Classical manipulation of a quantum system
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Spin Cooling and Polarization Transfer

In this chapter the spin polarization transfer among two non-interacting spatially separated spins coupled to a common heat bath is studied. The bath is modeled by an ensemble of harmonic oscillators. Under certain considerations the model is exactly solvable making it possible to derive the exact time-evolution of the spin components. It is shown that by introducing external forces in the form of short and strong pulses acting on one spin one can purify its state, i.e. can make relatively pure states from initially mixed ones in the presence of the other spin.

3.1 Introduction

It was shown in the previous chapter that it is possible to recover the initial state of a system by letting it interact with another quantum system. In this chapter two non-interacting quantum systems surrounded by a common environment are studied.

Quantum mechanics usually deals with the dynamics of isolated systems, but in the real world there are no perfectly closed systems. Quantum systems always interact with their surrounding environment, which are typically
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modeled by thermal baths. The interaction between a quantum system with one or a few degrees of freedom and a thermal bath with many degrees of freedom is the central concept in the theory of open quantum systems and quantum information [47,53,54]. Common approaches to study open quantum systems are based on system-plus-bath models so that the overall closed system can be described by a Hamiltonian of the general form

\[ \hat{H} = \hat{H}_S + \hat{H}_B + \hat{H}_I, \]  

(3.1)

where \( \hat{H}_S \) corresponds to the Hamiltonian of the system, \( \hat{H}_B \) represents the Hamiltonian of the bath, and \( \hat{H}_I \) stands for the interaction Hamiltonian between the system and the bath. It is worth noting that the Hilbert space belonging to \( \hat{H} \) is composed of the tensor product of the Hilbert space of the system \( S \) and the bath \( B: S \otimes B \).

Thus the state of \( S \) changes as a consequence of its internal dynamics as well as its interaction with the surrounding environment. Hence, in order to study the dynamics of an open quantum system, one has to consider the dynamics of its surroundings as well. In fact, the dynamics of the environment can imply substantial changes in the dynamics of the quantum system. For instance, the action of many variables of the bath on the system modifies the time-evolution of observables of the system by inclusion of random terms [55].

In order to describe only the dynamics of the system one usually traces out the environment degrees of freedom. This results in an effective description of the dynamics of the subsystem, the so-called reduced dynamics.

As for the quantum system one usually considers a two-level system. These are the simplest quantum systems to work with and the only physical systems whose Hilbert space corresponds exactly to that of a qubit in quantum information theory. Two-level systems describe many physical and chemical systems with discrete degrees of freedom such as spin-\( \frac{1}{2} \) particles, the polarization of a photon or a many level system where the two lowest levels are the only accessible ones. Systems having continuous degree of freedom subject to a potential energy function with two separated minima can also be modeled by two-level systems. Examples of such situations could be some type of chemical reaction involving electron transfer processes or the motion of defects in some crystalline solid [54]. Since two-level systems mathematically can be described by Pauli matrices for spin-\( \frac{1}{2} \) systems and, the bath is usually considered as an ensemble of bosons, the global model system has
been dubbed the \textit{spin-boson model}.

The Hamiltonian of a two-level system may be expressed by

\begin{equation}
\hat{H}_S = -\frac{\delta}{2} \hat{\sigma}_x + \frac{\varepsilon}{2} \hat{\sigma}_z, \tag{3.2}
\end{equation}

where $\hat{\sigma}_x$ and $\hat{\sigma}_z$ are the $x$ and $z$ components of the Pauli matrices. In the continuous degree of freedom example $\delta$ corresponds to the transition probability between wells of the potential and $\varepsilon$ stands for their energy difference. In the case of spin-$\frac{1}{2}$ systems, $\delta$ and $\varepsilon$ correspond to a static magnetic field acting on the $z$ and $x$ direction, respectively.

The bosonic bath is modeled by a set of harmonic oscillators, thus bosons. Harmonic oscillators can represent a bath of phonons to describe the electron spin resonance (ESR) \cite{53}. The set of harmonic oscillators is also used in quantum optics when one wants to describe a two-level atom interacting with a photonic bath \cite{56}. The most important properties of a bosonic bath are:

- The bath is a macroscopic entity in a stable equilibrium state with temperature $T$.
- The interaction between the system and the bath weakly perturbs the equilibrium state of the bath. Thus the system is influenced by the excitations, which can be considered as harmonic oscillator excitations. Hence the Hamiltonian of the bath is taken as

\begin{equation}
\hat{H}_B = \sum_k \hbar \omega_k \hat{a}_k \dagger \hat{a}_k, \quad [\hat{a}_k, \hat{a}_l \dagger] = \delta_{kl}, \tag{3.3}
\end{equation}

where $\hat{a}_k \dagger$ and $\hat{a}_k$ are bosonic creation and annihilation operators of the bath mode of wave vector $k$ with frequency $\omega_k$.

- The coupling of the system to the bath operator is linear in the bath harmonic oscillator operators. This corresponds to energy transfer to and from the bath by the absorption or emission of a bath quanta. This is due to the assumption of set of harmonic oscillators for the bath and the weak perturbation of the bath’s state by the system. As a result, a linear interaction is sufficient to bring the system in equilibrium state with the bath at temperature $T$ \cite{57,58}. Thus the
interaction Hamiltonian in the spin-boson model may be written as

$$\hat{H}_I = \frac{\hbar}{2} \sum_{\alpha=x,y,z} \sum_k \hat{\sigma}_\alpha \left( g^\alpha_k \hat{a}^\dagger_k + g^{\alpha\ast}_k \hat{a}_k \right), \quad (3.4)$$

where $g^\alpha_k$ is the coupling of the $\alpha$ component of the system to the bath mode of wave vector $k$ and $g_k^{\alpha\ast}$ is its complex conjugate. This interaction couples the bath to all spin components. It is usually simplified by considering the bath is only coupled to $\hat{\sigma}_z$.

- The spectrum of oscillator frequencies is smooth and dense. Thus the effect of the interaction between the bath and the system can be described by a single spectral function $J(\omega)$. This is the case when the thermodynamic limit is taken for the bath. As a result, all quantities involving the interaction with the bath will be composed of integrals of the spectral density.

- The coupling constant of the system to operators of the bath is a smooth function of the frequency of the oscillators. In the thermodynamic limit, it is not important to consider the coupling to each of the harmonic oscillators and a global description for the spin-bath interaction suffices to obtain all the interesting physics.

The couplings $g_k$ are parameterized via the spectral density function $J(\omega)$ as

$$J(\omega) = \sum_k |g_k|^2 \delta(\omega - \omega_k). \quad (3.5)$$

Within the above mentioned properties of the spin-boson model the problem is completely defined by the parameters $\varepsilon$ and $\delta$ and the function $J(\omega)$. However, it cannot be solved analytically. Here we study a simplification of this model that enables us to calculate the dynamics of the spin-$\frac{1}{2}$ system in a specific range of parameters. More precisely, we study the possibility of the polarization transfer between two spatially separated non-interacting spin-$\frac{1}{2}$ systems coupled to a common heat bath by means of an external perturbation of one of them. We show that if the initial polarization of both spins, i.e., the occupation of their ground states, is low, by applying short and strong pulses on one of them we can improve its final polarization (cool it down). Cooling spins, i.e., generating pure states from initially mixed ones is important in fields such as NMR spectroscopy and quantum information.
In NMR experiments the result depends strongly on the initial polarization of spins. Enhancing the polarization, enhances the output signal in the experiment. It is also important to have pure states in quantum information so that one can address them properly within a quantum algorithm.

The origin of cooling effect in the present model is in shifting the spins frequency by factors arising from the enhanced back reaction of the spin (via the pulses) on the collective coordinate of the bath. In the presence of another spin, this effect gets shared between the two spins.

On the other hand, the presence of the other spin yields the polarization transfer. If its initial polarization is considerable, it can be incompletely transferred to the initially unpolarized spin by applying external pulses on the latter. The existence of a thermal bath is crucial in our consideration and this process can take place due to the presence of the common thermal bath. Clearly, the effect could not survive the independent-bath approximation.

The content of this chapter is the following: In sections 3.2 we consider two independent spins at a distance $r$ from each other both immersed in a common heat bath. We show that under specific considerations the spin-boson model describing the situation is exactly solvable. Sections 3.3-3.5 are devoted to the study of the dynamics of spins observables as well as the dynamics of the bath. Then we introduce an external field in the form of sharp pulses acting on one spin in section 3.6. In section 3.7 we show how one can achieve the cooling and spin transfer via external perturbations of one spin. We also show that the results could not be achieved without the presence of the bath. This is in contrast to the usual belief that the bath is a serious hindrance one can not get rid of.

### 3.2 The model

Our model consists of two spatially separated non-interacting spin-$\frac{1}{2}$ systems (qubits) coupled to a common heat bath [59]. Spins $S_1$ and $S_2$ are subjected to static magnetic fields in the $z-$direction with the Zeeman Hamiltonian

$$\hat{H}_{Si} = \frac{\hbar \Omega_i}{2} \hat{\sigma}_z^{(i)}, \quad i = 1, 2$$  \hspace{1cm} (3.6)

where $\hat{\sigma}_x^{(i)}$, $\hat{\sigma}_y^{(i)}$, $\hat{\sigma}_z^{(i)}$ are Pauli matrices describing two spins.

The energy levels of each spin are then $\pm \frac{1}{2} \varepsilon_i$:

$$\varepsilon_i = \hbar \Omega_i, \quad i = 1, 2, \quad (3.7)$$
and where $\Omega_i$ is the frequency of the $i$-th spin. We consider $S_1$ and $S_2$ to be separated by a distance $r$ along the $z$-axis and be surrounded by a common bosonic bath with the Hamiltonian

$$\hat{H}_B = \sum_k \hbar \omega_k \hat{a}_k^\dagger \hat{a}_k.$$  \hspace{1cm} (3.8)

As for the interaction between the bath and spins we notice that the main role of a thermal bath is to derive spins initially in non-stationary state toward a stationary state. In this respect, for two-level systems (spin-$\frac{1}{2}$ systems) we distinguish two types of relaxation processes and the corresponding time-scales [60]:

- The $T_2$-time scale related to the relaxation of the average transverse components $\langle \hat{\sigma}_x^{(i)} \rangle$ and $\langle \hat{\sigma}_y^{(i)} \rangle$ of the $i$-th spin (decoherence), where $i = 1, 2$.

- The $T_1$-time scale related to the relaxation of $\langle \hat{\sigma}_z^{(i)} \rangle$, $i = 1, 2$.

It is customary to have situations where $T_2 \ll T_1$. \hspace{1cm} (3.9)

The physical reason of this assumption is that the transversal components (in the sense of Zeeman Hamiltonian for the spins) are not directly related to the energies of the spins.

