Classical manipulation of a quantum system

Mehmani, B.

Citation for published version (APA):
Mehmani, B. (2010). Classical manipulation of a quantum system

General rights
It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

Disclaimer/Complaints regulations
If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: http://uba.uva.nl/en/contact, or a letter to: Library of the University of Amsterdam, Secretariat, Singel 425, 1012 WP Amsterdam, The Netherlands. You will be contacted as soon as possible.
Quantum State Tomography

In this chapter the question of determining the unknown state of a quantum system is addressed. First, some of the interpretations of the notion of the quantum state is mentioned. Then quantum state tomography is introduced as a procedure of reconstructing the quantum state from measurements of observables of the system. This method aims to determine the unknown state of a quantum system from a linear transformation of a set of experimental data. Then the quantum state tomography of a two-level system or a “qubit” in the terminology of quantum information by simultaneous measurement of two commuting observables is studied. This can be done by letting the qubit interact with another qubit which is in a known initial state, or with a single mode of a quantized electromagnetic field. In the latter case, the interaction is studied within the Jaynes-Cummings Model. It is shown that it is possible to determine the unknown initial state of the qubit from two sets of measurements of commuting observables each belonging to one of the systems. In order to make sure that the reconstructed density matrix is a Hermitian, semipositive matrix with a unit trace, the maximum likelihood reconstruction method is applied. In this approach the density matrix that is most likely to have produced the measured data set is characterized by numerical optimization.
2.1 Introduction

The ability to determine and characterize the state of a quantum system is one of the most important areas in nowadays physical research specifically in quantum computation, quantum cryptography, and quantum communication. Given the state of a quantum system, one can calculate the expectation value of any observable of the system [4]. However, the inverse problem of determining the state by performing different measurements is not a trivial task. This problem was first discussed by Pauli in 1933 [5]. He raised the question of how to reconstruct the unknown wave function of an ensemble of identical spinless particles via the corresponding position and momentum probability densities. The interest in the state determination problem grew considerably since then, and is now a well-recognized subject [6–12].

In general, the process of reconstructing the quantum state (density matrix) of a system by means of performing measurements on different observables of the system is called quantum state tomography. In various experimental setups it is reasonably straightforward to reconstruct the state of a quantum system employing a linear tomographic technique. This way the elements of the density matrix can be linearly related to a set of measured quantities. But since different observables of a quantum system may not commute with each other, one often has to perform series of successive measurements of observables which cannot be done simultaneously. Simultaneous measurement of observables costs less time and energy and is more beneficial. However, there is one drawback in this method. The recovered state might not correspond to a physical state due to the experimental noise. For example, density matrices for any quantum system must be semipositive, Hermitian matrices with unit trace. The matrices resulting from a tomographic measurement may fail to be positive semidefinite. To avoid this issue the “maximum likelihood” method is adopted.

In section 2.2 we discuss some of the interpretations of quantum state and quantum measurement. Then we outline the strategy of quantum state tomography with simultaneous measurement of observables. Sections 2.3-2.4 are devoted to describe specific models we employ to this end. In section 2.7 we introduce the maximum likelihood method. This method is used to reconstruct the most proper density matrix based on a measured data set by numerical optimization. The numerical results are demonstrated in appendices A and B.
2.2 The state of a quantum system

In classical physics, the state of a system is characterized by specifying the values of all physical quantities, for instance the positions and the velocities of the particles that constitute the system. In quantum mechanics the situation is complicated by the fact that the physical quantities are mathematically represented by specific type of operators called observables, which in general are elements of a non-commutative algebra. Hence their values cannot be simultaneously specified, as emphasized in the Heisenberg’s uncertainty principle. Instead, the measurement results of each observable is characterized by a probability distribution, which involves statistical fluctuations. The “state of the system” is then represented by a mathematical notion that allows us to express the probability distribution of all the observables for an ensemble of identically prepared systems.

Various interpretations have been given to the concept of state in quantum mechanics. Here we list three of them [13–15]:

- According to the Schrödinger interpretation, a state is represented by a wave function or by a ket vector in the Hilbert space on which the observables of the considered system act. In this interpretation, the wavefunction is regarded as an intrinsic property of the system and it directly describes its physical properties. The probability distribution of the position of a particle, for instance, is obtained from the modulus square of its wave function. The distribution of the momentum is given by the modulus square of the Fourier transform of the wave function.

- In “information interpretation”, on the opposite point of view, the state does not pertain to the system itself, but only gathers the information we have on it [15]. The problem in this concept is that the wave function of a physical system would depend on the observer in analogy with classical probability [13]: If two observers have different information on the same physical system, they should use two different wave functions for describing it.

- In the statistical interpretation of quantum mechanics [14], to which we adhere, the state of a system is a mathematical object from which we can derive any probabilistic prediction about the physical quantities attached to this system. One typically imagines some experimental apparatus and procedure which “prepares” this quantum state; the
Quantum State Tomography

A mathematical object then reflects the setup of the apparatus. This way, the quantum state accounts for the full information available about the preparation of the system, from which we wish to derive consequences for future experiments. Since this knowledge is probabilistic it does not refer to a single system or single event. What we call a state, which is most of the time a mixed state, characterizes a statistical ensemble of systems of the same type, which are all prepared under identical physical considerations. The physical state is thus a mathematical representation of the result of a certain state preparation procedure; it accounts for our information about this preparation and upon knowing it we can elaborate consistent probabilistic predictions. It is thus a concept which merges objective and subjective aspects [16].

A standard tool to implement the statistical definition of state is the density matrix, which generalizes the pure state represented by a wave function. Indeed, there is no conceptual difference between wave function and density matrix which are both mathematical means for evaluating expectation values of the observables of the system or probabilities.

In the framework of the statistical interpretation, the laws of quantum mechanics can be summarized as follows:

- An observable $\hat{O}$ is represented by a self-adjoint linear operator acting on the Hilbert space pertaining to the system. It has a spectral representation, $\hat{O} = \sum_i o_i \hat{P}_i$ where $o_i$ are the eigenvalues of $\hat{O}$ and $\hat{P}_i$ are the orthogonal projection operators related to the orthonormal eigenvectors of $\hat{O}$, i.e., $\hat{P}_i = \sum_m |m, o_i\rangle \langle m, o_i|$. The parameter $m$ labels the degenerate eigenvectors of $\hat{O}$.

- The state of a system at a given time is represented by its density matrix, $\hat{\rho}$, which is a self-adjoint operator in Hilbert space with a unit trace. The density matrix should also be semipositive to ensure that any variance of the observables of the system is non-negative. Pure states correspond to the special case
  \[ \hat{\rho}^2 = \hat{\rho}. \]

- The dynamics of the system can be obtained by
  \[ \hat{\rho}(t) = \hat{U}(t, t_0) \hat{\rho}(t_0) \hat{U}^\dagger(t, t_0), \]
  where $\hat{U}(t, t_0)$ is the unitary time evolution operator.
Given the density matrix $\hat{\rho}$ of a system, one can find the expectation value of any observable $\hat{O}$ of the system in the considered situation as

$$\langle \hat{O} \rangle = \text{tr}[\hat{\rho} \hat{O}],$$

where $\text{tr}[\cdots]$ stands for the trace of a matrix.

Let us emphasize that throughout this thesis the operators are always distinguished by a $\hat{\cdot}$ sign.

As it was mentioned earlier, we wish to reconstruct the density matrix of a quantum system. Consider we are given an ensemble of systems $S$ which we don’t know its initial state. In other words, the probability to observe some result or another in the measurement of an observable is unknown. The following question then is of our interest. How can one determine the density matrix by identification of a set of observables, the measurement of which permits the precise determination of $\hat{\rho}$? In other words, how can one determine the quantum statistical operator that describes the preparation of the system?

Procedures of reconstructing the quantum state from measurements are known as quantum state tomography. Recently, they have found some applications in quantum information processing [17]. For example, in quantum cryptography one needs a complete specification of the qubit state both as it is emitted from the source and as it is received after transmission [18].

In the simplest example of a spin-$\frac{1}{2}$ system or equivalently any two-level quantum system the state is described by a $2 \times 2$ matrix. In the two-dimensional Hilbert space, any observable is a linear combination of the Pauli operators, which satisfy

$$\hat{\sigma}_\alpha^2 = \hat{1}, \quad \alpha = x, y, z,$$

$$\hat{\sigma}_x \hat{\sigma}_y = i \hat{\sigma}_z,$$

and are represented by the Pauli matrices

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ \hspace{1cm} (2.4)

A state is characterized by three real numbers: one for the diagonal elements of the $2 \times 2$ density matrix $\hat{\rho}$, and two for its off-diagonal elements. Equivalently, we can introduce the polarization vector, $\vec{r}$, the components of which are the expectation values of the Pauli matrices.

$$r_\alpha = \text{tr}(\hat{\rho} \hat{\sigma}_\alpha), \quad \alpha = x, y, z.$$ \hspace{1cm} (2.6)
Quantum State Tomography

Once we know the value of these parameters, we are able to determine the value of the density matrix, making use of the identity

$$ \hat{\rho} = \frac{1}{2} \left( \mathbf{1} + \vec{r} \cdot \vec{\sigma} \right). $$

(2.7)

Thus, according to the above argument, one has to perform three incompatible measurements for the unknown state determination, e.g., measuring the spin components along the x-, y- and z- axes via a Stern-Gerlach setup. However, during the measurement procedure of each component one loses the information about the two other components, since the spin operators in different directions do not commute. Thus, to determine the state of a spin-$\frac{1}{2}$ system, one needs to use three sets of Stern–Gerlach measurements performed along orthogonal directions. In this approach, the state of any two-level system, represented by a $2 \times 2$ density matrix $\hat{\rho}$, can be fully determined only through measurement of three linearly independent observables which do not commute and cannot be simultaneously measured.

