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An Optimal Investment Problem with Randomly Terminating Income

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***Abstract*— We investigate an optimal consumption and investment problem where we receive a certain fixed income stream that is terminated at a random time. It turns out that the optimal strategy and the value function for this problem differ considerably from the case where our income stream is certain to continue indefinitely. More specifically, the optimal consumption policy involves a function that is not analytic around the point that represents zero wealth.**

I. INTRODUCTION

In many papers which treat the problem of optimal investment and consumption strategies, the extra income that can be invested during the time period under consideration on top of the initial endowment is either treated as fixed (and often zero) or randomly fluctuating at all times. In this paper we want to treat the case where income is generated at a fixed rate up until a random time where this income stream suddenly stops. Since the income may stop at every possible moment, it is impossible to borrow an amount of cash now against the future income stream.

The problem of optimal investment and consumption without income goes back to the seminal work by Merton [10] in which this problem is formulated and solved for a model with stochastic dynamics in continuous time. Indeed, under the assumption that asset prices are Geometric Brownian Motion processes, and assuming a utility function for consumption that belongs to the class of Hyperbolic Absolute Risk Aversion (HARA) utility functions, closed-form solutions can be obtained for the optimal investment and consumption strategies and for the corresponding value function. The paper also treats the case of a deterministic income stream and the case where wages increase due to a Poisson process, but closed-form solutions can then only be found for the infinite horizon case and exponential utility functions for consumption.

These results have been extended to models where the price processes for the assets include jumps generated by Poisson processes [1] or even to the case where asset prices are only assumed to be Lévy processes [6]. For HARA utility functions, as well as exponential or logarithmic utilities, closed-form solutions are still possible for such generalized models. In papers by Huang & Pagès [4] and El Karoui & Jeanblanc [9], the optimal consumption and investment strategies are studied for an investor who receives a stochastic income stream. In both cases it is assumed that the income stream is spanned by the market assets so no new sources of uncertainty are introduced by this assumption. For HARA utility functions and an infinite time horizon, closed form solutions can be obtained. In Duffie et. al. [3] a similar problem is addressed, but here the stochastic income process is modelled using a new source of uncertainty so the market is incomplete. Using viscosity solution techniques the value function is shown to be a smooth solution of an Hamilton-Jacobi-Bellman equation. The opimal policy prescribes that at zero wealth a fixed fraction of income should be consumed.

In this paper we consider an infinite time horizon and assume that income is being paid at a fixed rate up until a random time. We believe this to be an interesting case, since the general problem with a perpetual income stream at a fixed rate can be solved by assuming that the whole income stream is exchanged for a fixed amount now (with the same present value) which is then used as extra initial wealth. Since this reduction of a problem with income to a problem without income is no longer possible if it is uncertain how long our income will last, we believe this to be an instructive case of optimal consumption problems in incomplete markets. More general results for random endowments in incomplete markets have been derived [2], [5], [8] but almost no non-trivial cases have been solved explicitly.

To find a solution to the problem we will use a dual formulation of the problem. The usefulness of dual formulations in optimal consumption problems is well established by now, see [7] for an extended overview of results. In particular we will show that certain numerical problems that arise when solving the Hamilton-Jacobi-Bellman equations for this problem can be overcome by working with the dual formulation, and it will allow us to characterize the rather interesting behaviour of the optimal investment and consumption strategies for his problem.

The paper is organised as follows. In the following section we formulate our model assumptions, and we derive the Hamilton-Jacobi-Bellman equation. The dual problem is then formulated in section 3 and in section 4 we solve a deterministic control problem based on this. In section 5 we give numerical results. In the last section we formulate conclusions and suggestions for further research.

II. PROBLEM FORMULATION

In our model, we will consider an investment and consumption problem where the only assets in which we can invest the part of our wealth that we do not consume are a bank account B and a single stock S. Trading can be done continuously, in all possible amounts, and there are no

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transaction costs. Let $B_t = e^{rt}$ be the deterministic value process of the bank account with fixed and known rate of return $r > 0$ and assume that the stock price process follows the stochastic differential equation

$$dS_t \;=\; \mu S_t dt \;+\; \sigma S_t dW_t$$

where $\mu > r$ and $\sigma > 0$ represent known constants which we call the drift and volatility of the stock respectively and $\{W_t,\ t \geq 0\}$ is a standard one-dimensional Brownian Motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The filtration $\{\mathcal{F}_t,\ t \geq 0\}$ is generated by the process W.