It is worth mentioning that there are experimentally realized examples of two-level systems with sufficiently long $T_2$ time scale properties with $T_1$ being several orders of magnitude larger than $T_2$. For example, for atoms in optical traps $T_2 \sim 1$ s, while the response time of the bath is $10^{-8}$ s [61]. For an electric spin injected or optically excited in a semiconductor $T_2 \sim 1 \mu$s [62] and for an exciton created in a quantum dot $T_2 \sim 10^{-9}$ s [63], where in both situations the response time of the bath is of order $10^{-9} - 10^{-13}$ s. Typical femtosecond ($10^{-15}$ s) laser pulses then are suitable for the pulsed dynamics that our model is based on. In NMR physics $T_2$ ranges between $10^{-6} - 10^3$ s, with bath’s response time of the order of one micro second and the duration of the pulses can vary between 1 ps and 1 $\mu$s [64]. In all these examples the

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$^1$ There is also a third relaxation time which has a different origin. It appears due to different energies or de-phasing of the non-interacting spins. In our model we assume that $\epsilon_1$ and $\epsilon_2$ are close to each other such that this time scale is large enough.
response time of the bath is much shorter than the internal time of spins. We restrict ourselves to times much less than the relaxation time $T_1$ of the longitudinal components $\langle \hat{\sigma}_z^{(i)} \rangle$ of the two spins. We choose the spin-bath interaction Hamiltonian such that the bath induces only transversal relaxation [65,66].

\[
\hat{H}_{\text{int}} = \sum_{i=1,2} \frac{\hbar}{2} \hat{X}^{(i)} \hat{\sigma}_z^{(i)}, \quad i = 1, 2
\]  

(3.10)

where $\hat{X}^{(1)}$ and $\hat{X}^{(2)}$ are collective coordinates of the bath seen by each spin and are defined as

\[
\hat{X}^{(1)} \overset{\text{def}}{=} \sum_k \left( g_k^1 \hat{a}_k^\dagger + g_k^{1*} \hat{a}_k \right),
\]  

(3.11)

\[
\hat{X}^{(2)} \overset{\text{def}}{=} \sum_k \left( g_k^2 \hat{a}_k^\dagger + g_k^{2*} \hat{a}_k \right),
\]  

(3.12)

In general the couplings are complex and out of phase with each other. We choose different couplings to the bath labeled by $g_k^1$ and $g_k^2$ since $S_1$ and $S_2$ have an explicit spatial separation. To make this evident, we assume two spins are separated by distance $r$ in the $z$–direction such that the local interaction of each spin with the bath can be represented by

\[
g_k^1 = g_k e^{(i/2) kr \cos \theta_k}, \quad g_k^2 = g_k e^{-(i/2) kr \cos \theta_k}
\]  

(3.13)

(3.14)

where $\theta$ is the polar angle measured against the $z$–axis in $k$–space, and

\[
|g_k^1| = |g_k^2| = g_k.
\]  

(3.15)

This form of coupling is capable of preserving the translational invariance of the system in the absence of an external potential. Couplings of the form (3.14-3.14) appear in the interaction of a particle with a fermionic bath [67] or in the polaron problem [54].

The bath spectral density function $J(\omega)$ is parameterized as

\[
J(\omega) = \sum_k |g_k|^2 \delta(\omega - \omega_k).
\]  

(3.16)

We notice that due to (3.15) it is the same for both spins. The thermodynamic limit of the bath will be taken later in section 3.5.
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The overall system can be studied within the spin-boson model \[57, 65, 66, 68–73\] with the Hamiltonian

\[
\hat{H} = \hat{H}_{S_1} + \hat{H}_{S_2} + \hat{H}_{B} + \hat{H}_{\text{int}}, \tag{3.17}
\]

\[
\hat{H} = \sum_{i=1}^{2} \hbar \Omega_{i} \hat{\sigma}^{(i)}_{z} + \sum_{k} \hbar \omega_{k} \hat{a}_{k}^{\dagger} \hat{a}_{k} + \frac{\hbar}{2} \sum_{i=1}^{2} \hat{X}^{(i)} \hat{\sigma}^{(i)}_{z}. \tag{3.18}
\]

It is seen that the \(z\)-components of both spins commute with \(\hat{H}\) and hence they are conserved, so the energy of the spins are constant of motion\(^2\). This is due to restricting the model to times much less than the relaxation time \(T_{1}\) and omitting the related terms form (3.18). On the other hand \(\langle \hat{\sigma}^{(i)}_{x} \rangle\) and \(\langle \hat{\sigma}^{(i)}_{y} \rangle\) do undergo an evolution. In other words, this model describes a purely decohering mechanism, where no energy exchange between the spins and the bath is present. In fact, energy exchange processes typically involve time scales much longer than the decoherence mechanisms.

In all the above mentioned examples the response time of the bath is much shorter than the internal time \(1/\Omega_{1}\) of the spin. Although this model now has a restricted validity, it is exactly solvable under the above imposed constraints. The last ingredient of this model is to introduce external fields acting on one of the spins. This will be discussed in section 3.6 where we formulate the external field as short pulses in order to keep the model analytically solvable.

### 3.3 Time evolution of the bath and spin operators

In this section we calculate the time evolution of the components of the spin of two systems as well as the time evolution of the bath collective coordinate using the Heisenberg equation:

\[
\dot{A} = \frac{i}{\hbar} [\hat{H}, \hat{A}], \tag{3.19}
\]

\(^2\)We notice that if the two spins were identical, i.e. \(\Omega_{1} = \Omega_{2}\) and they were sitting on top of each other, e.g. at \(r = 0\), besides the quantities \(\hat{\sigma}^{(1)}_{z}, \hat{\sigma}^{(2)}_{z}\), and \(\hat{\sigma}^{(1)}_{z} \otimes \hat{\sigma}^{(2)}_{z}\), the quantities \(\hat{\sigma}^{(1)}_{x} \otimes \hat{\sigma}^{(2)}_{x}\) and \(\hat{\sigma}^{(2)}_{x} \otimes \hat{\sigma}^{(1)}_{x}\) would be conserved as well. This means, any initial density matrix of the two spins that can be presented as a linear combination of these operators will remain unchanged in the course of time under the evolution given by (3.18).
3.3. Time evolution of the bath and spin operators

where $\hat{A}$ stands for any observable of the overall system and $\dot{\hat{A}}$ represents the time derivative of $\hat{A}$.

3.3.1 Time evolution of the bath operators

The dynamics of the bath annihilation operator is given by

$$
\dot{\hat{a}}_k = -i\omega_k \hat{a}_k - \frac{i}{2} \hat{\sigma}_z^{(1)} g_k e^{(i/2)kr \cos \theta_k} - \frac{i}{2} \hat{\sigma}_z^{(2)} g_k e^{-(i/2)kr \cos \theta_k}. \quad (3.20)
$$

This differential equation can be solved by a Laplace transformation and one gets

$$
\hat{a}_k(t) = e^{-i\omega_k t} \hat{a}_k(0) + \frac{g_k}{2\omega_k} \hat{\sigma}_z^{(1)} e^{(i/2)kr \cos \theta_k} (e^{-i\omega_k t} - 1)
+ \frac{g_k}{2\omega_k} \hat{\sigma}_z^{(2)} e^{-(i/2)kr \cos \theta_k} (e^{-i\omega_k t} - 1). \quad (3.21)
$$

The complex conjugate of (3.21) gives the dynamics of the creation operator. Thus for $\hat{X}^{(1)}(t)$ and $\hat{X}^{(2)}(t)$ given by (3.11, 3.12) we have [see appendix C]

$$
\hat{X}^{(1)}(t) = \hat{\eta}_r(t) - \hat{\sigma}_z^{(1)} G(t) - \hat{\sigma}_z^{(2)} G_r(t),
\quad (3.22)
$$

$$
\hat{X}^{(2)}(t) = \hat{\eta}_{-r}(t) - \hat{\sigma}_z^{(1)} G_{-r}(t) - \hat{\sigma}_z^{(2)} G(t). \quad (3.23)
$$

The functions $G(t)$ and $G_{\pm r}(t)$ are the response functions quantifying the back reaction of $S_1$ and $S_2$ on the collective coordinate operator of the bath.

$$
G_{\pm r}(t) = \sum_k \frac{g_k^2}{\omega_k} [\cos (kr \cos \theta_k) - \cos (\omega_k t \pm kr \cos \theta_k)]. \quad (3.24)
$$

While $G(t)$ is defined as $G_{r=0}(t)$

$$
G(t) = \sum_k \frac{|g_k|^2}{\omega_k} (1 - \cos \omega_k t). \quad (3.25)
$$

We define the time integral of $G_{\pm r}(t)$ by $F_{\pm r}(t)$, as the back reaction factor of spin at distance $\pm r$

$$
F_{\pm r}(t) \overset{\text{def}}{=} \int_0^t ds \ G_{\pm r}(s) = \sum_k \frac{|g_k|^2}{\omega_k^2} \left[ \omega_k t \cos (kr \cos \theta_k) - \sin (\omega_k t \pm kr \cos \theta_k) \right]. \quad (3.26)
$$
while $F(t)$, the back reaction factor of the very same spin under the study, is defined as

$$F(t) = \sum_k \frac{|g_k|^2}{\omega_k^2} [\omega_k t - \sin(\omega_k t)]. \quad (3.27)$$

In a similar fashion, when both spins are sitting on top of each other, $F_{r=0}(t) = F(t)$.

In the following sections we will see that the back reaction factor plays an important role in the cooling and polarization transfer process and the presence of the bath is essential in this scheme.

The operator $\hat{\eta}_r(t)$ in (3.22), and (3.23) is defined as

$$\hat{\eta}_r(t) \overset{def}{=} \sum_k g_k \left[ \hat{a}^\dagger_k(0) e^{i\frac{k}{2}c \cos \theta_k + \omega_k t} + \hat{a}_k(0) e^{-i\frac{k}{2}c \cos \theta_k + \omega_k t} \right]. \quad (3.28)$$

Setting $r = 0$ yields

$$\hat{\eta}(t) = \sum_k g_k \left[ \hat{a}^\dagger_k(0) e^{i\omega_k t} + \hat{a}_k(0) e^{-i\omega_k t} \right]. \quad (3.29)$$

$\hat{\eta}_r(t)$ is named quantum noise operator which acts as a random force on spins separated by distance $r$. We notice that $\hat{\eta}_r(t)$ is determined directly in terms of bath operators at the initial time $t = 0$. Therefore the nature of the initial state of the bath plays a significant role. It is also remarkable that the commutator of the noise operator is a $c-$numbered function of time [55] and it is independent of $r$ [see appendix D]

$$[\hat{\eta}_r(t), \hat{\eta}_r(t')] = -2i\text{sign}(t - t') \sum_k |g_k|^2 \sin[\omega_k(t - t')], \quad (3.30)$$

where sign($t - t'$) represents the sign function. This commutator can be written in terms of the back reaction factor $F(t)$ as

$$[\hat{\eta}_r(t), \hat{\eta}_r(t')] = -2i\ddot{F}(t - t'). \quad (3.31)$$

This is a straightforward consequence of the definition (3.28) and the creation and annihilation operator commutation relations.

