent.

Quantum scale dependence

The scale invariance found at the classical level generally does not extend to the quantum theory. Indeed, the regularization of the short-distance singularities always involves the introduction of a nontrivial scale and only in exceptional cases does this scale dependence completely disappear after the necessary counterterms are added and the regulator is removed. Let us therefore assume that the quantum partition function \( W \) also depends on an overall renormalization scale \( \mu \). Since \( \mu \) is the only dimensionful parameter in the theory, a global scale transformation of the form presented above should be equivalent to a change in \( \mu \). This means that:

\[ \mu \frac{\partial}{\partial \mu} W = - \int d^d x 2 \lambda g^{ij} \frac{\delta W}{\delta g^{ij}} + \int d^d x (\Delta I - \tilde{s} - d) \phi_I(0) \frac{\delta W}{\delta \phi_I(0)} . \]  

(1.52)
For a general CFT the anomalous transformation property of $W$ can be expressed as [18]:

$$
\delta_\lambda W = \int d^dx \lambda \beta^I[\phi(0)]\langle O_I \rangle_\phi + \int d^dx \lambda A_w[g_{ij}, \phi(0)],
$$  

(1.53)

where $A_w$ is a local function of $g_{ij}$ and $\phi(0)$ and is called the conformal anomaly or the Weyl anomaly of the theory. The $\beta^I$ are the beta functions of the theory, which may be nonzero for a CFT because of the presence of sources.

Let us demonstrate the relation between (1.53) and the ordinary renormalization group. To this end we consider the theory in flat space and sets some of the relevant or marginal sources to a constant nonzero background value which we denote $g^K$. Let us furthermore suppose that $A_w$ vanishes for this configuration. Consider now an $n$-point correlation function obtained by functional differentiation of $W$. Combining (1.51), (1.52) and (1.53) we find its scale dependence:

$$
\left( \mu \frac{\partial}{\partial \mu} + \beta^I[g^K] \frac{d}{dg^K} \right) \langle O_{I_1}(x_1)O_{I_2}(x_2)\ldots O_{I_n}(x_n) \rangle + \sum_{i=1}^n \Gamma_{I_i}^J \langle O_{I_1}(x_1)\ldots O_J(x_i)\ldots O_{I_n}(x_n) \rangle = 0,
$$  

(1.54)

with

$$
\Gamma_{I_i}^J = \frac{\partial \beta^J}{\partial \phi^I(0)}[g^K].
$$  

(1.55)

Equation (1.54) is the usual Callan-Symanzik equation. The beta functions are indeed given by the $\beta^I$ and $\Gamma_{I_i}^J$ is the matrix of anomalous dimensions.

**Example**

Consider a conformal Killing vector field $\zeta^i$. After a diffeomorphism along $\zeta^i$ the metric changes by $\delta g_{ij} = \frac{2}{d} \nabla^k \zeta_k g_{ij}$. We can combine this diffeomorphism with a Weyl rescaling with $\lambda = -\frac{1}{d} \nabla^k \zeta_k$ such that eventually $\delta g_{ij} = 0$. The variation of the sources under the combined diffeomorphism plus local rescaling is then:

$$
\delta \phi^I(0) = \left( \mathcal{L}_\zeta - \frac{1}{d}(\nabla^k \zeta_k)(\Delta_I - \tilde{s} - d) \right) \phi^I(0).
$$  

(1.56)

Let us again consider the case where the $O_I$ are scalars, so $\mathcal{L}_\zeta \phi^I(0) = \zeta^k \partial_k \phi^I(0)$. In the absence of anomalies $W$ should be invariant under this combined variation, which implies that:

$$
\int d^dx \sqrt{g(0)} \left( \frac{\Delta}{d}(\nabla^k \zeta_k) + \zeta^k \partial_k \right) \langle O_I \rangle_\phi = 0,
$$  

(1.57)

where we integrated by parts. This equation leads to the usual constraints on correlation functions along conformal Killing vector fields. For example in flat
space we may choose $\zeta^i = x^i$. By twice functionally differentiating (1.57) we find that the two-point function must satisfy:

$$0 = \left( x^i \frac{\partial}{\partial x^i} + y^j \frac{\partial}{\partial y^j} + \Delta_I + \Delta_J \right) \langle O_I(x) O_J(y) \rangle .$$

This is the familiar scaling behavior of a correlation function of primary operators.

### Conformal anomalies

It is well-known [17] that (1.58) together with translation, Lorentz and special conformal invariance fixes the flat-space two-point function to be of the form:

$$\langle O_I(x) O_J(y) \rangle = \frac{G_{IJ} \delta_{\Delta_I, \Delta_J}}{|x - y|^{2\Delta}} ,$$

where $G_{IJ}$ is a constant and $\Delta = \Delta_I = \Delta_J$. In Fourier space we find that:

$$\int d^d x e^{-ikx} \frac{1}{|x|^{2\Delta}} = \frac{\pi^{(d+1)/2} \Gamma(d - 2\Delta)}{\Gamma(\Delta) \Gamma(-\Delta + (d + 1)/2)} |k|^{2\Delta - d} .$$

However this equation cannot be valid for the special values

$$\Delta = l + \frac{d}{2} \quad l \in \{0, 1, 2, \ldots \} ,$$

since for these values the right-hand side of (1.60) has a pole. This implies that for these values the function $|x|^{-2\Delta}$ is an ill-defined distribution and not a valid quantum field theory correlation function. The aforementioned Ward identities cannot be exactly true and there must be some anomaly such that the physical answer is not just the one we obtained above from symmetry arguments.

The anomaly can be found by subtracting the divergence in $|x|^{-2\Delta}$. We shall use dimensional regularization in which:

$$d = 2\Delta - 2l - \epsilon .$$

By expanding (1.60) in $\epsilon$ we find

$$\int d^d x e^{-ikx} \frac{1}{|x|^{2\Delta}} = c_l |k|^{2k} \left( \frac{2}{\epsilon} + \ln(|k|^2 \mu^{-2}) + \ldots \right) ,$$

with

$$c_l = \frac{(-1)^l + 1}{\Gamma(l + 1) \Gamma(l + d/2)}$$

and we absorbed several terms in the definition of the renormalization scale $\mu$. After subtracting the leading-order divergence we are left with a well-defined renormalized distribution of the form:

$$\mathcal{R} \frac{1}{|x|^{2l + d}} \equiv c_l \int \frac{d^d k}{(2\pi)^d} e^{ikx} |k|^{2l} \ln(|k|^2 \mu^{-2})$$

19
1. The gauge/gravity dualities

Since we only subtracted an (infinite) contact term, this renormalized distribution agrees with the ordinary function away from $x = 0$. We note that such renormalized distributions would also be the outcome of standard perturbative quantum field theory computations.

Let us now return to the conformal anomaly. We can integrate back the two-point function for operators with a scaling dimension of the form \((1.61)\) to find that the corresponding term in \(W\) must be of the form:

\[
W = \frac{1}{2} \int d^{d}x \int d^{d}y \phi^{I}(y)\phi^{J}(y)G_{IJ} \mathcal{R} \frac{1}{|x - y|^{2d + 4}} + O[(\phi^{I}(0)^{3}] .
\]

From (1.65) we find the nontrivial scale dependence:

\[
\mu \frac{\partial}{\partial \mu} W = - \int d^{d}x G_{IJ} c_{I} \phi^{I}(x) (-\Box)^{\ell} \phi^{J}(x) + O[(\phi^{I}(0)^{3}] ,
\]

so we obtain the corresponding conformal anomaly density:

\[
\mathcal{A}_{w}(x) = - G_{IJ} c_{I} \phi^{I}(x) (-\Box)^{\ell} \phi^{J}(x) + O[(\phi^{I}(0)^{3}] .
\]

Notice that this is a flat-space result. As we shall see explicitly in section 1.6, nontrivial conformal anomalies also arise when we put the theory in curved space.

