Dynamics and the gauge/gravity duality

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Chapter 2

Real-time gauge/gravity duality

Since the advent of the AdS/CFT correspondence [9, 15, 16] a considerable amount of work has been devoted to developing holographic dualities leading to a very precise understanding of the ‘holographic dictionary’ that translates between bulk and boundary quantities. We sketched some aspects of this dictionary in the previous chapter and refer to the reviews [10, 11, 31] for more details. So far this dictionary was however always formulated in a purely ‘Euclidean’ regime, i.e. when the boundary theory is Wick-rotated and the corresponding bulk solution involves a positive-definite metric as well. While this suffices for many applications, there are also many reasons for obtaining a general real-time dictionary which can be used directly for Lorentzian signature backgrounds.

Indeed there is a wide range of applications for such a real-time prescription. To mention a few: one would like to understand better holography for time dependent backgrounds, to have a holographic description of non-equilibrium QFT and to be able to compute correlators in non-trivial states. Such a development would also be useful for applications of holography to modelling the dynamics of the quark-gluon plasma or condensed-matter systems, see [34, 35, 36] for reviews.

From a more theoretical perspective, one would like to understand better the interplay between causality and holography. Since bulk and boundary lightcones are different, it is not a priori clear that a bulk computation will produce the correct causal structure for boundary correlators, for example the correct $i\epsilon$ insertions. Conversely, one can ask how the bulk causal structure emerges from
boundary correlators. A related question is to understand how black hole horizons are encoded in boundary correlators. More generally one would like to study holographically the process of gravitational collapse.

In this chapter we develop a precise real-time gauge/gravity dictionary. Our formalism is applicable at the same level of generality as the corresponding Euclidean prescription and therefore constitutes an integral part of the definition of the holographic correspondence. More precisely, our setup is valid for all QFTs that have a holographic dual and is applicable for the holographic computation of arbitrary correlation functions of gauge-invariant operators in non-trivial states. Furthermore, the prescription is fully holographic, i.e. only boundary data and regularity in the interior are needed for the computations, and within the supergravity approximation all information is encoded in classical bulk dynamics.

There have been several earlier works discussing holography in Lorentzian signature, including [37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47]. One set of these papers is based on semi-classical quantization of the bulk fields around a classical bulk solution. For example, a case often discussed is that of the quantization of a free scalar field in AdS and the computation of the associated boundary 2-point function. Such results are clearly difficult to extend to cases where bulk interactions are essential because of the difficulty in quantizing the bulk gravitational theory. For example, higher point functions, correlators of the stress energy tensor and holographic RG flows are outside the remit of these works. Moreover, for the computation of the correlators in the supergravity limit one should not have to consider the quantization of the bulk theory at all – classical bulk dynamics should suffice.

A Lorentzian prescription that has been used widely in the literature is that of Son and Starinets [40]. This prescription leads to the computation of retarded 2-point functions and is based on imposing specific ‘ingoing’ boundary conditions in the interior of the spacetime. It leads to correct results (provided the infinities have been subtracted correctly) but is somewhat unsatisfactory from the holographic point of view, because it presumes the existence of a horizon in the bulk spacetime and ignores certain surface terms in the on-shell action. These issues can however be overcome once one embeds the prescription of [40] in our more general framework, as we explain in detail in chapter 4.

The remainder of this chapter is structured as follows. We begin with a review of real-time quantum field theory in section 2.1. Afterwards we present in section 2.2 the real-time gauge/gravity prescription. The holographic renormalization and computation of correlators in our prescription is discussed in detail for a scalar field in section 2.3 and for gravity in section 2.4. An appendix to this chapter contains further real-time quantum field theory results. In the next chapter we
apply the prescription to several concrete examples.

2.1 QFT preliminaries

In this section we review some aspects of real-time quantum field theory. In particular we discuss the specification of the initial and final states and the in-in formalism for non-equilibrium processes. The results in this section will be the springboard for the gauge/gravity prescription we present in the next section. For more details on real-time quantum field theory we refer to [48, 49, 50, 51].

2.1.1 Vacuum amplitudes

An essential ingredient for the computation of real-time correlation functions is the proper specification of the initial and final state. In this section we review how this is implemented for vacuum-to-vacuum amplitudes.

Consider a d-dimensional quantum field theory (QFT) defined on a Lorentzian spacetime with coordinates $(t, \vec{x})$ and with a set of fundamental fields collectively denoted as $\varphi(t, \vec{x})$. It is a standard result that the path integral on a segment $-T < t < T$ with the fields constrained to equal $\varphi_-(\vec{x})$ at $t = -T$ and $\varphi_+(\vec{x})$ at $t = T$ produces the transition amplitude $\langle \varphi_+, T | \varphi_-, -T \rangle$. This amplitude can be used as a building block for the vacuum amplitudes. Namely one multiplies this expression with the vacuum wave functions $\langle 0 | \varphi_+, T \rangle$ and $\langle \varphi_-, -T | 0 \rangle$ and integrates over $\varphi_+(\vec{x})$ and $\varphi_-(\vec{x})$ to produce the amplitude $\langle 0, -T | 0, T \rangle$. (As we review below, one may switch on sources between $-T$ and $T$ in order to obtain nontrivial amplitudes as well.) The vacuum wave functions can be computed using a path integral corresponding to an infinite evolution in imaginary time. Indeed, such an infinite evolution, starting for example at $-T$, corresponds to a transition amplitude $\lim_{\beta \to \infty} \langle \varphi_-, -T | e^{-\beta H} | \Psi \rangle$ for some state $| \Psi \rangle$. By inserting a complete set of energy eigenstates one finds that taking the limit projects onto the vacuum wave function $\langle \varphi_-, -T | 0 \rangle$, times a factor $\langle 0 | \Psi \rangle$ which does not depend on $\varphi_-$ and therefore only contributes to the overall normalization. We similarly obtain $\langle 0 | \varphi_+, T \rangle$ from an infinite evolution in imaginary time starting at $t = T$. The insertion of these wave functions is therefore equivalent to extending the fields in the path integral to live along a contour in the complex time plane as sketched in figure 2.1a.

These wave function insertions ultimately lead to the standard $i \epsilon$ factors in correlation functions. Namely one can deform the contour to a straight line that runs almost parallel to the real axis with only a slight downward slope in the complex
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\[ T - T \]

\[ C_0 \quad C_1 \quad C_2 \quad C_3 \]

\[ (a) \quad (b) \quad (c) \]

Figure 2.1: (a) Vacuum-to-vacuum contour. (b) In-in contour. (c) Real-time thermal contour. The circles reflect points that should be identified. The horizontal segments are on top of each other in the complex time plane, although they are separated in the figure.

time plane. This deformation does not affect the aforementioned projection onto the ground state. The downward slope in the contour is equivalent to replacing \( t \to t(1 - i\epsilon) \). In appendix 2.A we review that this is precisely the correct \( i\epsilon \) factor for Feynman propagators and also present a simple example where one may explicitly compute the vacuum wave functions.

2.1.2 In-in formalism

Let us now consider real-time correlation functions in nontrivial states, for example

\[ \langle \Psi | O(t) | \Psi \rangle = \langle \Psi | e^{i\hat{H}t} O e^{-i\hat{H}t} | \Psi \rangle \quad (2.1) \]

for some state \( |\Psi\rangle \) which is not an eigenstate of the Hamiltonian \( \hat{H} \). The expression on the right-hand side demonstrates that we not only have to evolve the state \( |\Psi\rangle \) forward in time before we insert the operator, but afterwards we also have to evolve back in time before we insert the final wave function. This results in the in-in or ‘closed time path’ formalism of [52, 53, 54, 55]. Extending the contour to go beyond the point \( t \), say to some point \( t' \), and then back again amounts to an insertion of the identity operator in the form \( \exp[i\hat{H}(t' - t)] \exp[-i\hat{H}(t' - t)] \). Such an extension of the contour will not change the overall amplitude.

In the previous subsection we used the Euclidean path integral to create the vacuum state which is then fed into the Lorentzian path integral as the initial and final state. More generally, one can use the Euclidean path integral to generate other states that can serve as initial/final states for the Lorentzian path integral. Indeed, in the context of a conformal field theory on \( \mathbb{R}^d \) the relation between Euclidean path integrals and states is the basis for the operator-state correspondence: inserting a local operator \( O \), say at the origin of \( \mathbb{R}^d \), and then performing the path integral over the interior of the sphere \( S^{d-1} \) that surrounds the origin results in
the corresponding quantum state $|\Psi_O\rangle$ on $S^{d-1}$. In particular, the vacuum state is generated by inserting the identity operator.

We are thus led to a prescription for computing real-time correlation functions in a given initial state involving a closed time contour as sketched in figure 2.1b. In the figure, the vertical pieces $C_0, C_3$ represent Euclidean path integrals, with the crosses representing operator insertions. As described in the previous paragraph, these segments create the chosen initial state $|\Psi\rangle$ at a certain time. To compute real-time correlation functions one evolves this state forward and backward in time following the horizontal segments $C_1$ and $C_2$ and inserts operators on these segments.

The in-in formalism can also be applied to quantum field theory at a finite temperature $T$. The expectation values of gauge-invariant operators then become traces,

$$\langle O(t) \rangle_\beta = \text{Tr}(\hat{\rho} O(t)), \quad (2.2)$$

with $\hat{\rho} = \exp(-\beta \hat{H})$ the thermal density matrix and $\beta = 1/T$. By a repetition of the above arguments leading to the in-in contour, one finds that for real-time thermal correlators one can use the closed time path contour in figure 2.1c. The vertical segment now represents the thermal density matrix and has length $\beta$. The circles indicate points that should be identified and reflect the fact that thermal correlators satisfy appropriate periodicity conditions in imaginary time (bosonic/fermionic variables are periodic/antiperiodic). As in the discussion in the previous paragraph, one can compute real-time correlation functions by inserting operators on the horizontal segments. Other density matrices, for example a thermal density matrix with chemical potentials, may be obtained in a similar manner.

Besides the examples in figure 2.1 one may also consider more general deformations of the contour in the complex time plane. In fact, one may deform the contour into any other direction in complex coordinate space. Such deformations are allowed as long as the contour does not run upward in the complex time plane, since the spectrum of the operator $\exp(-i\hat{H}\Delta t)$ is bounded only for $\text{Im}(\Delta t) \leq 0$. Similar restrictions apply for deformations in other directions in complex coordinate space. In general, the ‘metric’ along such a deformed contour would be complex.

### 2.1.3 Generating functional

For all of the contours defined above one can write a generating functional of correlation functions of gauge-invariant operators in nontrivial states with the
following path integral representation:

$$Z_{\text{QFT}}[\phi^I_{(0)}; C] = \int_C [D\varphi] \exp \left( i \int_C dt \int d^{d-1}x \sqrt{-g_{(0)}} \left( \mathcal{L}_{\text{QFT}}[\varphi] - \phi^I_{(0)} \mathcal{O}^I[\varphi] \right) \right).$$

(2.3)

Here $\varphi$ denotes again collectively all QFT fields, $\phi^I_{(0)}$ are sources that couple to gauge invariant operators $\mathcal{O}^I$ and $g_{(0)ij}$ is the spacetime metric (and also the source for the stress energy tensor $T_{ij}$). The path integral is defined for fields living on the contour $C$ in the complex time plane. Therefore, we think of $t$ as an complex time coordinate and $\int_C dt$ is then a contour integral. The partition function also depends on the shape of this contour.

For the piecewise horizontal and vertical contours like those in figure 2.1 one may split up the integral along $C$ into a sum of integrals corresponding to the different segments of the contour. Let us exemplify this using the contour of figure 2.1b. The vertical segments $C_0$ and $C_3$ in the contour are associated with Euclidean path integrals, as discussed above. In these segments we can parametrize the contour using $t = -i\tau$ with $\tau$ a real coordinate along $C$ and this substitution indeed results in the usual signs for a Euclidean path integral. (We demonstrate this in an example in appendix 2.A.) The segments $C_1$ and $C_2$ in figure 2.1b form a closed time path. It can be parametrized using a 'contour time' coordinate $t_c$ that increases monotonically along the contour. In the segment $C_1$ we can simply set $t = t_c$, where now $t_c$ ranges from 0 to $T$ (where $T$ may be $\infty$), and we can integrate along $C_2$ using $t = 2T - t_c$, with $T < t_c < 2T$. Notice that $dt = -dt_c$ on $C_2$, which gives rise to an extra sign for the action on this segment.