### 3.3.2 Time evolution of spin operators

We are interested in spin subsystems properties, in particular, in expectation values of the spin operators $S_1$ and $S_2$ at later times after coupling to the bath.
Choosing the Heisenberg representation we can exactly solve the Heisenberg equations of motion of the spin operators and then calculate their ensemble averages.

Let us first recall the following standard relations between the spin operators
\[
\hat{\sigma}^{(i)}_\pm = \hat{\sigma}^{(i)}_x \pm i \hat{\sigma}^{(i)}_y,
\]
\[i = 1, 2,
\]
(3.32)

\[
\left[ \hat{\sigma}^{(i)}_z, \hat{\sigma}^{(i)}_\pm \right] = \pm 2 \hat{\sigma}^{(i)}_\pm,
\]
(3.33)

\[
\hat{\sigma}^{(i)}_z \hat{\sigma}^{(i)}_\pm = \pm \hat{\sigma}^{(i)}_\pm.
\]
(3.34)

Since the overall Hamiltonian (3.18) commutes with \(\hat{\sigma}^{(i)}_z\), for \(i = 1, 2\), the Heisenberg equation for \(\hat{\sigma}^{(i)}_z\) reads
\[
\dot{\hat{\sigma}}^{(i)}_z = 0, \quad \hat{\sigma}^{(i)}_z(t) = \hat{\sigma}^{(i)}_z(0).
\]
(3.35)

It is easier to calculate the time evolution of the raising and lowering operators \(\hat{\sigma}^{(i)}_\pm\), which act on the energy eigenstates of spins rather than working out transversal components \(\hat{\sigma}^{(i)}_x\) and \(\hat{\sigma}^{(i)}_y\),
\[
\dot{\hat{\sigma}}^{(i)}_\pm = \frac{i}{\hbar} \left[ \hat{H}, \hat{\sigma}^{(i)}_\pm \right].
\]
(3.36)

Inserting \(\hat{H}\) from (3.18) and implying (3.33), for the spin \(S_1\) we have
\[
\dot{\hat{\sigma}}^{(1)}_\pm = \pm i \Omega_1 \hat{\sigma}^{(1)}_\pm \pm i \hat{X}^{(1)} \hat{\sigma}^{(1)}_\pm.
\]
(3.37)

Inserting \(\hat{X}^{(1)}(t)\) given by (3.22) into (3.37) and using (3.34) relations yields
\[
\dot{\hat{\sigma}}^{(1)}_\pm = i \left[ \pm \Omega_1 \pm \hat{n}_r(t) - G(t) \mp G_r(t) \hat{\sigma}^{(2)}_z \right] \hat{\sigma}^{(1)}_\pm.
\]
(3.38)

The equation (3.38) is a quantum Langevin-type equation with quantum noise \(\hat{n}_r(t)\), and back reactions \(G(t)\) and \(G_r(t)\).

Solving the differential equation (3.38) yields
\[
\hat{\sigma}^{(1)}_\pm(t) = \exp \left[ \pm i \Omega_1 t - i F(t) \right] \hat{\Pi}^\pm_r(0, t) \hat{\sigma}^{(1)}_\pm(0) \exp \left[ \mp i F_r(t) \hat{\sigma}^{(2)}_z \right],
\]
(3.39)

where \(\hat{\Pi}^\pm_r(t_0, t_1)\) is defined as
\[
\hat{\Pi}^\pm_r(t_0, t_1) \overset{\text{def}}{=} T \exp \left[ \pm i \int_{t_0}^{t_1} ds \hat{n}_r(s) \right],
\]
(3.40)
and where $\mathcal{T}$ stands for the time-ordering operator. The explicit expression for $\hat{\Pi}^{\pm}(t_0, t_1)$ will be given later.

The dynamics of $S_2$ located at distance $r$ from $S_1$ can be straightforwardly derived using the same analogy

$$\hat{\sigma}^{(2)}_{\pm}(t) = \exp[\pm i \Omega_2 t - i F(t)] \hat{\Pi}^{\pm}_r(0, t) \exp[\mp i F_{-r}(t) \hat{\sigma}^{(1)}_{\pm}(t)] \hat{\sigma}^{(2)}_{\pm}(0), \quad (3.41)$$

where $\hat{\Pi}^{\pm}_r(t_0, t_1)$ is defined as

$$\hat{\Pi}^{\pm}_r(t_0, t_1) \stackrel{\text{def}}{=} \mathcal{T} \exp \left[ \pm i \int_{t_0}^{t_1} ds \hat{\eta}_{-r}(s) \right]. \quad (3.42)$$

It is seen from (3.39) and (3.41) that there are three effects generated by the spin-bath interaction:

- random influences of the common bath on each spin due to the quantum noise operator; this is understandable since two non-interacting spins are open (not isolated) systems, their dynamics are not deterministic but rather contain stochastic elements due to the interaction with the thermal bath.
- A deterministic influence generated by the back reaction term of the spin under study,
- the influence generated by the back reaction term of the other spin at distance $r$ which initially assumed not to be directly interacting with the other one.

For later considerations, it is useful to calculate the time evolution of $\hat{\Pi}^{\pm}_r(t_0, t_1)$.

Defining $\mathcal{E}_t \hat{A}$ as the Heisenberg time evolution of an operator $\hat{A}$ over a period $t$,

$$\mathcal{E}_t \hat{A} \equiv e^{it \hat{H}/\hbar} \hat{A} e^{-it \hat{H}/\hbar}, \quad (3.43)$$

the Heisenberg dynamics of the quantum noise operator reads

$$\mathcal{E}_t \hat{\eta}_r(s) = \hat{\eta}_r(t + s) + \hat{\sigma}^{(1)}_z \sum_k \frac{|g_k|^2}{\omega_k} \left\{ \cos [\omega_k(t + s)] - \cos (\omega_k s) \right\} \quad (3.44)$$

$$+ \hat{\sigma}^{(2)}_z \sum_k \frac{|g_k|^2}{\omega_k} \left\{ \cos [\omega_k(t + s) + kr \cos \theta_k] - \cos (\omega_k s + kr \cos \theta_k) \right\}. \quad (3.44)$$
3.4. Factorized initial state

Inserting the expressions for the back reaction given by (3.24) in the expression for $\mathcal{E}_t \hat{\eta}_r(s)$ [see appendix D] we get

$$
\mathcal{E}_t \hat{\eta}_r(s) = \hat{\eta}_r(t + s) + [G(s) - G(t + s)] \hat{\sigma}_z^{(1)} + [G_r(s) - G_r(t + s)] \hat{\sigma}_z^{(2)}.
$$

By analogy,

$$
\mathcal{E}_t \hat{\eta}_{-r}(s) = \hat{\eta}_{-r}(t + s) + [G_{-r}(s) - G_{-r}(t + s)] \hat{\sigma}_z^{(1)} + [G(s) - G(t + s)] \hat{\sigma}_z^{(2)}.
$$

Thus, the of the operator $\hat{\Pi}_{\pm r}(0, t_1)$ reads

$$
\mathcal{E}_t \hat{\Pi}_{\pm r}(0, t_1) = \hat{\Pi}_{\pm r}(t, t + t_1) \exp \left[ \pm i \chi_{\pm r}(0, t_1, t) \hat{\sigma}_z^{(1)} \right] \exp \left[ \pm i \chi_r(0, t_1, t) \hat{\sigma}_z^{(2)} \right],
$$

where $\chi_{\pm r}(0, t_1, t)$ is defined as

$$
\chi_{\pm r}(0, t_1, t) \overset{\text{def}}{=} \int_0^t ds [G_{\pm r}(s) - G_{\pm r}(t_1 + s)],
$$

$$
= F_{\pm r}(t_1) + F_{\pm r}(t) - F_{\pm r}(t_1 + t),
$$

When deriving (3.50), we used the definition $F_{\pm r}(t_1) \equiv \int_0^{t_1} ds G_{\pm r}(s)$.

### 3.4 Factorized initial state

We assume that the spins are prepared independently from each other and the bath an then brought in contact with the bath at time $t = 0$. Thus at initial time $t = 0$ the common density matrix of the bath and spins, represented by $\hat{\rho}(0)$ is factorized:

$$
\hat{\rho}(0) = \hat{\rho}_B(0) \otimes \hat{\rho}_{S_1}(0) \otimes \hat{\rho}_{S_2}(0),
$$

where $\hat{\rho}_{S_1}(0)$ and $\hat{\rho}_{S_2}(0)$ are the initial density matrices of each spin. $\hat{\rho}_B$ stands for the Gibbs state of the bath, which is initially in equilibrium at inverse temperature $\beta$,

$$
\hat{\rho}_B = \frac{e^{-\beta \hat{H}_B}}{\text{tr}[e^{-\beta \hat{H}_B}]}.
$$
Spin Cooling and Polarization Transfer

Since the spins and the bath are initially independent, \( \hat{\rho}(0) \) can be written as

\[
\hat{\rho}(0) = e^{-\beta \hat{H}_0} \text{tr} \left[ e^{-\beta \hat{H}_0} \right],
\]

(3.53)

where \( \hat{H}_0 \) is the Zeeman Hamiltonian of each spin added to the bath Hamiltonian and is described by

\[
\hat{H}_0 = \frac{1}{2} \sum_{i=1,2} \hbar \Omega_i \hat{\sigma}_z^{(i)} + \sum_k \hbar \omega_k \hat{a}_k^\dagger \hat{a}_k.
\]

(3.54)

The factorized initial state (3.53) implies

\[
\langle \hat{\sigma}_z^{(i)}(0) \rangle = 0 \quad i = 1, 2,
\]

(3.55)

\[
\langle \hat{\sigma}_z^{(i)}(0) \rangle = -\tanh \left( \frac{\beta \hbar \Omega_i}{2} \right).
\]

(3.56)

We note that depending on the response of spins to an external static magnetic field, the initial polarization \( |\langle \hat{\sigma}_z \rangle| \) varies. This is best characterized by frequency/field ratio, which is for example 42 MHz/T for a proton. For an electron this ratio is \( 10^3 \) times larger due to the difference between atomic and nuclear Bohr magnetons. Thus at temperature \( T = 1 \) K and magnetic field \( B = 1 \) T, the equilibrium polarization of proton is only

\[
|\langle \hat{\sigma}_z \rangle| = \tanh \left( \frac{\hbar \mu B}{2 K_B T} \right) = 10^{-3},
\]

(3.57)

while for an electron it is 1000 times larger, \( |\langle \hat{\sigma}_z \rangle| \sim 1 \).