In what follows we show that the unknown density matrix of such a system $S$, in particular the full polarization vector of a spin-$\frac{1}{2}$ system, can be determined indirectly. This can be done by means of a single set of measurements performed simultaneously on both the system $S$ and another auxiliary system (assistant or ancilla) $A$, where $A$ starts its evolution from a known state [12,19–22]. The suggested strategy is the following: initially $S$ is in an unknown state that we wish to determine, while the state of the assistant $A$ is known. During some time lapse $S$ and $A$ interact in a known fashion. As a result their joint state is modified: it involves correlations and keeps memory of the initial state of $S$. Two commuting observables of the combined system $S+A$ are then simultaneously measured. Repeating this process provides then the necessary statistical data: the expectation values of the observables and also their correlation. We will show that one can infer the three components of the initial polarization vector of $S$, and hence the state of the system from these three sets of data. This way, performing one simultaneous measurement on observables of $S+A$ plays the same role as performing successive measurements on three non-commuting observables of $S$. This type of information transfer is remarkable. Initially, an unknown information was embedded in the matrix elements of $\hat{\rho}$, or equivalently in the components of the polarization vector of $S$. It had a quantum nature, and could not be represented by an ordinary probability distribution, due to the non-commutation of the three components of the spin operator. After the interaction between
2.2. The state of a quantum system

S and A this unknown initial information about S, together with the known information about A, is redistributed among the matrix elements of the joint density matrix of the overall system, S+A. However, the resulting classical joint probability distribution for the observables of the system S and the assistant A can keep full memory of the initial quantum information about S. The process on which we rely amounts to a transformation of quantum information into classical information, which can be gained by a classical type of measurement involving commuting observables only. This measurement modifies the state of S+A, but it can recover all the matrix elements of \( \hat{\rho} \).

The idea of transforming quantum into classical information by using an assistant system A was first proposed by D’Ariano [19] who showed the possibility of mapping the density matrix of S onto a single observable of S+A. It was explicitly implemented in a dynamical form by Allahverdyan et al. [12] of which the present work is a continuation. In particular, they showed that one can determine the unknown state of a spin-\( \frac{1}{2} \) system with a single apparatus by using another spin-\( \frac{1}{2} \) system as an assistant. This idea was recently implemented by Peng et al. [21] who used pulses to induce the proper dynamics of the interaction between the spin-\( \frac{1}{2} \) system and its assistant. They verified the initial state of the system obtained from this procedure with the result of the direct measurement of the three components of the spin vector of the system. Later it was shown that one can employ a single mode of coherent light as an assistant in order to reconstruct the initial state of a two-level system [20].

In the next section 2.3 we briefly review the proposed procedure by Allahverdyan et al [12] about determination of the state of a spin-\( \frac{1}{2} \) system employing another spin-\( \frac{1}{2} \) system as an assistant. Then we specifically show that the unknown density matrix of an ensemble of two-level systems (atom or spin) can be determined via interaction with a single mode of the electromagnetic field. The atom-field interaction is studied within the Jaynes-Cummings model (JCM) [23]. The unknown state of a two-level system is characterized by repeated measurement of two commuting observables: the population difference of the system \( \hat{\sigma}_z \), and the photon number of the field \( \hat{a}^\dagger \hat{a} \). This measurement supplies three averages: \( \langle \hat{\sigma}_z \rangle \), \( \langle \hat{a}^\dagger \hat{a} \rangle \), and \( \langle \hat{\sigma}_z \hat{a}^\dagger \hat{a} \rangle \), which will be linearly related to the elements of the initial density matrix of the ensemble of the two-level systems. Note that since \( \hat{\sigma}_z \) and \( \hat{a}^\dagger \hat{a} \) commute, \( \langle \hat{\sigma}_z \hat{a}^\dagger \hat{a} \rangle \) is recovered from the gathered data of \( \hat{\sigma}_z \) and \( \hat{a}^\dagger \hat{a} \) by counting the number of coincidences.
Consider a two-level system, $S$, the state of which we wish to determine. The aim is to find an indirect procedure involving only measurements of commuting observables, which therefore can be performed by means of a single apparatus. To this end, we let the system $S$ be coupled to an auxiliary two-level system $A$. $A$ is in a known state.

Let us recall the general form of an unknown state of $S$ from (2.7)

$$\hat{\rho} = \frac{1}{2} \left( 1 + \vec{r} \cdot \vec{\sigma} \right), \quad (2.8)$$

where the polarization vector $\vec{r}$ is defined as

$$\vec{r} = \text{tr} [\hat{\rho} \hat{\sigma}_\alpha], \quad \alpha = x, y, z. \quad (2.9)$$

The state is called pure if $|\vec{r}| = 1$. $|\vec{r}| < 1$ represents a mixed state, and $|\vec{r}| > 1$ is physically excluded.

We choose the state of the assistant, represented by $\hat{R}$ as

$$\hat{R} = \frac{1}{2} \left( 1 + \lambda \hat{s}_z \right), \quad 0 \leq \lambda \leq 1 \quad (2.10)$$

where $\hat{s}_x$, $\hat{s}_y$, and $\hat{s}_z$ are the Pauli matrices in the Hilbert space belonging to the assistant $A$.

Initially there is no interaction between $S$ and $A$. Therefore the initial state of the overall system, $\hat{\Omega}_0$, can be written as

$$\hat{\Omega}_0 = \hat{R} \otimes \hat{\rho} = \frac{1}{4} \begin{pmatrix} (1 + \lambda) \left( 1 + \vec{r} \cdot \vec{\sigma} \right) & 0 \\ 0 & (1 - \lambda) \left( 1 + \vec{r} \cdot \vec{\sigma} \right) \end{pmatrix}. \quad (2.11)$$

Now we let the two systems interact for some time. The interaction can be described with the help of a $4 \times 4$ unitary matrix $\hat{U} = e^{-i\hat{H}}$, where we set $t = 1$. Here, we don’t specify our Hamiltonian and consider a general unitary matrix and we parametrize it such that it generates a proper time-evolved overall density matrix at later time $t = 1$, given by $\hat{\Omega}_f$ such that the initial state of $S$ can be read off easily. The observables of which the measurements yields the determination of the initial state of $S$ are the final polarization of each spin of the overall system $S+A$ [12]. They can be measured simultaneously and the correlation of the two can be derived from the gathered data. We
show that it is possible to read off the initially unknown state of the system S from the three above mentioned sets of data. Let us decompose $\hat{U}$ into the following $2 \times 2$ block matrix,

$$\hat{U} = \begin{pmatrix} \hat{A} & \hat{C} \\ \hat{B} & \hat{D} \end{pmatrix},$$

and express the unitarity of $\hat{U}$ in terms of the $2 \times 2$ matrices $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ in the Hilbert space of S. The polar decomposition of $\hat{A}$ and $\hat{B}$ yields

$$\hat{A} = \hat{v}\hat{k}, \quad \hat{B} = \hat{w}\hat{k}',$$

where $\hat{v}$, and $\hat{w}$ are unitary matrices while $\hat{k}$ and $\hat{k}'$ are semi-positive Hermitian matrices. Since $\hat{v}$, and $\hat{w}$ are unitary, it is easy to see that $\hat{k}$ and $\hat{k}'$ are the non-negative square roots of $\hat{A}^\dagger\hat{A}$ and $\hat{B}^\dagger\hat{B}$, respectively. If $\hat{k}$ and $\hat{k}'$ have a vanishing eigenvalue, these representations of $\hat{A}$ and $\hat{B}$ still hold but are no longer unique. We shall restrict ourselves to the case where $\hat{k}$ and $\hat{k}'$ are strictly positive.

The condition $\hat{U} \hat{U}^\dagger = 1$ implies

$$\hat{C}\hat{C}^\dagger = 1 - \hat{A}\hat{A}^\dagger, \quad \hat{D}\hat{D}^\dagger = 1 - \hat{B}\hat{B}^\dagger, \quad (2.14)$$
$$\hat{A}\hat{B}^\dagger + \hat{C}\hat{D}^\dagger = 0, \quad (2.15)$$

while $\hat{U}^\dagger\hat{U} = 1$ implies

$$\hat{A}^\dagger\hat{A} + \hat{B}^\dagger\hat{B} = 1, \quad \hat{C}^\dagger\hat{C} + \hat{D}^\dagger\hat{D} = 1, \quad (2.16)$$
$$\hat{A}^\dagger\hat{C} + \hat{B}^\dagger\hat{D} = 0. \quad (2.17)$$

Implementing (2.16) on the polar decomposition of $\hat{A}$ and $\hat{B}$ given by (2.13) yields

$$\hat{k}' = \sqrt{1 - \hat{k}^2}. \quad (2.18)$$

Thus $\hat{C}\hat{C}^\dagger$ and $\hat{D}\hat{D}^\dagger$ can be simplified as

$$\hat{C}\hat{C}^\dagger = \hat{v}\hat{k}'\hat{v}^\dagger, \quad \hat{D}\hat{D}^\dagger = \hat{w}\hat{k}'\hat{w}^\dagger. \quad (2.19)$$

Since $\hat{k}$ and $\hat{k}'$ are strictly positive and $\hat{v}$ and $\hat{w}$ are unitary matrices, we can define unitary matrices $\hat{x}$ and $\hat{y}$ such that $\hat{C}$ and $\hat{D}$ have the form

$$\hat{C} = \hat{v}\hat{k}'\hat{x}, \quad \hat{D} = \hat{w}\hat{k}'\hat{y}. \quad (2.20)$$
Quantum State Tomography

The remaining unitary condition \( \hat{A}^\dagger \hat{C} + \hat{B}^\dagger \hat{D} = 0 \) reads \( \hat{k}\hat{k'}(\hat{x} + \hat{y}) = 0 \). Again, since \( \hat{k} \) and \( \hat{k'} \) are strictly positive this implies \( \hat{y} = -\hat{x} \), which fixes \( \hat{y} \) in a unique way.