For more general contours one finds an equivalent result. Instead of using a single complex $t$ coordinate as in (2.3) one may always choose to parametrize the contour using real coordinates like $\tau$ and $t_c$. The action in (2.3) then reduces to the Euclidean action along the vertical segments of the contour and to the Lorentzian action along the horizontal segments, with an extra minus sign for backward-going horizontal segments.

**Contour time-ordering**

Upon functional differentiation with respect to the sources one may obtain correlation functions from the path integral (2.3). From the usual slicing arguments that lead to the path-integral representation it follows that the operators in these correlation functions are **contour-time-ordered**. That is, if we pick a real contour time parameter $t_c$ that increases monotonically along the entire contour, then the operators in the correlation functions obtained from a path integral along this contour are ordered from small to large $t_c$. 

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To see the implications of contour time-ordering let us consider two-point functions for the in-in contour in figure 2.1b. Suppose that the source $\phi(0)$ in (2.3) is nonzero only along the Lorentzian segments $C_1$ and $C_2$. We parametrize this segment using the contour time coordinate $t_c$ introduced above. The source-operator coupling can then be split in two parts and the partition function is defined as:

$$Z[\phi(0)[1], \phi(0)[2]] = \langle \exp \left( -i \int_0^T d t_c d t_c^{-1} x \phi(0)[1] O[1] + i \int_T^{2T} d t_c d t_c^{-1} x \phi(0)[2] O[2] \right) \rangle,$$

(2.4)

where a subscript in square brackets denotes the segment on which the field lives. (We will adopt such a notation throughout this work.) The expectation values of course depend on the ensemble or state that is specified at $t = 0$, but we will not write this explicitly.

Via functional differentiation one obtains four possible two-point functions,

$$\langle T_c O[i](t, x) O[j](t', x') \rangle = (-1)^{\delta_{ij}} \frac{\delta^2 Z}{\delta \phi(0)[i](t, x) \delta \phi(0)[j](t', x')} \bigg|_{\phi(0)[1] = \phi(0)[2] = 0},$$

(2.5)

with $T_c$ denoting contour-time-ordering. Along the first segment contour-time-ordering coincides with normal time-ordering,

$$\langle T_c O[1](t, x) O[1](t', x') \rangle = \langle T O(t, x) O(t', x') \rangle.$$

(2.6)

Along the second, backward-running segment, contour-time-ordering coincides with anti-time ordering, denoted by $\bar{T}$,

$$\langle T_c O[2](t, x) O[2](t', x') \rangle = \langle \bar{T} O(t, x) O(t', x') \rangle.$$

(2.7)

If one puts one argument on the forward contour and the other on the backward contour, the latter one will always be later in contour time than the former and we get the Wightman functions:

$$\langle T_c O[1](t, x) O[2](t', x') \rangle = \langle O[2](t', x') O[1](t, x) \rangle = \langle O(t', x') O(t, x) \rangle,$$

$$\langle T_c O[2](t, x) O[1](t', x') \rangle = \langle O[2](t, x) O[1](t', x') \rangle = \langle O(t, x) O(t', x') \rangle.$$

(2.8)

Notice that the in-in path is therefore suitable to obtain vacuum-to-vacuum Wightman functions from a path integral as well.

## 2.2 Prescription

We are now ready to present the real-time gauge/gravity prescription. In the previous section we discussed that the computation of real-time correlation functions requires a proper specification of the initial and final state, which is often
implemented via a contour in the complex time plane. In particular, for vacuum amplitudes the specification of the contour led to the proper \( i\epsilon \) insertions in the correlation functions. Our starting point in the holographic description is that this contour dependence should be reflected in the bulk string theory, and in the low energy approximation it should be part of the supergravity description. Within the saddle-point approximation, our prescription is to associate supergravity solutions with QFT contours, or, more figuratively, to ‘fill in’ the entire QFT contour with a bulk solution. We have sketched several examples of such a construction in figure 3.1 on page 85.

One can think of the field theory contour \( C \) as a \( d \)-dimensional subspace of a complexified boundary spacetime. In most cases, as we saw above, this would be a line in the complexified time plane times a real space, \( \mathbb{R} \times \Sigma^{d-1} \). The corresponding bulk solution should have \( C \) as its conformal boundary and the bulk fields \( \Phi^I \) should satisfy boundary conditions parametrized by fields \( \phi^I(t, \vec{x}) \) living on \( C \). In particular this holds for the bulk metric \( G_{\mu\nu} \) whose boundary values should be given in terms of the boundary metric \( g_{(0)ij} \) on the various segments of the contour. Since the signature of \( G_{\mu\nu} \) is the same as that of \( g_{(0)ij} \), we see that horizontal segments of \( C \) will be filled in with Lorentzian solutions, while vertical segments will be filled in with Euclidean solutions.\(^1\) We denote the total bulk manifold consisting of all these segments as \( M_C \). The signature of the metric on \( M_C \) then jumps at certain ‘corner’ hypersurfaces where the solution corresponding to a vertical segment meets with a horizontal one. These hypersurfaces end on the corners of the contour. Below we show how appropriate matching conditions control the behavior of the fields at these hypersurfaces.

Note that the bulk manifold is not necessarily of the form \( \mathbb{R} \times X^d \) with \( \partial X^d = \Sigma^{d-1} \). Instead, we can have more general bulk solutions that may ‘interpolate’ between various parts of the contour. An important example is the eternal BTZ black hole we consider below.

Given such a solution \( M_C \) that fills in the entire field theory contour \( C \), the next step in the prescription is to compute the corresponding on-shell supergravity action. We may again write it as a single integral along a contour in complexified coordinate space:

\[
I[\phi_{(0)}; C] = \int_{M_C} d^{d+1}x \sqrt{-G} L_{\text{on-shell}}^{\text{bulk}}
\]

Just as in field theory one may pass from the complex coordinate \( t \) to a real contour time variable \( \tau \) or \( t_c \). The vertical segments of the contour then involve the

\(^1\)We use the word ‘Euclidean’ for solutions that are obtained after some form of Wick rotation in the equations of motion. Normally the metric on these solutions is positive definite and they should be more properly called ‘Riemannian’. We will see in later sections, however, that the ‘Euclidean’ solutions can also involve a complex metric.
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Euclidean on-shell action and horizontal segments the Lorentzian on-shell action, with factors of $i$ and signs becoming standard in this description. Notice that this discussion does not require that $t_c$ and $\tau$ extend globally on $M$, as the asymptotic analysis suffices to fix all signs.

The partition function of the quantum field theory is then given by a formula analogous to (1.31), namely:

$$Z_{QFT}[\phi(0); C] = \exp(iI[\phi(0); C]).$$

Equation (2.10) is a bare relation, since both the bare QFT partition function and the bare on-shell supergravity action are divergent. We described in chapter 1 the procedure of holographic renormalization which is needed to render a Euclidean on-shell supergravity action finite. In section 2.2.2 we will describe how this procedure can be extended to on-shell actions integrated along a contour as in (2.10).

Notice that the Euclidean bulk solution which is associated with the initial state on the QFT side can also be thought of as providing a Hartle-Hawking wave function [56] for the bulk theory. Thus our prescription is not only QFT inspired but also in line with standard considerations on wave functions in quantum gravity; see also [39, 44] for related discussions. There has been considerable discussion in the literature over the choice of contour in the Euclidean path integrals and the reality conditions of the semi-classical saddle point evaluation, see for example [57]. In our case, the bulk reality conditions are dictated by the boundary theory and, in particular, for a generic complex boundary contour the bulk manifold would have a complex metric (but in all cases the boundary correlators would satisfy standard reality conditions).

2.2.1 Corners

Piecewise straight contours have corners, where either a horizontal and a vertical segment meet or two horizontal segments join. These corners extend to hypersurfaces $S$ in the bulk. The signature of the metric changes at the hypersurface corresponding to a corner of a horizontal and a vertical segment, but otherwise it remains unchanged. Modulo subtleties at the boundary of $S$, which we discuss in the next section, we impose the following two matching conditions at $S$:
1. We impose continuity of the fields across $S$. That is, we require the induced metric, the values of the scalars, and induced values of the other fields to be continuous;

2. If the contour passes from a segment $M_-$ to $M_+$, then we impose appropriate continuity of the conjugate momenta across $S$:

$$\pi_- = \eta \pi_+ ,$$  \hspace{1cm} (2.11)

where $\pi_\pm$ denote collectively the conjugate momenta of all fields on the two sides $M_\pm$ of $S$ (defined using a real time coordinate like $t_c$ or $\tau$), and $\eta = -i$ when we consider a Euclidean to a Lorentzian corner like for example from $C_0$ to $C_1$ in figure 2.1b, whereas $\eta = -1$ if we have a (non-trivial) Lorentzian to Lorentzian corner as from $C_1$ to $C_2$ in figure 2.1b. In all cases, the matching condition is equivalent to

$$\hat{\pi}_+ = \hat{\pi}_- ,$$  \hspace{1cm} (2.12)

where $\hat{\pi}$ is defined using the complex time variable $t$. In other words, if we use analytic continuation of the fields in the complex $t$ coordinate to smooth out the corner by bending the contour, then the matching conditions dictate that the solution would be at least $C^1$. In chapter 3 we illustrate with examples how these matching conditions determine the bulk solution for a given contour.

The on-shell supergravity action can be regarded as the saddle point approximation of the ‘bulk path integral’ which can also be used to justify the matching conditions in the following way. Recall that a path integral for fields living on a certain manifold can always be split in two by cutting the manifold in two halves and imposing boundary conditions for the fields on the cut surface. Afterwards, one can glue the pieces back together by imposing the same boundary condition on either side and then integrate over these boundary conditions.

The saddle-point approximation can similarly be performed in steps. After cutting the manifold, one first finds a saddle-point approximation on either side with arbitrary initial data at the cut surface. This replaces the partial path integrals on either side by an on-shell action which in particular depends on the initial data. Then, one imposes continuity of the initial data, which is the first matching condition, and performs a second saddle-point approximation with respect to the initial data. Since the first variation of an on-shell action with respect to boundary data yields the conjugate momentum, this second saddle-point precisely yields (2.11). The matching conditions should then be viewed as an equation determining the initial data. One may verify that the signs come out right, too.
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2.2.2 Holographic renormalization

The fundamental holographic relation (2.10) is a bare relation because both sides are divergent: there are UV divergences on the QFT side and IR infinite volume divergences on the gravitational side. So appropriate renormalization is needed to make this relation well-defined. Below we will show that the procedure of holographic renormalization for the spaces under consideration is a priori more complicated, but that none of these complications enter in the final result. Therefore formulas like for example (1.117) on page 32 for the holographically computed one-point functions remain valid in the context of our real-time prescription as well. As the precise derivation of this result is not essential for the rest of the paper, the reader may wish to skip the remainder of this chapter on a first reading and proceed directly to the examples of chapter 3.

As we explained in the previous chapter, the holographic renormalization in the Euclidean case is done by introducing a set of local covariant boundary counterterms which render the on-shell action finite. In the Lorentzian setup, in addition to the infinities due to the non-compactness of the radial direction, there are also new infinities because of the non-compactness of the time direction. Correspondingly, in checking the variational problem one now has to deal both with boundary terms at spatial infinity and also at timelike infinity. Thus, in generalizing the Euclidean analysis to the Lorentzian case there are two issues to be discussed. First, one has to check that the Euclidean analysis that leads to the radial counterterms goes through when we move from Euclidean to Lorentzian signature. This is indeed the case because all steps involved in the derivation of the radial counterterms are algebraic and hold irrespectively of the signature of spacetime. The second issue one needs to analyze are the infinities due to the non-compactness of the time directions and the new boundary terms at timelike infinity. One possible approach to this issue would be to analyze the asymptotic structure of the solutions near timelike infinity, which as far as we know has not been performed. Our prescription bypasses this problem by gluing in Euclidean AdS manifolds near timelike infinity. This effectively pushes the asymptotic region to the (radial) boundary of the Euclidean AdS manifold, whose asymptotic structure is well known.