Since the bath is initially in a Gibbs state, employing (3.51) we have

\[
\langle \hat{a}_k(0) \rangle = \langle \hat{a}_k^\dagger(0) \rangle = 0,
\]

(3.58)

\[
\langle \hat{a}_k(0) \hat{a}_k^\dagger(0) \rangle = \langle n_k \rangle + 1 \delta_{kk'},
\]

(3.59)

\[
\langle \hat{a}_k^\dagger(0) \hat{a}_{k'}(0) \rangle = \delta_{kk'} \langle n_k \rangle,
\]

(3.60)

\[
\langle \hat{a}_k^\dagger(0) \hat{a}_k(0) + \hat{a}_k(0) \hat{a}_k^\dagger(0) \rangle = \coth \left( \frac{\beta \hbar \omega_k}{2} \right),
\]

(3.61)

where \( \langle n_k \rangle = \left[ e^{\beta \hbar \omega_k} - 1 \right]^{-1} \) is the thermal occupation of mode \( k \).

The relation (3.58) implies that the quantum noise has the property of

\[
\langle \hat{\eta}_r(t) \rangle = 0.
\]

(3.62)
We recall that \( \hat{\eta}_r(t) \) is determined via \( \hat{a}_k(0) \) and \( \hat{a}^\dagger_k(0) \).

Taking the average of the commutator of the noise given by (3.31) over the initial state of the bath yields

\[
\langle [\hat{\eta}_r(t), \hat{\eta}_r(t')] \rangle = -2i\dot{G}(t - t') = -2i\ddot{F}(t - t'),
\]

(3.63)

where the back reaction factor \( F(t) \) is given by (C.22).

We notice that thermal state of the bath and the anticommutation rules between the creation and annihilation operators of the bath implies (3.63) which is independent of the distance \( r \) between two spins.

Implying (3.61) we can also calculate the average of the anticommutator of the noise operator [see the appendix D] as

\[
\langle \{\hat{\eta}_r(t), \hat{\eta}_r(t')\} \rangle = 2\sum_k |g_k|^2 \coth\left(\frac{\beta\hbar\omega_k}{2}\right) \cos[\omega_k(t - t')],
\]

(3.64)

which is independent of the separation distance \( r \) between two spins due to the initial thermal state of the bath and the anti-commutation relations between the bath creation and annihilation operators.

From (3.64) the symmetrized correlation function of quantum noise operator \( K(t - t') \) reads

\[
K(t - t') = \frac{1}{2} \langle \{\hat{\eta}_r(t), \hat{\eta}_r(t')\} \rangle.
\]

(3.65)

Then, the time-order correlation function of the noise operator defined by

\[
K_T(t - t') \overset{\text{def}}{=} \langle T (\hat{\eta}_r(t) \hat{\eta}_r(t')) \rangle
\]

(3.66)

can be described in terms of the symmetrized correlation function given by (3.64) and the back reaction forces as

\[
K_T(t - t') = K(t - t') - i\dot{G}(t - t'),
\]

\[
= K(t - t') - i\ddot{F}(t - t').
\]

(3.67)

Since \( \langle \hat{\eta}_r(t) \rangle = 0 \), we can use Wick’s theorem for decomposing higher-order products of the noise operator [74]. This is important since our ultimate goal is to calculate the ensemble average of the time-evolved spin operators. Wick’s theorem is related to the fact that the commutator of the quantum noise operator is a \( c \)-number. According to this theorem if \( \langle \hat{\eta}_r(t) \rangle = 0 \), any correlation of an odd number of \( \hat{\eta}_r(t) \) vanishes. A correlation of an
even number of \( \hat{n}_r(t) \) is equal to the sum of products of pair correlations, the sum being taken over all pairings. Thus the Wick’s decomposition of 
\( \langle T(\hat{n}_r(t_1) \cdots \hat{n}_r(t_{2k})) \rangle \) will be a sum of \((2k−1)!! = (2k−1)(2k−3)\cdots3\) terms. These are the characteristic properties of a classical stationary Gaussian stochastic process [75]. Therefore, the operators \( \hat{n}_r(t) \) are called Gaussian operators [76].

We employ this property in order to calculate the time-ordered exponential of the time integral of the quantum noise which appears in (3.48) and (3.40) [66]:

\[
\langle \hat{\Pi}_r^\pm(t_0,t_1) \rangle = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \int_{t_0}^{t_1} \cdots \int_{t_0}^{t_1} ds_1 \cdots ds_2k \langle T(\hat{n}_r(s_1) \cdots \hat{n}_r(s_{2k})) \rangle \\
= \exp \left[-\frac{1}{2} \int_{t_0}^{t_1} \int_{t_0}^{t_1} ds_1 ds_2 K_T(s_1 - s_2) \right] \\
= \exp \left[-\xi(t_1 - t_0) + i F(t_1 - t_0) \right]. \tag{3.68}
\]

The function \( \xi(t_1 - t_0) \) is defined in terms of the symmetrized correlation function of the quantum noise as

\[
\xi(t_1 - t_0) = \frac{1}{2} \int_{t_0}^{t_1} \int_{t_0}^{t_1} ds_1 ds_2 K(s_1 - s_2). \tag{3.69}
\]

Since the commutator and the anticommutator of the noise operator is independent of \( r \), in the same fashion

\[
\langle \hat{\Pi}_r^\pm(t_0,t_1) \rangle = \exp \left[-\xi(t_1 - t_0) + i F(t_1 - t_0) \right]. \tag{3.70}
\]

having the expressions for the time-evolved transversal components of the spins given by (3.39), (3.41), we can calculate their ensemble averages as following:

\[
\langle \sigma_\pm^{(1)}(t) \rangle = e^{\pm i\Omega_1 t - i F(t)} \langle \Pi_+^r(0,t) \rangle \langle \sigma_\pm^{(1)}(0) \rangle \langle e^{\mp i F_r(t)} \delta^{(2)}_\pm \rangle, \tag{3.71}
\]

\[
\langle \sigma_\pm^{(2)}(t) \rangle = e^{\pm i\Omega_2 t - i F(t)} \langle \Pi_-^r(0,t) \rangle \langle e^{\mp i F_{-r}(t)} \delta^{(1)}_\pm \rangle \langle \sigma_\pm^{(2)}(0) \rangle. \tag{3.72}
\]

Inserting the averaged time-ordered operator from (3.68), we see that the back reaction forces \( F(t) \) cancel out. As a result the transversal components of spins decay due to the interaction with the bath as

\[
\langle \sigma_\pm^{(1)}(t) \rangle = e^{\pm i\Omega_1 t - \xi(t)} \langle \sigma_\pm^{(1)}(0) \rangle \langle e^{\mp i F_r(t)} \delta^{(2)}_\pm \rangle, \tag{3.73}
\]

\[
\langle \sigma_\pm^{(2)}(t) \rangle = e^{\pm i\Omega_2 t - \xi(t)} \langle e^{\mp i F_{-r}(t)} \delta^{(1)}_\pm \rangle \langle \sigma_\pm^{(2)}(0) \rangle. \tag{3.74}
\]
Thus the factor $e^{-\xi(t)}$ with $\xi(t)$ defined by (3.69) leads to decoherence for a general factorized initial state. In section 3.6 we show that $F(t)$ which disappears from (3.74) plays an important role when $S_1$ undergoes a perturbation by implying short pulses.

3.5 Ohmic spectrum of the bath

For the spin-bath interaction, we shall consider the ohmic regime [57]. The most studied ohmic case corresponds to an environment which induces a dissipative force linear in the velocity of a Brownian particle moving in it. In the ohmic regime the spectral density function reads

$$J(\omega) = \gamma \omega e^{-\omega/\Gamma}, \quad (3.75)$$

where $\gamma$ is a dimensionless coupling constant, and where $\Gamma$ (usually much larger than $\Omega_1$ and $\Omega_2$) is the maximal characteristic frequency of the bath’s response.

For the inverse dispersion relation we take the most natural one

$$k = \frac{\omega_k}{c}, \quad (3.76)$$

where $c$ is the phonon velocity in the bath.

The symmetrized correlation function of the quantum noise operator in ohmic case is given by

$$K(t) = \int_0^\infty d\omega \, J(\omega) \coth\left(\frac{\beta h\omega}{2}\right) \cos(\omega t)$$

$$= \gamma \int_0^\infty d\omega \, e^{-\omega/\Gamma} \omega \coth\left(\frac{\beta h\omega}{2}\right) \cos(\omega t), \quad (3.77)$$

where we have inserted (3.16) and (3.75) into the expression for the quantum noise correlator given by (3.64).

We notice that the decay factor $\xi(t)$ is related to $K(t)$ via Eq. (3.69). Thus we can get an exact expression for $\xi(t)$ [66]

$$\xi(t) = \gamma \ln \left[ \frac{\Gamma^2 (1+\Theta) \sqrt{\Gamma^2 + \Gamma^2 t^2}}{\Gamma (1+i\Theta \Gamma t) \Gamma (1+\Theta + i\Theta \Gamma t)} \right], \quad (3.78)$$
where $\Gamma$ is Euler’s gamma function, and $\Theta$ is defined as
\[
\Theta \overset{\text{def}}{=} \frac{1}{\beta \hbar \Gamma},
\] (3.79)
which is called a dimensionless temperature [65]. This implies that in low temperatures the decay behaves as the following power-law expression
\[
\Theta \ll 1 : \quad e^{-\xi(t)} = \left(1 + \Gamma^2 t^2\right)^{-\gamma/2}.
\] (3.80)
For $\Theta \gtrsim 1 \ e^{-\xi(t)}$ starts as a Gaussian, but continues as $e^{-t/T_2}$ with $T_2 = \hbar/2\beta \gamma$ [66,69,70,72].