1.4 Asymptotic behavior

We shall now consider the gravity theory in more detail. As we discussed above, the bulk spacetime \((M, G)\) must have some sort of asymptotically AdS form where the asymptotic \((i.e.\ near-boundary)\) behavior of the bulk fields is given in terms of the field theory sources. In this section we shall define the precise notion of an ‘Asymptotically locally AdS spacetime’ and consider the proper specification of boundary data for the various bulk fields in such spacetimes.

One particular bulk field is the metric \(G_{\mu\nu}\) itself and its boundary data should be given in terms of the field theory metric \(g_{ij}\) on \(X\). We shall from now on denote this boundary metric as \(g_{(0)ij}\) to indicate that it corresponds to field theory data, much like the \(\phi^{I}(0)\).

1.4.1 Example

As a first example we will demonstrate how to define the boundary data for a scalar field \(\Phi\) propagating on a fixed \(\text{AdS}_{d+1}\) background. This spacetime has the metric

\[
G_{\mu\nu} dx^{\mu} dx^{\nu} = \frac{dz^{2}}{z^{2}} + \frac{1}{z^{2}} \delta_{ij} dx^{i} dx^{j}
\]

Notice that this is a flat-space result. As we shall see explicitly in section 1.6, nontrivial conformal anomalies also arise when we put the theory in curved space.

1.4 Asymptotic behavior

We shall now consider the gravity theory in more detail. As we discussed above, the bulk spacetime \((M, G)\) must have some sort of asymptotically AdS form where the asymptotic \((i.e.\ near-boundary)\) behavior of the bulk fields is given in terms of the field theory sources. In this section we shall define the precise notion of an ‘Asymptotically locally AdS spacetime’ and consider the proper specification of boundary data for the various bulk fields in such spacetimes.

One particular bulk field is the metric \(G_{\mu\nu}\) itself and its boundary data should be given in terms of the field theory metric \(g_{ij}\) on \(X\). We shall from now on denote this boundary metric as \(g_{(0)ij}\) to indicate that it corresponds to field theory data, much like the \(\phi^{I}(0)\).

1.4.1 Example

As a first example we will demonstrate how to define the boundary data for a scalar field \(\Phi\) propagating on a fixed \(\text{AdS}_{d+1}\) background. This spacetime has the metric

\[
G_{\mu\nu} dx^{\mu} dx^{\nu} = \frac{dz^{2}}{z^{2}} + \frac{1}{z^{2}} \delta_{ij} dx^{i} dx^{j}
\]

Notice that this is a flat-space result. As we shall see explicitly in section 1.6, nontrivial conformal anomalies also arise when we put the theory in curved space.
1.4. Asymptotic behavior

where the $x^i$ span $\mathbb{R}^d$ and $z > 0$. Its boundary is the $d$-dimensional plane at $z \to 0$, plus a point at $z \to \infty$ which compactifies this plane to a sphere. The point at infinity induces some subtleties which we may however safely ignore in this chapter. The boundary data for $\Phi$ is then defined solely on the plane at $z \to 0$. We will suppose that $\Phi$ satisfies the massive Klein-Gordon equation:

$$\Box_G \Phi - m^2 \Phi = 0,$$

which in the metric (1.69) becomes:

$$z^2 \partial_z^2 \Phi + (1 - d)z \partial_z \Phi + z^2 \delta^{ij} \partial_i \partial_j \Phi - m^2 \Phi = 0. \quad (1.71)$$

We will ignore the backreaction from the field on the spacetime.

Solving (1.71) for small $z$ we find that the equation of motion has the asymptotic solution:

$$\Phi = \phi_0(x^k)z^{d-\Delta} + \ldots + \phi_{(2\Delta-d)}(x^k)z^\Delta + \ldots \quad (1.72)$$

with

$$\Delta = \frac{d}{2} + \frac{1}{2} \sqrt{d^2 + 4m^2} \quad (1.73)$$

and we ignored the special case $\Delta = d/2$. From (1.72) we see that for $\Delta \neq d$ the scalar field either diverges or vanishes near $z \to 0$ so we cannot specify standard Dirichlet boundary data. Rather the boundary data for $\Phi$ is determined by the specification of the leading term in the radial expansion, which is $\phi_0(x^k)$ for the case at hand. Indeed it is not hard to find a unique solution to (1.71) for general $\phi_0(x^k)$ so the Dirichlet problem phrased in this way is well-posed. As our notation already indicates we will interpret $\phi_0(x^k)$ as the source of the dual field theory operator. We shall furthermore see below that the scaling dimension of this operator is precisely the $\Delta$ defined in (1.73).

For the metric (1.69) we similarly see a divergence as $z \to 0$ so we should specify its boundary conditions also in terms of the leading term in a certain radial expansion. However in a theory of gravity there is a priori no distinguished radial coordinate in which we can expand and we therefore face an ambiguity in the specification of the boundary data. In the remainder of this section we will parametrize this ambiguity and demonstrate how it is in fact dual to the freedom to make Weyl rescalings in the boundary theory.

Reviews of the mathematical aspects discussed here can be found in [19, 20].

1.4.2 Conformally compact manifolds

We begin with a proper definition of the boundary of the spacetimes under consideration. To this end we suppose $(M, G)$ is a conformally compact manifold-
with-metric, which is defined as follows. Let \( M \) be the interior of a manifold \( \bar{M} \) with boundary \( \partial M = X \). Suppose there exists a smooth, non-negative defining function \( z \) on \( \bar{M} \) such that \( z(\partial M) = 0 \), \( dz(\partial M) \neq 0 \) and the metric

\[
\tilde{G} = z^2 G
\]

extends smoothly to a non-degenerate metric on \( \bar{M} \). We then say that \((M, G)\) is conformally compact and the choice of a defining function determines a conformal compactification of \((M, G)\). Notice that for example (1.69) defines such a conformally compact manifold with a defining function equal to the coordinate \( z \).

The metric \( \tilde{G} \) induces a regular metric \( g_{(0)} \) on \( X \) which however depends on the defining function. For example if we picked \( z \) as a defining function in (1.69) we would obtain \( \delta_{ij} \) as the boundary metric, whereas if we picked \( e^{\sigma} z \) for some function \( \sigma(x^i) \), the boundary metric would be \( e^{2\sigma} \delta_{ij} \). It follows that the pair \((M, G)\) determines an equivalence class of boundary metrics on \( X \), where two metrics are equivalent if they differ by a Weyl rescaling. Such an equivalence class is called a conformal structure on \( X \). We shall denote the equivalence class of \( g_{(0)} \) as \([g_{(0)}]\) and \((X, [g_{(0)}])\) is then called the conformal infinity or conformal boundary of \((M, G)\). This construction is same as the Penrose method of compactifying spacetime by introducing conformal infinity.

If we compute the Riemann tensor of \( G \), we find that near \( \partial M \) it has the form:

\[
R_{\mu\nu\rho\sigma} = -\tilde{G}^{\kappa\lambda} \nabla_\kappa z \nabla_\lambda z (G_{\mu\rho} G_{\nu\sigma} - G_{\nu\rho} G_{\mu\sigma}) + O(z^{-3}).
\]  

(1.75)

Notice that the leading term is order \( z^{-4} \) as \( G \) is order \( z^{-2} \). Taking its trace we obtain that:

\[
R = -D(D - 1) \tilde{G}^{\kappa\lambda} \nabla_\kappa z \nabla_\lambda z + O(z).
\]  

(1.76)

If we now additionally impose that the spacetime has constant negative curvature

\[
R = -D(D - 1),
\]

(1.77)

then we find to leading order:

\[
\tilde{G}^{\kappa\lambda} \nabla_\kappa z \nabla_\lambda z = 1.
\]

(1.78)

The Riemann curvature of such a metric thus approaches that of AdS space with cosmological constant \( \Lambda = -(D - 1)(D - 2)/2 \), for which one may check explicitly that \( R_{\mu\nu\rho\sigma} = -G_{\mu\rho} G_{\nu\sigma} + G_{\nu\rho} G_{\mu\sigma} \) holds exactly. A conformally compact manifold whose metric also satisfies \( R = -D(D - 1) \) is therefore also called an Asymptotically locally AdS manifold. Notice that we added the word ‘local’ because we have not put any requirements on global issues like the topology of \( X \), which may very well be different from the sphere at conformal infinity of global AdS.
1.4. Asymptotic behavior

1.4.3 Fefferman-Graham metric and Weyl rescalings

A main result of Fefferman and Graham [21] is that in a finite neighborhood of \( \partial M \) the metric of an AlAdS spacetime can always be cast in the form:

\[
ds^2 = \frac{dz^2}{z^2} + \frac{1}{z^2} g_{ij} dx^i dx^j,
\]

(1.79)

where the conformal boundary is at \( z = 0 \) and the metric \( g \) induces a regular metric at \( \partial M \), so:

\[
g_{ij}(x^k, z) = g_{ij}(0) + \ldots,
\]

(1.80)

where the dots represent terms that vanish as \( z \to 0 \). Just as for the scalar field we interpret the leading term \( g_{ij}(0) \) in the expansion (1.80) as the field theory metric.