The unitary matrix \( \hat{U} \) then becomes:

\[
\hat{U} = \begin{pmatrix}
\hat{v} & 0 \\
0 & \hat{w}
\end{pmatrix}
\begin{pmatrix}
\hat{k} & \hat{k'} \\
\hat{k'} & -\hat{k}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & \hat{x}
\end{pmatrix}.
\]

In order to get a more symmetric form for \( \hat{U} \), we introduce a unitary matrix \( \hat{X} \) such that \( (\hat{X}^\dagger)^2 = \hat{x} \) and define the matrices \( \hat{V} = \hat{v}\hat{X}^\dagger \) and \( \hat{W} = \hat{w}\hat{X}^\dagger \), \( \hat{K} = \hat{X}\hat{k}\hat{X}^\dagger \), \( \hat{K'} = \hat{X}\hat{k'}\hat{X}^\dagger \). We can then write the 4×4 unitary transformation operator \( \hat{U} \) as

\[
\hat{U} = \begin{pmatrix}
\hat{V} & 0 \\
0 & \hat{W}
\end{pmatrix}
\begin{pmatrix}
\hat{K} & \hat{K'} \\
\hat{K'} & -\hat{K}
\end{pmatrix}
\begin{pmatrix}
\hat{X} & 0 \\
0 & \hat{X}^\dagger
\end{pmatrix}.
\]

Having the unitary matrix given by (2.22) we can calculate the state of the overall system at later time as

\[
\hat{\Omega}_f = \hat{U}\hat{\Omega}_0\hat{U}^\dagger,
\]

where \( \hat{\Omega}_0 \) is given by (2.11). The observables that can be simultaneously measured by means of the same apparatus on the state \( \hat{\Omega}_f \) are the \( z \)-components of the spin of each system. In case \( S \) and \( A \) are generally two-level systems, the \( z \) component of quasi-spin is related to the level occupation and thus the energy of the system. This corresponds to the following averages

\[
\begin{align*}
\langle \hat{s}_z \rangle &= \text{tr} \left[ \hat{\Omega}_f \hat{s}_z \right], \\
\langle \hat{\sigma}_z \rangle &= \text{tr} \left[ \hat{\Omega}_f \hat{\sigma}_z \right], \\
\langle \hat{s}_z \hat{\sigma}_z \rangle &= \text{tr} \left[ \hat{\Omega}_f (\hat{s}_z \hat{\sigma}_z) \right].
\end{align*}
\]

We notice that the correlation \( \langle \Omega_f (\hat{s}_z \hat{\sigma}_z) \rangle \) can be recovered form the gathered data of \( \hat{\sigma}_z \) and \( \hat{s}_z \) via the number of coincidences.

Inserting the unitary time-evolution operator in the expectation values (2.24)
we get a linear relation between the gathered data of measurements of \( \hat{s}_z, \hat{\sigma}_z \) and their correlation on one hand and the the elements of the initial density matrix of \( S \) given by \( r_x, r_y \) and \( r_z \) on the other hand. But before calculating the above mentioned expectation values, let us first parametrize the unitary time evolution operator \( \hat{U} \).

Since \( \hat{K} \) is a Hermitian matrix with \( 0 \leq \hat{K} \leq 1 \), we can parametrize it as

\[
\hat{K} = \cos \theta \cos \phi + \sin \theta \sin \phi (\vec{\chi} \cdot \vec{\sigma}),
\]

where \( \vec{\chi} \) is a unit vector and \( 0 < \phi \leq \theta \leq \frac{\pi}{2} - \phi \). It is straightforward to see that \( \hat{K}' \) is given by

\[
\hat{K}' = \sin \theta \cos \phi - \cos \theta \sin \phi (\vec{\chi} \cdot \vec{\sigma}).
\]

Since the initial overall density matrix \( \hat{\Omega}_0 \) is block diagonal, multiplication of the unitary matrix \( \hat{X} \) by a phase factor does not affect \( \hat{U} \hat{\Omega}_0 \hat{U}^\dagger \) although it modifies \( \hat{U} \). Therefore, we can parametrize \( \hat{X} \) as

\[
\hat{X} = e^{i\psi(\vec{\xi} \cdot \vec{\sigma})} = \cos \psi + i(\vec{\xi} \cdot \vec{\sigma}) \sin \psi,
\]

where \( \vec{\xi} \) is a unit vector which we assume to be perpendicular to \( \vec{\chi} \) for simplicity, and \( 0 \leq \psi \leq \pi \).

Parametrization of \( \hat{V} \) and \( \hat{W} \) can be done due to the fact that we are not interested in the off-diagonal block elements of \( \hat{\Omega}_f \). In other words, the three expectation values (2.24) do not require the determination of the off-diagonal elements of the overall density matrix and it would be sufficient to determine the action of \( \hat{V} \) and \( \hat{W} \) on the \( \hat{\sigma}_z \):

\[
\hat{V}^\dagger \hat{\sigma}_z \hat{V} = \vec{\eta} \cdot \vec{\sigma}, \quad \hat{W}^\dagger \hat{\sigma}_z \hat{W} = \vec{\zeta} \cdot \vec{\sigma},
\]

where \( \vec{\eta} \) and \( \vec{\zeta} \) are three dimensional unit vectors.

Inserting the expression for \( \hat{\Omega}_f \) from (2.23) into (2.24) using the parametrization introduced by (2.25)-(2.28) yields

\[
\langle \hat{s}_z \rangle = \lambda \cos 2\theta \cos 2\phi + \lambda (\vec{\chi} \cdot \vec{r}) \sin 2\theta \sin 2\phi \cos 2\psi \\
+ \left[ (\vec{\xi} \times \vec{\chi}) \cdot \vec{r} \right] \sin 2\theta \sin 2\phi \sin 2\psi,
\]

(2.29)
Quantum State Tomography

\[
\langle \hat{\sigma}_z \rangle + \langle \hat{s}_z \hat{\sigma}_z \rangle = \lambda (\vec{x} \cdot \vec{\eta}) \sin 2\theta \sin 2\phi + (\vec{x} \cdot \vec{\eta}) (\vec{x} \cdot \vec{r}) (1 - \lambda \cos 2\theta)(1 - \cos 2\phi) \cos 2\psi \\
+ (\vec{x} \cdot \vec{\eta}) \left[ (\vec{\xi} \times \vec{\chi}) \cdot \vec{r} \right] (\lambda - \cos 2\theta)(1 - \cos 2\phi) \sin 2\psi \\
+ (\vec{\xi} \cdot \vec{\eta}) \left( \vec{\xi} \cdot \vec{r} \right) (\cos 2\phi + \lambda \cos 2\theta)(1 - \cos 2\psi) \\
+ (\vec{\eta} \cdot \vec{r}) (\cos 2\phi + \lambda \cos 2\theta) \cos 2\psi \\
+ \left[ (\vec{\xi} \times \vec{\eta}) \cdot \vec{r} \right] (\lambda \cos 2\phi + \cos 2\theta) \sin 2\psi. \tag{2.30}
\]

Finally, \( \langle \hat{\sigma}_z \rangle - \langle \hat{s}_z \hat{\sigma}_z \rangle \) can be obtained by transforming \( 2\theta \) to \( 2\theta + \pi \) and replacing \( \vec{\eta} \) with \( \vec{\zeta} \) in (2.30).

For the sake of simplicity, we assume that \( \vec{\xi} \) is the unit vector in the \( x \)-direction and that the unit vector \( \vec{\chi} \) lies in the \( y \)-direction:

\[
\vec{\xi} = (1, 0, 0), \quad \vec{\chi} = (0, 1, 0), \quad \vec{\xi} \times \vec{\chi} = (0, 0, 1). \tag{2.31}
\]

Therefore the components of the two vectors \( \vec{\eta} \) and \( \vec{\zeta} \) on \( \vec{\xi} \) and \( \vec{\chi} \) can be defined as

\[
\eta_x \overset{\text{def}}{=} \vec{\xi} \cdot \vec{\eta}, \quad \eta_y \overset{\text{def}}{=} \vec{\chi} \cdot \vec{\eta}, \quad \eta_z = [\vec{\xi} \times \vec{\chi}] \cdot \vec{\eta} \\
\zeta_x \overset{\text{def}}{=} \vec{\xi} \cdot \vec{\zeta}, \quad \zeta_y \overset{\text{def}}{=} \vec{\chi} \cdot \vec{\zeta}, \quad \zeta_z = [\vec{\xi} \times \vec{\chi}] \cdot \vec{\zeta}. \tag{2.32}
\]