The amount that can be consumed and invested depends on our wealth process, which we denote by X. We assume that our rate of consumption c and percentage investment in stock π are functions of our current wealth only i.e. $c(X_t)$ and $\pi(X_t)$ with $c : \mathbb{R}^+ \to \mathbb{R}^+$ and $\pi : \mathbb{R}^+ \to \mathbb{R}$ functions that need to be chosen optimally[1]. Once these consumption and investment strategies have been chosen we can define

$$\begin{aligned} dX_t^{c,\pi,x} \;&=\; (r + (\mu - r)\pi(X_t^{c,\pi,x})X_t^{c,\pi,x}dt \\ &\quad + \pi(X_t^{c,\pi,x})X_t^{c,\pi,x}\sigma dW_t \\ &\quad + (a_t - c(X_t^{c,\pi,x}))dt \end{aligned}$$

with $X_0^{c,\pi,x} = x > 0$ the initial wealth. Here a_t is an income process that will be specified later.

A control pair $(c, \pi) : \mathbb{R}^+ \to \mathbb{R}^+ \times \mathbb{R}$ will be called *admissible* if this stochastic differential equation for $X^{c,\pi,x}$ with initial condition has a unique non-exploding solution for all $t \geq 0$. We denote the class of all admissible control laws for a given initial condition x by $\mathcal{A}_x$.

We assume that the utility $U : \mathbb{R}^+ \to \mathbb{R}$ for a certain consumption rate is given. Our objective is to maximize the expected discounted utility of consumption

$$J^{c,\pi}(x) \;=\; \mathbb{E}\int_0^\infty e^{-\delta t} U(c(X_t^{c,\pi,x}))dt$$

where $\delta > r$ is an a priori given discount rate. We are looking for a strategy (c^*, π^*) such that[2]

$$u(x) \;:=\; J^{c^*,\pi^*}(x) \;=\; \sup_{(c,\pi)\in\mathcal{A}_x} J^{c,\pi}(x), \qquad \forall\, x \in \mathbb{R}^+$$

It is well known that for $U(x) = \ln x$ and $a_t = a \geq 0$ the value function $u_1(x)$ which gives the optimal value of J when starting with an initial capital x, equals $u_1(x) = u_0(x + \frac{a}{r})$ where

$$u_0(x) \;=\; \frac{r + \frac{1}{2}\phi^2 - \delta + \delta\ln(\delta x)}{\delta^2}, \qquad \phi = \frac{\mu - r}{\sigma}$$

while the corresponding optimal strategy is given by

$$\pi_0(x) \;=\; \frac{\phi}{\sigma}\,\frac{x + \frac{a}{r}}{x}, \qquad c_0(x) \;=\; \delta\,(x + \tfrac{a}{r}).$$

[1] Since we treat the infinite horizon case only in this paper, one may show that the fact that we do not allow control laws to change over time is no restriction.

[2] A remark on notation: upper case letters represent utility functions and their duals, lower case letters represent value functions, U and u represent primal variables and V and v their duals. The subscript 0 corresponds to cases without income, the subscript 1 to cases involving income.

The case $a = 0$ can be treated by solving the Hamilton-Jacobi-Bellman equation and verifying that the solution meets all the requirements to be optimal. The case $a > 0$ then follows by a simple translation of the solution. The economic intuition behind this is easily understood. If we are guaranteed to receive an income stream equal to a and the riskfree rate of return r is known to be constant as well, then we may as well sell the income stream to someone else in return for immediately receiving its present value a/r. This value can then simply be added to our optimal current wealth x and the problem is then reduced to a problem without income. Note that this is still feasible if we have a negative[3] wealth x satisfying $-a/r < x \leq 0$.

We now consider this case of randomly terminating income. Let τ be a stochastic variable which has an exponential distribution with parameter $\eta > 0$ under $\mathbb{P}$ and is independent of the Brownian Motion W. We then define the income process by

$$a_t \;=\; aN_t, \qquad N_t = \mathbf{1}_{t\leq\tau}$$

where $a \geq 0$ is a given constant. We denote the corresponding value function by u_1 and let c_1 and π_1 be the functions which describe our strategy before the income has been terminated. After the income has been terminated we are back in the case without income, so the strategies c_0 and π_0 can be used from that time on.