In order to calculate the time derivative of the back reaction factor $G_r(t)$ in the ohmic regime, we first average $G_r(t)$ given by (3.24) over all $\theta_k$ and denote it by $\bar{G}_r(t)$
\[
\bar{G}_r(t) = \frac{1}{2} \int_0^\pi d\theta_k \sin \theta_k G_r(t)
\] (3.81)
\[
= \sum_k \frac{g_k^2}{2\omega_k} \frac{1}{kr} \left[2 \sin(\omega_k t) - \sin (\omega_k t + kr) + \sin (\omega_k t - kr)\right].
\]
We notice that
\[
G_r(t) = G_{-r}(t).
\] (3.82)
Implying $k = \omega_k/c$ and inserting the bath spectral density in (3.81) we get
\[
\bar{G}_r(t) = \int_0^\infty d\omega \frac{J(\omega)}{2\omega^2} \{2 \sin \omega \tilde{t} - \sin [\omega(t + \tilde{t})] \sin [\omega(t - \tilde{t})]\},
\] (3.83)
where we define $\tilde{t}$ as the time spent by phonons to travel between $S_1$ and $S_2$
\[
\tilde{t} \overset{\text{def}}{=} \frac{r}{c}.
\] (3.84)
Considering the ohmic regime and inserting (3.75) into (3.87) we get
\[
\bar{G}_r(t) = \frac{\gamma}{2t} \int_0^\infty d\omega \frac{\omega}{\omega^2} e^{-\omega/\Gamma} \left\{2 \sin \omega \tilde{t} - \sin [\omega(t + \tilde{t})] \sin [\omega(t - \tilde{t})]\right\},
\] (3.85)
which can be worked out [see appendix C] as
\[
\bar{G}_r(t) = \frac{\gamma}{2t} \left\{2 \arctan(\Gamma \tilde{t}) - \arctan \left[\Gamma(t + \tilde{t})\right] + \arctan \left[\Gamma(t - \tilde{t})\right]\right\}.
\] (3.86)
3.6. Pulsed dynamics

Let us recall $G(t)$ given by (3.25)

$$G(t) = \int_0^\infty d\omega \frac{J(\omega)}{\omega} (1 - \cos \omega t). \quad (3.87)$$

In the ohmic regime this function reads

$$G(t) = \gamma \int_0^\infty d\omega \, e^{-\omega/\Gamma} (1 - \cos \omega t)$$

$$= \gamma \Gamma \left( 1 - \frac{1}{1 + \Gamma^2 t^2} \right). \quad (3.88)$$

We notice that in the limit where $r$ approaches zero, $\tilde{G}_r(t)$ corresponds to $G(t)$. Calculation of the back reaction factor $F_r(t)$ in the ohmic regime yields [see appendix C]

$$\tilde{F}_r(t) = \frac{\gamma}{2\bar{\tau}} \left\{ \frac{1}{2\Gamma} \ln \left( \frac{1 + \left[ \Gamma(t + \tilde{\tau}) \right]^2}{1 + \left[ \Gamma(t - \tilde{\tau}) \right]^2} \right) + 2\tilde{\tau} \arctan (\Gamma\tilde{\tau}) \right\}.$$

We notice that setting $r = 0$ in (3.86) reproduces the result for a single spin coupled to a heat bath in the ohmic regime [66]. Moreover, for fixed $r$, at the characteristic time $1/\Gamma$, $\tilde{G}_r(t)$ and $\tilde{F}_r(t)$ become constant. It is remarkable that the time-scale of the back reaction factor $\tilde{F}_r(t)$ is temperature-independent while the decoherence time does depend on $T$.

3.6 Pulsed dynamics

In this section we study the case where one of the spins, say $S_1$, is acted on by two successive pulses. The aim is to see how this would result to a higher polarized spin state for $S_1$. The reason for applying two pulses and not just one lies in the no-cooling principle [77]. According to the no-cooling principle which is related to the second law of thermodynamics, an equilibrium system cannot be cooled by means of a cyclic external field. One cannot achieve cooling by implying a single pulse since it sees the initial local equilibrium state of the spin, and then according to the no-cooling principle
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it can only heat the spins state up. Thus we have to employ at least two pulses [65].

The external field acting on $S_1$ is described by a time-dependent Hamiltonian as

$$\hat{H}_P = \frac{1}{2} \sum_{\alpha=x,y,z} h_{\alpha}(t) \hat{\sigma}^{(1)}_{\alpha}, \quad (3.90)$$

with magnitudes $h_{\alpha}(t)$. We consider $H_P$ to be a pulse. A pulse of duration $\delta$ is defined by sudden switching on the external field at some time $t > 0$, and then suddenly switching off at time $t + \delta$.

Adding $\hat{H}_P$ to the Hamiltonian $\hat{H}$ of the overall system given by (3.18) makes the total Hamiltonian time-dependent

$$\hat{H}(t) = \hat{H} + \hat{H}_P(t). \quad (3.91)$$

In the pulsed regime [68] $\vec{h}(t)$ differs from zero only for a very short time interval $\delta$ being there very large, $\vec{h}(t)\delta \sim 1$, to achieve a finite effect. This kind of interaction was used to describe spin-echo phenomena [78], which deals with the refocusing of the precessing of nuclear spin magnetization. It is also implied in the processes of switching off undesired interactions, such as those causing decoherence [73].

It is well known that during a sudden switching on and switching off, the density matrix of the system does not change [79] while the Hamiltonian gets a finite change. For the moment, we keep an arbitrary form of the external field Hamiltonian in the time interval $(t, t + \delta)$. The operator for the Hamiltonian $\hat{H}(t)$ given by (3.91) in the time interval $(0, t + \tau)$, where $\tau > \delta$ reads

$$\mathcal{T} \exp \left[ -\frac{i}{\hbar} \int_0^{t+\tau} ds \hat{H}(s) \right] =$$

$$e^{-i[t+\tau-(t+\delta)]\hat{H}/\hbar} \mathcal{T} \exp \left[ -\frac{i}{\hbar} \int_t^{t+\delta} ds \hat{H}(s) \right] e^{-i\hat{H}t/\hbar}, \quad (3.92)$$

where $\mathcal{T}$ represents the time-ordering operator. In (3.92) we have separated out the time intervals $(0, t)$ and $(t + \delta, t + \tau)$, at which the system evolves freely and the actual time-dependence of the Hamiltonian appears only in $(t, t + \delta)$.

We denote the pulse evolution operator $\hat{U}_P(t)$ as

$$\hat{U}_P(t) \overset{\text{def}}{=} e^{i\delta \hat{H}/\hbar} \mathcal{T} \exp \left[ -\frac{i}{\hbar} \int_t^{t+\delta} ds \hat{H}(s) \right], \quad (3.93)$$
and want to show that it can be represented by a rotation operator acting on the spin of \( S_1 \).

We notice that the expression (3.93) satisfies the same first-order differential equation in \( \delta \) with the same boundary condition \( \delta = 0 \) as the following expression:

\[
T \exp \left[ -\frac{i}{\hbar} \int_0^{\delta} ds \, e^{is\hat{H}/\hbar} \hat{H}_P(s + t) e^{-is\hat{H}/\hbar} \right].
\]  

(3.94)

This can be shown by calculating the derivative of (3.93) and (3.94) with respect to \( \delta \). For the derivative of (3.93) with respect to \( \delta \) we have

\[
\frac{\partial}{\partial \delta} e^{i\delta \hat{H}/\hbar} T \exp \left[ -\frac{i}{\hbar} \int_t^{t+\delta} ds \hat{H}(s) \right] = \\
\frac{i}{\hbar} \left[ \hat{H} - e^{i\delta \hat{H}/\hbar} \hat{H}(t + \delta) e^{-i\delta \hat{H}/\hbar} \right] e^{i\delta \hat{H}/\hbar} T \exp \left[ -\frac{i}{\hbar} \int_t^{t+\delta} ds \hat{H}(s) \right] \\
- \frac{i}{\hbar} e^{i\delta \hat{H}/\hbar} \hat{H}_P(t + \delta) e^{-i\delta \hat{H}/\hbar} e^{i\delta \hat{H}/\hbar} T \exp \left[ -\frac{i}{\hbar} \int_t^{t+\delta} ds \hat{H}(s) \right].
\]  

(3.95)

where we have used \( \hat{H}(t) = \hat{H} + \hat{H}_P(t) \).

On the other hand the derivative of (3.94) with respect to \( \delta \) reads

\[
\frac{\partial}{\partial \delta} T \exp \left[ -\frac{i}{\hbar} \int_0^{\delta} ds \, e^{is\hat{H}/\hbar} \hat{H}_P(s + t) e^{-is\hat{H}/\hbar} \right] = \\
-\frac{i}{\hbar} e^{i\delta \hat{H}/\hbar} \hat{H}_P(t + \delta) e^{-i\delta \hat{H}/\hbar} T \exp \left[ -\frac{i}{\hbar} \int_0^{\delta} ds \, e^{is\hat{H}/\hbar} \hat{H}_P(s + t) e^{-is\hat{H}/\hbar} \right].
\]  

(3.96)

Thus \( \hat{U}_P(t) \) can be written as

\[
\hat{U}_P(t) = e^{i\delta \hat{H}/\hbar} T \exp \left[ -\frac{i}{\hbar} \int_t^{t+\delta} ds \hat{H}(s) \right] \\
= T \exp \left[ -\frac{i}{\hbar} \int_t^{t+\delta} ds \, e^{i(s-t)\hat{H}/\hbar} \hat{H}_P(s) e^{i(t-s)\hat{H}/\hbar} \right] \\
= T \exp \left[ -\frac{i}{\hbar} \int_0^\delta ds \, e^{is\hat{H}/\hbar} \hat{H}_P(s + t) e^{-is\hat{H}/\hbar} \right].
\]  

(3.97)

Now we show that by considering very short pulses, we can mathematically represent them with unitary operators. When the pulses acting on \( S_1 \) are
Spin Cooling and Polarization Transfer

very short so that during the time interval \((t, t + \delta)\) the terms containing \(\hat{\sigma}^{(1)}_z\) in the Hamiltonian can be neglected, we can take the first term in the Taylor expansion

\[
e^{i\delta \hat{H}/\hbar} \hat{H}_P(s + t) e^{-i\delta \hat{H}/\hbar} \approx \hat{H}_P(s + t) + \frac{i\delta}{\hbar} \hat{H}, \hat{H}_P(s + t) + \cdots
\]

(3.98)

Thus for the pulse evolution operator we get

\[
\hat{U}_P(t) = \mathcal{T} \exp \left[ -\frac{i}{\hbar} \int_0^\delta ds \hat{H}_P(s + t) \right].
\]

(3.99)

Thus a very short pulse can be represented by a unitary operator in the Hilbert space of the spin it is acting on. This means that a pulse rotates the Bloch vector \(\langle \hat{\sigma}^{(1)} \rangle\). We parameterize pulses by coefficients \(c_{n,ab}\) as

\[
\hat{P}_a \hat{\sigma}^{(1)}_a \hat{P}_b \hat{\sigma}^{(1)} = \hat{U}_P(t)^\dagger \hat{\sigma}^{(1)}_a \hat{U}_P(t) = \sum_{b=\pm,z} c_{n,ab} \hat{\sigma}^{(1)}_b, \quad a = \pm, z, \ n = 1, 2,
\]

(3.100)

where \(n = 1, 2\) counts the number of pulses.