The important issue in this part of parametrization is to consider the vectors \( \vec{\chi} \) and \( \vec{\xi} \) are perpendicular to each other which substantially simplifies the calculation of the expectation values. Within the above choice of the unit vectors we can relate the measured values of the population difference of the two energy-levels of \( A \) and \( S \) to the initial state of the system \( S \) as

\[
\begin{pmatrix}
\langle \hat{s}_z \rangle \\
\langle \hat{\sigma}_z \rangle \\
\langle \hat{s}_z \hat{\sigma}_z \rangle
\end{pmatrix}
= \mathcal{C}
\begin{pmatrix}
r_x \\
r_y \\
r_z
\end{pmatrix}
+ \mathcal{F}, \tag{2.33}
\]
where \( \mathbf{C} \) is a \( 3 \times 3 \) matrix of the coefficients whose elements are given by

\[
\begin{align*}
c_{11} &= 0, \\
c_{12} &= \lambda \sin 2\theta \sin 2\phi \cos 2\psi, \\
c_{13} &= \sin 2\theta \sin 2\phi \sin 2\psi, \\
c_{21} &= (\eta_x + \zeta_x) \cos 2\phi + \lambda (\eta_x - \zeta_x) \cos 2\theta, \\
c_{22} &= (\eta_y + \zeta_y) \cos 2\psi + \lambda (\eta_y - \zeta_y) \cos 2\theta \cos 2\phi \\
&\quad - \sin 2\psi \left[ \lambda (\eta_x + \zeta_x) \cos 2\phi + (\eta_x - \zeta_x) \cos 2\theta \right], \\
c_{23} &= \lambda (\eta_y + \zeta_y) \sin 2\phi + (\eta_y - \zeta_y) \cos 2\theta \sin 2\psi \\
&\quad + \cos 2\psi \left[ (\eta_x + \zeta_x) \cos 2\phi + \lambda (\eta_x - \zeta_x) \cos 2\theta \right], \\
c_{31} &= (\eta_x - \zeta_x) \cos 2\phi + \lambda (\eta_x + \zeta_x) \cos 2\theta, \\
c_{32} &= (\eta_y - \zeta_y) \cos 2\phi + \lambda (\eta_y + \zeta_y) \cos 2\theta \sin 2\phi \\
&\quad - \sin 2\phi \left[ \lambda (\eta_x - \zeta_x) \cos 2\phi + (\eta_x + \zeta_x) \cos 2\theta \right], \\
c_{33} &= \lambda (\eta_y - \zeta_y) \sin 2\phi + (\eta_y + \zeta_y) \cos 2\theta \cos 2\phi \\
&\quad + \cos 2\psi \left[ (\eta_x - \zeta_x) \cos 2\phi + \lambda (\eta_x + \zeta_x) \cos 2\theta \right],
\end{align*}
\]

(2.34)

and \( \mathbf{F} \) is the vector of constants given by:

\[
\mathbf{F} = \lambda \begin{pmatrix} 
\cos 2\theta \cos 2\phi \\
(\eta_x - \zeta_x) \sin 2\theta \sin 2\phi \\
(\eta_y + \zeta_y) \sin 2\theta \sin 2\phi 
\end{pmatrix}.
\]

(2.35)

The elements of the initially unknown density matrix of \( S \) which are encoded by \( \mathbf{r} \) are related to these expectation values, so they can be recovered if and only if the determinant of the coefficient matrix \( \mathbf{C} \) is non-zero. With some algebra we can calculate the determinant of the coefficient matrix, represented by \( D \), as

\[
\frac{8D}{\sin 2\theta \sin 2\phi} = \frac{(1 - \lambda^2) \sin 4\psi}{2} \left[ (\cos 2\phi + \lambda \cos 2\theta) \eta_x \zeta_y \\
- (\cos 2\phi - \lambda \cos 2\theta) \eta_y \zeta_x \right] \\
+ \eta_x \zeta_x (\cos 2\phi - \lambda \cos 2\theta) \lambda \cos 2\phi \\
+ \cos 2\theta (\lambda^2 \cos^2 2\psi + \sin^2 2\psi) \\
- \eta_x \zeta_x (\cos 2\phi + \lambda \cos 2\theta) \lambda \cos 2\phi \\
- \cos 2\theta (\lambda^2 \cos^2 2\psi + \sin^2 2\psi). \]

(2.37)
Thus the initial state of the system \( S \) can be determined from \( \langle \hat{S}_z \rangle \), \( \langle \hat{\sigma}_z \rangle \) and \( \langle \hat{S}_z \hat{\sigma}_z \rangle \) provided that the determinant \( D \) is non-zero.

In what follows we consider two limiting cases: \( i \) when the assistant is initially in a completely disordered state \( (\lambda = 0) \), and \( ii \) when it starts its evolution from a pure state, i.e. \( \lambda = 1 \). Then we maximize the value of \( D \) over the parameters of \( \hat{U} \) and reconstruct the initial state of \( S \).

### 2.3.1 Assistant with completely disordered initial state

Inserting \( \lambda = 0 \) in the expression for the determinant given by (2.37) yields

\[
D = \frac{1}{16} \sin 2\theta \sin 4\phi \sin 2\psi [\cos 2\psi (\eta_x \zeta_y - \eta_y \zeta_x) + \cos 2\theta \sin 2\psi (\eta_z \zeta_x + \eta_x \zeta_z)].
\]  

(2.38)

It is clear that this determinant is maximized over the parameter \( \phi \) if \( \phi = \pm \frac{\pi}{8} \).

Furthermore, the maximum of \( D = \frac{1}{16} \sin 2\theta \sin 2\psi [\cos 2\psi (\eta_x \zeta_y - \eta_y \zeta_x) + \cos 2\theta \sin 2\psi (\eta_z \zeta_x + \eta_x \zeta_z)] \) over \( \vec{\eta} \) and \( \vec{\zeta} \) is reached when

\[
\vec{\eta} = \vec{\zeta} = \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right).
\]  

(2.39)

Thus we have

\[
D = \frac{1}{16} \sin 2\theta \sin 2\psi \sqrt{1 - \sin^2 2\theta \sin^2 2\psi}.
\]  

(2.40)

The determinant (2.40) reaches its maximum value \( 1/32 \) for \( \theta = \pi/8 \) and \( \psi = \pi/4 \). Such non-zero determinant guarantees the procedure of inversion and characterizing the initial state of the system. Inserting the above values of the parameters in the expressions for the expectation values of the \( z \)-component of spins, (2.29), (2.30) we can reconstruct the initial density matrix of \( S \):

\[
\begin{align*}
  r_x &= 2\langle \hat{S}_z \rangle, \\
  r_y &= -2\langle \hat{S}_x \hat{S}_z \rangle, \\
  r_z &= 2\langle \hat{S}_z \rangle.
\end{align*}
\]  

(2.41)

We see that for a suitable choice of the evolution operator \( \hat{U} \) it is possible to determine the initial state of a spin-1/2 system implying an assistant which is initially in completely disordered state.
2.3. Assistant with a pure initial state

Considering the assistant starts its evolution from a pure state is equivalent to set \( \lambda = 1 \) in the general expression for the determinant given by (2.37), which yields

\[
D = \frac{1}{8} \sin 2\theta \sin 2\phi (\cos^2 \phi - \cos^2 \theta)(\eta_x\zeta_x - \eta_z\zeta_z).
\] (2.42)

The maximum value of the determinant, \(|D| = 1/12\sqrt{3}\), in this case is reached when \(\vec{\eta}\) and \(\vec{\zeta}\) are perpendicular to each other

\[
\vec{\eta} = \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right), \quad \vec{\zeta} = \left( \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right),
\] (2.43)

\(\phi = \pm \frac{\pi}{4}\), and \(\sin^2(2\theta) = 1/3\) while \(\psi\) which determines a phase in the unitary operator remains an arbitrary parameter.

Thus the initial state of \(S\) can be determined as

\[
\begin{align*}
r_x &= \sqrt{3}\langle \hat{s}_z \hat{\sigma}_z \rangle, \\
r_y &= \sqrt{3}\left( \cos 2\psi \langle \hat{s}_z \rangle - \sin 2\psi \langle \hat{\sigma}_z \rangle \right), \\
r_z &= \sqrt{3}\left( \sin 2\psi \langle \hat{s}_z \rangle + \cos 2\psi \langle \hat{\sigma}_z \rangle \right).
\end{align*}
\] (2.44)

If we choose the phase \(\psi = \pi/4\), we get

\[
\begin{align*}
r_x &= \sqrt{3}\langle \hat{s}_z \hat{\sigma}_z \rangle, \\
r_y &= -\sqrt{3}\langle \hat{\sigma}_z \rangle, \\
r_z &= \sqrt{3}\langle \hat{s}_z \rangle.
\end{align*}
\] (2.45)

We conclude this section by mentioning that one can fully determine the unknown state of a spin-\(1/2\) system by simultaneous measurements of the population difference of the system and the assistant provided that the determinant \(D\) is non-zero. Very small determinant means the errors made in the measurement procedures are substantial and one cannot infer the initial state from the gathered data. Comparing the maximum values of \(D\) when the assistant is in pure state and when it is in completely random state, shows that there is not much gain in using an assistant in a pure state. In other words, it is possible to characterize the initial state of \(S\) by coupling it to an assistant being in completely disordered state. Such assistants are much easier to produce from a technical point of view.
2.4 The Jaynes-Cummings model

In previous section we did not restrict ourselves to a specific Hamiltonian in order to describe the interaction between S and A. The next section will be devoted to the case where a single coherent mode of electromagnetic field plays the role of the assistant. In this case the interaction Hamiltonian is known as the Jaynes-Cummings model. We will first introduce the model and then implement it for characterizing the initial state of a two-level atom by simultaneous measurements.

The Jaynes-Cummings model (JCM) [23] plays an important role in quantum optics and atomic physics [24–26]. This model describes the interaction of a two-level atom with a single mode of electromagnetic field, and it was employed by Jaynes and Cummings for studying the quantum features of spontaneous emission. Later on, the model generated several non-trivial theoretical predictions such as collapse and revivals of the atomic population that are related to the discretness of the photon [27, 28]. These predictions were successfully tested in experiments [29]. In particular, the model explains experimental results on one-atom masers [29], and on the passage of (Rydberg) atoms through cavities [30–33]. JCM is also used for describing quantum correlation and formation of macroscopic quantum states. It was recently employed in quantum information theory [34, 35]. More recently, the JCM has found applications in semiconductors [36], and in Josephson junctions [37–39]. JCM has denoted a family of models, since the original model of Jaynes and Cummings. It has been generalized several times for more adequate description of the atom-field interaction (e.g., multi-mode fields, multi-level atoms, damping) [40–42]. We shall however study the simplest original realization of JCM that involves a two-level atom interacting with a single mode of electromagnetic field. In particular, we neglect the effects of noise and dissipation. This situation has direct experimental realizations [29, 43, 44]. For instance with superconducting microcavities one can achieve $\sim 0.1\text{s}$ for the average lifetime of the cavity photon. This is much larger than the typical field-atom interaction time $\sim 100 - 500\mu\text{s}$ [30, 33].