Theorem 2.1: The value function $u_1(x)$ is finite in $x = 0$. If u_1 is twice continuously differentiable and strictly concave then $u_1'(0) = \infty$ and

$$\lim_{x\downarrow 0} \frac{\pi_1(x)}{\frac{\phi}{\sigma}\ln\frac{1}{x}} = 1, \qquad \lim_{x\downarrow 0} \frac{c_1(x)}{\frac{a\delta}{\eta\ln\frac{1}{x}}} = 1.$$

Proof. Clearly, the value function u_1 must satisfy

$$u_0(x) \leq u_1(x) \leq u_0(x + \tfrac{a}{r})$$

for all $x \geq 0$ so $u_1(0) < \infty$. If we have zero wealth we can still consume half of our income as long as it is there (so we take $c_1(x) = \frac{1}{2}a, \pi_1(x) = 0$) and we will then have a wealth of $\frac{1}{2}a(e^{r\tau} - 1)$ the moment our income terminates, and therefore a finite expected utility value. This shows that $u_1(0) > -\infty$.

The function u_1 should satisfy the HJB-equation

$$\begin{aligned} 0 \;=\; &\sup_{\pi_1,c_1}\Bigg(U(c_1) + \tfrac{1}{2}\pi_1^2 x^2\sigma^2\frac{d^2u_1}{dx^2} \\ &\qquad + \frac{du_1}{dx}[-c_1 + a + x(r + (\mu - r)\pi_1)]\Bigg) \\ &-\delta u_1 - \eta(u_1 - u_0) \end{aligned}$$

Carrying out the optimization for $U(x) = \ln x$ gives the optimal strategies for the amounts to invest and consume:

$$x\pi_1(x) \;=\; -\frac{\phi u_1'(x)}{\sigma u_1''(x)}, \qquad c_1(x) \;=\; \frac{1}{u_1'(x)}.$$

[3] See [9] for a consumption and investment problem with the explicit condition that the current wealth should remain positive at all times.

Note that this implies that for all $x > 0$

$$x\pi_1(x)\frac{c_1'(x)}{c_1(x)} = \frac{\phi}{\sigma}. \qquad (1)$$

The HJB equation then reduces to

$$\begin{aligned} 0 &= \eta u_0(x) - \ln(u_1'(x)) - \tfrac{1}{2}\phi^2\frac{(u_1'(x))^2}{u_1''(x)} \\ &\quad -1 + u_1'(x)(a+rx) - (\eta+\delta)u_1(x) \qquad (2) \end{aligned}$$

or, equivalently,

$$\begin{aligned} 0 &= \tfrac{\eta}{\delta^2}(r + \tfrac{1}{2}\phi^2 - \delta + \delta\ln(\delta x)) + \ln(c_1(x)) - 1 \\ &\quad + \frac{a + rx + \frac{1}{2}(\mu - r)x\pi_1(x)}{c_1(x)} - (\eta+\delta)u_1(x). \end{aligned}$$

Taking the righthand side limit for x going to zero from above gives $\lim_{x\downarrow 0} c_1(x) = 0$ since we already established that $\lim_{x\downarrow 0} u_1(x)$ is finite.

Define the function X_1 as the inverse of c_1 (this inverse exists since u_1 is strictly concave so c_1 is strictly increasing and continuous). We then find

$$\begin{aligned} 0 &= \eta u_0(x) + \ln c_1(x) + \tfrac{1}{2}\phi^2\frac{1}{c_1'(x)} - 1 \\ &\quad + \frac{a+rx}{c_1(x)} - \alpha u_1(x) \end{aligned}$$

or

$$\begin{aligned} 0 &= \eta u_0(X_1(c)) + \ln c + \tfrac{1}{2}\phi^2 X_1'(c) - 1 \\ &\quad + \frac{a + rX_1(c)}{c} - \alpha u_1(X_1(c)) \end{aligned}$$

for all $c > 0$, and hence

$$\begin{aligned} -\eta\delta^{-2}(r + \tfrac{1}{2}\phi^2 - \delta + \delta\ln\delta) + 1 + \alpha u_1(X_1(c)) &= \\ \frac{\eta}{\delta}\ln X_1(c) + \ln c + \frac{a + rX_1(c)}{c} + \tfrac{1}{2}\phi^2 X_1'(c). \end{aligned}$$