The Fefferman-Graham expansion for a given AlAdS manifold is however not unique. Indeed, one may apply an infinitesimal coordinate transformation along a vector field \( \zeta^\mu \) given by:

\[
\zeta^\mu \partial_\mu = -z \lambda \partial_z + \frac{1}{2} z^2 (\partial_i \lambda) g_{ij}(0) \partial_j,
\]

(1.81)

where \( \lambda \) is an arbitrary function of the boundary coordinates \( x^i \). This coordinate transformation retains the Fefferman-Graham form of the metric (1.79), but it changes the boundary metric by a Weyl rescaling,

\[
\delta g_{ij}(0) = 2 \lambda g_{ij}(0).
\]

(1.82)

This Weyl rescaling freedom reflects the fact that a conformally compact manifold induces a conformal structure rather than a metric on the boundary. The choice for a specific representative \( g_{ij}(0) \) within the class \( [g_{ij}(0)] \) is equivalent to picking a specific Fefferman-Graham coordinate system.

Notice that the same diffeomorphism changes the leading-order term in the expansion (1.72) by:

\[
\delta \phi(0) = \lambda (\Delta - d) \phi(0).
\]

(1.83)

The transformation of the field theory sources (1.82) and (1.83) is in fact precisely the one we found when we discussed scale invariance in section 1.3.3, see the equations (1.46) and (1.50). Weyl rescalings in the CFT are thus implemented by diffeomorphisms in an AlAdS metric [22, 23] and the freedom to change the Fefferman-Graham expansion is precisely dual to the freedom to perform local Weyl rescalings in the field theory. We will see below how these diffeomorphisms can induce conformal anomalies in the field theory as well.

We may also consider the case of constant \( \lambda \) for which the diffeomorphism along (1.81) reduces to a rescaling of the \( z \) coordinate. In the field theory we interpret
this as a rescaling of the renormalization scale, cf. equation (1.52). The coordinate \( z \) is therefore often said to be dual to the renormalization scale of the dual field theory, with small \( z \) corresponding to large \( \mu \), so the UV of the theory, and vice versa. We shall use this intuition below.

Notice finally that any field theory that is described by gravity in an AlAdS spacetime has this asymptotic Weyl invariance at least up to anomalies. This explains the statement made in the second paragraph of section 1.2 where we claimed that the field theory should behave like a CFT at least at high energies.

Summary

Let us summarize the results of this subsection. Within the class of gauge/gravity dualities we consider here the corresponding bulk spacetimes must have an AlAdS form. Near the boundary of such spacetimes the metric and other fields diverge. It is however always possible to pick a so-called Fefferman-Graham coordinate system with a distinguished radial coordinate \( z \) and to define the boundary data as the leading-order coefficient in the expansion of the fields in this coordinate \( z \). We denoted these coefficients \( \phi(0) \) and \( g(0) \) above and they are interpreted as sources in the dual field theory. The Fefferman-Graham coordinate system is not unique but picking a different coordinate \( z \) corresponds to a Weyl rescaling in the dual field theory.

1.4.4 Holographic renormalization

With the boundary data specified we can try to find a bulk solution to the equations of motion and compute its on-shell action. Since the solution of the equations of motion is a function of \( \phi(0) \) and \( g(0) \), so is the corresponding on-shell action \( I[\phi(0), g(0)] \). According to (1.31) we may interpret it as the generating functional of connected correlation functions in the dual field theory.

However, the naive on-shell action is always infinite, essentially because of the infinite volume of AlAdS spacetimes. In order to obtain finite answers we need to regularize and then renormalize the computation of the on-shell action. This holographic renormalization procedure [24, 25, 26, 27, 28, 29] depends crucially on the asymptotic properties of an AlAdS metric and this is the place where the above framework finds a practical application.

The procedure of holographic renormalization is implemented as follows. The divergences in the on-shell action \( I[\phi(0), g(0)] \) all arise from integrals that diverge as we send the Fefferman-Graham coordinate \( z \to 0 \). To regulate these divergences one therefore introduces a cutoff by restricting the spacetime integrals to some
1.4. Asymptotic behavior

small but finite $z_0$. The regulated on-shell action then contains a number of terms that would diverge as $z_0 \to 0$. These divergences are then cancelled by adding counterterms to the action, which are boundary terms defined at the cutoff surface $z = z_0$. These counterterms are chosen in such a way that the combined action is finite as one removes the cutoff. This results in a finite renormalized on-shell action which one may use to compute field theory correlation functions.

The precise counterterm action is far from arbitrary. First of all the counterterm action has to be a boundary action in order for the counterterms not to affect the bulk equations of motion. To maintain covariance under transformations of the boundary coordinates $x^i$ these counterterms should furthermore be functionals of the induced fields on the slice given by $z = z_0$. They should also be local functionals of these fields in order not to change the nonlocal, dynamical, part of the on-shell action. Finally, in order to respect the variational principle for all finite values of $z_0$ the counterterms have to be defined in terms of the fields themselves and not involve their conjugate momenta, i.e. their radial derivatives. As a sidenote we remark that once the cutoff is removed the counterterms in fact precisely modify the variational principle such that it is appropriate for AlAdS spacetimes [30].

The precise counterterm action for AlAdS spacetimes depends on the bulk theory under consideration. On the other hand it does not depend on the specific solution to the equations of motion. Instead the counterterms are universal and make a given action finite for any solution to the corresponding equations of motion. This is dual to the statement that the renormalizability of the field theory can be demonstrated solely by analyzing its UV properties, i.e. independently of the specific form of the correlation functions.

Furthermore the procedure of holographic renormalization also leads to certain constraints which the renormalized on-shell action $I$ must satisfy. These constraints precisely correspond to the field theory Ward identities of section 1.3. They are again independent of the specific form of the bulk solution which reflects the fact that the Ward identities are also a property of the UV of the dual field theory.

In the next sections we will present the holographic renormalization procedure in two simple examples. We will show how to find the counterterms that make the action finite and how the holographic Ward identities arise. We will also find an explicit solution to the equation of motion which we use to compute a two-point function in the boundary theory.

For a general introduction to holographic renormalization we refer to [31].
1.5 Scalar field in AdS\(_{d+1}\)

We will again consider the example of a free massive scalar field \(\Phi\) in an AdS\(_{d+1}\) spacetime. Its action is given by:

\[
S = \frac{1}{2} \int d^{d+1}x \sqrt{G} \left( \partial_{\mu} \Phi \partial^{\mu} \Phi + m^2 \Phi^2 \right).
\] (1.84)

We shall in this section use a new coordinate \(r = -\log(z)\) in which the metric (1.69) becomes:

\[
G_{\mu\nu} dx^\mu dx^\nu = dr^2 + \gamma_{ij} dx^i dx^j, \quad \gamma_{ij} = e^{2r} \delta_{ij}.
\] (1.85)

Notice that the conformal boundary of the spacetime is now at \(r \to \infty\). In this coordinate the equation of motion (1.70) takes the form:

\[
\ddot{\Phi} + d\dot{\Phi} + \Box_{\gamma} \Phi - m^2 \Phi = 0,
\] (1.86)

where a dot denotes a radial derivative and \(\Box_{\gamma} = e^{-2r} \Box_0\) with \(\Box_0 = \delta_{ij} \partial_i \partial_j\) the \(d\)-dimensional flat space Laplacian. We presented in equation (1.72) the leading-order form of the solution near the boundary. In the coordinate \(r\) it reads:

\[
\Phi(r, x^i) = e^{(\Delta - d)r} \left( \phi_{(0)}(x^i) + \ldots + e^{-(2\Delta - d)r} \phi_{(2\Delta - d)}(x^i) + \ldots \right),
\] (1.87)

where we recall that \(\Delta = \frac{1}{2}(d + \sqrt{d + 4m^2})\) as given in (1.73). We shall assume that \(\Delta > d/2\). Notice that there are asymptotically two independent solutions. We interpret the leading-order term \(\phi_{(0)}(x)\) as the source in the dual field theory.