2.4.1 Atom-field interaction Hamiltonian

The Jaynes-Cummings Hamiltonian is an specific type of a general atom-field interaction Hamiltonian. In this section we cast the JC Hamiltonian from this interaction Hamiltonian.
2.4. The Jaynes-Cummings model

The interaction of a radiation field $\hat{E}$ with an atom within the dipole approximation can be written as

$$\hat{H} = \hat{H}_A + \hat{H}_F - e\hat{\vec{g}} \cdot \hat{\vec{E}}. \quad (2.46)$$

Here $\hat{H}_A$ and $\hat{H}_F$ represent the Hamiltonian of the atom and field, respectively, in the absence of interaction. $\vec{g}$ is the position of the electron and $\hat{\vec{E}}$ represents the electric field.

In the dipole approximation the atom size is considered to be much smaller than the wavelength of the radiation field. Hence, the field is assumed to be uniform over the whole atom.

The energy of the free field is given in terms of the bosonic creation $\hat{a}_k^\dagger$ and annihilation $\hat{a}_k$ operators, where $k$ is the number of modes. Neglecting the zero-point energy we have

$$\hat{H}_F = \sum_\lambda \sum_k \hbar \nu_{k\lambda} \hat{a}_k^\dagger \hat{a}_k \lambda, \quad (2.47)$$

where $\nu_k$ is the frequency of the $k$–th mode, and $\lambda$ is the polarization index.

$\hat{H}_A$ and $e\hat{\vec{r}}$ can be expressed by the atom transition operator

$$\hat{\sigma}_{ij} = |i\rangle \langle j|, \quad (2.48)$$

where $\{|i\rangle\}$ represents a complete set of atomic energy eigenstates, i.e.,

$$\sum_i |i\rangle \langle i| = 1. \quad (2.49)$$

It then follows

$$\hat{H}_A = \sum_i E_i |i\rangle \langle i| = \sum_i E_i \hat{\sigma}_{ii}, \quad (2.50)$$

also

$$e\hat{\vec{g}} = \sum_{i,j} e |i\rangle \langle i| \hat{\vec{g}} |j\rangle \langle j| = \sum_{i,j} \vec{D}_{ij} \hat{\sigma}_{ij}. \quad (2.51)$$

The coefficient $\vec{D}_{ij} = e \langle i| \hat{\vec{g}} |j\rangle$ is the electric-dipole transition matrix element.

In the dipole approximation, the electric field operator is evaluated at the position of the point atom. For the atom being at the origin it the follows

$$\hat{\vec{E}} = \sum_{k,\lambda} \vec{\varepsilon}_{k\lambda} \vec{\varepsilon}_{k\lambda} (\hat{a}_{k\lambda} + \hat{a}_{k\lambda}^\dagger). \quad (2.52)$$
where $\mathcal{E}_{k\lambda} = (\hbar \nu_{k\lambda}/2\epsilon_0 V)^{1/2}$. Here $V$ represents the volume. $\vec{\varepsilon}_{k\lambda}$ are the real unit vectors of the linear polarization basis.

Inserting $\hat{H}_A, \hat{H}_F, e\vec{r}$, and $\vec{E}_k$ from Eqs. (2.47), (2.50), (2.51), and (2.52) into the total Hamiltonian given by (2.46) for a polarized field we get

$$\hat{H} = \sum_k \hbar \nu_k \hat{a}_k^\dagger \hat{a}_k + \sum_i E_i \hat{\sigma}_{ii} + \hbar \sum_{i,j} \sum_k g_{ij}^k \hat{\sigma}_{ij} (\hat{a}_k + \hat{a}_k^\dagger),$$

(2.53)

where

$$g_{ij}^k = -\frac{\vec{D}_{ij} \cdot \vec{\varepsilon}_k \mathcal{E}_k}{\hbar}.$$  

(2.54)

We now proceed to the case of a two-level atom and single mode radiation field.

We denote the eigenstates of the two-level atom by $|+\rangle$, and $|-\rangle$ with the eigenenergies $E_+$, and $E_-$, respectively.

We notice that in this case $g_{ij}^k$ reduces to a single scalar parameter, known as the atom-field coupling constant:

$$g_{ij}^k = g.$$  

(2.55)

The energy of the two-level system reads as

$$\hat{H}_A = E_+ |+\rangle \langle + | + E_- |-\rangle \langle - |.$$  

(2.56)

Defining $\hat{\sigma}_z$ as

$$\hat{\sigma}_z = \frac{1}{2} (|+\rangle \langle + | - |-\rangle \langle - |),$$  

(2.57)

the Hamilton of the atom up to an irrelevant constant energy reads

$$\hat{H}_A = \hbar \omega \hat{\sigma}_z,$$  

(2.58)

where $\omega = (E_+ - E_-)/\hbar$ is the atom frequency and a shift $(E_+ + E_-)/2$ has been omitted.

Therefore, the Hamiltonian for the interaction of a two-level system with a single mode field reads

$$\hat{H} = \hbar \omega \hat{\sigma}_z + \hbar \nu \hat{a}_a^\dagger \hat{a}_a + \hbar g (\hat{a}_a^\dagger + \hat{a}_a) \hat{\sigma}_x.$$  

(2.59)

34
In quantum optical realizations of JCM the coupling constant $g$ is normally much smaller than $\omega$ and $\nu$, e.g., it is typical to have $\nu \sim \omega \sim g \times 10^5$ ($\omega \sim \nu = 10\text{GHz}$, $g \sim \Delta = 10\text{–}100\text{KHz}$). Therefore the subsequent reasoning based on the interaction representation is legitimate. We note that in the interaction representation the coupling term reads:

$$ \hbar g (\hat{a}^\dagger e^{-i\nu t} + \hat{a} e^{i\nu t}) (\hat{\sigma} - e^{-i\omega t} + \hat{\sigma}_+ e^{i\omega t}), $$

where we introduced the raising and lowering spin operators

$$ \hat{\sigma}_+ = \hat{\sigma}_x + i\hat{\sigma}_y, \quad \hat{\sigma}_- = \hat{\sigma}_x - i\hat{\sigma}_y,$$

with the following commutation rules:

$$ [\hat{\sigma}_\pm, \hat{\sigma}_z] = \mp \hat{\sigma}_\pm, \quad [\hat{\sigma}_+, \hat{\sigma}^-] = 2\hat{\sigma}_z, \quad \hat{\sigma}_+ \hat{\sigma}^- + \hat{\sigma}^- \hat{\sigma}_+ = 1. $$

We now apply the rotating wave approximation to (2.59): the atom and field frequencies are assumed to be close to each other, therefore the factors proportional to $e^{\pm it(\nu + \omega)}$ in (2.59) oscillate in time much stronger than those proportional to $e^{\pm it(\nu - \omega)}$. Thus the rapidly oscillating terms can be neglected within this approximation and we arrive at

$$ \hat{H} = \hbar \omega \hat{\sigma}_z + \hbar \nu \hat{a}^\dagger \hat{a} + \hbar g (\hat{\sigma}_+ \hat{a} + \hat{\sigma}^- \hat{a}^\dagger), $$

which is called the JC Hamiltonian We shall denote

$$ \Delta \overset{\text{def}}{=} \omega - \nu, $$

for the detuning parameter. For our future purposes we note that $\Delta$ is a tunable parameter. Within the atom-cavity realizations of the JCM, the detuning $\Delta$ can be controlled by changing the shape of the cavity and this changes the mode frequency $\nu$. Alternatively, $\Delta$ can be changed via the atom frequency $\omega$ by applying an electric field across the cavity [45]. Then $\omega$ is modified due to the Stark effect.

The above standard derivation of (2.61) is based on small detuning $\Delta$ and weak atom-mode coupling $g$:

$$ \Delta \ll \min(\omega, \nu), \quad g \ll \min(\omega, \nu). $$

Both these conditions are normally satisfied for quantum optical realizations of JCM.
Quantum State Tomography

There are however situations—especially for the solid state physics applications of the Hamiltonian (2.59)—where the atom-field interaction constant $g$ is not small. It is useful to know that sometimes the counter-rotating terms $\propto e^{\pm i(\nu + \omega)}$ vanish due to specific selection rules, and then JCM applies in the strong-coupling situation as well. This can be achieved by proper choice of circular polarization basis for the electromagnetic field [46].

In the next section we use the fact that the Hamiltonian (2.61) is exactly solvable and derive the corresponding unitary time evolution operator. Having that at hand, we would be able to calculate the expectation value of the observables of the overall system at any time $t$.

### 2.4.2 The unitary time evolution operator

In this section, we show that the time evolution operator of the Jaynes-Cummings (JC) Hamiltonian can be calculated exactly.

We begin with rewriting the JC Hamiltonian as the sum of two commuting terms:

$$\hat{H} = \hat{H}_1 + \hat{H}_2,$$

where

$$\hat{H}_1 = \hbar \nu \hat{\sigma}_z + \hbar \nu \hat{a}^\dagger \hat{a},$$

$$\hat{H}_2 = \hbar \Delta \hat{\sigma}_z + \hbar g (\hat{\sigma}_+ \hat{a} + \hat{\sigma}_- \hat{a}^\dagger).$$

Since the two parts of the JC Hamiltonian commute with each other, the unitary time evolution operator can be factorized:

$$\hat{U}(t, 0) = e^{-i\hat{H}t/\hbar} = e^{-i\hat{H}_1t/\hbar} e^{-i\hat{H}_2t/\hbar}. \tag{2.64}$$

The first factor in the expression (2.64) is diagonal:

$$\hat{U}_1(t) = e^{-i\hat{H}_1t/\hbar} = e^{-i\nu t \hat{a}^\dagger \hat{a}} \left( \begin{array}{cc} e^{-i\nu t/2} & 0 \\ 0 & e^{i\nu t/2} \end{array} \right). \tag{2.65}$$

In order to calculate $\hat{U}_2$, we expand the expression $e^{-i\hat{H}_2t/\hbar}$:

$$\hat{U}_2(t) = e^{-i\hat{H}_2t/\hbar} = \sum_{n=0}^{\infty} \frac{(-it/\hbar)^n}{n!} (\hat{H}_2)^n \tag{2.66}$$

$$= \sum_{n=0}^{\infty} (-i)^n \frac{tn}{n!} \left( \begin{array}{cc} \frac{\Delta}{2} & g \hat{a}^\dagger \\ g \hat{a} & -\frac{\Delta}{2} \end{array} \right)^n.$$
2.4. The Jaynes-Cummings model

where we have inserted the expression (2.63) for $\hat{H}_2$.