The lefthand side converges to a finite value when $c \downarrow 0$, and we call the limit K. If $\lim_{c\downarrow 0} X_1'(c) = d > 0$ then the righthand side becomes ∞ for $c \downarrow 0$, so we must have that $\lim_{c\downarrow 0} X_1'(c) = 0$. But then we find

$$X_1(c) = c^{-\frac{\delta}{\eta}} e^{-\frac{a\delta}{c\eta} + \frac{\delta}{\eta}K}(1 + o(1)).$$

The result for c_1 now follows, and the result for π_1 can then be derived by (1) and L'Hôpital's Rule. ■

Note that this result means that the optimal consumed and invested wealth $c_1(x)$ and $x\pi_1(x)$ converge to zero for x to zero, while the consumed and invested *percentages* of wealth $c_1(x)/x$ and $\pi_1(x)$ go to infinity.

Even if the ordinary differential equation does indeed have a solution which is twice continuously differentiable on $]0, \infty[$ it is not clear how to find numerical approximations. The value of $u_1(0)$, or any midpoint condition should be chosen in such a way that the solution u has the desired properties, but numerical integration of the ODE is problematic due to the pathological behaviour around zero.

In the following section we will formulate a dual problem for the optimization which results in a deterministic optimal control problem that can be solved more explicitly, and this will help us to find a midpoint condition for the ordinary differential equation which makes it possible to find numerical solutions. We will see, however, that the method proposed there only works when $\phi = 0$; there is a duality gap when $\phi > 0$.

III. DUAL FORMULATION

To formula an appropriate dual problem, we consider processes Y that are defined using the following stochastic differential equation

$$\begin{aligned} dY_t &= Y_{t-}(\beta_t dt + \gamma_t dW_t + \epsilon_t d\tilde{N}_t) \\ Y_0 &= y \end{aligned}$$

where $\tilde{N} = N_t - \eta\int_0^t (1 - N_s)ds$ is a martingale with respect to the filtration generated by the process N, and β, γ and ϵ are càdlàg processes, adapted to the filtration generated by N and W and such that $\epsilon_t > -1$ for all t. We write $c_t = c(X_t^{c,\pi,x})$ and $\pi_t = \pi(X_t^{c,\pi,x})$ for given strategies c and π and use the abbreviation $X_t = X_t^{c,\pi,x}$. We have

$$\begin{aligned} &\mathbb{E}_x\left[\int_0^T e^{-\delta s}U(c_s)ds\right] \\ &= \mathbb{E}_x\left[\int_0^T e^{-\delta s}U(c_s)ds - X_T Y_T + X_0 Y_0 + \right. \\ &\qquad \left.\int_0^T Y_{s-}dX_s + \int_0^T X_{s-}dY_s + [X,Y]_T\right] \\ &= \mathbb{E}_x\left[\int_0^T e^{-\delta s}U(c_s)ds - X_T Y_T + xy\right. \\ &\qquad \left. + \int_0^T Y_{s-}(r + \beta_s + (\mu - r + \gamma_t\sigma)\pi_s)X_s + a_s - c_s)ds\right]. \end{aligned}$$

Since X and Y are positive we immediately see that if we want to optimize this expression over c and π and end up with a finite value, we must take $\beta_t = -r$ and $\gamma_t = -\phi$ for all t. Optimizing over c then gives $c_t = V(e^{\delta t}Y_t)$ where V is the Legendre transform of U:

$$V(y) = \sup_{x\in\mathbb{R}^+}(U(x) - xy).$$

Taking the limit for $T \to \infty$ then gives the dual relationship

$$\begin{aligned} u(x) &\le xy + \mathbb{E}_x\left[\int_0^\infty (e^{-\delta s}V(e^{\delta s}Y_s) + a_s Y_s)ds\right] \\ &:= xy + v_1(y) \end{aligned}$$

where

$$Y_t = ye^{-rt - \phi W_t - \frac{1}{2}\phi^2 t - \eta\int_0^t \epsilon_s(1-N_s)ds + \int_0^t \ln(1+\epsilon_s)dN_s}.$$

From now on we will use the notation $\epsilon_t = (\lambda(t)/\eta) - 1$.