We notice that there is no need to neglect the bath and \(S_2\) Hamiltonian during application of pulses, since external fields are acting on \(S_1\) only and the influence of the bath and \(S_2\) Hamiltonian disappear automatically. For a more detailed application, we will need the explicit form of \(\hat{U}_P(t)^\dagger\) given by (3.99). We parameterize the pulse by a \(2 \times 2\) unitary matrix as

\[
\hat{U}_P(t)^\dagger = \begin{pmatrix} e^{-i\varphi} \cos \vartheta & -e^{-i\psi} \sin \vartheta \\ e^{i\psi} \sin \vartheta & e^{i\varphi} \cos \vartheta \end{pmatrix},
\]

(3.101)

where

\[
\varphi \geq 0, \quad \psi \leq 2\pi, \quad 0 \leq \vartheta \leq \frac{\pi}{2}.
\]

Such parameterizations are common in experiments where the spin is rotated in certain degrees over a well-defined axis \([78]\). For this specific form of parametrization of the pulse, the coefficients \(c_{k,ab}\) read

\[
c_{n,+z} = -e^{i(\psi_n - \varphi_n)} \sin 2\vartheta_n, \quad c_{n,++} = e^{-2i\varphi_n} \cos^2 \vartheta_n, \\
c_{n,+z} = -e^{2i\varphi_n} \sin^2 \vartheta_n, \quad c_{n,z} = \cos 2\vartheta_n, \\
c_{n,z+} = \frac{1}{2} e^{-i(\psi_n + \varphi_n)} \sin 2\vartheta_n, \quad c_{n,z-} = \frac{1}{2} e^{i(\psi_n + \varphi_n)} \sin 2\vartheta_n,
\]

(3.103)

where \(n = 1, 2\) stands for the index of the pulses.
3.7 Cooling and polarization transfer

The problem we address in this section is set up as follows. Two non-interacting spins \( S_1 \) and \( S_2 \) separated at distance \( r \) in the \( z \)-direction are coupled to a common thermal bath. Suddenly at time \( t \), which we set to be much larger than the response time of the thermal bath, \( S_1 \) undergoes two successive pulses one at time \( t \) and the other at time \( t + \tau \). As a result \( S_1 \) cools down. Since the spins are coupled to the same bath, we observe polarization transfer between two spins. In order to study spin cooling and polarization transfer, we calculate the final polarization of both spins as well as the final averaged transversal components of spins.

The final expressions for \( \sigma_z^{(1)} \) after applying two pulses \( P_1 \) at time \( t \) and \( P_2 \) at time \( t + \tau \) reads

\[
\sigma_z^{(1)}(t + \tau) = \mathcal{E}_t P_1 \mathcal{E}_t P_2 \hat{\sigma}_z^{(1)}. \tag{3.104}
\]

Implying (3.100) for \( P_2 \hat{\sigma}_z^{(1)} \) we get

\[
\sigma_z^{(1)}(t + \tau) = \mathcal{E}_t P_1 \mathcal{E}_t \left[ c_{2,zz} \hat{\sigma}_z^{(1)} + 2 \Re \{ c_{2,z+} \hat{\sigma}_+^{(1)} \} \right]. \tag{3.105}
\]

The operator \( \mathcal{E}_\tau \) acting on \( \hat{\sigma}_z^{(1)} \) does not change it while for \( \hat{\sigma}_+^{(1)}(\tau) \) we can insert the expression (3.39) into (3.105) and get

\[
\sigma_z^{(1)}(t + \tau) = \mathcal{E}_t P_1 \left[ c_{2,zz} \hat{\sigma}_z^{(1)} \right. \\
+ \left. 2 \Re \left\{ c_{2,z+} e^{i(\Omega_1 \tau - F(\tau)) \hat{\Pi}_r^+ (0, \tau)} \hat{\sigma}_+^{(1)} e^{-iF_r(\tau) \hat{\sigma}_z^{(2)}} \right\} \right] \tag{3.106}
\]

where, for brevity, we drop the initial time \( t = 0 \) argument of \( \hat{\sigma}_z^{(i)} \) and \( \hat{\sigma}_z^{(i)} \).

Operation of the pulse \( P_1 \) on \( S_1 \) results in

\[
\sigma_z^{(1)}(t + \tau) = c_{2,zz} \mathcal{E}_t \left[ c_{1,zz} \hat{\sigma}_z^{(1)} + 2 \Re \{ c_{1,z+} \hat{\sigma}_+^{(1)} \} \right] \\
+ 2 \mathcal{E}_t \Re \left\{ c_{2,z+} e^{i(\Omega_1 \tau - F(\tau)) \hat{\Pi}_r^+ (0, \tau)} \times \right.
\]

\[
\times \left[ c_{1,z+} \hat{\sigma}_z^{(1)} + c_{1,z+} \hat{\sigma}_+^{(1)} + c_{1,z-} \hat{\sigma}_-^{(1)} \right] e^{-iF_r(\tau) \hat{\sigma}_z^{(2)}} \right\}. \tag{3.107}
\]
Applying the time evolution operator $\mathcal{E}_t$ on the spin components using (3.39), (3.49), (3.69), and (3.49) finally yields

$$\hat{\sigma}_z^{(1)}(t + \tau) = c_{2,zz} c_{1,zz} \hat{\sigma}_z^{(1)} + 2 c_{2,zz} \Re \left\{ c_{1,z+} e^{i[\Omega_1 t - F(t)]} \hat{\Pi}_+^z(0, t) \hat{\sigma}_+^{(1)} e^{-iF_r(t) \hat{\sigma}_z^{(2)}} \right\} + 2 \Re \left\{ c_{2,z+} c_{1,z+} e^{i[\Omega_1 t - F(t)]} \hat{\Pi}_+^z(t, t + \tau) \hat{\sigma}_z^{(1)} e^{i\chi_1(t, \tau) \hat{\sigma}_1^{(1)}} e^{i[\hat{\chi}_r(t, \tau) - \hat{F}_r(t)] \hat{\sigma}_2^{(2)}} \right\} + 2 \Re \left\{ c_{2,z+} c_{1,z+} e^{i[\Omega_1 t + \tau - F(t + \tau)]} \hat{\Pi}_+^z(t, t + \tau) \hat{\Pi}_+^- (0, t) \hat{\sigma}_+^{(1)} e^{-i\hat{F}_r(t + \tau) \hat{\sigma}_z^{(2)}} \right\} + 2 \Re \left\{ c_{2,z+} c_{1,z+} e^{i[\Omega_1 t + \tau - F(t + \tau)]} \hat{\Pi}_+^- (t, t + \tau) \hat{\Pi}_-^z (0, t) \hat{\sigma}_-^{(1)} \times e^{i\hat{\sigma}_z^{(2)}} \hat{F}_r(t) \frac{\hat{F}_r(t)}{\hat{F}_r(t)} - \hat{F}_r(t) \frac{\hat{F}_r(t)}{\hat{F}_r(t)} \right\},$$

where we have used the definition of $\hat{\chi}_r(t, \tau)$ as the averaged value of $\chi_1(t, \tau)$ over $\theta_k$, which is given by

$$\hat{\chi}_r(t, \tau) = \hat{F}_r(t) + \hat{F}_r(t) - \hat{F}_r(t),$$

where $\hat{F}_r(t)$ is given by (3.90). We notice that the back reaction factor $F(t)$ shifts the frequency $\Omega_1$ of $S_1$. In other words, applying pulses amounts to enhancement of the back reaction force of the spin on the collective coordinates of the bath.

Averaging (3.108) over the bath and both spin states results the final polarization of $S_1$ as

$$\langle \hat{\sigma}_z^{(1)}(t + \tau) \rangle = c_{1,zz} c_{2,zz} \langle \hat{\sigma}_z^{(1)} \rangle + 2 e^{-\xi(t)} \Re \left\{ c_{2,z+} c_{1,z+} e^{i\Omega_1 t} \langle e^{i\chi_1(t, \tau) \hat{\sigma}_1^{(1)}} \hat{\sigma}_z^{(1)} \rangle \langle e^{i[\hat{\chi}_r(t, \tau) - \hat{F}_r(t)] \hat{\sigma}_2^{(2)}} \rangle \right\},$$

where implied the initial condition (3.56) and set $\langle \hat{\sigma}_z^{(1)} \rangle = 0$ in deriving (3.110).

We notice that there are two factors that come from the bath:

- $e^{-\xi(t)}$ which amounts to the decoherence of the transversal spin components, of the system located at the origin, in the time period $\tau$ between the two pulses. Note that the transversal terms are generated by the first pulse.
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- \( \chi(\tau, t) \) and \( \chi_r(\tau, t) \), defined by (3.50), representing back reaction factors of both spins on the collective coordinate operator of the bath. As we saw in section 3.4, Eqs. (3.74–3.74), this effect is not relevant for decoherence, but it is crucial in this context.

We now work out the second term of the expression (3.110) for the final polarization of the pulsed spin after two successive pulses.

\[
2e^{-\xi(\tau)} \Re \left\{ c_{2,z+}c_{1,z+} e^{i\Omega_1 \tau} \langle e^{i\chi(\tau, t)\sigma_z^{(1)}} \rangle \langle e^{i[\chi_r(\tau, t)-F_r(\tau)]\sigma_z^{(2)}} \rangle \right\}
\]

\[
= 2e^{-\xi(\tau)} \Re \left\{ c_{2,z+}c_{1,z+} e^{i\Omega_1 \tau} P_r \right\},
\]

where

\[
P_r \defeq i \sin [\chi(\tau, t)] \cos [\chi_r(\tau, t) - F_r(\tau)]
+ i \langle \sigma_z^{(1)} \rangle \langle \sigma_z^{(2)} \rangle \cos [\chi(\tau, t)] \sin [\chi_r(\tau, t) - F_r(\tau)]
+ \langle \sigma_z^{(1)} \rangle \cos [\chi(\tau, t)] \cos [\chi_r(\tau, t) - F_r(\tau)]
- \langle \sigma_z^{(2)} \rangle \sin [\chi(\tau, t)] \sin [\chi_r(\tau, t) - F_r(\tau)].
\]

For the following results we set \( \Gamma t \gg 1 \) which means we wait long enough so the systems and the bath reach the equilibrium. Then we apply two successive pulses on the spin \( S_1 \). This guarantees the independency of the outcome result on the details of the initial state preparation. In this limit \( \chi(\tau, t) \) and \( \chi_r(\tau, t) - F_r(\tau) \) read [see appendix C]

\[
\chi_r(\tau, t) - F_r(\tau) = -\frac{\gamma T c}{r} \arctan \left( \frac{\Gamma r}{c} \right),
\]

\[
\chi(\tau, t) = -\gamma \arctan(\Gamma \tau).
\]