What remains to analyze is whether there are any problems at the ‘corners’, i.e. at the hypersurfaces where the Lorentzian and Euclidean solutions are joined. In principle, there can be new corner divergences which would require new counterterms. In the next two sections we show that such corner divergences are absent in two examples: a free massive scalar field in a fixed background and pure gravity. We expect such corner divergences to be absent in general.


2.3 Scalar field

This section serves to illustrate the holographic renormalization for the prescription (2.10), and we will therefore adopt the simplest possible setting. As indicated in figure 2.2, we consider here a single corner where the contour makes a right angle, passing from a vertical segment to a horizontal segment. The ‘space’ part of the boundary manifold is taken to be $\mathbb{R}^{d-1}$. In the absence of sources, we explain below how this contour can be ‘filled’ with empty AdS$_{d+1}$, with a metric that jumps along a spacelike hypersurface from Euclidean to Lorentzian. On this background, we consider a massive scalar field which propagates freely and without backreaction and we compute the renormalized one-point function of the dual operator.

2.3.1 Background manifold

For the bulk manifold under consideration, we take one copy $M_1$ of empty Lorentzian AdS$_{d+1}$ in the Poincaré coordinate system $(r, x^i)$ with the metric

$$ds^2 = dr^2 + e^{2r} \eta_{ij} dx^i dx^j,$$

and one copy $M_0$ of empty Euclidean AdS in similar coordinates and metric

$$ds^2 = dr^2 + e^{2r} \delta_{ij} dx^i dx^j. \tag{2.14}$$

We will take $x^0$ to be the time coordinate, denoting it by $t$ on $M_1$ and $\tau$ on $M_0$. We use the notation $x^a$ for the other boundary coordinates, so for example $x^i = (t, x^a)$ on $M_1$, and we also introduce $x^A = (r, x^a)$. The conformal boundaries of the spacetimes lie at $r \to \infty$ and are denoted $\partial_r M_1$ and $\partial_r M_0$.

Next, we perform the gluing and the matching. To this end, we cut off the space-times across the surface $t = 0$ and $\tau = 0$ such that $t > 0$ and $\tau < 0$, and glue them together along the cut surface which we call $\partial_t M$. This surface is the extension of the corner in the boundary to the bulk. The induced metric on $\partial_t M$ is the same on both sides,

$$h_{AB} dx^A dx^B = dr^2 + e^{2r} \delta_{ab} dx^a dx^b, \tag{2.15}$$

and the extrinsic curvature $K_{AB}$ vanishes on both sides. Therefore, both the conjugate momentum $\pi_{AB} = K h_{AB} - K_{AB}$ as well as the induced metric $h_{AB}$ are continuous across $\partial_t M$ and all the matching conditions of section 2.2.1 are satisfied for this background. (We elaborate on the matching conditions for gravity in the next subsection.) The unit normals to $\partial_t M$ on either side are given by

$$n_{[1]}^\mu dx^\mu = -e^r dt, \quad n_{[0]}^\mu dx^\mu = e^r d\tau. \tag{2.16}$$
2.3. Scalar field

\[ \partial_r M_0 \]
\[ \partial_r M_1 \]
\[ t \]

**Figure 2.2:** A single corner in the contour in the complex time plane. We use this part of a field theory contour to illustrate the holographic renormalization.

where we used subscripts in square brackets to indicate whether we are on \( M_1 \) or on \( M_0 \).

Notice that the contour of figure 2.2 is not complete, since there is no out state specified at the right end of the contour. This should be remedied, for example by gluing a Euclidean segment at \( t = T \) which would result in the vacuum-to-vacuum contour of figure 2.1a. In the bulk, this incompleteness means that we should also glue another solution to some ‘final’ hypersurface lying in \( M_1 \). To obtain the contour of figure 2.1a, for example, one should glue in half a Euclidean solution \( M_2 \). With \( \mathbb{R}^{d-1} \) as the spacelike manifold, this would result in a spacetime as sketched in figure 3.4a on page 98.

In this section we will focus on a single corner. We will therefore omit any contributions from such an \( M_2 \), as well as some terms defined on the final matching surface for \( M_1 \). Since the matching between \( M_1 \) and \( M_2 \) is a word-for-word repetition of the matching between \( M_0 \) and \( M_1 \), these terms can be easily reinstated.

### 2.3.2 Scalar field setup

In the background we just described we consider a scalar field \( \Phi \) of mass \( m \), dual to a scalar operator \( \mathcal{O} \) of dimension \( \Delta \) such that \( m^2 = \Delta(\Delta - d) \). We will consider the case where \( \Delta = d/2 + k \) with \( k \in \{1, 2, 3, \ldots \} \), and sometimes we will specialize to \( k = 2 \). The actions for \( \Phi \) on \( M_1 \) and \( M_0 \) are respectively given by:

\[
S_1 = \frac{1}{2} \int_{M_1} \sqrt{-G}(-\partial_\mu \Phi \partial^\mu \Phi - m^2 \Phi^2),
\]

\[
S_0 = \frac{1}{2} \int_{M_0} \sqrt{G}(\partial_\mu \Phi \partial^\mu \Phi + m^2 \Phi^2). \tag{2.17}
\]

Suppose \( \Phi \) is a solution on \( M_0 \) and \( M_1 \) of the equations of motion derived from these actions, with asymptotic value corresponding to the radial boundary data
and furthermore satisfies the aforementioned matching conditions (which we discuss in more detail below) on the gluing surface. Our aim is then to compute the corresponding on-shell action,

\[ iS_1 - S_0, \quad (2.18) \]

while using the method of holographic renormalization to make it finite. As we mentioned above, equation (2.18) can alternatively be written as:

\[ \frac{i}{2} \int_C dt \int dr \, d^{d-1}x \sqrt{-G} (-\partial_\mu \Phi \partial^\mu \Phi - m^2 \Phi^2), \quad (2.19) \]

with a path \( C \) in the complex time plane as in figure 2.2, which goes down at first (yielding \(-S_0\) after substituting \( t = -i\tau \)) and then makes a corner and lies along the real \( t \) axis. We will not use this notation in this example, but it will be relevant when we consider gravity below.

Let us repeat the essential ingredients of the holographic renormalization for the scalar field in Euclidean signature which we presented in section 1.5. The holographic renormalization relies on the fact that the solution \( \Phi \) can (both on \( M_0 \) and on \( M_1 \)) be written as a Fefferman-Graham expansion:

\[ \Phi = e^{(k-d/2)r}(\phi(0) + e^{-2r}\phi(2) + \ldots + e^{-2kr}[\phi(2k) + \tilde{\phi}(2k) \log e^{-2r}] + \ldots). \quad (2.20) \]

In this expansion, the radial boundary data is given by specification of \( \phi(0)(x^i) \). As one can verify using the equation of motion for \( \Phi \), the coefficients \( \phi(2n) \) with \( 2n < 2k \), as well as \( \tilde{\phi}(2k) \), are locally determined by \( \phi(0) \). For example, for \( k \neq 1 \), we find

\[ \phi(2) = \frac{\Box \phi(0)}{4(k-1)}, \quad (2.21) \]

with \( \Box \) the Laplacian of the boundary metric on \( \partial_r M \), which in the case at hand is either \( \eta_{ij} \) or \( \delta_{ij} \). Similarly, all coefficients \( \phi(2n) \propto \Box^n \phi(0) \) for \( n < k \) and \( \tilde{\phi}(2k) \propto \Box^k \phi(0) \), all with some \( k \)-dependent coefficients. The coefficient \( \phi(2k) \) is normally nonlocally determined by \( \phi(0) \), but in our case it also depends on the initial data that one may specify at \( \partial_r M \). As we demonstrated in equation (1.117) in the previous chapter, in Euclidean backgrounds without corners this coefficient (times a factor \(-2k\)) is precisely the renormalized one-point function. Below, we show this is still the case in Lorentzian signature and in the presence of corners.

### 2.3.3 Matching conditions

Let us first discuss the matching conditions of subsection 2.2.1 for the scalar field. Consider two solutions \( \Phi_1 \) and \( \Phi_0 \) on \( M_1 \) and \( M_0 \) that satisfy the given radial
boundary data, but have arbitrary initial data. The first matching condition is continuity of \( \Phi \) across \( \partial_\mathcal{L} \), that is:

\[
\Phi_0(\tau = 0, r, x^a) = \Phi_1(t = 0, r, x^a) \tag{2.22}
\]

for all \( r \) and \( x^a \).

To derive the second matching condition, we compute the on-shell action for \( \Phi_0 \) and \( \Phi_1 \) satisfying the equation of motion and the first condition (2.22). This action is divergent and we regulate it by cutting off the radial integrals at some large but finite \( r_0 \). We then consider the variation of the regulated version of the total action (2.18) as we vary the initial data \( \Phi(t = 0, r, x^a) \) and obtain

\[
\delta(iS_1 - S_0) = \int_{\partial_\mathcal{L}} \sqrt{he^{-r}( -i\partial_t \Phi_1 - \partial_\tau \Phi_0)} \delta \Phi_1 , \tag{2.23}
\]

where we used that \( \delta \Phi_1 = \delta \Phi_0 \) by (2.22). As explained in section 2.2, we then request that the total action is also at an extremum with respect to the initial data. The second matching condition thus becomes:

\[
i\partial_t \Phi_1 + \partial_\tau \Phi_0 = 0 \quad \text{on} \quad \partial_\mathcal{L}. \tag{2.24}
\]

As we mentioned before (and as one may check easily using \( t = -i\tau \)), this second matching condition can be read as \( C^1 \)-continuity in the complex time plane of \( \Phi \) across the corner. In the remainder of this section, whenever we write \( C^m \)-continuity, we always mean continuity in the complex time plane.

Now let us substitute the Fefferman-Graham expansion (2.20) of \( \Phi_1 \) and \( \Phi_2 \) in the matching conditions (2.22) and (2.24). The matching conditions imply the \( C^1 \)-continuity of all coefficients \( \phi(2l) \), which, in turn, implies higher-order continuity of the source \( \phi(0) \). For example, the first matching condition for \( \phi(2) \) becomes, via (2.21),

\[
\Box_{[1]} \phi(0)[1] = \Box_{[0]} \phi(0)[0] \quad \text{on} \quad \partial_\mathcal{L} , \tag{2.25}
\]

which shows that \( \phi(0) \) has to be at least \( C^2 \)-continuous across the matching surface. Notice that this is again continuity in the complex time plane, since \( \Box_{[1]} \) is not equal to \( \Box_{[0]} \). Next, the second matching condition applied to \( \phi(2) \) actually implies \( C^3 \)-continuity for \( \phi(0) \):

\[
i\partial_t \Box_{[1]} \phi(0)[1] + \partial_\tau \Box_{[0]} \phi(0)[0] = 0 . \tag{2.26}
\]

A similar story holds for the subsequent terms. Since the highest number of derivatives is always in \( \phi(2k) \propto \Box^k \phi(0) \), applying the second matching condition to this term results eventually in a \( C^{2k+1} \)-continuity condition for \( \phi(0) \) in the complex time plane. Below, we will see the relevance of these high-order continuity conditions.
A comment considering this smoothness condition for \( \phi(0) \) is in order. Namely, this continuity condition essentially follows from the requirement of the existence of a Fefferman-Graham expansion at the matching surface. In that light, this higher-order smoothness condition for \( \phi(0) \) is not surprising, since without it the Fefferman-Graham expansion would fail even in the case without a corner. Although it would be interesting to study what happens for discontinuous boundary data, such an investigation can be undertaken independently of the presence of corners and shall not be pursued here.

### 2.3.4 Holographic renormalization

The on-shell action (2.18), evaluated on the solution that satisfies the matching conditions, is of the form:

\[
iS_1 - S_0 = -\frac{i}{2} \int_{\partial_r M} \sqrt{-\gamma} \Phi_1 \partial_r \Phi_1 - \frac{1}{2} \int_{\partial_r M} \sqrt{\gamma} \Phi_0 \partial_r \Phi_0 - \frac{1}{2} \int_{\partial_t M} \sqrt{h}[\partial_t \Phi_1 \partial_\tau \Phi_1 + \Phi_0 \partial_\tau \Phi_0].
\]

(2.27)

The contributions from \( \partial_t M \), i.e. the second line in (2.27), vanish by virtue of the matching conditions. Recall that we are omitting the contribution from any ‘final’ surface for \( M_1 \), which will however by the same mechanism cancel against a matching solution.