Decomposing the matrix power series into even and odd powers, it is straightforward to see that for any integer number $l$, the even powers read

$$\left( \frac{\Delta}{2g\hat{a}^\dagger} \ - \frac{\Delta}{2} \right)^{2l} = \begin{pmatrix} (\hat{\varphi} + g^2)^l & 0 \\ 0 & \hat{\varphi}^l \end{pmatrix}$$

(2.67)

where

$$\hat{\varphi} = g^2\hat{a}^\dagger\hat{a} + \frac{\Delta^2}{4}.$$  

(2.68)

Note that $\hat{\varphi}$ is not commuting with $\hat{a}$ and $\hat{a}^\dagger$. We notice that

$$\hat{a}\hat{\varphi} = (\hat{\varphi} + g^2)\hat{a}.$$  

(2.69)

It then follows that for the odd powers we get

$$\left( \frac{\Delta}{2g\hat{a}^\dagger} \ - \frac{\Delta}{2} \right)^{2l+1} = \begin{pmatrix} \frac{\Delta}{2}(\hat{\varphi} + g^2)^l & g(\hat{\varphi} + g^2)^l\hat{a} \\ g\hat{\varphi}^l\hat{a}^\dagger & -\frac{\Delta}{2}\hat{\varphi}^l \end{pmatrix},$$

(2.70)

which then yields

$$\hat{U}_2(t) = \begin{pmatrix} \cos[t\sqrt{\hat{\varphi} + g^2}] - i\frac{\Delta}{2}\sin[t\sqrt{\hat{\varphi} + g^2}] \sqrt{\hat{\varphi} + g^2} & -ig\frac{\sin[t\sqrt{\hat{\varphi} + g^2}]}{\sqrt{\hat{\varphi} + g^2}}\hat{a} \\ -ig\frac{\sin[t\sqrt{\hat{\varphi} + g^2}]}{\sqrt{\hat{\varphi} + g^2}}\hat{a}^\dagger & \cos[t\sqrt{\hat{\varphi} + g^2}] + i\frac{\Delta}{2}\sin[t\sqrt{\hat{\varphi} + g^2}] \end{pmatrix}. \quad (2.71)$$

Equations (2.65) and (2.71) determine time evolution operator $\hat{U} = \hat{U}_1\hat{U}_2$.

In the eigenbasis of the two-level system $\hat{U}$ reads:

$$\hat{U}(t) = e^{-i\omega t(\hat{a}^\dagger\hat{a} + \frac{1}{2})} \begin{pmatrix} \cos[t\sqrt{\hat{\varphi} + g^2}] - i\frac{\Delta}{2}\sin[t\sqrt{\hat{\varphi} + g^2}] \sqrt{\hat{\varphi} + g^2} & -i\Delta\sin[t\sqrt{\hat{\varphi} + g^2}] \sqrt{\hat{\varphi} + g^2} \hat{a} \\ -i\Delta\sin[t\sqrt{\hat{\varphi} + g^2}] \sqrt{\hat{\varphi} + g^2} \hat{a}^\dagger & \cos[t\sqrt{\hat{\varphi} + g^2}] + i\frac{\Delta}{2}\sin[t\sqrt{\hat{\varphi} + g^2}] \end{pmatrix} |+\rangle\langle+|$$

$$-ige^{-i\omega t(\hat{a}^\dagger+\frac{1}{2})}\sin[t\sqrt{\hat{\varphi} + g^2}] \sqrt{\hat{\varphi} + g^2} \hat{a} |+\rangle\langle-|$$

$$-ige^{-i\omega t(\hat{a}\hat{a}^\dagger-\frac{1}{2})}\sin t\sqrt{\hat{\varphi} + g^2} \hat{a}^\dagger |+\rangle\langle-|$$

$$+e^{-i\omega t(\hat{a}^\dagger\hat{a}-\frac{1}{2})}\left( \cos t\sqrt{\hat{\varphi} + \frac{\Delta}{2}\sin t\sqrt{\hat{\varphi}}} |+\rangle\langle-| + |\hat{a}|\right), \quad (2.72)$$

where $|\pm\rangle$ are the eigenstates of $\hat{\sigma}_z$ with eigenenergies $E_\pm$. 

37
The unitarity of $\hat{U}(t)$ is satisfied because of the identities

$$\sin \left[ t \sqrt{\hat{\varphi}} + g^2 \right] \hat{a} = \hat{a} \sin \left[ t \sqrt{\hat{\varphi}} \right],$$

$$\cos \left[ t \sqrt{\hat{\varphi}} + g^2 \right] \hat{a} = \hat{a} \cos \left[ t \sqrt{\hat{\varphi}} \right].$$

Having $\hat{U}$ at hand, we can calculate any property of $S + A$ we wish.

### 2.5 Initial states

We consider the most general form of the initial state for the atom. This is described by some general mixed density matrix $\hat{\rho}_S$:

$$\hat{\rho}_S = \left( \frac{1}{2} + r_z \right) |+\rangle\langle +| + (r_x - ir_y)|+\rangle\langle -| + (r_x + ir_y)|-\rangle\langle +| + \left( \frac{1}{2} - r_z \right) |+\rangle\langle -|,$$

where we have written the initial state of $S$ in the eigen-basis of $\hat{\sigma}_z$, and where $\vec{r}$ defines unknown elements of the initial state of $S$.

For the assistant, we shall assume that the single cavity mode starts its evolution from a coherent state with a known parameter $\alpha$:

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle,$$

where $|\alpha\rangle$ is the eigenvector of the annihilation operator $\hat{a}$,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle,$$

and where $|n\rangle$ is the eigenvector of the photon number operator $\hat{a}^\dagger \hat{a}$,

$$\hat{a}^\dagger \hat{a}|n\rangle = n|n\rangle.$$

The assumption (2.75) on the initial state of the field is natural since these are the kinds of fields produced by classical currents [47], and also, to a good approximation, by sufficiently intense laser fields.
2.6 Commuting observables

We assume the system and the assistant are initially separated and do not interact with each other. As a result, the overall initial density matrix is factorized,

\[ \hat{\rho}(0) = \hat{\rho}_S \otimes |\alpha\rangle \langle \alpha| . \] (2.76)

The initial state of the field is given by

\[ |\alpha\rangle \langle \alpha| = e^{-|\alpha|^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha^n \alpha^* m}{\sqrt{n!} \sqrt{m!}} |n\rangle \langle m|. \] (2.77)

The state of \( S + A \) at a later time \( t \) can be calculated with the help of the unitary operator (2.72) calculated in section 2.4.2:

\[ \hat{\rho}(t) = \hat{U}(t) \hat{\rho}(0) \hat{U}^\dagger(t). \] (2.78)

Then the expectation value of any observable \( \hat{O} \) of the overall system at time \( t \) is

\[ \langle \hat{O} \rangle = \text{tr} \left[ \hat{\rho}(t) \hat{O} \right] . \] (2.79)

2.6 Commuting observables

The simplest set of two commuting observables of \( S + A \) with which we can build up the initial state of the atom are the energies of each system, which are described by the atom population difference \( \hat{\sigma}_z \) and \( \hat{a}^\dagger \hat{a} \). Using (2.72), (2.74), (2.77), and (2.78), the atom population difference at later time \( t \), denoted by \( \langle \hat{\sigma}_z \rangle_t \), reads

\[ \langle \hat{\sigma}_z \rangle_t = \frac{g^2}{2} \sum_{n=0}^{\infty} (n + 1)(c_{n+1} - c_n) \frac{\sin^2 (\Omega_n t/2)}{(\Omega_n t/2)^2} \]

\[ + 4g r_x \sum_{n=0}^{\infty} c_n \frac{\sin (\Omega_n t/2)}{\Omega_n} \Re \{ \chi_n(t) \} \]

\[ + 4g r_y \sum_{n=0}^{\infty} c_n \frac{\sin (\Omega_n t/2)}{\Omega_n} \Im \{ \chi_n(t) \} \]

\[ + r_z \left\{ 1 - g^2 \sum_{n=0}^{\infty} (n + 1)(c_{n+1} + c_n) \frac{\sin^2 (\Omega_n t/2)}{(\Omega_n/2)^2} \right\} , \]
Quantum State Tomography

where $r_x$, $r_y$, and $r_z$, are the unknown elements of the initial atom density matrix, which we want to find out. $\Re$ and $\Im$ stand for the real and the imaginary parts, respectively. The parameters $\chi_n(t)$, $c_n$ are defined as

$$
\chi_n(t) \overset{\text{def}}{=} \alpha \left[ \cos \left( \frac{\Omega_n t}{2} \right) + i \Delta \frac{\sin (\Omega_n t/2)}{\Omega_n} \right],
$$

and

$$
c_n \overset{\text{def}}{=} e^{-|\alpha|^2} \frac{\alpha^{2n}}{n!},
$$

where the corresponding Rabi frequency, $\Omega_n$, is defined as

$$
\Omega_n \overset{\text{def}}{=} \sqrt{4(n + 1) g^2 + \Delta^2}.
$$

The average number of photons in the cavity, $\langle \hat{a}^\dagger \hat{a} \rangle_t$, can be calculated in a similar way

$$
\langle \hat{a}^\dagger \hat{a} \rangle_t = \sum_{n=0}^{\infty} n c_n - \frac{g^2}{2} \sum_{n=0}^{\infty} (n + 1)(c_{n+1} - c_n) \frac{\sin^2(\Omega_n t/2)}{\Omega_n^2} - 4 g r_x \sum_{n=0}^{\infty} c_n \frac{\sin (\Omega_n t/2)}{\Omega_n} \Im \{\chi_n(t)\}