Inserting (3.113) into the expression of \( P_r \) given by (3.112) yields

\[
P_r = -i \sin [\gamma \arctan(\Gamma \tau)] \cos \left[ \frac{\gamma T c}{r} \arctan \left( \frac{\Gamma r}{c} \right) \right]
- i \langle \sigma_z^{(1)} \rangle \langle \sigma_z^{(2)} \rangle \cos [\gamma \arctan(\Gamma \tau)] \sin \left[ \frac{\gamma T c}{r} \arctan \left( \frac{\Gamma r}{c} \right) \right]
+ \langle \sigma_z^{(1)} \rangle \cos [\gamma \arctan(\Gamma \tau)] \cos \left[ \frac{\gamma T c}{r} \arctan \left( \frac{\Gamma r}{c} \right) \right]
- \langle \sigma_z^{(2)} \rangle \sin [\gamma \arctan(\Gamma \tau)] \sin \left[ \frac{\gamma T c}{r} \arctan \left( \frac{\Gamma r}{c} \right) \right].
\]
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Inserting (3.112) and the parameters describing the pulses from (3.103) into (3.110), the final polarization of the pulsed spin reads

\[
\langle \sigma_z^{(1)}(t + \tau) \rangle = \cos 2\vartheta_1 \cos 2\vartheta_2 \langle \sigma_z^{(1)} \rangle - \sin 2\vartheta_1 \sin 2\vartheta_2 e^{-\xi(t)} \Re \left\{ e^{i\Omega_1 \tau} e^{i\zeta} P_r \right\},
\]

where \( P_r \) is defined by (3.115) and

\[
\zeta \overset{\text{def}}{=} \psi_1 - \psi_2 - \varphi_1 - \varphi_2.
\]

This expression is one of our main results. We notice that due the presence of the first term in the expression for \( P_r \) given by (3.115) the final polarization of the pulsed spin can be non-zero even if both spins are initially completely unpolarized, i.e. in the case where \( \langle \sigma_z^{(1)} \rangle = 0 = \langle \sigma_z^{(2)} \rangle \). In this case the expression for \( P_r \) given by (3.115) reads

\[
P_r = -i \sin \left[ \gamma \arctan(\Gamma \tau) \right] \cos \left[ \frac{\gamma c}{r} \arctan \left( \frac{\Gamma \tau}{c} \right) \right].
\]

Thus the final polarization of \( S_1 \) will be

\[
\langle \sigma_z^{(1)}(t + \tau) \rangle = -e^{-\xi(t)} \sin 2\vartheta_1 \sin 2\vartheta_2 \times \sin \left[ \gamma \arctan(\Gamma \tau) \right] \cos \left[ \frac{\gamma c}{r} \arctan \left( \frac{\Gamma \tau}{c} \right) \right] \Im \{ e^{i\Omega_1 \tau} e^{i\zeta} \}.
\]

Since \( \Omega_1 \) is negligible due to the initial conditions,

\[
\langle \sigma_z^{(1)}(t + \tau) \rangle = -e^{-\xi(t)} \sin 2\vartheta_1 \sin 2\vartheta_2 \times \sin \left[ \gamma \arctan(\Gamma \tau) \right] \cos \left[ \gamma \tau \cos (kr \cos \theta) \right] \Im \{ e^{i\zeta} \}.
\]

This expression can be maximized over the pulse parameters by choosing

\[
\vartheta_1 = \vartheta_2 = \frac{\pi}{4}, \quad \zeta = \frac{\pi}{2}.
\]

In terms of pulses this means to apply a \( \frac{\pi}{2} \) pulse along the \( x \)-axis at time \( t \) followed by another \( -\frac{\pi}{2} \) pulse along the \( y \)-direction at time \( t + \tau \).

We represent a \( \frac{\pi}{2} \) pulse along the \( x \)-axis as [66]

\[
P_1 \left( x, \frac{\pi}{2} \right),
\]

\[74\]
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Figure 3.1: The final polarization of $S_1$ after two successive pulses in terms of $\Gamma \tau$ for different temperatures when the dimensionless bath coupling constant $\gamma = 2$ and $\gamma = 0.1$. We compare these two cases when both spins are initially unpolarized and are located at the same place, $r = 0$. 

where

$$\mathcal{P}_1 \left( x, \frac{\pi}{2} \right) \hat{\sigma}_z \equiv e^{i\hat{\sigma}_x \pi/4} \hat{\sigma}_z e^{-i\hat{\sigma}_x \pi/4}. \tag{3.123}$$

The first $\frac{\pi}{2}$ pulse in the $x-$direction applied on $\hat{\sigma}_z^{(1)}$ can be described by

$$\mathcal{P}_1 \left( x, \frac{\pi}{2} \right) \hat{\sigma}_z^{(1)} = \frac{1}{2i} \hat{\sigma}_+^{(1)} - \frac{1}{2i} \hat{\sigma}_-^{(1)}, \tag{3.124}$$

which in terms of the pulse coefficients $c_{n,ab}$ means

$$c_{1,zz} = 0, \quad c_{1,z+} = \frac{1}{2i}. \tag{3.125}$$

Therefore

$$\psi_1 + \varphi_1 = \frac{\pi}{2}. \tag{3.126}$$

Applying $-\frac{\pi}{2}$ pulse in the $y-$direction on $\hat{\sigma}_z^{(1)}$ gives

$$\mathcal{P}_2 \left( y, -\frac{\pi}{2} \right) \hat{\sigma}_z^{(1)} = e^{-i\hat{\sigma}_y^{(1)}\pi/4} \hat{\sigma}_z^{(1)} e^{i\hat{\sigma}_y^{(1)}\pi/4}, \tag{3.127}$$

which yields

$$\mathcal{P}_2 \left( y, -\frac{\pi}{2} \right) \hat{\sigma}_z^{(1)} = \frac{1}{2} \hat{\sigma}_+^{(1)} + \frac{1}{2} \hat{\sigma}_-^{(1)}. \tag{3.128}$$

Therefore the pulse coefficients read

$$c_{2,zz} = 0, \quad c_{2,z+} = \frac{1}{2}. \tag{3.129}$$
\[ c_{2,z+} = \frac{1}{2} \text{ implies } \psi_2 + \varphi_2 = 0. \] (3.130)

As a result, the final polarization of \( S_1 \) is

\[ \langle \sigma^{(1)}_z(t + \tau) \rangle = -e^{-\xi(\tau)} \sin [\gamma \arctan(\Gamma_\tau)] \cos \left[ \frac{\gamma \tau c}{r} \arctan \left( \frac{\Gamma r}{c} \right) \right]. \] (3.131)

The physical reason of getting non-zero final polarization is the back reaction of both spins which is now shared between them. The generation of coherence by the first \( \pi/2 \) pulse couples \( S_1 \) to the bath and \( S_2 \). The polarization of \( S_1 \) changes under the shifted frequency. We notice that the final polarization of \( S_1 \) depends on the distance between the two spins and the time \( \tau \) between two pulses. The final polarization decays with the factor \( e^{-\xi(\tau)} \), which describes the decoherence of the transversal terms produced by the first pulse in the time interval between the two pulses. Thus the time interval between two pulses should be such that it does not let the decoherence overcome the influence of the first pulse. The factor \( e^{-\xi(\tau)} \) implies that the final value of the polarization of \( S_1 \) decreases with \( \gamma \) (weaker back reaction) and \( 1/T \) (larger decoherence).

When we set \( r \to \infty \) which means we consider two separates spins each with its own bath, \( \cos \left[ \frac{\gamma \tau c}{r} \arctan \left( \frac{\Gamma r}{c} \right) \right] = 1 \) and we get the same result as in the single qubit case \([65]\):

\[ \langle \sigma^{(1)}_z(t + \tau) \rangle = -e^{-\xi(\tau)} \sin [\gamma \arctan(\Gamma_\tau)]. \] (3.132)

While considering both spins sitting on top of each other, \( r = 0 \) yields

\[ \langle \sigma^{(1)}_z(t + \tau) \rangle = -e^{-\xi(\tau)} \sin [\gamma \arctan(\Gamma_\tau)] \cos (\gamma \Gamma_\tau). \] (3.133)

Another interesting phenomenon is the case of spin transfer, which takes place if the initial polarization of \( S_2 \) is larger than the initial polarization of \( S_1 \). This can be seen from the last term in the expression (3.116) for \( P_r \) given by

\[ -\langle \sigma^{(2)}_z \rangle \sin [\gamma \arctan(\Gamma_\tau)] \sin \left[ \frac{\gamma \tau c}{r} \arctan \left( \frac{\Gamma r}{c} \right) \right]. \] (3.134)

Thus when \( S_2 \) is initially highly polarized while \( S_1 \) is unpolarized, \( P_r \) given
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Figure 3.2: The final polarization of $S_1$ after two successive pulses in terms of $\Gamma \tau$ for different temperatures when the dimensionless bath coupling constant $\gamma = 2$ and $\gamma = 0.1$. We compare these two cases when both spins are initially unpolarized and are located at the same place, $r \gg 0$.

by (3.115) reads

$$ P_r = -i \sin[\gamma \arctan(\Gamma \tau)] \cos \left[ \frac{\gamma TC}{r} \arctan \left( \frac{\Gamma r}{c} \right) \right] $$

$$ -\langle \hat{\sigma}_z^{(2)} \rangle \sin[\gamma \arctan(\Gamma \tau)] \sin \left[ \frac{\gamma TC}{r} \arctan \left( \frac{\Gamma r}{c} \right) \right], $$

and the final polarization of an initially unpolarized spin reads

$$ \langle \hat{\sigma}_z^{(1)}(t + \tau) \rangle = -e^{-\xi(\tau)} \sin 2\vartheta_1 \sin 2\vartheta_2 R \{ e^{i\varsigma} P_r \}, $$

where $P_r$ is given by (3.135).

The polarization transfer from $S_2$ to $S_1$ takes place when we consider to following pulse parameters

$$ \vartheta_1 = \vartheta_2 = \frac{\pi}{4}, \quad \varsigma = 0. $$

The above condition can be fulfilled by applying a $-\frac{\pi}{2}$ pulse in the $x-$direction followed by a $-\frac{\pi}{2}$ pulse in the $y-$direction after a time period of $\tau$. A $-\frac{\pi}{2}$ pulse in the $x-$direction means

$$ \mathcal{P}_1 \left( x, -\frac{\pi}{2} \right) \hat{\sigma}_x^{(1)} = e^{-i\theta_x^{(1)} \pi/4} \hat{\sigma}_x^{(1)} e^{i\theta_x^{(1)}}, $$
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Figure 3.3: The final polarization of $S_1$ after two successive pulses in terms of the dimensionless distance $\Gamma r/c$ for different pulse durations $\Gamma \tau$. The dimensionless bath coupling constant $\gamma = 2$, and $T_\text{He} = 0.1$. Both spins are initially unpolarized. The dotted curve: $\Gamma \tau = 1.5$, the dashed curve: $\Gamma \tau = 2$, and the solid curve: $\Gamma \tau = 2.5$.

which yields to

$$P_1 \left(x, -\frac{\pi}{2}\right) \hat{\sigma}_z^{(1)} = -\frac{1}{2i} \hat{\sigma}_+^{(1)} + \frac{1}{2} \hat{\sigma}_-^{(1)}.$$  \hspace{1cm} (3.139)

Therefore

$$\psi_1 + \varphi_1 = \frac{3\pi}{4}.$$  \hspace{1cm} (3.140)

Thus the final polarization of $S_1$ in this case reads

$$\langle \sigma_z^{(1)}(t + \tau) \rangle =$$

$$e^{-\xi(\tau)} \langle \hat{\sigma}_z^{(2)} \rangle \sin [\gamma \arctan(\Gamma \tau)] \sin \left[\frac{\gamma \tau c}{P} \arctan \left(\frac{\Gamma r}{c}\right)\right].$$  \hspace{1cm} (3.141)

Thus we see the initial polarization of $S_2$ is partially transferred to $S_1$ without modifying the initial polarization of $S_2$. This is not in contradiction with the no-cloning theorem, which states that no well-defined state can be attributed to a subsystem of an entangled state. Since in our case the quantum states are represented by commuting density matrices, they can be cloned (copied) exactly.