**Asymptotic solution**

Let us first expand the asymptotic solution to higher order. From the equation of motion we find that the subleading terms have the form of a power series:

\[
\Phi = e^{(\Delta - d)r} \left( \phi_{(0)}(x^i) + e^{-2r} \phi_{(2)}(x^i) + \ldots + e^{-2kr} \phi_{(2k)}(x^i) + \ldots \right),
\] (1.88)

whose coefficients are given by:

\[
\phi_{(2)} = \frac{\Box_0 \phi_{(0)}}{2(2\Delta - d - 2)}
\]

\[
\phi_{(2k)} = \frac{\Box_0 \phi_{(2k-2)}}{2k(2\Delta - d - 2k)}.
\] (1.89)

For generic values of \(\Delta\) this expansion continues indefinitely and a corresponding expansion exists for the second branch of the solution in (1.87). However if:

\[
\Delta = l + \frac{d}{2} \quad l \in \{1, 2, 3, \ldots\}
\] (1.90)
we find that equation (1.89) cannot be satisfied for \( k = l \) and we have to modify the expansion:

\[
\Phi = e^{(\Delta-d)r} \left( \phi_0(x^i) + e^{-2r} \phi_2(x^i) + \ldots + e^{-2kr} \phi_{2k}(x^i) + \ldots + e^{-2lr} \tilde{\phi}_{2l}(x^i) + e^{-2lr} \phi_{2l}(x^i) + \ldots \right). \tag{1.91}
\]

We then find in addition to (1.89) the new relation for \( k = l \):

\[
\tilde{\phi}_{2l} = \frac{1}{l} \Box_0 \phi_{2l-2} \tag{1.92}
\]

and we also find that \( \phi_{2l} \) is not determined by the asymptotic expansion. We shall see how it is determined in terms of \( \phi_0 \) once we find the full solution to the equations of motion below.

Notice that (1.90) implies that \( m^2 = l^2 - d^2/4 \). One often obtains precisely such masses by a Kaluza-Klein decomposition of the fields on the compact part of the geometry like we did in section 1.1.2. We will therefore focus on these cases from now on.

**On-shell action and divergences**

We now substitute the asymptotic solution (1.91) into the action (1.84). After an integration by parts we find that the bulk term vanishes by the equations of motion and the on-shell action reduces to a boundary term:

\[
I = \frac{1}{2} \int_{r_0}^{\infty} d^d x \sqrt{\gamma} \Phi \dot{\Phi}. \tag{1.93}
\]

This term would in principle be defined the surface \( r \to \infty \) in the metric (1.85). However plugging in the above solution and using \( \sqrt{\gamma} = e^{dr} \) we find:

\[
I = \int_{r_0}^{\infty} d^d x e^{(2\Delta-d)r} \left( (\Delta-d)\phi_0^2 + e^{-2r} 2(\Delta-d-1)\phi_0 \phi_2 + \ldots - dre^{-2\Delta-d)r} \tilde{\phi}_{2l} \phi_0 + (\tilde{\phi}_{2l} - d\phi_{2l}) \phi_0 + \ldots \right) \tag{1.94}
\]

and we find a number of divergences as we let \( r \to \infty \). We therefore impose a cutoff and put the spatial integral in (1.93) at some large but finite \( r_0 \). This regulates the divergences and ensures that the bare action is finite.

The divergent terms in (1.94) involve precisely all the terms that were determined by the radial expansion above. Indeed the first term involving the undetermined term \( \phi_{2l} \) is precisely a finite term and is therefore unimportant as far as the construction of the counterterms in concerned. This explicitly demonstrates the
The dilatation operator

There exist various systematic procedures to find the exact form of the counterterm action. The method we describe below follows [29] and relies on the following observation. Since the solution to the equations of motion is completely determined by the boundary data, so is the on-shell action $I$. For the regulated action these boundary data are the induced values of the bulk fields on the cutoff surface. For example, the on-shell action for the scalar field depends on the induced metric $\gamma_{ij}$ and the boundary values of the scalar field $\Phi$ at this cutoff surface. We may therefore write:

$$I[\gamma_{ij}, \Phi]$$  \hspace{1cm} (1.96)

This in particular implies that all the dependence on the cutoff is completely implicit in the dependence of $I$ on the induced fields. The radial derivative of $I$ is therefore given by a simple application of the chain rule:

$$\dot{I} = \int d^d x \gamma_{ij} \frac{\delta I}{\delta \gamma_{ij}} + \int d^d x \Pi \frac{\delta I}{\delta \Phi} = 2 \int d^d x \gamma_{ij} \frac{\delta I}{\delta \gamma_{ij}} + \int d^d x \Pi \frac{\delta I}{\delta \Phi}. \hspace{1cm} (1.97)$$
Since the background metric is fixed we substituted its background value $\dot{\gamma}_{ij} = 2\gamma_{ij}$ and we also introduced $\Pi = \dot{\Phi}$. Notice that the dot in (1.97) denotes a derivative with respect to the cutoff. We shall henceforth denote the cutoff by $r$ rather than $r_0$ since it is the only radial coordinate entering in this section.

From the expansion (1.91) we also know that
\[ \Pi = (\Delta - d)\Phi + \ldots \]  
(1.98)
where the dots represent subleading terms as $r \to \infty$. We therefore find that:
\[ \dot{I} = \delta_D I + \ldots \]  
(1.99)
with $\delta_D$ the dilatation operator:
\[ \delta_D = 2 \int d^d x \gamma_{ij} \frac{\delta}{\delta \gamma_{ij}} + \int d^d x (\Delta - d)\Phi \frac{\delta}{\delta \Phi}. \]  
(1.100)
Notice that it is appropriately named because when we send $r \to \infty$ it becomes precisely the field theory dilatation operator (1.51). It is now however a covariant operator which acts on arbitrary functionals of the fields $\Phi$ and $\gamma_{ij}$ evaluated at the cutoff surface.

We now recall that
\[ \Pi = \frac{1}{\sqrt{\gamma}} \frac{\delta I}{\delta \Phi}. \]  
(1.101)
This result follows either from explicit variation of the action (1.84) or from general Hamilton-Jacobi theory where one treats the radial coordinate as the ‘time’ coordinate. Since $I$ was a functional of $\gamma_{ij}$ and $\Phi$, then from (1.101) so is $\Pi$. Therefore the radial derivative acting on $\Pi$ can also be written in the form (1.97):
\[ \dot{\Pi} = 2 \int d^d x \gamma_{ij} \frac{\delta \Pi}{\delta \gamma_{ij}} + \int d^d x \Pi \frac{\delta \Pi}{\delta \Phi} \]  
(1.102)
and again the radial derivative is asymptotically equal to the dilatation operator.