If we assume that $\lambda(t)$ is *deterministic*, and in particular independent of the filtration generated by W, then we can simplify the expression the inequality for u and v_1 using tedious but straightforward calculations that are given in the

Appendix, see (6). There it is shown that for our choice of utility function $U(x) = \ln x$ we have $v_1 = v_0 + \tilde{v}$ where

$$\begin{aligned} v_0(y) &= \frac{r - 2\delta + \frac{1}{2}\phi^2}{\delta^2} + \tfrac{-\ln y}{\delta}, \\ \tilde{v}(y) &= ay \int_0^\infty e^{-rt - \int_0^t \lambda(s)ds} dt \\ &\quad + \tfrac{\eta}{\delta} \int_0^\infty e^{-(\delta+\eta)t} \left(\tfrac{\lambda(t)-\eta}{\eta} - \ln \tfrac{\lambda(t)}{\eta} \right) dt. \end{aligned}$$

We recognize v_0 as the dual value function for the no-income case, and the function $\tilde{v}(y)$ as the extra term due to the income stream. This last term indeed equals zero if $a = 0$ since taking $\lambda(t) = \eta$ for all t is optimal in that case; this recovers the well-known result for the dual of the value function when there is no income. However, if there is income, we now need to choose $\lambda(t)$ optimally, and this will be the subject of the next section.

IV. A DETERMINISTIC OPTIMAL CONTROL PROBLEM

To find the optimal process λ we write

$$\frac{d}{dt}x(t) = -x(t)\,(r - \alpha + \lambda(t)), \qquad x(0) = 1$$

and we then aim to minimize the value of the functional

$$\tilde{v}(y) = \frac{\eta}{\delta} \int_0^\infty e^{-\alpha t}[g(\lambda(t)) + Kx(t)]dt$$

where we use the notation $\alpha = \eta + \delta$ and $K = ay\delta/\eta$ for strictly positive constants and where the function

$$g(\lambda) = \frac{\lambda - \eta}{\eta} - \ln \frac{\lambda}{\eta}$$

has a unique global minimum zero in $\lambda = \eta$ since $g'(\lambda) = 1/\eta - 1/\lambda$ and $g''(\lambda) = 1/\lambda^2$ while g goes to infinity for $\lambda \downarrow 0$ and for $\lambda \to \infty$. In the sequel we will use the notation

$$\Lambda(t) = \int_0^t \lambda(s)ds$$

We can now prove the following result.

Theorem 4.1: The optimal control function $\lambda_y(t)$ which minimizes $\tilde{v}(y)$ satisfies

$$\frac{1}{\eta} - \frac{1}{\lambda_y(t)} = \frac{ay\delta}{\eta} e^{\alpha t} \int_t^\infty e^{-rs - \int_0^s \lambda_y(u)du} ds$$

and the optimal initial conditions

$$L(y) := \lambda_y(0)$$

satisfy the differential equation

$$\frac{dL}{dy} = \begin{cases} \frac{L}{L-\alpha+r} \left(\frac{a\delta}{\eta} L - \frac{\alpha}{y\eta}(L - \eta) \right) & y \neq y^* \\ \frac{a\delta(\alpha-r)}{2(\delta-r)} \left(\sqrt{1 + \frac{4(\delta-r)(\alpha-r)}{\alpha\eta}} - 1 \right) & y = y^* \end{cases}$$

where $y^* = \frac{\alpha(\delta-r)}{a\delta(\alpha-r)}$, with the midpoint condition that

$$L(y^*) = \alpha - r.$$

The minimal value of the functional $\tilde{v}(y)$ then equals

$$\tilde{v}^*(y) = \tfrac{\eta}{\delta\alpha} \left[\frac{ay\delta}{\eta} + g(L(y))) + (\alpha - r - L(y))g'(L(y)) \right]$$

Proof. To find the optimal control function, define for a fixed $y > 0$ the Hamiltonian

$$H = (g(\lambda) + Kx)e^{-\alpha t} - px(r - \alpha + \lambda)$$

The Hamiltonian equation gives

$$\dot{p} = -\frac{\partial H}{\partial x} = -Ke^{-\alpha t} + p(r - \alpha + \lambda)$$

with boundary condition $\lim_{t\to\infty} p(t) = 0$ which implies that

$$\frac{d}{dt}(xp) = x\dot{p} + p\dot{x} = -Kxe^{-\alpha t} = -Ke^{-rt-\Lambda(t)}$$

so

$$p(t)x(t) = K \int_t^\infty e^{-rs-\Lambda(s)} ds$$

The optimal control function now satisfies

$$0 = \frac{\partial H}{\partial \lambda} = g'(\lambda)e^{-\alpha t} - xp$$

so we have that

$$\frac{1}{\eta} - \frac{1}{\lambda_y(t)} = Ke^{\alpha t} \int_t^\infty e^{-rs-\Lambda_y(s)} ds \qquad (3)$$