- 4 g r_y \sum_{n=0}^{\infty} c_n \frac{\sin (\Omega_n t/2)}{\Omega_n} \Re \{\chi_n(t)\}

+ g^2 r_z \sum_{n=0}^{\infty} (n + 1)(c_{n+1} + c_n) \frac{\sin^2(\Omega_n t/2)}{\Omega_n^2}.
$$

The correlation of the two observables, $\langle \hat{\sigma}_z \hat{a}^\dagger \hat{a} \rangle_t$, which amounts to the number of coincidences, reads

$$
\langle \hat{\sigma}_z \hat{a}^\dagger \hat{a} \rangle_t = \frac{g^2}{4} \sum_{n=0}^{\infty} (n + 1) \left[ (2n + 3)c_{n+1} - (2n + 1)c_n \right] \frac{\sin^2(\Omega_n t/2)}{\Omega_n^2} + 2 g r_x \sum_{n=0}^{\infty} c_n (2n + 1) \frac{\sin (\Omega_n t/2)}{\Omega_n} \Im \{\chi_n(t)\}

+ 2 g r_y \sum_{n=0}^{\infty} c_n (2n + 1) \frac{\sin (\Omega_n t/2)}{\Omega_n} \Re \{\chi_n(t)\}

+ r_z \left\{ \sum_{n=0}^{\infty} n c_n - \frac{(n + 1)g^2}{2} \left[ (2n + 3)c_{n+1} + (2n + 1)c_n \right] \frac{\sin^2(\Omega_n t/2)}{\Omega_n^2} \right\}.
$$
Expectedly, these three quantities, i.e., the atom population difference \( \langle \hat{\sigma}_z \rangle_t \), the average number of photons \( \langle \hat{a}^\dagger \hat{a} \rangle_t \), and the correlation of these two observables \( \langle \hat{\sigma}_z \hat{a}^\dagger \hat{a} \rangle_t \) are linearly related to the three unknown parameters \( r_x, r_y, r_z \) of the initial atom density matrix:

\[
\begin{pmatrix}
\langle \hat{\sigma}_z \rangle_t \\
\langle \hat{a}^\dagger \hat{a} \rangle_t \\
\langle \hat{\sigma}_z \hat{a}^\dagger \hat{a} \rangle_t
\end{pmatrix} = \mathcal{M} \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} + \mathcal{B}, \quad \mathcal{B} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.
\] (2.86)

The elements of the \( 3 \times 3 \) matrix \( \mathcal{M} \) and the vector \( \mathcal{B} \) are read off from Eqs. (2.80) – (2.85). They depend on the parameter \( \alpha \) of the initial assistant state, on the detuning parameters \( \Delta \), coupling \( g \) of the JC Hamiltonian, and on the interaction time \( t \). Thus, if the matrix \( \mathcal{M} \) is non-singular, i.e., the determinant of \( \mathcal{M} \) is not zero, one can invert \( \mathcal{M} \) and express the unknown parameters of the initial atom density matrix via known quantities. Although the elements of \( \mathcal{M} \) are complicated, the determinant itself is much simpler. It takes the explicit form

\[
D(t) \overset{\text{def}}{=} \det[\mathcal{M}] = 4\Delta g^2 e^{-2|\alpha|^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{|\alpha|^{2(n+m+1)}}{n!m!} (n - m) \times \left[ \sin^2 \left( \frac{\Omega_n t}{2} \right) \sin \Omega_m t - \sin^2 \left( \frac{\Omega_m t}{2} \right) \sin \Omega_n t \right].
\] (2.87)

We note that the determinant \( D(t) \) is real. At the initial time \( t = 0 \), \( D(0) \) is naturally zero, since the initial state of the overall system is factorized. According to the expression (2.87) a non-zero detuning \( \Delta \) is essential for a non vanishing determinant. Thus some non-zero detuning is crucial for the present scheme of the state determination. Although in the resonant case, i.e., when the frequency of the two-level system is equal to the cavity mode frequency, it is not possible to determine the initial state of a two-level system by measuring the the energies of the system and the assistant, however, this scheme is still applicable for spin-\( \frac{1}{2} \) systems because in order to recover the initial state of the spin-\( \frac{1}{2} \) system one can measure the transversal component of spin (\( x \) or \( y \) component) instead of the \( z \) component [48]. But in general, for two-level systems other than spins, it is rather difficult to measure the transversal component since it cannot be defined well. While the \( z \) component of quasi-spin is related to the level occupation and thus the energy of the two-level system. The crucial point in this case is that

41
Quantum State Tomography

Figure 2.1: The dynamics of the determinant of the matrix $M$ in the Jaynes-Cummings model for different values of the mean photon number in the cavity $\bar{n} = 2, 5, 10$ with two detuning parameters: $\Delta = 10$ KHz and $\Delta = 100$ KHz. The coupling constant $g = 50$ KHz in all different cases.

there should be a detuning between the frequency of the field and that of the system of interest in order to invert the relevant relations between simultaneously measured observables in one hand and the elements of the initial density matrix of the two-level system on the other hand [20]. It is seen in Figs. 2.1(a)–2.1(f) that for a non-zero detuning, the determinant $D(t)$ is non-zero for a certain initial period $t > 0$. On the other hand, large
2.6. Commuting observables

$D(t)$ implies that the state of the atom and the field are entangled [49]. Comparing figures Fig. 2.1(a) and Fig. 2.1(c) we see that although higher initial photon numbers $\bar{n}$ lead to bigger values for the determinant, they cause rapid oscillations in the value of the determinant. This makes the measurement process more difficult. (Note in this context that the determinant depends on the absolute value of $\alpha$ and $\bar{n} = |\alpha|^2$ is the average number of photons.)

If the average number of photons $\bar{n} = |\alpha|^2$ in the initial state of the field is sufficiently large, the determinant is nearly zero for intermediate times; see Figs. 2.1(e) and 2.1(f). This collapse can be understood by looking at (2.87). It has the same origin as the collapse of the atomic population difference well known for the JCM [26]. Each term in the right hand side of (2.87) oscillates with a different frequency. With time these oscillations get out of phase and $D(t)$ vanishes (collapses). However, since the number of significant oscillations in $D(t)$ is finite, they partially get in phase for later times producing the revival of $D(t)$, as seen in the Figs. 2.1(e) and 2.1(f).

It is seen that $D$ does not depend on separate frequencies $\omega$ and $\nu$ of the two-level system and the field, only their difference $\Delta = \omega - \nu$ is relevant. This is due to the choice of the measurement basis—see the left hand side of (2.86)—that involves quantities which are constants of motion for $g \to 0$. The value of $D(t)$ changes by varying the detuning parameter $\Delta$. Comparing the figures Fig. 2.1(a) with Fig. 2.1(b), Fig. 2.1(c) with Fig. 2.1(d), and Fig. 2.1(e) with Fig. 2.1(f) one observes that the value of the highest peak of $D(t)$ increases by an order of magnitude when the detuning parameter changes from 10kHz to 100kHz. Note that in Eq. (2.87) for the determinant $D(t)$ the contribution from the diagonal $n = m$ matrix elements of the assistant initial state $|\alpha\rangle\langle\alpha|$ cancels out. Thus, it is important to have an initial state of the assistant with non-zero diagonal elements in the $\{|n\rangle\}$ basis. In other words, if we consider an initial Gibbsian state for the electromagnetic field, i.e., a thermal bath at equilibrium with temperature $T$, the determinant vanishes and we cannot deduce the initial state of the atom by performing simultaneous measurements of the bath’s photon number and the population difference of the atom beam.

The principal message of this section is that the determinant $D(t)$ is not zero for a realistic range of the parameters. This means that the initial unknown state of the two level system can be determined by specifying the average atom population difference $\langle \hat{\sigma}_z \rangle_t$, the average number of photons $\langle \hat{a}^\dagger \hat{a} \rangle_t$, and their correlation $\langle \hat{\sigma}_z \hat{a}^\dagger \hat{a} \rangle_t$. These quantities are obtained from
measuring two commuting observables: the atom population difference $\hat{\sigma}_z$ and the photon number $\hat{a}^\dagger \hat{a}$. Having at hand the proper measurement data for these two observables, one can calculate $\langle \hat{\sigma}_z \rangle_t$, $\langle \hat{a}^\dagger \hat{a} \rangle_t$, and find $\langle \hat{\sigma}_z \hat{a}^\dagger \hat{a} \rangle_t$ via the number of coincidences.

### 2.6.1 Random interaction time

We saw in the previous sections that the success of the presented scheme is to a large extent determined by the ability to select properly the interaction time $t$, since this ultimately should ensure a non-zero (and sufficiently large) determinant $D(t)$ (It is clear that a small determinant will amplify numerical errors. This is illustrated in appendix B.