**Eigenfunction expansion**

The next step in the Hamiltonian holographic renormalization is to expand $\Pi$ in eigenfunctions of the dilatation operator $\delta_D$ defined in (1.100). We write:
\[ \Pi = \Pi_{(d-\Delta)} + \Pi_{(d-\Delta+2)} + \Pi_{(d-\Delta+4)} + \ldots \]  
(1.103)
where by definition
\[ \delta_D \Pi_{(s)} = -s \Pi_{(s)}. \]  
(1.104)
Notice that since the radial derivative is asymptotically equal to the dilatation operator we obtain that:

\[ \dot{\Pi}(s) = -s\Pi(s) + \ldots \]  \hspace{1cm} (1.105)

However the organization in eigenfunctions of the dilatation operator is not precisely a radial expansion. Indeed, this expansion organizes the various divergences in \( \Pi \) in a covariant fashion, that is in terms of \( \Phi \) rather than \( \phi_{(0)} \).

From (1.98) we immediately find that \( \Pi_{(d-\Delta)} = (\Delta - d)\Phi \) which is in fact how we obtained the leading-order weight \( (d - \Delta) \) in (1.103). To find an expression for the subleading terms we use the equation of motion (1.86):

\[ \dot{\Pi} + d\Pi - \Box_\gamma \Phi - \Delta(\Delta - d)\Phi = 0. \]  \hspace{1cm} (1.106)

We then replace \( \dot{\Pi} \) with (1.102), substitute (1.103) for all instances of \( \Pi \) and use (1.104) to collect terms of equal dilatation weight. This results in the following expressions for the subleading terms:

\[ \Pi_{(d-\Delta+2)} = \frac{1}{2\Delta - d - 2} \Box_\gamma \Phi \]
\[ \Pi_{(d-\Delta+4)} = \frac{-1}{(2\Delta - d - 4)(2\Delta - d - 2)^2} \Box_\gamma^2 \Phi \]  \hspace{1cm} (1.107)
\[ \Pi_{(d-\Delta+2k)} = \frac{c(k, \Delta, d)}{(2\Delta - d - 2k)^k} \Box_\gamma^k \Phi \]

where the coefficients \( c(k, \Delta, d) \) can be determined recursively. The above expansion continues up to the terms with \( k = l \), so the terms with weight \( -\Delta \), where we find that the equation of motion (1.106) cannot be satisfied when we substitute the expansion (1.103). Just as for the radial expansion (1.91) we have to include an inhomogeneous term:

\[ \Pi = \Pi_{(d-\Delta)} + \Pi_{(d-\Delta+2)} + \ldots + \Pi_{(\Delta+2)} + r\Pi_{(\Delta)} + \Pi_{(\Delta)} + \ldots \]  \hspace{1cm} (1.108)

whose properties under dilatations take the form:

\[ \delta_D(\Pi_{(\Delta)} + r\Pi_{(\Delta)}) = -\Delta(\Pi_{(\Delta)} + r\Pi_{(\Delta)}) + \Pi_{(\Delta)}. \]  \hspace{1cm} (1.109)

Notice again the similarity between the dilatation operator and the radial derivative. By iterating the equations of motion we find that:

\[ \Pi_{(\Delta)} = (-1)^l 2^{2l-2} \Gamma(l) \Box_\gamma^{2l} \Phi, \]  \hspace{1cm} (1.110)

whereas \( \Pi_{(\Delta)} \) is not determined by this procedure.
1.5. Scalar field in AdS$_{d+1}$

Renormalized action and one-point function

We recall that the bare action (1.93) is given by:

$$I_{\text{bare}} = \frac{1}{2} \int r \, d^d x \sqrt{g} \Phi \Pi .$$  \hspace{1cm} (1.111)

Substituting now the expansion (1.103) we directly find the divergences organized in a covariant expansion in $\Phi$ rather than in a radial expansion as given in (1.94). This means that we can directly cancel them with the counterterm action:

$$I_{\text{ct}} = - \frac{1}{2} \int r \, d^d x \sqrt{g} \Phi \left( \sum_{(d-\Delta) \leq s < \Delta} \Pi_{(s)} + r \tilde{\Pi}_{(\Delta)} \right)$$  \hspace{1cm} (1.112)

and the renormalized action is given by:

$$I_{\text{ren}} = \lim_{r \to \infty} - \frac{1}{2} \int r \, d^d x \sqrt{g} \Phi \Pi_{\Delta} ,$$  \hspace{1cm} (1.113)

which is finite as $r \to \infty$.

Notice the presence of the explicit $r$ in the last counterterm which is necessary to cancel the logarithmic divergence in (1.94). This explicit coordinate dependence leads to anomalous transformation properties under translations in the $r$ direction. We mentioned in section 1.4.3 that these translations map to Weyl rescalings in the boundary theory and one may therefore expect a corresponding Weyl anomaly. This indeed turns out to be the case.

To find the anomaly we recall that the dilatation operator reduces to the global rescaling defined in (1.51) for $\lambda = 1$. We can therefore act with $\delta_D$ on the renormalized action and compare with (1.53) to find the corresponding anomaly $A_w$. Using (1.109), (1.110) and (1.113) we obtain:

$$\delta_D I_{\text{ren}} = - \frac{1}{2} \lim_{r \to \infty} \int r \, d^d x \sqrt{g} \tilde{\Pi}_{(\Delta)} = \int d^d x (-1)^l + 2^{l-1} \Gamma(l) \phi(0) \Box_l \phi(0) ,$$  \hspace{1cm} (1.114)

whose form precisely matches the conformal anomaly for operators of dimension (1.90) which we discussed in section 1.3.3. Notice that we obtained this anomaly without having to find a complete solution to the equations of motion: just as in the boundary theory it follows purely from the UV structure of the theory.

Correlation functions

From the renormalized action (1.113) we may obtain the one-point function in the presence of sources. Using (1.31) we find:

$$\langle \mathcal{O}(x^i) \rangle_{\Phi} = \frac{1}{\sqrt{g(0)}} \frac{\delta I_{\text{ren}}}{\delta \phi(0)(x^i)} = \lim_{r \to \infty} \frac{e^{\Delta r}}{\sqrt{g}} \frac{\delta}{\delta \Phi(x^i, r)} (I_{\text{bare}} + I_{\text{ct}}) ,$$  \hspace{1cm} (1.115)
1. The gauge/gravity dualities

where the extra factor $e^{\Delta r}$ converts $\Phi$ to $\phi(0)$ and $\gamma$ to $g(0)$ as $r \to \infty$. We may compute this explicitly by substituting the values for $\Pi(s)$ and $\tilde{\Pi}(\Delta)$ in (1.112) and using (1.101). We find:

$$\langle O(x^i) \rangle_\phi = \lim_{r \to \infty} e^{\Delta r} \Pi(\Delta)(r, x^i). \quad (1.116)$$

Substituting now (1.91) we obtain [27]:

$$\langle O(x^i) \rangle_\phi = -2l \phi(2l)(x^i) + C[\phi(0)(x^i)], \quad (1.117)$$

where the $C[\phi(0)]$ are local contact terms which are scheme-dependent.

In order to proceed from here we have to determine the full solution to the equation of motion (1.86) for given boundary data $\phi(0)$. Indeed the full solution allows one to express $\phi(2l)$ as a function of $\phi(0)$ so we can differentiate (1.117) once more with respect to $\phi(0)$ to obtain a nontrivial two-point function.