The remainder of the action is defined on the cutoff surface \( r = r_0 \) and it would diverge if \( r_0 \to \infty \). Therefore, a counterterm action has to be added before removing the cutoff. Since the radial terms in (2.27) have a familiar form, one can use precisely the same procedure of holographic renormalization as in section 1.5 to find the counterterm action. In particular, we found the counterterm action (1.112) for a scalar field with general \( k \). Let us focus on the case \( k = 2 \), for which

\[
S_{ct} = \frac{1}{2} \int_{\partial_r M} \sqrt{|\gamma|} \left( (k - \frac{d}{2})\Phi^2 + \frac{\Phi \Box_\gamma \Phi}{2(1 - k)} + \frac{1}{4} \Phi \Box^2 \Phi \log e^{-r} \right)
\]

(2.28)

is the explicit counterterm action. The first two terms are actually valid for any \( k \geq 2 \) and we used the notation \( \Box_\gamma \) for the Laplacian of the induced metric \( \gamma \) at \( r = r_0 \). In our case, we simply have \( \Box_\gamma = e^{-2r} \Box_0 \), both on \( M_1 \) and on \( M_0 \). Taking care of the signs, we find that

\[
iS_1 - S_0 + iS_{ct,1} + S_{ct,0}
\]

(2.29)

is finite as \( r_0 \to \infty \). We see that the usual procedure of holographic renormalization yields a finite on-shell action and possible initial or final terms (which might have caused corner divergences) are absent exactly by the matching conditions.
2.3.5 One-point functions

One-point functions are computed by taking variational derivatives of the on-shell action with respect to the boundary data. Let us compute the one-point function $\langle O_{[1]}(x) \rangle$, where the subscript indicates that $x$ lies on $\partial_r M_1$. In QFT on a background with a Lorentzian metric $g(0)_{ij}$, the coupling between a source $\phi(0)$ and an operator $O$ in the partition function is as in (2.3). Therefore, the one-point function is

$$\langle O_{[1]}(x) \rangle = \frac{i}{\sqrt{-g(0)}} \frac{\delta}{\delta \phi_{[1]}(x)} Z[\phi(0)].$$

(2.30)

In our case, the partition function $Z[\phi(0)]$ is given by the renormalized on-shell supergravity action. The easiest way to take care of the divergences is by taking the functional derivative before removing the regulator, resulting in:

$$\langle O_{[1]}(x) \rangle = \lim_{r_0 \to \infty} i e^{(k+\frac{d}{2})r_0} \frac{\delta}{\delta \Phi_1(x,r_0)} \left[ iS_1 - S_0 + iS_{ct,1} + S_{ct,0} \right].$$

(2.31)

where the extra factor $e^{(k+\frac{d}{2})r_0}$ converts $\Phi$ to $\phi(0)$ and $\gamma$ to $g(0)$ as $r_0 \to \infty$.

In performing this computation, we see that the presence of corners gives rise to corner terms, which arise from the integration by parts that is necessary in varying the counterterm action (2.28). For example, for the variation of the second term in (2.28) we obtain:

$$\delta \left( \frac{1}{2} \int_{\partial_r M} \sqrt{\gamma} \frac{\Phi_{\gamma} \Phi_{\gamma}}{2(1-k)} \right) = \int_{\partial_r M} \sqrt{\gamma} \frac{\delta \Phi_{\gamma} \Phi_{\gamma}}{2(1-k)} + \frac{1}{2} \int_{C_1} \sqrt{\sigma} e^{-2r} (\partial_t \Phi \delta \Phi - \Phi \partial_t \delta \Phi) \frac{1}{2(1-k)}.$$  

(2.32)

The second term on the right hand side is a corner contribution. However, a similar corner term arises in $S_{ct,0}$, and in the total action (2.29) these two corner terms cancel each other precisely by the matching conditions.

The subsequent terms in the counterterm action are all of the form $\sqrt{\gamma} \Phi^{n} \Phi$ for $n < k$, plus a log term of the form $\sqrt{\gamma} \Phi^{k} \Phi \log e^{-r_0}$. After the integration by parts, these all give corner terms as well, which involve a higher number of derivatives of $\Phi$. More precisely, the corner expressions that one obtains from such terms are of the form

$$\int_{C} \sqrt{\gamma} e^{-2r} \Phi \partial_t \Phi^{n-1} \Phi,$$

(2.33)

and equivalent terms with some of the derivatives shifted to the first $\Phi$.

Let us now systematically show that all such terms cancel against a matching solution, using the higher-order smoothness of $\phi(0)$ that we derived before. First of
all, recall that the matching conditions imply that $\phi(0)$ should actually be $C^{2k+1}$-continuous. This in turn means that $\phi(2)$ is $C^{2k-1}$ continuous, $\phi(4)$ is $C^{2k-3}$-continuous, etc., up to $\tilde{\phi}(2k)$ and $\phi(2k)$, which are just $C^1$-continuous. Substituting this in the Fefferman-Graham expansion (2.20), we see that $\Phi$ is not only $C^1$-continuous by the matching conditions, but also $C^3$-continuous up to terms of order $e^{-(k+d/2-2)n}r$, and $C^5$ continuous up to terms of order $e^{(-k-d/2+2)n}r$, etc.

We now rewrite the leading piece of (2.33) as

$$\int_C e^{(k+d/2-2n)r} \sqrt{g^{(0)}(0)\phi^{(0)}(0)} \partial_t \Box^{n-1} \Phi + \ldots \quad \text{(2.34)}$$

A complete cancellation of this term between $M_1$ and $M_0$ takes place if $\Phi$ is $C^{2n-1}$-continuous up to and including terms of order $e^{-(k+d/2-2n)n}r$. However, the previous argument shows that $C^{2n-1}$-continuity for $\Phi$ holds up to terms of order $e^{-(k+d/2-2n+4)n}r$, and the continuity condition is satisfied indeed, for all $n < k$. Therefore, as $r_0 \to \infty$, the terms coming from $M_0$ and $M_1$ cancel indeed and no corner contributions to the one-point functions arise. A similar argument shows that there is no problem with the log term with $n = k$ either.

Having shown the absence of corner contributions in (2.31), one finds that the expression for the one-point function becomes of the standard form, given for example for $k = 2$ by:

$$\langle O[1](x) \rangle = \lim_{r_0 \to \infty} e^{k+d/2r_0} \left[ \partial_r \Phi(x) - (k - d/2) \Phi(x) - \frac{\Box \gamma \Phi(x)}{2(1 - k)} - \frac{1}{2} \Box^2 \gamma \Phi(x) \log e^{-2r} \right]_{r = r_0} \quad \text{(2.35)}$$

Substitution of the expansion (2.20) yields:

$$\langle O[1](x) \rangle = -2k \phi(2k)[1](x) \quad \text{(2.36)}$$

which is precisely the same result as (1.117) in section 1.5. Equation (2.36) is actually valid for all nonzero $k$.

Finally, consider the one-point function on $M_2$, where we should use the Euclidean version of the source-operator coupling, $-\int \sqrt{g(0)\phi(0)}[0] O[0]$. Repeating the above procedure, we find again exactly the same result as in (1.117):

$$\langle O[0](x) \rangle = -2k \phi(2k)[0](x) \quad \text{(2.37)}$$

Since $\phi(2k)(x)$ is continuous across the matching surface by the first matching condition, and since localized corner terms are absent, the one-point function is continuous across the corner as well.
2.4 Gravity

For gravity the procedure requires modification and becomes more involved. We therefore begin with an outline of the steps taken below.

The first step is to establish the variational principle for the Einstein-Hilbert action for a manifold whose boundary has corners. Recall that in the Euclidean setup a well-defined variational problem requires the addition of the boundary counterterms [30] and the variational derivatives w.r.t. boundary data lead to the boundary correlators. In the Lorentzian setup the variational derivatives w.r.t. initial and final data are also important and lead to matching conditions. The analysis of the variational problem is done in subsection 2.4.2. We will find that there is a need for a special corner term.

The next step is to understand how to glue the various pieces together. Given a corner in the boundary contour there should exist a corresponding bulk hypersurface across which the various bulk pieces are matched. So we need to understand the possible bulk extensions of the boundary contour. This is analyzed in subsection 2.4.3 where we show that the extensions are parametrized by a single function $f(r, x^a)$ with a certain asymptotic expansion.

Using these results we then derive the matching conditions in subsection 2.4.4 and find their implications for the radial expansion of the bulk fields near the corner in 2.4.5. These are all the data we need to analyze whether there are any new contributions to the on-shell action and the one-point function from the matching surfaces. This is done for the on-shell action in subsection 2.4.6 and for the 1-point functions in subsection 2.4.7. We find that there are possible contributions from each segment but the matching conditions imply complete cancellation between the contributions of the two pieces that one glues to each other.

The upshot of the discussion is therefore very similar to the scalar field: we will show that no localized corner terms arise and that the one-point function of the stress energy tensor is (appropriately) continuous across the corner.

2.4.1 Setup

As we mentioned earlier, we consider manifolds $M_C$ consisting of a number of segments $M_j$ where the metric is Lorentzian or Euclidean. To simplify the computation of the on-shell action for these spacetimes, we use the notation involving the complex coordinate $t$ as discussed in section 2.2. In this notation the Einstein-
Hilbert action $S_j$ for each separate segment $M_j$ is always written as

$$S_j = \frac{1}{2\kappa^2} \int_{M_j} d^{d+1}x \sqrt{-G}(R - 2\Lambda),$$

(2.38)

where $\kappa^2 = 8\pi G_{d+1}$ and $\Lambda = -d(d-1)/(2\ell^2)$ with $\ell$ the AdS radius. Throughout this chapter, we set $\ell = 1$. In (2.38) the square root is defined with a branch cut just above the real axis. For example, we obtain a Euclidean metric using $t = e^{-i\pi/2}\tau$, so $\sqrt{-G} = \sqrt{e^{-i\pi}|G|}$ which with our choice of branch cut becomes $-i\sqrt{|G|}$. This leads to $iS = -S_E$ with $S_E = \int \sqrt{|G|}(-R + 2\Lambda)$ the correct Euclidean action. Similarly, for a Lorentzian metric on a backward-going contour we obtain an extra minus sign since we are on the other branch of the square root. (To see this, notice that the time coordinate $t_c$ on this segment is given by $t = e^{i\pi}t_c$. If $G,G_c$ denote the metric determinant in the $t,t_c$ coordinate system, respectively, then $G_c = e^{2\pi i}G$, and we make a full turn indeed.) The advantage of this formalism is that the total Einstein-Hilbert action $S_{EH}$ for $M_C$ becomes

$$iS_{EH} = iS_0 + iS_1 + \ldots$$

(2.39)

for all vertical or horizontal segments $M_0, M_1, \ldots$ We see that all the signs are absorbed in the volume element. This action for $M_C$ needs to be supplemented with various surface terms which we define in due course.

Although we will not discuss this in detail, this prescription can be extended to general complex metrics, allowing for the ‘filling’ of more general QFT contours that are not just built up from horizontal and vertical segments in the complex time plane. In such cases the bulk metric $G_{\mu\nu}$ may be complex, but it should always be non-degenerate for the scalar curvature to be well-defined. Allowing for a complex metric implies that one has to allow for complex diffeomorphisms as well, for example to bring the metric to a Fefferman-Graham form. Complex diffeomorphisms are discussed in some detail in [57]. For such cases, our choice for the branch cut in the volume element is then precisely consistent with the requirement that a QFT contour cannot go upward in the complex time plane.

### 2.4.2 Finite boundaries

In equation (2.39), we split the on-shell action for $M_C$ as a sum over the various segments $M_i$. Just as for the scalar field, we will find the matching conditions via a saddle-point approximation which involves taking functional derivatives of the on-shell action with respect to the initial and final data. This only works if we have a well-defined variational principle for each segment separately, which is what we investigate in this subsection.
Consider a single Asymptotically locally AdS (AlAdS) manifold $M$ with a (possibly complex) metric $G_{\mu\nu}$ and two ‘initial’ and ‘final’ boundaries which we denote here as $\partial_{\pm} M$. The manifold $M$ also has a radial conformal boundary, which we denote as $\partial_r M$, and the corners where $\partial_{\pm} M$ meets $\partial_r M$ are denoted as $C_{\pm}$. We pick coordinates $(r, x^i)$ on $M$, with $x^i = (t, x^a)$, and we will also use $x^A = (r, x^a)$. The conformal boundary is again at $r \to \infty$. We regulate the computation of the on-shell action by imposing $r < r_0$. In this subsection we consider the variational principle in the case where one keeps $r_0$ finite throughout.