To quantify the robustness of the presented scheme it is reasonable to assume that there is no perfect control in choosing the interaction time. To this end let us assume that the interaction time $t$ is a random, Gaussian distributed quantity centered at $t_0$ with a dispersion $\sigma$ and that an ensemble of measurements is performed to map out this spread. The corresponding probability distribution $P(t)$ of thus reads

$$P(t) = \frac{1}{2\pi\sigma} e^{-\frac{(t-t_0)^2}{2\sigma}}.$$  

(2.88)

We notice that the expectation value of each observable as it is described in section 2.2, is calculated by making an its ensemble average. Now we have to take into account that the repeated measurement of counting the number of the photon in the cavity and the population difference of atoms are performed at a random $t$ in each set of measurement, which obeys the Gaussian distribution. Thus we have to perform a time-average in the relevant time window as well. Since we just want to get a rough estimation about the consequence of such way of measurement on the value of the determinant, we avoid the tedious time-averaging calculation of $\langle \hat{a}^\dagger \rangle$, $\langle \hat{\sigma}_z \rangle$, and their correlation by making a shortcut and perform the time-averaging of the determinant itself.

Averaging the determinant $D(t)$ over this distribution yields

$$\overline{D}(t_0) = 4\Delta g^2 e^{-2|\alpha|^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{|\alpha|^{2(n+m+1)}}{n!m!} \times$$

$$\times (n - m) [w(\Omega_n, \Omega_m; t_0) - w(\Omega_m, \Omega_n; t_0)],$$  

(2.89)
2.7 Maximum likelihood method

In section 2.5 we have shown how one determines the initial spin density matrix of a spin-$\frac{1}{2}$ system given the three averages $\langle \hat{\sigma}_z \rangle_t$, $\langle \hat{a}^\dagger \hat{a} \rangle_t$, and $\langle \hat{\sigma}_z \hat{a}^\dagger \hat{a} \rangle_t$. However, there is one important issue about this method: the recovered state from the above mentioned three averages might not correspond to a physical state because of experimental noise. By a physical state we mean a density matrix which is Hermitian, semipositive matrix with unit trace. In order to

Figure 2.2: The time averaged determinant $\bar{D}$ in the Jaynes-Cummings model as a function of $t_0$ when the mean photon number in the cavity is $\bar{n} = 2$, and $g = 50$ KHz for different values of $\Delta$ and $\sigma$; see Eqs. (2.88)–(2.90).

where

$$w(\Omega_n, \Omega_m; t_0) = \frac{1}{4\Omega_n^2\Omega_m^2} \left\{ 2e^{-\frac{\sigma}{2} \Omega_m^2} \sin[t_0\Omega_m] \right. \left. - e^{-\frac{\sigma}{2} (\Omega_m + \Omega_n)^2} \sin[t_0(\Omega_m + \Omega_n)] \right. \left. - e^{-\frac{\sigma}{2} (\Omega_m - \Omega_n)^2} \sin[(\Omega_m - \Omega_n)t_0] \right\}. \quad (2.90)$$

It is seen that the oscillations of $D(t)$ turn after averaging into exponential factors $e^{-\frac{\sigma}{2}(\Omega_m \pm \Omega_n)^2}$ and $e^{-\frac{\sigma}{2} \Omega_n^2}$ in (2.89, 2.90), due to which the averaged determinant $\bar{D}(t_0)$ gets suppressed for a sufficiently large “indeterminacy” $\sigma$. This suppression is illustrated in Fig. 2.2(a) and Fig. 2.2(b). By comparing Fig. 2.2(a) and Fig. 2.2(b) we realize that when the dispersion $\sigma$ grows by one order of magnitude, the value of the averaged determinant drops dramatically.

2.7 Maximum likelihood method

It is seen that the oscillations of $D(t)$ turn after averaging into exponential factors $e^{-\frac{\sigma}{2}(\Omega_m \pm \Omega_n)^2}$ and $e^{-\frac{\sigma}{2} \Omega_n^2}$ in (2.89, 2.90), due to which the averaged determinant $\bar{D}(t_0)$ gets suppressed for a sufficiently large “indeterminacy” $\sigma$. This suppression is illustrated in Fig. 2.2(a) and Fig. 2.2(b). By comparing Fig. 2.2(a) and Fig. 2.2(b) we realize that when the dispersion $\sigma$ grows by one order of magnitude, the value of the averaged determinant drops dramatically.

2.7 Maximum likelihood method

In section 2.5 we have shown how one determines the initial spin density matrix of a spin-$\frac{1}{2}$ system given the three averages $\langle \hat{\sigma}_z \rangle_t$, $\langle \hat{a}^\dagger \hat{a} \rangle_t$, and $\langle \hat{\sigma}_z \hat{a}^\dagger \hat{a} \rangle_t$. However, there is one important issue about this method: the recovered state from the above mentioned three averages might not correspond to a physical state because of experimental noise. By a physical state we mean a density matrix which is Hermitian, semipositive matrix with unit trace. In order to
Quantum State Tomography

avoid this problem, we employ the method of “maximum likelihood” recon-
struction [50,51] in the following section. In this approach the density matrix
that is “mostly likely” to have produced the above mentioned measured data
set is determined by numerical optimization.

Our scheme operates by measuring only the commuting variables. As
a result, for the approximate state reconstruction we do not need anything
beyond the most standard (classical) Maximum Likelihood (ML) method.
Since one measures the number of photons and the spin direction along the
$z$-axes (these quantities are represented by the operators $\hat{a}^\dagger \hat{a}$ and $\hat{\sigma}_z$, respec-
tively), the incomplete data in our case means that we are given frequencies
$\nu_a(m)$ of events, where one registered $m$ photons ($m = 0, 1, \ldots$), and where,
simultaneously, the spin component assumed values $a = \pm 1$. In the ML
method the probabilities $p_a(m)$ (given the frequencies $\nu_a(m)$) are obtained
by maximizing the likelihood function over $p_a(m)$

$$L[p_a(m)] = \sum_{a=\pm 1} \sum_{m=0}^{\infty} \nu_a(m) \ln [p_a(m)].$$  \hfill (2.91)

This maximization over $p_a(m)$ is to be carried out in the presence of relevant
constraints. For our case the initial spin density matrix $\hat{\rho}_S$ must be a positive-
definite, normalized matrix. Thus we get a single constraint

$$r_x^2 + r_y^2 + r_z^2 \leq 1.$$  \hfill (2.92)

Working out (2.86) we write this constraint as a function of the probabilities
$p_a(m)$:

$$(u - B)^T C (u - B) \leq 1,$$  \hfill (2.93)

where $^T$ means the transpose of a matrix.

$$C \overset{\text{def}}{=} (\mathcal{M} \mathcal{M}^T)^{-1}.$$  \hfill (2.94)

The matrix $\mathcal{M}$ and the vector $\mathcal{B}$ are defined in (2.86), and where finally

$$u = \left( \frac{\sum_{a=\pm 1} \sum_{m=0}^{\infty} a p_a(m)}{\sum_{a=\pm 1} \sum_{m=0}^{\infty} m p_a(m)}, \frac{\sum_{a=\pm 1} \sum_{m=0}^{\infty} m p_a(m)}{\sum_{a=\pm 1} \sum_{m=0}^{\infty} a m p_a(m)} \right).$$  \hfill (2.95)

\footnote{Equivalently, one can minimize over $p_a(m)$ the relative entropy
$\sum_{a=\pm 1} \sum_{m=0}^{\infty} \nu_a(m) \ln \frac{\nu_a(m)}{p_a(m)}$. This measure of distinguishability between $p_a(m)$
and $\nu_a(m)$ is equal to zero if and only if $p_a(m) = \nu_a(m)$.}
2.8. Conclusion

If the constraint (2.93) is satisfied automatically, the maximization of \( \mathcal{L}[p_a(m)] \) in (2.91) produces

\[
p_a(m) = \nu_a(m),
\]

(2.96)
i.e., that the sought probabilities are equal to the frequencies, as one would expect intuitively [52]. However, in general this constraint is not satisfied automatically and has to be included explicitly in the maximization of \( \mathcal{L}[p_a(m)] \) over \( p_a(m) \). Indeed, looking at (2.91) and (2.93) we may deduce qualitatively that the constraint (2.93) will be satisfied automatically by (2.96), if the frequencies are not very far from the actual probabilities (the ones that would be obtained in the perfect experiment) and, simultaneously, the determinant \( \det[M] \) is not very close to zero.

2.8 Conclusion

In this chapter we described a method for quantum state tomography. The usual way of solving this inverse problem of quantum mechanics is to make measurements of non-commuting quantities. Single apparatus tomography proceeds differently employing controlled interaction and measuring commuting observables. This is done via coupling the system of interest to an auxiliary system (assistant) that starts its evolution from a known state. The essence of the method is that the proper coupling is able to transfer the information on the initial state of the system to a commuting basis of observables for the composite system (system + assistant).

It is important to implement the single-apparatus tomography for a situation with a physically transparent measurement base and with a realistic system-assistant interaction. Here we carried out this program for a two-level atom (system) interacting with a single mode of electromagnetic field (assistant). The atom-field interaction is given by the Jaynes-Cummings Hamiltonian, which has direct experimental realizations in quantum optics [25,29–33], superconductivity [37–39], semiconductor physics [36], etc. As the measurement base we have taken the simplest set of observables related to the energies of the atom and field: population difference of the atoms \( \hat{\sigma}_z \) and the number of photons \( \hat{a}^\dagger \hat{a} \) in the field. We have shown that one can determine the unknown initial state of the atom via post-interaction values of 1) the average atomic population difference \( \langle \hat{\sigma}_z \rangle \), 2) the average number of photons
Quantum State Tomography

$\langle \hat{a}^\dagger \hat{a} \rangle$ and 3) the correlation of these quantities $\langle \hat{\sigma}_z \hat{a}^\dagger \hat{a} \rangle$. The third quantity does not need a separate measurement, since it can be recovered from the simultaneous measurement of the two basic observables $\hat{\sigma}_z$ and $\hat{a}^\dagger \hat{a}$.

Since our scheme is based on measuring commuting observables, we can apply the classical Maximum Likelihood setup for an approximate reconstruction of the unknown density matrix given the incomplete (noisy) measurement data.