In Fourier space one easily obtains the two linearly independent solutions of (1.86). These are given in terms of modified Bessel functions:

$$e^{-(d/2)r} K_l(|k|e^{-r}) \quad e^{-(d/2)r} I_l(|k|e^{-r}) \quad (1.118)$$

For $r \to -\infty$ the solutions behave as:

$$e^{-(d/2)r} K_l(|k|e^{-r}) = \sqrt{\frac{\pi e^{-(d-1)r}}{2|k|}} \exp(-|k|e^{-r})(1 + O(e^r)) \quad (1.119)$$

$$e^{-(d/2)r} I_l(|k|e^{-r}) = \sqrt{\frac{e^{-(d-1)r}}{2\pi|k|}} \left[ \exp(|k|e^{-r}) + \exp(-|k|e^{-r} - (l + \frac{1}{2})\pi i) \right]$$

$$\times (1 + O(e^r))$$

We see that the solution involving $I_l$ blows up and therefore should be discarded. For large $r$ we obtain:

$$e^{-(d/2)r} K_l(|k|e^{-r}) = \Gamma(l) \frac{2^{l+1}e^{-(d/2-l)r}}{|k|^l}(1 + O(e^{-r})) \quad (1.120)$$

The full solution to the equations of motion therefore takes the form:

$$\Phi(x, k) = \int \frac{d^d k}{(2\pi)^d} e^{ikx} \phi(0)(k) \frac{|k|^l}{2^{l+1}\Gamma(l)} e^{-(d/2)r} K_l(|k|e^{-r}) \quad (1.121)$$

where $K_l$ is the modified Bessel function of the second kind and we Fourier transformed the source $\phi(0)$. One may expand the Bessel function to recover precisely
1.6 Einstein gravity

the expansion (1.91), including the inhomogeneous term. The full solution also
determines the final term in (1.91):

$$\phi(2l) = \int \frac{d^d k}{(2\pi)^d} e^{i k x} \phi(0)(k) \left( \frac{-1}{2l \Gamma(l) \Gamma(l + 1)} \right) \left( \log(|k|^2) - \log(4) + \frac{1}{2} \psi(l + 1) - \gamma \right)$$  \hspace{1cm} (1.122)

where $$\psi(x) = d \ln \Gamma(x)/dx$$ is the digamma function and $$\gamma$$ the Euler-Mascheroni constant. We can plug this into (1.117) and differentiate once more with respect to $$\phi(0)$$ to obtain:

$$\langle O(k) O(-k) \rangle = \left( \frac{-1}{2l + 1} \right) \left| k \right|^{2l} \log\left( \frac{\left| k \right|^2}{\mu^2} \right) + C(k^2)$$  \hspace{1cm} (1.123)

where the $$C$$ denote contact terms that we can safely ignore and we also introduced a scale $$\mu$$ on dimensional grounds. In position space we may use (1.65) to find:

$$\langle O(x) O(y) \rangle = \frac{2l \pi^{-d} \Gamma(l + d/2)}{\pi^{d/2} \Gamma(l)} R \frac{1}{|x|^{2\Delta}}$$  \hspace{1cm} (1.124)

We indeed reproduced the renormalized correlation function, which was to be expected from the anomalous conformal Ward identity we found above.

1.6 Einstein gravity

In this section we will work out the holographic renormalization for $$(d + 1)$$-dimensional Einstein gravity with a negative cosmological constant. The proper action functional is given by the Einstein-Hilbert term plus the Gibbons-Hawking boundary term:

$$S = \int \frac{d^{d+1} x}{2\kappa^2} \sqrt{G} (-R + 2\Lambda) - \frac{1}{\kappa^2} \int d^d x \sqrt{\gamma} K.$$  \hspace{1cm} (1.125)

This form of the action results in a well-defined variational principle if one keeps the metric fixed on a boundary that is at finite distance from the interior of the space [32]. We work in $$(d + 1)$$ dimensions and will specialize to $$d \leq 4$$ below. The bulk metric is denoted $$G_{\mu\nu}$$, the boundary metric $$\gamma_{ij}$$ and $$K_{ij}$$ is the extrinsic curvature of the boundary. The equations of motion obtained from (1.125) are the Einstein equations:

$$R_{\mu\nu} - \frac{1}{2} R G_{\mu\nu} + \Lambda G_{\mu\nu} = 0$$  \hspace{1cm} (1.126)

and we use the following conventions for the curvatures:

$$R_{\mu\nu}^\sigma = \partial_\nu \Gamma^\sigma_{\mu\rho} + \Gamma^\lambda_{\mu\rho} \Gamma^\sigma_{\nu\lambda} - (\mu \leftrightarrow \nu), \quad R_{\mu\rho} = R_{\mu\sigma\rho}^\sigma.$$  \hspace{1cm} (1.127)

In terms of the AdS radius of curvature $$\ell$$ one has $$\Lambda = -\frac{1}{4} d(d - 1) \ell^{-2}$$. We will set $$\ell = 1$$ in what follows. Notice that one then obtains $$R = -d(d + 1)$$ from the trace of (1.126).
AlAdS solution

We would like to renormalize the on-shell action for a solution to (1.126) which is AlAdS. In section 1.4.3 we described that near the conformal boundary the metric for such a solution can be put in the Fefferman-Graham form:

\[ G_{\mu\nu}dx^\mu dx^\nu = dr^2 + \gamma_{ij}(r,x^k) dx^i dx^j \]

\[ \gamma_{ij}(r,x^k) = e^{2r}g_{ij}(0) + \ldots \]

(1.128)

In these coordinates the extrinsic curvature \( K_{ij} \) for a hypersurface of constant \( r \) takes the simple form:

\[ K_{ij} = \frac{1}{2} \dot{\gamma}_{ij} \]

(1.129)

where the dot denotes a radial derivative. The nonzero connection coefficients for the metric (1.128) are

\[ \Gamma^r_{ij} = -K_{ij}, \quad \Gamma^i_{rj} = K^i_j, \quad \Gamma^i_{jk}[G] = \Gamma^i_{jk}[\gamma], \]

(1.130)

and the Einstein equations become:

\[ 0 = R - K^2 + K_{ij}K^{ij} + d(d-1) \]
\[ 0 = \nabla_i K^i_j - \nabla_j K \]
\[ 0 = \partial_r(K^i_j) + KK^i_j - R^i_j - d\delta^i_j. \]

(1.131)

Here and below we write \( R_{ij} \) and \( \nabla_i \) for the curvature and the covariant derivative associated with the metric \( \gamma_{ij} \). Similarly indices are always raised with \( \gamma^{ij} \) unless indicated otherwise.

We may use the equations (1.131) to obtain an asymptotic expansion of the metric. For example for the first subleading term we find:

\[ \gamma_{ij} = e^{2r}\left( g_{ij} + e^{-2r}g^{(2)}_{ij} + \ldots \right) \]

(1.132)

with, for \( d > 2 \),

\[ g^{(2)}_{ij} = \frac{1}{d-2} \left( R_{(0)ij} - \frac{1}{2(d-1)} R_{(0)}g_{(0)ij} \right) \]

(1.133)

where \( R_{(0)ij} \) is the Ricci tensor for the metric \( g_{(0)ij} \). Notice that this term is a local function of the boundary data \( g_{(0)ij} \). Iterating this procedure one finds that all the subleading terms up until \( g^{(d)ij} \) are locally determined by \( g_{(0)ij} \), but at order \( e^{-dr} \) the procedure breaks down and one finds that \( g^{(d)ij} \) is not (completely) determined by the asymptotic solution to the equations of motion. Furthermore, for \( d \) even and greater than two one also has to insert a term of the form \( e^{-dr}g^{(2^d)}_{ij} \) in the asymptotic expansion. The resulting asymptotic structure is then very similar to
that of the scalar field presented in the equations (1.89) and (1.91) above. Furthermore, just as for the scalar field we will see below that the nonlocally determined term $g_{(d)ij}$ will eventually enter in the one-point function (as in equation (1.117)) and that the inhomogeneous term $\tilde{g}_{(d)ij}$ is related to a conformal anomaly (as in equation (1.114)).

Divergences and constraint equations

Let us now analyze the divergences in the on-shell action. Using the above expressions for the bulk curvature and cosmological constant one finds that the on-shell action reduces to:

$$I = \frac{d}{\kappa^2} \int d^{d+1}x \sqrt{G} - \frac{1}{\kappa^2} \int d^d x \sqrt{\kappa} K.$$  \hspace{2cm} (1.134)

Upon substitution of the Fefferman-Graham expansion one finds a number of divergences. For example, from (1.128) and (1.129) one finds at leading order:

$$I_{\text{bare}} = \frac{1}{\kappa^2} \int_{r_0} d^d x \left( \sqrt{g_0} e^{dr} + \ldots \right), \hspace{2cm} (1.135)$$

where we introduced a cutoff at a large but finite $r_0$ to regulate the divergences.