A well-defined variational principle for Dirichlet boundary conditions in the presence of corners requires the Einstein-Hilbert action to be supplemented not only with the usual Gibbons-Hawking boundary terms on $\partial_r M$ and $\partial_{\pm} M$, but also with special corner terms defined on $C_{\pm}$ [58, 59, 60]. To find these corner terms, we choose coordinates such that $\partial_r M$ is given by $r = r_0$ and $\partial_{\pm} M$ by $t = t_{\pm}$. The metric near the corners can be put in the following two ADM-forms:

$$G_{\mu\nu} dx^\mu dx^\nu = (\hat{H}^2 + \hat{H}_i \hat{H}^i) dr^2 + 2 \hat{H}_i dx^i dr + \hat{\gamma}_{ij} dx^i dx^j, \tag{2.40}$$

$$\hat{\gamma}_{ij} dx^i dx^j = (-\hat{M}^2 + \hat{M}_a \hat{M}^a) dt^2 + 2 \hat{M}_a dx^a dt + \sigma_{ab} dx^a dx^b,$$

as well as

$$G_{\mu\nu} dx^\mu dx^\nu = (-M^2 + M_A M^A) dt^2 + 2 M_A dx^A dt + h_{AB} dx^A dx^B, \tag{2.41}$$

$$h_{AB} dx^A dx^B = (H^2 + H_a H^a) dr^2 + 2 H_a dx^a dr + \sigma_{ab} dx^a dx^b.$$

Relating the two metrics, we find

$$H^2 = \frac{\hat{H}^2 M^2}{M^2 + (Mr)^2 \hat{H}^2}, \quad \hat{H}_t = M_r \tag{2.42}$$

$$\hat{M}^2 = M^2 - (Mr)^2 H^2, \quad -\frac{Mr}{M^2} = \frac{\hat{M}^2}{\hat{H}^2}.$$

For a real Lorentzian metric $M^2$ and $\hat{M}^2$ are positive, whereas they are negative for a Euclidean metric. We will henceforth assume that $\sigma$, the determinant of $\sigma_{ab}$, is real and positive. This will simplify the discussion and is sufficient for all the examples below.

The standard Gibbons-Hawking-York surface terms involve the extrinsic curvature $\pm K_{AB}$ of $\partial_{\pm} M$ and $\hat{K}_{ij}$ of $\partial_r M$, which we will define using the (possibly complex) unit normals,

$$\partial_r M : \hat{n}_\mu dx^\mu = \frac{\sqrt{-\hat{G}}}{\sqrt{-\hat{\gamma}}} dr \quad \rightarrow \quad \hat{K}_{ij} = \frac{\sqrt{\hat{H}^2 M^2}}{2 \sqrt{M^2 \hat{H}^2}} (\hat{D}_i \hat{H}_j + \hat{D}_j \hat{H}_i - \partial_r \hat{\gamma}_{ij}), \tag{2.43}$$

$$\partial_{\pm} M : \pm n_\mu dx^\mu = \pm \frac{\sqrt{-G}}{\sqrt{h}} dt \quad \rightarrow \quad \pm K_{AB} = \pm \frac{\sqrt{H^2 M^2}}{2 \sqrt{H^2 M^2}} (D_A M_B + D_B M_A - \partial_t h_{AB}).$$

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Adding the Gibbons-Hawking-York terms, we define the bare action as:

\[ S_b = \frac{1}{2\kappa^2} \left[ \int d^{d+1}x \sqrt{-G} (R - 2\Lambda) + 2 \int_{\partial \pm M} d^d x \sqrt{\bar{n}} \pm \mathcal{K} + 2 \int_{\partial_r M} d^d x \sqrt{-\bar{\gamma}} \hat{K} \right], \quad (2.44) \]

where here and below the summation over \( \partial \pm M \) is implicit and we use the conventions given above for the square roots. For a real Lorentzian metric all the above terms are real, but for a real Euclidean metric all terms in (2.44) are purely imaginary (because \( \sqrt{-G} \) and \( \sqrt{-\gamma} \) are then imaginary, from which it follows that \( \pm n_\mu dx^\mu \) and therefore \( \pm \mathcal{K} \) are imaginary as well). As one may verify explicitly, in the latter case our choice of branch cut for the square roots in the volume elements implies that \( iS_b = -S_E \) with \( S_E \) the Euclidean action with the correct Gibbons-Hawking-York terms.

In the case of corners, (2.44) is not the correct action to use for Dirichlet boundary conditions. This is because we cannot perform a diffeomorphism at the corner mixing \( t \) and \( r \) without changing the definition of the two slices and therefore \( \hat{H}, M_r, M^2 \) and \( \hat{H}^2 \) are no longer pure gauge at the corner. With this in mind, the variation of the bare action (2.44) is given by the equations of motion, the conjugate momenta for all the various boundaries, plus a corner term

\[ \delta S_b = \frac{1}{2\kappa^2} \int_{C_\pm} d^{d-1}x \sqrt{\sigma} \delta X_\pm + \ldots, \quad (2.45) \]

with \( X \) given implicitly by

\[ \delta X_\pm = \pm 2 \sqrt{H^2 M^2} \delta M^r. \quad (2.46) \]

To find an explicit form of \( X_\pm \), we have to integrate \( \delta X \) for fixed \( \hat{M}^2 \) and \( H^2 \), using the relations (2.42). If the metric is completely real and \( H^2 \) and \( \hat{M}^2 \) are positive, then we find

\[ \delta X_\pm = \pm 2 \delta \arcsinh \left( \frac{H M^r}{\hat{M}} \right), \quad (2.47) \]

whereas if \( \hat{M}^2 \) is negative and \( H^2 \) and \( M^r \) are positive we get

\[ \delta X_\pm = \mp 2i \delta \arccos \left( \frac{H M^r}{\sqrt{-\hat{M}^2}} \right). \quad (2.48) \]

We can rewrite these expressions in a covariant form using the unit normals defined in (2.43). Their inner product is given by:

\[ \pm n^\mu \hat{n}_\mu = \pm \frac{\sqrt{H^2}}{\sqrt{M^2}} M^r. \quad (2.49) \]
2.4. Gravity

For real $M^2$, $\sqrt{H^2}$ and $\dot{M}^2$, we can therefore write without branch cut ambiguities:

$$X_\pm = \begin{cases} 
2 \arcsinh(\pm n^\nu \hat{n}_\mu) & \dot{M}^2 > 0 \\
-2i \arcsin(\pm n^\nu \hat{n}_\mu) & \dot{M}^2 < 0.
\end{cases} \quad (2.50)$$

In the more general case, the required corner term has the same structure but one needs to be careful about the branch cuts. Notice that $X$ is defined up to a local piece, for example a constant.

Following [58, 59, 60], we aim for a variational principle that is well-defined for a fixed induced metric on the boundaries, i.e. for fixed $\hat{\gamma}_{ij}$ and $h_{AB}$. In that case, we should add a corner term to cancel the unwanted variation $\delta X$ in (2.45). Such a corner term is given by

$$S_{C_\pm} = -\frac{1}{2\kappa^2} \int_{C_\pm} d^{d-1}x \sqrt{\sigma} X_\pm. \quad (2.51)$$

Adding corner terms to the action (2.44) defines an improved (but still bare) action $S_I$,

$$S_I = S_b + S_{C_\pm} = \frac{1}{2\kappa^2} \left[ \int_{\partial_r M} d^{d+1}x \sqrt{-\hat{g}} (R - 2\Lambda) + 2 \int_{\partial_{\pm} M} d^{d}x \sqrt{h} \pm K \right] + 2 \int_{\partial_r M} d^{d}x \sqrt{-\hat{g}} \hat{K} - \int_{C_\pm} d^{d-1}x \sqrt{\sigma} X_\pm, \quad (2.52)$$

whose variation is of the form

$$\delta S_I = \frac{1}{2\kappa^2} \left[ \int_{\partial_r M} \sqrt{-\hat{g}} (\hat{\gamma}^{ij} \hat{K} - \hat{K}^{ij}) \delta \hat{\gamma}_{ij} + \int_{\partial_{\pm} M} \sqrt{h} (h^{AB} \pm K - \pm K^{AB}) \delta h_{AB} \\
- \int_{C_\pm} d^{d-1}x \delta (\sqrt{\sigma}) X_\pm \right], \quad (2.53)$$

which is the correct variation for Dirichlet boundary conditions indeed. We will henceforth use this improved action as the bare action and drop the subscript $I$.

2.4.3 Fefferman-Graham coordinates

The above discussion was valid for a general spacetime whose boundary has corners. Since we are interested in $\text{AdS}$ spacetimes where the metric diverges near the radial boundary, we will run into divergences as we let $r_0 \to \infty$. To investigate these divergences, we pick a coordinate system in which the metric is of the Fefferman-Graham form,

$$ds^2 = dr^2 + \gamma_{ij} dx^i dx^j, \quad (2.54)$$
with the radial expansion

\[ \gamma_{ij} = e^{2r}(g_{(0)ij} + e^{-2r}g_{(2)ij} + \ldots + e^{-dr}[g_{(d)ij} + \tilde{g}_{(d)ij} \log e^{-2r}] + \ldots). \]  

(2.55)

As we explained in section 1.6, from the Einstein equations we find that all coefficients \( g_{(2n)ij} \) with \( 2n < d \), as well as \( \tilde{g}_{(d)ij} \), are locally determined by \( g_{(0)ij} \) and involve up to \( 2n \) or \( d \) derivatives of \( g_{(0)ij} \). The term \( g_{(d)ij} \) is not locally determined (except for its trace and its divergence) and this term directly enters in the one-point function of the stress energy tensor (see equation (1.169) for \( d = 2 \) and [27] for the explicit expressions in other dimensions).

The disadvantage of the Fefferman-Graham form of the metric is that one can generally no longer pick a coordinate \( t \) such that the surfaces \( \partial_{\pm}M \) are given by slices of constant \( t \). On the other hand, one can use the leftover gauge freedom to make sure that \( \partial_{\pm}M \) are asymptotically given by:

\[ \partial_{\pm}M : t = f_{\pm}(r, x^a), \]  

(2.56)

with

\[ \lim_{r \to \infty} f_{\pm}(r, x^a) = t_{\pm} \]  

(2.57)

and \( t_{\pm} \) constants. We will discuss the asymptotic behavior of \( f_{\pm} \) more precisely below.

Let us consider a single initial or final boundary. Dropping for now the subscript \( \pm \), we write an ADM-decomposition of \( \gamma_{ij} \) near the corner:

\[ \gamma_{ij} dx^i dx^j = (-N^2 + N^a N_a) dt^2 + 2 N_a dtdx^a + \tau_{ab} dx^a dx^b. \]  

(2.58)

We may pick boundary Gaussian normal coordinates centered at the corner, so that \( N_a \sim O(1) \). Furthermore, \( N^2 = e^{2r} N_{(0)}^2 + N_{(2)}^2 + \ldots \) and \( \tau_{ab} = e^{2r} \tau_{(0)ab} + \tau_{(2)ab} + \ldots \). We can relate this ADM-decomposition to the double ADM-decomposition (2.41) of the previous subsection by introducing a new coordinate

\[ t' = t - f(r, x^a), \]  

(2.59)

after which the initial slice is given by \( t' = 0 \). In the new coordinates, the metric is of the form (2.41), with \( t \) replaced by \( t' \), and with the components

\[
\begin{align*}
-M^2 + M_A M^A &= -N^2 + N_a N^a \\
M_r &= (-N^2 + N_a N^a) \partial_r f \\
M_a &= N_a + (-N^2 + N_c N^c) \partial_a f \\
H^2 + H_a H^a &= 1 + (-N^2 + N_a N^a)(\partial_r f)^2 \\
H_a &= (-N^2 + N_c N^c) \partial_a f \partial_r f + N_a \partial_r f \\
\sigma_{ab} &= \tau_{ab} + (-N^2 + N_c N^c) \partial_a f \partial_b f + N_a \partial_b f + N_b \partial_a f,
\end{align*}
\]  

(2.60)
where indices are raised with the appropriate metric. We use these equations below to write down a radial expansion of the components on the left-hand side in terms of the Fefferman-Graham expansion and a radial expansion of $f$.