From this we calculate the scaled value function $\frac{\delta}{\eta}\tilde{v}(y)$ as

$$\begin{aligned} &\int_0^\infty (e^{-\alpha t}[\tfrac{\lambda(t)-\eta}{\eta} - \ln \tfrac{\lambda(t)}{\eta}] \; + \; Ke^{-rt-\Lambda(t)})dt \\ &= [\tfrac{e^{-\alpha t}}{-\alpha}(\tfrac{\lambda(t)-\eta}{\eta} - \ln \tfrac{\lambda(t)}{\eta})]_{t=0}^{t=\infty} \\ &\quad - \int_0^\infty \tfrac{e^{-\alpha t}}{-\alpha} \dot{\lambda}(t)(\tfrac{1}{\eta} - \tfrac{1}{\lambda(t)})dt \; + \; (\tfrac{1}{\eta} - \tfrac{1}{\lambda(0)}) \\ &= \tfrac{1}{\alpha}(\tfrac{\lambda(0)-\eta}{\eta} - \ln \tfrac{\lambda(0)}{\eta}) \\ &\quad + \tfrac{K}{\alpha} \int_0^\infty \dot{\lambda}(t) \int_t^\infty e^{-rs-\Lambda(s)} dsdt \; + \; (\tfrac{1}{\eta} - \tfrac{1}{\lambda(0)}) \\ &= \tfrac{1}{\alpha} g(\lambda(0)) \; + \; \tfrac{K}{\alpha} \int_0^\infty \dot{\lambda}(t) \int_t^\infty e^{-rs-\Lambda(s)} dsdt \\ &\quad + g'(\lambda(0)) \end{aligned}$$

where we have used (3). We change the order of integration in the remaining integral:

$$\begin{aligned} &\tfrac{K}{\alpha} \int_0^\infty e^{-rs-\Lambda(s)} \int_0^s \dot{\lambda}(t)dtds \\ &= \tfrac{K}{\alpha} \int_0^\infty e^{-rs}(\lambda(s) - \lambda(0))e^{-\Lambda(s)} ds \\ &= \tfrac{K}{\alpha}[-e^{-rs}e^{-\Lambda(s)}]_{s=0}^{s=\infty} - \tfrac{K}{\alpha} \int_0^\infty (-r)e^{-rs}(-e^{-\Lambda(s)})ds \\ &\quad - \tfrac{\lambda(0)}{\alpha} g'(\lambda(0)) \\ &= \tfrac{K}{\alpha} - \tfrac{r}{\alpha} K \int_0^\infty e^{-rs-\Lambda(s)})ds - \tfrac{\lambda(0)}{\alpha} g'(\lambda(0)) \\ &= \tfrac{K}{\alpha} - \tfrac{r}{\alpha} g'(\lambda(0)) - \tfrac{\lambda(0)}{\alpha} g'(\lambda(0)) \end{aligned}$$

so

$$\tilde{v}(y) = \tfrac{\eta}{\delta\alpha} \, [\, K + g(\lambda(0)) + (\alpha - r - \lambda(0))g'(\lambda(0)) \,]$$

as claimed. To analyze the optimal strategy λ we differentiate (3) to find

$$\frac{\dot\lambda_y(t)}{\lambda_y(t)} = \alpha(\frac{\lambda_y(t)}{\eta} - 1) - K\lambda_y(t)e^{(\alpha-r)t-\Lambda_y(t)}$$

so if we define

$$\begin{aligned} M(t) &= -\ln K + \int_0^t (\lambda_y(s) - (\alpha - r))ds \\ m(t) &= \lambda_y(t) - (\alpha - r) \end{aligned}$$

then the dynamics for M and m do no longer contain explicit references to time t and the variable y:

$$\begin{aligned} \dot M(t) &= m(t) \\ \dot m(t) &= (m(t) + \alpha - r)\left(\alpha \frac{m(t) + \alpha - r - \eta}{\eta}\right) \\ &\quad - (m(t) + \alpha - r)^2 e^{-M(t)} \end{aligned}$$

apart from the initial conditions

$$(M(0), m(0)) = (-\ln K, \lambda_y(0) - \alpha + r).$$

This dynamical system has only one equilibrium point

$$M^* = -\ln \frac{\alpha(\delta - r)}{\eta(\alpha - r)}, \qquad m^* = 0$$

and the linearized system matrix in the equilibrium point equals

$$\begin{bmatrix} 0 & 1 \\ \frac{\alpha(\delta-r)(\alpha-r)}{\eta} & \alpha \end{bmatrix}$$

which has eigenvectors $(1, f_+)$ and $(1, f_-)$ for the eigenvalues

$$f_\pm = \tfrac{1}{2}\alpha \pm \tfrac{1}{2}\sqrt{\alpha^2 + 4\alpha(\delta - r)(\alpha - r)/\eta}$$