The divergences should be cancelled with a suitable counterterm action which can be obtained as follows. Just as for the scalar field we begin by introducing the conjugate momentum as the variation of the on-shell action:

$$\pi_{ij} = \frac{2\kappa^2}{\sqrt{\gamma}} \frac{\delta I}{\delta \gamma_{ij}}.$$  \hspace{2cm} (1.136)

From (1.125) one may easily compute it to be:

$$\pi_{ij} = K_{ij} - K \gamma_{ij}, \hspace{2cm} (1.137)$$

which can be inverted to $K_{ij} = \pi_{ij} + \pi_{ik} \gamma_{kj} / (d-1)$. In terms of $\pi_{ij}$ the first equation in (1.131) then becomes:

$$R + \pi^j_i \pi^i_j - \frac{(\pi^k_k)^2}{d-1} + d(d-1) = 0 \hspace{2cm} (1.138)$$

If one then substitutes (1.136) this equation becomes a functional differential equation for $I$ which is precisely the Hamilton-Jacobi equation for the on-shell gravity action. We will shortly see how this single equation is sufficient to completely determine the divergent part of the on-shell action.
Dilatation weight expansions

Let us again introduce the dilatation operator which we introduced in (1.100) above. In this case the metric is the only dynamical field and the dilatation operator therefore takes the form:

\[ \delta D = 2 \int d^d x \gamma_{ij} \frac{\delta}{\delta \gamma_{ij}}. \] (1.139)

Just as in the previous section we notice that the regulated on-shell action \( I \) as well as the conjugate momentum \( \pi^{ij} \) are functionals of the boundary metric \( \gamma_{ij} \) alone. By a repetition of the discussion around equation (1.99) one may conclude that the action of the dilatation operator on these quantities is therefore asymptotically equal to taking a radial derivative. To isolate the divergent pieces it is then again convenient to use an expansion in eigenfunctions of the dilatation operator.

Notice that we may always write the on-shell action as an integral over the cutoff surface:

\[ I = \frac{1}{2\kappa^2} \int d^d x \sqrt{\gamma} \lambda \] (1.140)

for some (a priori complicated) function \( \lambda[\gamma_{ij}] \). We will now expand \( \lambda \) in eigenfunctions of the dilatation operator. Using (1.135) and the fact that \( \sqrt{\gamma} = e^{dr} \sqrt{g(0)} + \ldots \) we see that the leading-order weight of \( \lambda \) has to be zero. We therefore write:

\[ \lambda = \lambda(0) + \lambda(2) + \ldots \] (1.141)

with by definition:

\[ \delta D \lambda(s) = -s \lambda(s) \] (1.142)

and from (1.135) we immediately find \( \lambda(0) = 2(1 - d) \). We shall shortly see that one needs to add inhomogeneous terms to the expansion (1.141) when \( d \) is even, just as in equation (1.108) for the conjugate momentum of the scalar field.

Let us now consider the expansion of \( \pi^{ij} \). Notice that since \( \gamma_{ij} \) has weight two we find that lowering or raising an index on a tensor increases or decreases its dilatation weight by two, respectively. We will choose to expand \( \pi^i_j \) in eigenfunctions of the dilatation operator. To leading order one finds from (1.137) that

\[ \pi^i_j = (1 - d) \delta^i_j + \ldots , \] (1.143)

so the corresponding expansion for \( \pi^i_j \) also begins with a term of weight zero. (This is consistent with (1.136) and the fact that \( \lambda \) also begins with a term of weight zero.) We therefore write:

\[ \pi^i_j = \pi^{(0)}_{(0)j}^i + \pi^{(2)}_{(2)j}^i + \ldots , \] (1.144)
1.6. Einstein gravity

with again:

\[ \delta_D \pi_{(s)}^i_j = -s \pi_{(s)}^i_j \]  

(1.145)

and with the leading-order term:

\[ \pi_{(0)}^i_j = (1 - d) \delta^i_j. \]  

(1.146)

We will explain below that it again becomes necessary to add an inhomogeneous term to the expansion for certain values of \( d \).

**Computation of the expansion coefficients**

We can now explicitly construct all the divergent terms in the expansion (1.141) using the constraint equation (1.138). Just as for the scalar field all these terms will be algebraically determined and the resulting expressions will be local and covariant functions of the boundary metric \( \gamma_{ij} \).

The first step is to realize that for any variation of \( \gamma_{ij} \) we may use (1.136) and (1.140) to find:

\[ \sqrt{\gamma} \pi^i_j \delta \gamma_{ij} = \delta (\sqrt{\gamma} \lambda) \]  

(1.147)

at least up to a total divergence. We may however use the total divergence ambiguity in \( \lambda \) to make sure that the above equation holds exactly, see [29] for details.

We may then set \( \delta = \delta_D \) and find that

\[ 2 \pi^k_k = (d + \delta_D) \lambda \]  

(1.148)

and upon substitution of the above expansions for \( \lambda \) and \( \pi^i_j \) we obtain:

\[ \lambda_{(s)} = \frac{2}{d - s} \pi^k_k (s). \]  

(1.149)

We may furthermore obtain \( \pi^i_j (s) \) from substitution of the expansions in (1.136). This results in:

\[ \sqrt{\gamma} \pi^i_j (s) = \gamma^{jk} \frac{\delta}{\delta \gamma_{ik}} \int d^d x \sqrt{\gamma} \lambda_{(s)}. \]  

(1.150)

Notice that we may substitute (1.149) into (1.150) to recover \( \pi^i_j (s) \) completely by knowing only its trace, at least for \( s \neq d \). This allows for a recursive solution of (1.138) as we now illustrate by performing the first few steps.

First of all, at dilatation weight zero equation (1.138) reduces to an equation that is trivially satisfied upon substitution of (1.146). At dilatation weight two we use (1.146) to obtain:

\[ 2 \pi^k_k (2) = -R \]  

(1.151)

and using (1.149):

\[ \lambda_{(2)} = -\frac{1}{d - 2} R, \]  

(1.152)

37
1. The gauge/gravity dualities

at least for \( d \neq 2 \). Furthermore, from (1.150) we find:

\[
\pi_{ij}^{(2)} = \frac{1}{d-2} \left( R_{ij} - \frac{1}{2} R \gamma_{ij} \right),
\]

again for \( d \neq 2 \).

Next, at dilation weight 4 we obtain:

\[
\pi_{k}^{(4) k} = \frac{\left( \pi_{k}^{(2) k} \right)^2}{2(d-1)} - \frac{1}{2} \pi_{j}^{(2) j} \pi_{i}^{(2) i}
\]

\[
= -\frac{1}{8(d-2)^2} \left( \frac{d}{d-1} R^2 + 4 R_{ij} R_{ij} \right),
\]

and for \( d \neq 2, 4 \) we may again use (1.149) to find

\[
\lambda_{(4)} = -\frac{1}{4(d-4)(d-2)^2} \left( \frac{d}{d-1} R^2 + 4 R_{ij} R_{ij} \right)
\]

and so on.

In general the above procedure breaks down at dilatation weight \( s = d \). For odd \( d \) we find that \( \pi_{i}^{(d) k} = 0 \) and that \( \lambda_{i}^{(d)} \) is undetermined by this procedure. For even \( d \) the expansions for \( \lambda \) and \( \pi_{i}^{(d)} \) given above need to be modified to:

\[
\pi_{i}^{j} = \pi_{i}^{(0) j} + \ldots + r \tilde{\pi}_{i}^{(d) j} + \pi_{i}^{(d) j} + \ldots
\]

\[
\lambda = \lambda_{(0)} + \ldots + r \tilde{\lambda}_{i}^{(d) j} + \lambda_{(d)} + \ldots
\]

with:

\[
\delta_D (r \tilde{\pi}_{i}^{(d) j} + \pi_{i}^{(d) j}) = -dr \pi_{i}^{(d) j} - d\pi_{i}^{(d) j} + \pi_{i}^{(d) j}
\]

and similarly for \( \lambda_{(d)} \) and \( \tilde{\lambda}_{(d)} \). Repeating then the above procedure we find first of all that \( \pi_{i}^{(s) j} \) and \( \lambda_{i}^{(s)} \) are unmodified for \( s < d \). Furthermore, for \( s = d \) we find from (1.138) the traces of the new terms, namely \( \pi_{k}^{(d) k} = 0 \) and \( \pi_{k}^{(d) k} \) is the same as above. Using (1.148) one also obtains that

\[
\tilde{\lambda}_{i}^{(d)} = \pi_{i}^{(d) k}
\]

and \( \lambda_{(d)} \) is again undetermined. Finally we use (1.150) to obtain:

\[
\tilde{\pi}_{i}^{(d) j} = \frac{1}{\sqrt{\gamma_{jk}}} \delta \frac{\gamma_{ik}}{\delta \gamma_{jk}} \int d^d x \sqrt{\gamma} \pi_{i}^{(d) k},
\]

but since \( \lambda_{(d)} \) is undetermined we cannot determine \( \pi_{i}^{(d) j} \) from (1.150).