For AlAdS spacetimes the Dirichlet boundary data are given by $g_{(0)ij}$ and $h_{AB}$. Asymptotically, $g_{(0)ij}$ determines a Fefferman-Graham radial coordinate as well as the subleading coefficients up to $g_{(d)ij}$ in the Fefferman-Graham expansion of the metric. Of course, the initial and final metric $h_{AB}$ should be such that $\partial_{\pm} M$ can be embedded in the asymptotic metric dictated by $g_{(0)ij}$ and this condition constrains the asymptotic form of $h_{AB}$. To be precise, $h_{AB}$ should have a radial expansion that is compatible with the last three equations in (2.60) for a certain $f$. However, as long as $f$ is unspecified, $h_{AB}$ is not to any order determined in terms of $g_{(0)ij}$.

We remark that the last three lines in (2.60) signify constraints on $h_{AB}$ only. Therefore, they are different from the usual constraints on the initial data in a Hamiltonian formalism of general relativity, which also involve the extrinsic curvature. These usual constraints are satisfied if the extrinsic curvature of the initial slice is computed using the embedding of the initial slice as a hypersurface in the solution. Therefore, they are automatically satisfied if we compute the extrinsic curvature using the first three lines of (2.60). Since this is how we compute the extrinsic curvature below, we will not worry about these constraints.

### 2.4.4 Gluing and matching conditions

In the previous subsections, we found an improved action (2.52) and discussed the Fefferman-Graham expansion for a single AlAdS spacetime with corners. We now take two of such spacetimes and glue them together along the initial and final hypersurfaces $\partial\pm M$.

We will denote the two segments by $M_0$ and $M_1$ and we glue $\partial_+ M_0$ to $\partial_- M_1$, which we from now on we denote as $\partial_t M$. The corner, i.e. the intersection of $\partial_0 M$ with $\partial_t M_0$ and $\partial_t M_1$, is denoted by $C$. As before, a subscript (sometimes in square brackets) indicates the manifold under consideration. We make no assumptions about the signature of the metric on $M_0, M_1$ and in fact the metric may even be complex. We write the total action as

$$iS_0 + iS_1 ,$$

with the individual actions given by (2.52). We recall that we use the conventions of subsection 2.4.1, so extra factors of $i$ might be included in the volume elements and extrinsic curvatures. As we did for the scalar field, we will henceforth ignore
the contribution from other segments than $M_0$ and $M_1$ as well as the contribution to the on-shell actions of $M_0$ and $M_1$ that may arise from other matching surfaces. Let us now find the precise matching conditions that the metrics on $M_0$ and $M_1$ have to satisfy near $\partial_t M$. The first matching condition is continuity of the initial and final Dirichlet data. For gravity, this becomes continuity of the induced metric:

$$ h_{[0],AB} = h_{[1],AB}. \hspace{1cm} (2.62) $$

The second matching condition is obtained from the variation of the on-shell regularized action with respect to the data on $\partial_t M$. Let us first suppose the variation vanishes at the corner $C$. In that case, we read off from (2.53) that the second matching condition becomes:

$$ \mathcal{K}_{[0],AB} + \mathcal{K}_{[1],AB} = 0. \hspace{1cm} (2.63) $$

We can also consider a variation that does not vanish at $C$, for which (2.53) shows that

$$ (X_{[1]} + X_{[0]}) \delta(\sqrt{\sigma}) = 0, \hspace{1cm} (2.64) $$

where we included the $\delta(\sqrt{\sigma})$ because of the following reason. Notice that this is a corner matching condition which is therefore not valid to all orders in $r$. However, since $\sqrt{\sigma} \sim e^{-(d-1)r}$, (2.64) is actually divergent as $r_0 \to \infty$. Therefore, it only vanishes completely if the $X$’s match to high order in their radial expansion. If there are no log terms, then we find

$$ X_{[1]} + X_{[0]} = O(e^{-dr}). \hspace{1cm} (2.65) $$

Equation (2.65) is the corner analogue of the bulk matching condition (2.63). Notice that such a corner condition was absent when we discussed the scalar field discussed above. Its implications will be investigated in the next subsection.

We showed before that $\mathcal{K}_{AB}$ and $X$ are imaginary for a Euclidean metric. Therefore, although it is not transparent in our notation, the matching conditions (2.63) and (2.65) do contain factors of $i$ when joining a Lorentzian to a Euclidean metric.

### 2.4.5 Imposing the matching conditions

For the scalar field, the matching conditions were crucial in demonstrating the cancellation of corner divergences and the absence of localized corner contributions to the one-point function. A similar cancellation will occur for gravity, but imposing the three matching conditions (2.62), (2.63) and (2.65) will not be as straightforward as for the scalar field.
In this subsection we shall impose the matching conditions order by order in a radial expansion of $h_{AB}$, $K_{AB}$ and $X$. We start with a detailed analysis of the leading-order terms in the matching conditions. We then discuss continuity in the complex time plane of the boundary metric. Just as for the scalar field, the higher-order continuity is related to the continuity of the subleading terms in the Fefferman-Graham expansion of the bulk metric. Afterwards, we show that our leading-order results extend to the higher-order terms as well.

**Leading order matching conditions**

We will work in the Fefferman-Graham coordinates, with the matching surface $\partial_t M$ given by (2.56). Without loss of generality, we assume that the corner is given by $t = 0$ on $\partial_r M_1$ and $\tau = 0$ on $\partial_r M_0$, so $\lim_{r \to \infty} f(r, x^a) = 0$ on either side. We suppose that $f$ behaves asymptotically as

$$f = e^{-r} f^{(1)}(x^a) + \ldots$$

(2.66)

This is generally the leading asymptotic behavior of $f$, since any slower falloff near $r \to \infty$ would yield a non-spacelike induced metric on $\partial_t M$ in a real Lorentzian spacetime. Substituting the expansion (2.66) and the leading-order terms in the ADM-decomposition (2.58) of $\gamma_{ij}$ in (2.60), we find the leading behavior of $H^2$, $M^2$ and $M^r$. The inner product between the unit normals, given in (2.49), becomes to leading order:

$$\pm n^\mu \hat{n}_\mu = \mp \frac{\sqrt{N^2_{(0)} f^{(1)}}}{\sqrt{1 - N^2_{(0)} f^{2(1)}}}.$$ (2.67)

Since continuity of $X_\pm$ follows from continuity of $\pm n^\mu \hat{n}_\mu$, the corner matching condition (2.65) becomes to leading order:

$$\sqrt{N^2_{(0)[0]} f^{(1)[0]}} = \sqrt{N^2_{(0)[1]} f^{(1)[1]}},$$ (2.68)

where we reinstated the subscripts to indicate the manifold under consideration.

Let us work out the consequences of this condition. Recall that we absorbed factors of $i$ in the square roots of the metric determinant, and therefore (2.68) is not necessarily a relation between real quantities. For example, if we match a Lorentzian to a Euclidean solution, then $N^2_{(0)}$ changes sign across the corner and the square root on the Euclidean side of (2.68) becomes imaginary. On the other hand, the square root on the Lorentzian side is real, and so is $f^{(1)}$ since we use real coordinates. This means that in that case we must have:

$$f^{(1)[0]} = f^{(1)[1]} = 0,$$ (2.69)
which more generally holds in all cases for which the phase of $N^2_{(0)}$ is discontinuous across the corner. Actually, this phase is only continuous when we match two solutions with the same signature (recall that we chose boundary coordinates in which $N_a(0) = 0$). This happens either if we have no corner at all, or if the corner makes a 180-degree turn. In the first case, we can pick boundary coordinate systems in which $N^2_{(0)}$ is continuous across the corner and (2.68) becomes simply

$$f^{(1)[0]} = f^{(1)[1]}.$$  

(2.70)

Since we just artificially split a spacetime in two parts, it is natural that there is no further constraint on $f$. The case in which the corner makes a full turn is slightly more involved. First of all, the two boundary segments ending on the corner must be straight horizontal lines in the complex time plane, since the boundary contour cannot go up in this plane. We may again assume that $N^2_{(0)}$ is continuous across the corner, but that does not mean that the square roots in (2.68) are. Namely, one of the segments is backward-going in the complex time plane and in subsection 2.4.1 we already mentioned that the square root for a backward-going contour results in a minus sign. The matching condition for a full turn therefore becomes

$$f^{(1)[0]} = -f^{(1)[1]}.$$  

(2.71)

This implies that, at least at this order, we can freely move the hypersurface $\partial_t M$ up and down in the bulk, as long as we move it by the same amount on both components and keep the location of the corner fixed. We have sketched this in figure 2.3.

We have worked out the leading order term in the corner matching condition in three cases, corresponding to three different corners. We emphasize that our
formalism of subsection 2.4.1 allowed us to summarize all three cases in the single equation (2.68). We will see below that the subleading behavior of \( f \) is constrained in an analogous way.

As a sidenote, let us also compute the leading order term in the radial expansion of the second matching condition (2.63). If we use (2.43) to expand the trace of the extrinsic curvature \( \kappa_{AB} \), the leading order term becomes:

\[
\kappa = \pm d \sqrt{\frac{N^2_{(0)} f_{(1)}}{1 - N^2_{(0)} f_{(1)}^2}}.
\]

The trace part of (2.63) therefore results to leading order again in (2.68). It is plausible that for AAdS spacetimes the corner matching condition (2.65) follows from (2.63) and does not need to be imposed separately. This would be related to the fact that the asymptotics of the bulk metric are completely determined by the Fefferman-Graham data, but a more complete analysis is required to settle this issue completely. This will not be attempted here and we will instead continue to treat (2.65) as an additional condition.

**Continuity in the complex time plane**

Just as for the scalar field, the Fefferman-Graham expansion relates subleading terms in the matching conditions to higher-order continuity in the complex time plane of the sources. Before proceeding with the subleading terms in the matching conditions, let us therefore first discuss the notion of smoothness in the complex time plane for the boundary metric.

Consider a contour in the complex time plane with a corner. We will define \( C^k \)-smoothness for the boundary metric as the condition that the \( k \)-th order \( t \)-derivatives of the metric components exist and are continuous at the at the corner of the contour. Although this is a natural definition, in our notation a complication arises because we do not work directly with a complex time coordinate on for example the vertical segments. Instead, we rather use a contour time like \( t_c \) or \( \tau \) which is real on a particular segment of the contour and for such parametrizations the continuity condition has a different form. We may find this new form by regarding these local parameters as related to \( t \) via a complex diffeomorphism, for example \( t = -i \tau \) or \( t = 2T - t_c \). If we use these parameters to express continuity of the metric, then we need to take care of the transformation properties under the diffeomorphism as well. For example, \( C^0 \) continuity of \( g_{(0)ij} \) across the corner of figure 2.2, where \( t = -i \tau \), becomes the condition that at the corner

\[
\begin{align*}
g_{(0)[0][0]} &= -g_{(0)[1][1]} t, & g_{(0)[0][1]} &= -i g_{(0)[1][2]} t, & g_{(0)[0][a]} &= g_{(0)[1][a]},
\end{align*}
\]

(2.73)
Similarly, $C^1$ continuity in the complex time plane becomes
\[ \partial_\tau (g(0)[0]_{ab}) = -i \partial_t (g(0)[1]_{ab}), \quad \partial_\tau (g(0)[0]_{\tau\tau}) = i \partial_t (g(0)[1]_{tt}). \] (2.74)

The extension to higher orders and other components is analogous. As an example, take $ds^2_0 = d\tau^2 + \delta_{ab} dx^a dx^b$ and $ds^2_1 = -dt^2 + \delta_{ab} dx^a dx^b$. Although there is an apparent discontinuity in the metric components, with our definitions the metric is $C^\infty$ at the corner.

We will from now on assume that the boundary metric at the corner is $C^d$ continuous in the complex time plane. The reason for this smoothness condition is the same as that for the scalar field: it guarantees the existence of a Fefferman-Graham expansion of the metric at the corner, and the locally determined coefficients in this expansion are then automatically continuous across the corner as well. Since we continue to use real coordinates like $\tau$, we will always need to supplement the continuity condition with the transformation under the complex diffeomorphism.