Thus the existence of $S_2$ improves the cooling effect via polarization transfer provided it is initially sufficiently polarized.
Now we calculate $\langle \hat{\sigma}_\pm^{(1)}(t + \tau) \rangle$ to see how the transversal components of $S_1$ which initially are zero evolve after applying two successive pulses on $S_1$. Following the same analogy as we employed in deriving the final polarization of $S_1$ we have

$$\hat{\sigma}_\pm^{(1)}(t + \tau) = \mathcal{E}_i \mathcal{P}_1 \mathcal{E}_\tau \mathcal{P}_2 \hat{\sigma}_\pm^{(1)},$$

which then yields

$$\begin{align*}
\hat{\sigma}_\pm^{(1)}(t + \tau) &= c_{2,\pm z} c_{1,zz} \hat{\sigma}_z^{(1)} \\
&+ 2 c_{2,\pm z} \Re \left\{ c_{1,zz} e^{i[\Omega t - F(t)]} \hat{\Pi}_r^+(0, t) \hat{\sigma}_z^{(1)} e^{-iF_r(t)\delta_z^{(2)}} \right\} \\
&+ c_{2,\pm z} e^{i[\Omega t - F(t)]} \hat{\Pi}_r^+(t, t + \tau) e^{i\chi(t, t)\delta_z^{(1)}} \hat{\sigma}_z^{(1)} e^{i[\hat{\chi}_r(t, t) - F_r(t)]\delta_z^{(2)}} \\
&+ c_{2,\pm} e^{-i[\Omega t + F(t)]} \hat{\Pi}_r^-(t, t + \tau) e^{-i\chi(t, t)\delta_z^{(1)}} \hat{\sigma}_z^{(1)} e^{-i[\hat{\chi}_r(t, t) - F_r(t)]\delta_z^{(2)}} \\
&+ c_{2,\pm} e^{i[\Omega t + F(t)]} \hat{\Pi}_r^+(t, t + \tau) \hat{\Pi}_r^-(0, 0) \hat{\sigma}_r^{(1)} e^{-iF_r(t)\delta_z^{(2)}} \\
&+ c_{2,\pm} e^{-i[\Omega t + F(t)]} \hat{\Pi}_r^-(t, t + \tau) \hat{\Pi}_r^+(0, 0) \hat{\sigma}_r^{(1)} e^{-iF_r(t)\delta_z^{(2)}} \\
&\times \hat{\sigma}_z^{(1)} e^{i[F_r(t) - 2F_r(t)]\delta_z^{(2)}} \\
&+ c_{2,\pm} e^{i[\Omega t - F(t)]} \hat{\Pi}_r^+(t, t + \tau) \hat{\Pi}_r^-(0, 0) \hat{\sigma}_r^{(1)} e^{-iF_r(t)\delta_z^{(2)}} \\
&\times \hat{\sigma}_z^{(1)} e^{-i[F_r(t) - 2F_r(t)]\delta_z^{(2)}}.
\end{align*}$$

Taking the ensemble averages and implying the initial condition

$$\langle \hat{\sigma}_\pm^{(1)} \rangle = 0,$$

yields

$$\begin{align*}
\langle \hat{\sigma}_\pm^{(1)}(t + \tau) \rangle &= c_{2,\pm z} c_{1,zz} \langle \hat{\sigma}_z^{(1)} \rangle \\
&+ c_{2,\pm z} e^{i\Omega t - \xi(\tau)} \langle e^{ix(t, t)\delta_z^{(1)}} \hat{\sigma}_z^{(1)} e^{i[\hat{\chi}_r(t, t) - F_r(t)]\delta_z^{(2)}} \rangle \\
&+ c_{2,\pm} e^{-i\Omega t - \xi(\tau)} \langle e^{-ix(t, t)\delta_z^{(1)}} \hat{\sigma}_z^{(1)} e^{-i[\hat{\chi}_r(t, t) - F_r(t)]\delta_z^{(2)}} \rangle.
\end{align*}$$

Inserting the pulse coefficients from (3.103) and employing the definition of $P_r$ from (3.115) for $\langle \hat{\sigma}_+^{(1)}(t + \tau) \rangle$ we get

$$\begin{align*}
\langle \hat{\sigma}_+^{(1)}(t + \tau) \rangle &=
-e^{-i(\varphi_2 - \varphi_2)} \sin 2\vartheta_2 \cos 2\vartheta_1 \langle \hat{\sigma}_z^{(1)} \rangle - \sin 2\vartheta_1 e^{-\xi(\tau)} \times \\
&\times \{ \cos^2 \vartheta_2 e^{i(\varphi_1 - \varphi_1 - 2\varphi_2)} e^{i\Omega \tau} P_r - \sin^2 \vartheta_2 e^{-i(\varphi_1 - \varphi_1 - 2\varphi_2)} e^{-i\Omega \tau} P_r \},
\end{align*}$$

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where \( P_r^* \) is the complex conjugate of \( P_r \).

Let us now consider the case where both spins are initially unpolarized; \( \langle \hat{\sigma}^{(1)}_z \rangle = 0 = \langle \hat{\sigma}^{(1)}_z \rangle \). Applying a \( \frac{\pi}{2} \) pulse along the \( x \)-direction followed by a \( -\frac{\pi}{2} \) pulse along the \( y \)-direction on \( S_1 \) with the pulse parameter

\[
\vartheta_1 = \varphi_2 = \frac{\pi}{4}, \quad \varsigma = \frac{\pi}{2},
\]

results in

\[
P_1 \left( \frac{\pi}{2}, x \right) \hat{\sigma}^{(1)}_+ = \frac{1}{2} \hat{\sigma}^{(1)}_+ + \frac{1}{2} \hat{\sigma}^{(1)}_- - i \hat{\sigma}^{(1)}_z
\]

\[
P_2 \left( -\frac{\pi}{2} \right) \hat{\sigma}^{(1)}_+ = \frac{1}{2} \hat{\sigma}^{(1)}_+ - \frac{1}{2} \hat{\sigma}^{(1)}_- - \hat{\sigma}^{(1)}_z.
\]

Thus inserting \( \vartheta_1 = \varphi_2 = \frac{\pi}{4} \) in the expressions for \( c_{1,+z}, c_{1,++}, \) and \( c_{2,+z} \) given by (3.103) we get

\[
\psi_1 = 0, \quad \varphi_1 = \frac{\pi}{2} \quad \psi_2 = \varphi_2 = 0.
\]

Now the expression for \( \langle \hat{\sigma}^{(1)}_+(t + \tau) \rangle \) reads

\[
\langle \hat{\sigma}^{(1)}_+(t + \tau) \rangle = -i e^{-\xi(t)} \mathbb{R}\{P_r\},
\]

where we take into account that \( \Omega_1 \tau \) is negligible. We notice that \( P_r \) given by the expression (3.118) is a purely imaginary expression and thus

\[
\langle \hat{\sigma}^{(1)}_x(t + \tau) \rangle = 0 = \langle \hat{\sigma}^{(1)}_y(t + \tau) \rangle.
\]

Thus while the polarization of initially unpolarized \( S_1 \) increases by applying a \( \pi \) pulse in the \( x \)-direction and a \( -\frac{\pi}{2} \) in the \( y \)-direction, its transversal components will remain the same when \( S_2 \) is also initially unpolarized.

On the other hand, when \( S_2 \) is initially highly polarized, \( |\langle \hat{\sigma}^{(2)}_z \rangle| \sim 1 \), following the same lines of calculation we get a non-zero final transversal component in the \( y \)-direction by applying two successive \( -\frac{\pi}{2} \) pulses in the \( x \) and \( y \) direction. In this case we have

\[
\psi_1 = \varphi_1 = \frac{3\pi}{4}, \quad \psi_2 = \varphi_2 = 0.
\]
Therefore

\[
\langle \hat{\sigma}_+^{(1)}(t + \tau) \rangle = -ie^{-\xi(\tau)}\Im\{P_r\},
\]

(3.154)

where \( P_r \) is given by (3.135). Thus for the final transversal components of the \( S_1 \) spin we have

\[
\langle \hat{\sigma}_x^{(1)}(t + \tau) \rangle = 0,
\]

(3.155)

\[
\langle \hat{\sigma}_y^{(1)}(t + \tau) \rangle = e^{-\xi(\tau)} \sin [\gamma \arctan (\Gamma \tau)] \cos \left[ \frac{\gamma \tau c}{r} \arctan \left( \frac{\Gamma r}{c} \right) \right].
\]

We recall that since there are no external forces applying on \( S_2 \), the process does not affect its state.

### 3.8 Conclusion

Spin-1/2 systems are quantum systems for which the Hilbert space is two-dimensional and thus can be considered as qubits. Recently, nuclear spins as examples of spin-1/2 systems have been suggested as good candidates for realizing quantum information processing [80–82]. Before any quantum algorithm can be executed, the qubits themselves must be initialized into a well-defined state which in most cases should be pure. Unfortunately, the nuclear spin systems are surrounded by the environment. That is to say, they are usually found in a highly mixed state and thus unpolarized. However, there are several techniques for increasing the polarization of nuclear spin. The most common ones are optical pumping [83] in which light is used to enhance the polarization and dynamic nuclear polarization [84], which is based on transferring the spin polarization of electrons to nuclei in a coupled two-spin system with the help of radio frequency pulses applied on both spins. Here we describe another polarization enhancement method based on the back reaction of the environment. In this scheme we consider two spatially non-interacting qubits coupled to a common bath. By applying two successive pulses on one of spins, we can increase its polarization (cool it down) even if the initial polarization of both spins is negligible. When one spin is initially highly polarized, applying two successive pulses on the initially unpolarized spin results in polarization transfer, hence enhancement of polarization of an initially unpolarized spin. We studied this situation within the spin-boson model. We showed that under certain considerations the model can be exactly solvable. For the spectral density of the bath we considered the ohmic
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regime. We showed that the origin of polarization enhancement mechanism
lies in the shared back reaction of spins to the common bath. Thus the pres-
ence of the bath is necessary. In fact, it is the only interactive component
between the spins which intermediates the polarization transformation. By
applying strong pulses on two non-interacting spins coupled to a common
bath, it is possible to reach final non-zero polarization even when both spins
are initially unpolarized. In case the pulsed spin is initially unpolarized while
the other spin has high initial polarization we obtain rather good polarization
transfer.