Let us consider explicitly the dimensions \( d = 2 \) and \( d = 4 \). For \( d = 2 \) we find

\[
\tilde{\lambda}_{(2)} = \pi_{i}^{(2) k} = -\frac{1}{2} R,
\]

\[
\tilde{\pi}_{i}^{(4) j} = \frac{1}{\sqrt{\gamma_{jk}}} \delta \frac{\gamma_{ik}}{\delta \gamma_{jk}} \int d^4 x \sqrt{\gamma} \pi_{i}^{(4) k},
\]

38
and then from (1.159) one obtains
\[ \tilde{\pi}^i_{(2)j} = 0, \] (1.161)
since the metric variation of the topological invariant \( \int d^2x \sqrt{\gamma} R \) vanishes completely. For \( d = 4 \) we obtain:
\[ \tilde{\lambda}^{(4)} = \pi^{(4)k}_k = -\left( \frac{1}{24} R^2 + \frac{1}{8} R^i_j R^j_i \right), \] (1.162)
which we may substitute in (1.150) to find an expression for \( \tilde{\pi}^{(4)i}_j \) that we shall not write down here.

Counterterms and renormalization

The counterterm action is given by the terms with positive dilatation weight. Remembering that the metric determinant has weight \( d \), we find for odd dimensions that
\[ I_{ct} = -\frac{1}{2\kappa^2} \int d^d x \sqrt{\gamma} \left( \lambda^{(0)} + \lambda^{(2)} + \ldots + \lambda^{(d-1)} \right), \] (1.163)
and for even dimensions that
\[ I_{ct} = -\frac{1}{2\kappa^2} \int d^d x \sqrt{\gamma} \left( \lambda^{(0)} + \lambda^{(2)} + \ldots + r \tilde{\lambda}^{(d)} \right). \] (1.164)
For all dimensions the renormalized on-shell action is given by:
\[ I_{ren} = \lim_{r \to \infty} (I_{bare} + I_{ct}) = \lim_{r \to \infty} \frac{1}{2\kappa^2} \int d^d x \sqrt{\gamma} \lambda^{(d)}, \] (1.165)
and we immediately obtain the renormalized one-point function of the energy-momentum tensor as:
\[ \langle T_{ij}(x) \rangle_{g(0)} = \frac{2}{\sqrt{g(0)(x)}} \frac{\delta}{\delta g^{ij}_{(0)}(x)} I_{ren} \]
\[ = \lim_{r \to \infty} \frac{-e^{(d-2)r}}{\kappa^2 \sqrt{\gamma(x)}} \frac{\delta}{\delta \gamma_{ij}(x)} \int d^d x \sqrt{\gamma} \lambda^{(d)} \] (1.166)
\[ = -\lim_{r \to \infty} e^{(d-2)r} \frac{1}{\kappa^2} \pi^{(d)ij}, \]
where we added a subscript \( g(0)ij \) on the left-hand side to indicate the nontrivial curved background metric. An explicit expression for \( \pi^{(d)ij} \) can be found by using the Fefferman-Graham expansion. For example for \( d = 2 \) we have:
\[ \gamma_{ij} = e^{2r} g_{(0)ij} + g_{(2)ij} + \ldots \] (1.167)
1. The gauge/gravity dualities

\[ \pi(2)_{ij} = \pi_{ij} - \pi(0)_{ij} + \ldots \]
\[ = K_{ij} - K\gamma_{ij} + \gamma_{ij} + \ldots \]  
\[ = -g(2)_{ij} + g(0)_{ij} \text{tr}(g^{-1}(0)g(2)) + \ldots \]  
\[ = -\frac{1}{2} R \]  
\[ = -\frac{1}{2} e^{-2r} R(0) + \ldots \]  
\[ = \text{tr}(g^{-1}(0)g(2)) - \frac{1}{2} R(0). \]  
\[ (1.168) \]

where the dots represent terms that vanish as \( r \rightarrow \infty \). Furthermore we may use the fact that \( \pi(2)_{k} = -\frac{1}{2} R = -\frac{1}{2} e^{-2r} R(0) + \ldots \) to find that \( \text{tr}(g^{-1}(0)g(2)) = -\frac{1}{2} R(0). \)

Our final result for \( d = 2 \) is then [27]:

\[ \langle T_{ij} \rangle_{g(0)} = \frac{1}{2\kappa^2}(2g(2)_{ij} + R(0)g(0)_{ij}). \]  
\[ (1.169) \]

We will use this result in the upcoming chapters.

Ward identities and anomalies

In section 1.3 we formulated the conformal Ward identities in terms of the trace of the one-point function of the energy-momentum tensor. In particular, from (1.51) and (1.53) we find that in the absence of any other sources:

\[ A_w[g(0)] = \langle T^i_i \rangle_{g(0)}. \]  
\[ (1.170) \]

Explicit expressions can be found by taking the trace of (1.166) and substituting the above results for \( \pi_{(d)k} \). We obtain [24, 25]:

\[ \langle T^i_i \rangle_{g(0)} = \frac{1}{\kappa^2} \lim_{r \rightarrow \infty} e^{dr} \pi_{(d)k} \]
\[ = \begin{cases} 
0 & \text{d odd} \\
\frac{1}{2\pi} R(0) & \text{d = 2} \\
\frac{1}{\kappa^2}(\frac{1}{24} R^2(0) + \frac{1}{8} R(0)_{ij} R(0)_{ij}) & \text{d = 4} 
\end{cases} \]  
\[ (1.171) \]

The final expressions were first obtained in [24]. They satisfy the Wess-Zumino consistency conditions and are therefore valid conformal anomalies. Notice that the asymptotic analysis was sufficient to find these holographic Weyl anomalies and we did not need to completely solve the equations of motion, in agreement with the general discussion of section 1.4.4.

For \( d = 2 \) we may recover the standard Virasoro central charge which in our conventions can be obtained from \( \langle T^i_i \rangle_{g(0)} = \frac{c}{24\pi} R. \) We find:

\[ c = \frac{12\pi}{\kappa^2} = \frac{3\ell}{2G_N}, \]  
\[ (1.172) \]

where we reinstated the AdS radius of curvature \( \ell \) and used \( \kappa^2 = 8\pi G_N. \) This result agrees with [33]. Notice that the supergravity limit only applies when gravity
is weak and the AdS radius is large, so only when \( c \gg 1 \). This is the two-dimensional analogue of the large \( N \) limit discussed in section 1.1.

Finally we may also obtain the diffeomorphism Ward identity. This identity follows from the second equation of (1.131) which may be rewritten as:

\[
\nabla_i \pi^i_j = 0 \quad (1.173)
\]

Substituting now the expansion in terms of the dilatation operator we find that this directly leads to the conservation of the energy-momentum tensor:

\[
\nabla_{(0)i} \langle T^{i}_j \rangle_{g_{(0)}} = 0, \quad (1.174)
\]

where \( \nabla_{(0)i} \) is the covariant derivative with respect to \( g_{(0)ij} \). This is precisely (1.41) in the absence of sources. Notice again that the asymptotic analysis was sufficient to obtain this Ward identity.