**Higher order matching conditions**

We showed above that the leading order matching conditions imply that $f^{(1)}$ usually vanishes, except in special cases when $N^2(0)$ does not change across the corner. In this subsection, we show that the matching conditions and the $C^d$ continuity of the boundary metric fix the higher-order terms in $f$ to behave just as $f^{(1)}$, at least up to terms that vanish faster than $e^{-dr}$.

We first assume that the leading order term in $f$ is:
\[ f(r, x^a) = e^{-nr} f_n(x^a). \] (2.75)

One may easily check that in this case
\[ \pm n^\mu \hat{n}_\mu = \mp \sqrt{N^2_{(0)} f_n e^{(1-n)r}} + \ldots \] (2.76)

and a repetition of the previous analysis shows that, for $n \leq d$, the leading order term in (2.65) becomes equivalent to
\[ \sqrt{N^2_{(0)[0]} f_n[0]} = \sqrt{N^2_{(0)[1]} f_n[1]}. \] (2.77)

Therefore, if the phase of $\sqrt{N^2_{(0)}}$ is discontinuous across the corner, we find that not only $f^{(1)}$ but all terms up to and including order $e^{-dr}$ in $f(r, x^a)$ vanish as well.

If $N^2_{(0)}$ is continuous, then $f^{(1)}$ does not necessarily vanish, equation (2.75) no longer holds, and the above derivation for the subleading terms is no longer valid.
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However, the $C^d$ continuity of the boundary metric implies that the locally determined terms in the Fefferman-Graham expansion (2.55) are continuous across the matching surface as well and the metric is thus the same to high order on either side (up to the complex diffeomorphism discussed above). A discontinuity in (2.55) may appear at the earliest for the nonlocally determined term $g_{(d)ij}$, which is at overall order $e^{(2-d)r}$ in the radial expansion of the bulk metric. By substituting this radial expansion in the fourth equation in (2.60), and using the continuity to all orders of $H^2 + H_a H^a$, we find that $f$ has to be continuous across the corner up to and including terms of order $e^{-dr}$. (Notice that the fourth equation in (2.60) is invariant under $f \leftrightarrow -f$, but we fixed the overall sign already at leading order.)

This finishes our discussion about imposing the matching conditions: the previous two paragraphs show that $f$ ‘matches’ up to and including terms of order $e^{-dr}$ for all three cases. Up to this order, we find that $f_0 = -f_1$ for a full turn, that $f_0$ and $f_1$ both vanish for any other corner, and that $f_0 = f_1$ if there is no corner at all. In the next subsection, we will use these conditions to demonstrate the absence of localized (divergent) corner contributions to the on-shell action, in order to eventually show the continuity of the one-point function of the stress energy tensor around the corner.

### 2.4.6 Computation of the on-shell action

The bare on-shell action (2.52) has the usual Gibbons-Hawking-York contribution from $\partial_t M$ as well as an extra corner contribution. However, the matching conditions directly imply that these terms cancel between the two spacetimes. The total action (2.61) becomes:

$$iS_0 + iS_1 = \frac{i}{2\kappa^2} \int_{M_0} d^{d+1}x \sqrt{-G} (R - 2\Lambda) + \frac{i}{\kappa^2} \int_{\partial_r M_0} d^d x \sqrt{-\gamma} K$$

$$+ \frac{i}{2\kappa^2} \int_{M_1} d^{d+1}x \sqrt{-G} (R - 2\Lambda) + \frac{i}{\kappa^2} \int_{\partial_r M_1} d^d x \sqrt{-\gamma} K. \quad (2.78)$$

This action can be renormalized with the usual radial counterterms which we found explicitly in section 1.6, except for a subtlety involving the bulk integrals in this action. Namely, the $t$-integrals do not run between fixed endpoints, say $t = 0$ and $t = T$, but now rather end on $t = f_\pm (r, x^a)$. The usual radial counterterms, however, assume an $r$-independent limit on the bulk integral and the radial counterterms may not exactly cancel all divergences.

We will now show that these extra divergences also cancel between the two matching solutions. To first order, the cancellation can be shown very explicitly. Namely,
if \( f \) is of the form (2.66), then we can radially expand the volume element as:

\[
\int_{M_0} \sqrt{-G} \, d^{d+1}x = \int_{r_0}^r \, dr \int dx^a \int dt \sqrt{N^2 \sigma} (d-1)
\]

The first term has an \( r \)-independent lower limit on the \( t \)-integral and so all divergences in this term are dealt with by integrating the usual radial counterterms also until \( f(r_0, x^a) \). The second term is not cancelled by counterterms and may lead to new divergences. However, in (2.78) a similar term comes from the expansion of the action \( S_1 \) for \( M_1 \) and by the corner matching condition (2.68) the terms exactly cancel each other. Notice that an extra sign on \( M_1 \) arises because we expand the upper rather than the lower limit of the \( t \)-integral.

For higher orders, we recall that \( f \) is continuous or vanishing up to and including terms of order \( e^{-dr} \). Using also the higher-order continuity of the bulk metric, a continuation of the expansion (2.79) shows that the corrections cancel up to finite terms. This means that no extra divergences arise from the discrepancy between the limits of the \( t \)-integration.

Having eliminated all possible sources of corner divergences, we may conclude that the usual radial counterterms are sufficient to make the total on-shell action finite. For example, in \( d = 4 \) we can easily construct the explicit counterterm action from the discussion of section 2.4. It takes the form:

\[
S_{ct} = \frac{1}{2 \kappa^2} \int d^d x \sqrt{-\gamma} \left( 3 + \frac{1}{4} R + \frac{1}{4} \log e^{-r_0} \left[ \frac{1}{4} R_{ij} R_{ij} - \frac{1}{6} R^2 \right] \right), \tag{2.80}
\]

where the curvatures are those of the boundary metric \( \gamma_{ij} \) at \( r = r_0 \). This counterterm action is valid for all signatures if we define \( \sqrt{-\gamma} \) in the same way as \( \sqrt{-G} \) above, \( i.e. \) with the branch cut above the positive real axis.

### 2.4.7 Continuity of the one-point function

We have shown that the on-shell action can be holographically renormalized with the usual counterterms in the presence of corners. It remains to show that the one-point function is appropriately continuous around the corners as well.

The renormalized one-point function of the stress energy tensor is obtained by varying the renormalized on-shell action with respect to radial boundary data. As for the scalar field, the integration by parts in the variation of a counterterm action like (2.80) may result in localized corner contributions to the one-point
function. However, a similar analysis as for the scalar field shows that the higher-order continuity of the boundary metric in the complex time plane ensures that such contributions again cancel between two matching solutions.

Let us explicitly show the cancellation of the first corner term that arises from the integration by parts of the radial counterterms, which originates from the second term in (2.80). This is just an Einstein-Hilbert like term and it cancels against the matching solution if the extrinsic curvature of the corner, which we denote $K_{(0)ab}$, is continuous across the corner:

$$K_{(0)[0]ab} + K_{(0)[1]ab} = 0.$$  \hspace{1cm} (2.81)

Cancellation of the next term gives a higher-order continuity condition. Explicitly, the variation of these terms gives

$$\delta \int_{\partial_r M} d^d x \sqrt{-\gamma} \left[ R_{ij} R^{ij} - \frac{d R^2}{4(d-1)(d-2)} \right] = \int_{\partial_r M} d^d x \sqrt{-\gamma} (\ldots) \delta \gamma_{ij}$$

$$+ \int_C d^{d-1} x \sqrt{\sigma} \left[ n_i P^{ij} (\nabla^l \delta \gamma_{lj} - \gamma^{kl} \nabla_j \delta \gamma_{kl}) + (\nabla_i P^{ij})(n_j \gamma^{kl} \delta \gamma_{kl} - n^k \delta \gamma_{kj}) \right],$$  \hspace{1cm} (2.82)

where

$$P^{ij} = - \frac{d R \gamma^{ij}}{4(d-1)(d-2)} + \frac{R^{ij}}{(d-2)^2}$$  \hspace{1cm} (2.83)

and $n^i$ is an appropriately defined unit normal for the corner as a submanifold of $\partial_r M$. From (2.82) we explicitly see that the higher-order continuity condition involves up to three derivatives of the metric in $d = 4$.

By the absence of initial or corner contributions, the holographic expression for the one-point function of the stress-energy tensor is completely analogous to the Euclidean case. We may therefore repeat the result (1.166) of section 1.6, namely

$$\langle T_{ij} \rangle_{g(0)} = - \lim_{r \to \infty} e^{(d-2)r} \frac{1}{\kappa^2} \pi_{(d)ij},$$  \hspace{1cm} (2.84)

where $\pi_{(d)ij}$ is the term of dilatation weight $d$ in the expansion of the radial canonical momentum in eigenfunctions of the dilatation operator. Upon substitution of the radial expansion one finds that the one-point function is expressed directly in terms of $g_{(d)ij}$ and terms that are determined locally by $g_{(0)ij}$. For example, in $d = 2$ we obtained (1.169) and in $d = 4$ we find up to scheme-dependent terms that

$$\langle T_{ij} \rangle = \frac{2}{\kappa^2} \left( g_{(4)ij} - \frac{1}{8} \left[ (\text{Tr} g_{(2)})^2 - \text{Tr} g_{(2)}^2 \right] - \frac{1}{2} g_{(2)}^2_{ij} + \frac{1}{4} g_{(2)ij} \text{Tr} g_{(2)} \right),$$  \hspace{1cm} (2.85)

see [27] for the exact expressions in other dimensions.
Since by assumption all locally determined terms in the Fefferman-Graham expansion of the metric are continuous, continuity of the one-point function will follow from continuity of $g_{(d)ij}$ across the corner. Fortunately, the continuity of $g_{(d)ij}$ follows directly if we substitute the expansion (2.55) in the last equation of (2.60). The left-hand side in this equation is continuous to all orders by the first matching condition. On the other side, we know that $f$ is continuous up to and including terms of order $e^{-dr}$, and we know that all $g_{(2n)ij}$ with $2n < d$ as well as $\tilde{g}_{(d)ij}$ are continuous since they are locally determined by $g_{(0)ij}$. Collecting terms of overall order $e^{(2-d)r}$ then establishes that $g_{(d)ij}$ has to be continuous as well. (As shown in [30], there is no diffeomorphism freedom at this order if we fix a boundary coordinate system and a boundary metric, so continuity of $g_{(d)ab}$ implies continuity of $g_{(d)ij}$ indeed.) We have thus established that the vev of the stress-energy tensor is continuous across the corner (in the sense discussed in subsection 2.4.5).

We end this section with a remark about the function $f(r, x^a)$. Recall that we could in some cases freely specify this function at the corner, provided it was the same on both sides (possibly up to a sign). On the other hand, this function has no place in the QFT, and therefore holographically computed QFT correlators should be independent of $f$. Our prescription passes this test, since the one-point function we obtain is indeed independent of $f$.

## 2.5 Conclusion

We have presented a general prescription to holographically compute real-time correlation functions within the supergravity approximation. The main challenge in developing such a real-time prescription, relative to Euclidean methods, was to understand in detail how to deal with initial data. Our prescription is a direct ‘holographic lift’ of QFT real-time techniques to the gravitational setting, namely there is a gravitational counterpart of all QFT steps involved in such computations. In more detail, in QFT one typically chooses a contour in the complex time plane which usually consists of a sequence of horizontal (real) and vertical (imaginary) segments, the latter being related to the choice of density matrix or initial/final state. On the gravitational side, we construct solutions that directly correspond to such QFT contours. Typically, horizontal segments are associated with Lorentzian solutions and vertical segments with Euclidean solutions, with appropriate matching conditions imposed on the joining surface. The Euclidean parts encode the initial and final state in the field theory and this is reflected in the bulk, where they can be thought of as Hartle-Hawking wave functions [56]. We will see in the next chapter how these wave functions also provide the necessary initial and final data for the perturbations around a given supergravity background.
For the prescription to well defined, one must establish that one can remove all infinities through a process of (holographic) renormalization. Relative to the Euclidean discussion, new infinities can appear at timelike infinity. In our setup the analysis boils down to analyzing possible new contributions from the joining surfaces. We show that no new counterterms are needed and the holographic 1-point functions are continuous across the matching surface. The continuity of the 1-point functions is an important consistency condition of the entire setup: as mentioned above, the Euclidean parts of the solution are directly related to the initial/final state but as is also well known the 1-point functions encode the same information, too.

As a sidenote, the holographic nature of the prescription also nicely shows up in the following issue that we encountered when demonstrating the renormalization. Starting from a boundary state defined at a boundary Cauchy surface, say the surface $t = t_0$, one can extend this surface to the bulk, $t = f(r, \vec{x})$ with $f(r, \vec{x}) \to t_0$ as $r \to \infty$, but clearly there is a certain amount of freedom of how this is done, parametrized by the subleading behavior of $f(r, \vec{x})$. These extensions are not part of the boundary theory, so the renormalized theory should be independent of them. We explicitly find that possible dependence on $f(r, \vec{x})$ drops out indeed.

Having set up the prescription, we will in the next chapter move on to demonstrate how to apply it in a variety of examples that each illustrate different points.

### 2.A Real-time quantum field theory

In this appendix we discuss some aspects of real-time quantum field theory relevant for our discussion. The material presented here is not new and it is included to make this work self-contained.

#### 2.A.1 Vacuum wave function insertions

In this section we will analyze how the vacuum wave function insertions in the path integral lead to $i\epsilon$ insertions. In the main text, we mentioned how the wave functions can be obtained as path integrals along vertical segments in the complex time plane, leading ultimately to a contour as in figure 2.1a. Let us begin by an explicit computation of these wave functions in a relatively simple case.
Computation of the wave functions

We will consider a real free massive boson on flat Minkowski space \( \mathbb{R}^{1,d-1} \). As explained in the main text, the initial wave function \( \langle \phi_-, -T | 0 \rangle \) is computed via the projection:

\[
\lim_{\beta \to \infty} \langle \phi_-, -T | e^{-\beta \hat{H}} | \Psi \rangle = \langle \phi_-, -T | 0 \rangle \langle 0 | \Psi \rangle.
\]

(2.86)

For simplicity, we shift the time coordinate such that \( -T \to 0 \) and we will take \( | \Psi \rangle = | \phi_\beta, i\beta \rangle \) for some spatial field configuration \( \phi_\beta(x) \).

Since we take the field to be free, the path integral is Gaussian and can be computed exactly. Let \( \hat{\phi}(t,x) \) be the solution to the Euclidean equation of motion satisfying \( \hat{\phi}(i\beta, x) = \phi_\beta(x) \) and \( \hat{\phi}(0, x) = \phi_-(x) \). We then obtain

\[
\langle \phi_-, 0 | 0 \rangle = \lim_{\beta \to \infty} N e^{-S_E[\hat{\phi}]},
\]

(2.87)

with \( N \) a normalization that does not depend on \( \phi_- \) and \( S_E \) the Euclidean on-shell action for the boson. The Lorentzian action for the boson is:

\[
S_L[\phi] = \frac{1}{2} \int dt d^{d-1}x \left( -\eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 \right)
\]

(2.88)

and we find after substitution of \( t = -i\tau \) that \( iS_L \to -S_E \) with the Euclidean action given by:

\[
S_E[\phi] = \frac{1}{2} \int_{-\beta}^{0} d\tau \int d^{d-1}x \left( \delta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2 \right).
\]

(2.89)

where we also set the limits of the \( \tau \) integrals to the appropriate values. On-shell this action reduces to a surface integral,

\[
\langle \phi_-, 0 | 0 \rangle = \lim_{\beta \to \infty} N \exp \left( -\frac{1}{2} \int d^{d-1}x \left[ \hat{\phi}(\tau, x) \partial_\tau \hat{\phi}(\tau, x) \right]_{\tau=-\beta}^{0} \right).
\]

(2.90)

Finding \( \hat{\phi} \) is not hard and in the limit \( \beta \to \infty \) we find that all dependence on \( \phi_\beta \) can be absorbed in a shift of \( N \) and we recover the usual Gaussian wave function [49], written in Fourier space as

\[
\langle \phi_-, 0 | 0 \rangle = N' \exp \left( -\frac{1}{2} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \phi_-(k) \omega_k \phi_-(k) \right),
\]

(2.91)

with \( \omega_k = \sqrt{k^2 + m^2} \). The conjugate final wave function \( \langle 0 | \phi_+, T \rangle \) can be computed using the same procedure, leading to exactly the same result.

If interactions are switched on, the wave functions receive corrections. However, as long as these interactions can be switched off adiabatically for large times, the
corrections can also be ignored in the limit $t_i, t_f \to \infty$. The analogous case in thermal field theory, which we discuss below, is briefly discussed in [51, section 2.4.1]. For massless field theories there are subtleties, but these considerations are not directly relevant for us and they will not be discussed here. A computation of the ground state wave function for electromagnetism and linearized gravity can be found in [61].

**Effect of the wave function insertions**

Let us now show how the wave function insertions determine $i\epsilon$-insertions in the propagator. To this end, we introduce a source $J$ for $\phi$ and compute

$$Z[J] = \langle 0 | e^{-i\int J \phi} | 0 \rangle.$$  

(2.92)

We suppose that the source vanishes smoothly at the endpoints $t = \pm T$ of the Lorentzian segment. Again via the usual slicing arguments, the path-integral representation one obtains is

$$Z[J] = \int [D\phi] \exp \left( iS[\phi] - i \int dtd^{-1}x J \phi - \frac{1}{2} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \phi_-(k) \omega_k \phi_-(k) \right),$$

(2.93)

where $\phi_\pm(k)$ is the Fourier transform of $\phi(\pm T, x)$ with respect to the spatial coordinates. Notice that the boundary values for the path integral $\int [D\phi]$ are not fixed.

To compute the path integral, we shift the integrand $\phi = \chi + \psi$, where $\chi$ satisfies $\Box \chi - m^2 \chi = J$ and $\psi$ is the new integration variable. Notice that $\chi$ is not uniquely defined unless we specify some boundary conditions. To find these, notice that the aim of this shift is to get all the factors involving $J$ and $\chi$ to come out in front of the path integral, resulting in

$$Z[J] = \mathcal{N} \exp \left( -\frac{i}{2} \int d^d x J \chi \right),$$

(2.94)

from which we would directly obtain the propagator as is shown below. However, an analysis of the boundary terms shows that such a factorization only occurs if one imposes additionally the two extra constraints:

$$-i \int d^{d-1}x \psi_-(x) \partial_t \chi(-T, x) - \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \psi_-(k) \omega_k \chi_-(k) = 0,$$

$$+i \int d^{d-1}x \psi_+(x) \partial_t \chi(T, x) - \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \psi_+(k) \omega_k \chi_+(k) = 0,$$

(2.95)
which should hold for all values of $\psi_{\pm}$. These conditions provide the boundary conditions for $\chi$. Since the source vanishes at the endpoints, $\chi$ is homogeneous for $t = \pm T$, and therefore has a Fourier expansion involving only modes of the form $e^{\pm i\omega t + i k x}$. The boundary conditions that one derives from these constraints are then simply that $\chi(-T, x)$ should contain only negative frequencies ($i.e.$ modes of the form $e^{-i\omega t + i k x}$ with $\omega < 0$) and $\chi(T, x)$ should contain only positive frequencies. But this uniquely fixes $\chi$ to be of the form

$$\chi = \int dt' d^{d-1}x' \Delta_F(t - t', x - x')J(x'), \quad (2.96)$$

with $\Delta_F$ the Feynman propagator,

$$\Delta_F(t, x) = \int \frac{dtd^{d-1}x}{(2\pi)^d} \frac{e^{-i\omega t + i k x}}{-\omega^2 + k^2 - m^2 - i\epsilon}. \quad (2.97)$$

As one may verify by contour deformation, one indeed obtains only positive/negative frequencies to the future/past of the source.

We can now take the limit $T \to \infty$. Assuming that the source and any perturbatively added interactions vanish slowly at late times, the propagator and the wave functions are unmodified and all that is left are the $i\epsilon$-insertions which enter in the perturbative expansion, which is precisely what we wanted to show.

Different (equivalent) arguments that translate wave functions to $i\epsilon$ insertions can be found in the textbooks [49] and [48]. In the main text we already discussed the method of [48] where the contour of figure 2.1a is deformed to a straight line that runs almost parallel to the real time axis, from $-T(1 - i\epsilon)$ to $T(1 - i\epsilon)$, with $T \to \infty$. The projection property is left unchanged and this way one still obtains vacuum-to-vacuum amplitudes. The contour should always go downward or horizontal in the complex time plane so that the operator $\exp(-i\hat{H}\Delta t)$ remains finite.

Finally, notice that the saddle-point $\chi$ is actually a complex solution, although we started with a real scalar field and a real source $J(x)$. This can be viewed as a contour deformation in field space before taking the saddle-point approximation. Such a deformation is very explicit when one discretizes the path integral. Nevertheless, the usual hermiticity constraints of $n$-point functions are still satisfied. The fact that a saddle-point approximation may involve complex fields holds for gravity as well. In the context of holography, it is the hermiticity of the boundary stress energy tensor and its correlators that restricts the allowed complex metrics.
2.A.2 Analyticity properties of two-point functions

In this section, we briefly review the analytic properties of two-point functions and corresponding $i\epsilon$-insertions.

We start with the Wightman function

$$\langle \psi(x) \psi(0) \rangle, \tag{2.98}$$

which is analytic in the lower half of the complex $t$ plane [62]. The Wightman function can be obtained by the replacement $-i\tau = t - i\epsilon$ in the Euclidean correlator, because then the poles along the real $t$ axis are shifted into the upper half of the complex $t$ plane. Its Fourier transform,

$$\int dtd^{-1}xe^{i\omega t - ikx} \langle \psi(x) \psi(0) \rangle, \tag{2.99}$$

vanishes for negative frequencies, since we can close the contour for the $t$-integral via the lower half plane. Positivity of the spectral density also implies that the Fourier transform is a real and positive distribution for positive frequencies [62]. The Fourier transform thus maps a function (or distribution) that is analytic in a upper or lower half plane to a function that vanishes on the right or the left real axis.

Next, the time-ordered two-point function is defined as

$$\langle T \psi(x) \psi(0) \rangle = \theta(t) \langle \psi(x) \psi(0) \rangle + \theta(-t) \langle \psi(0) \psi(x) \rangle, \tag{2.100}$$

which can be obtained from the Euclidean correlator by the replacement $-i\tau = t - i\epsilon t$. Its poles are shifted into the upper half of the complex $t$ plane for Re $t > 0$ and in the lower half plane for Re $t < 0$. To obtain the Fourier transform, we close the contour in the appropriate half plane in the complex time plane. Picking up the poles, we find a sum over positive frequencies for $t > 0$ and one over negative frequencies for $t < 0$. This implies that we need the usual Feynman contour around the poles to define the inverse Fourier transform. One may replace $\omega \rightarrow \omega + i\epsilon\omega$ in the Fourier-space expression to explicitly indicate such a contour. Obviously

$$-(\omega + i\epsilon\omega)^2 = -\omega^2 - i\epsilon$$ \hspace{1cm} (2.101)

and for example the propagator (2.97) indeed has the required analyticity properties.

Finally, the retarded two-point function is defined as:

$$\Delta_R(x', x) = -i\theta(x' - x)\langle [\mathcal{O}(x'), \mathcal{O}(x)] \rangle. \tag{2.102}$$
Causality of the field theory determines that it vanishes completely outside the future lightcone. We may write it as an inverse Fourier transform:

\[ \Delta_R(x, 0) = \frac{1}{(2\pi)^d} \int d\omega d^{d-1}k e^{-i\omega t + ikx} \Delta_R(\omega, k). \]  

Notice that \( \Delta_R(x, 0) \) vanishes for \( t < 0 \). Since we can then close the \( \omega \) integral in (2.103) in the upper half plane, we find that \( \Delta_R(\omega, k) \) must be analytic in the upper half of the complex frequency plane. Finally, the advanced two-point function is the reversed retarded two-point function:

\[ \Delta_A(x, x') = \Delta_R(x', x). \]  

It therefore vanishes outside of the past lightcone and is analytic in the lower half of the complex frequency plane.