Since $f_- < 0$ and $f_+ > 0$ the equilibrium point is unstable which means that for almost all starting points $(M(0), m(0))$, $\lambda_y(t)$ converges to zero or infinity. Since this is clearly never optimal (the function $g(\lambda_y(t))$ over which we integrate to find the functional $\tilde v$ that we try to minimize would become infinitely large) the trajectories $(M(t), m(t))$ must be one of the only three stable trajectories which converge to (M^*, m^*): the equilibrium point itself, a trajectory for $M(0) < M^*$ and one for $M(0) > M^*$. The last two trajectories are characterized by

$$\begin{aligned} \frac{dm}{dM} = \frac{\dot m}{\dot M} &= \frac{m + \alpha - r}{m}\left(\tfrac{\alpha}{\eta}(m + \alpha - r - \eta)\right) \\ &\quad - \frac{(m + \alpha - r)^2}{m} e^{-M} \end{aligned}$$

and in particular we must have that the initial conditions

$$(M(0), m(0)) = (-\ln \tfrac{ay\delta}{\eta}, L(y) - \alpha + r)$$

are on these trajectories for all possible y. Transforming back to our original coordinates we find that

$$-y\frac{dL}{dy} = \frac{L}{L - \alpha + r}\left(\tfrac{\alpha}{\eta}(L - \eta) - \tfrac{ay\delta}{\eta}L\right)$$

We have a singularity in the equilibrium point which corresponds to

$$L(y^*) = \alpha - r, \qquad y^* = \frac{\alpha(\delta - r)}{(\alpha - r)\delta a}$$

but we know that the derivative there equals

$$\begin{aligned} \left.\frac{dL}{dy}\right|_{y=y^*} &= \frac{1}{-y}\left.\frac{dm}{dM}\right|_{M=M^*} = \frac{f_-}{-y^*} \\ &= \frac{\frac{1}{2}\alpha - \frac{1}{2}\sqrt{\alpha^2 + 4\alpha(\delta - r)(\alpha - r)/\eta}}{-\alpha(\delta - r)/((\alpha - r)\delta a)} \\ &= \frac{a\delta(\alpha - r)}{2(\delta - r)}\left(\sqrt{1 + \tfrac{4(\delta-r)(\alpha-r)}{\eta\alpha}} - 1\right) \end{aligned}$$

This proves the result. ∎

We now substitute this result into our expression for the dual optimization problem.

$$\begin{aligned} u_1(x) &\le \inf_y (xy + v_1(y)) = \inf_y (xy + v_0(y) + \tilde v(y)) \\ &= \inf_y \Bigg[xy + \frac{r - 2\delta + \frac{1}{2}\phi^2}{\delta^2} + \frac{-\ln y}{\delta} + \frac{ay}{\alpha} \\ &\quad + \frac{\eta}{\delta\alpha}\left(-\ln L(y) - \frac{\alpha - r}{L(y)} + \ln\eta + \frac{\alpha - r}{\eta}\right)\Bigg]. \end{aligned}$$

But since

$$\begin{aligned} \frac{d}{dy}v_1(y) &= \frac{-1}{y\delta} + \frac{a}{\alpha} - \frac{\eta}{\delta\alpha}\frac{\frac{d}{dy}L(y)}{L(y)^2}(L(y) - \alpha + r) \\ &= -\frac{\eta}{\delta y L(y)} \end{aligned} \tag{4}$$

(note that this formula is also correct in the singular point $y = y^*$) this means that $u_1(x) = xy + v_1(y)$ when $x = \frac{\eta}{\delta y L(y)}$ so

$$\begin{aligned} u_1(\tfrac{\eta}{\delta y L(y)}) &\le \frac{\eta}{\delta L(y)} + v_0(y) + \frac{ay}{\alpha} \\ &\quad + \frac{\eta}{\delta\alpha}\left(-\ln\frac{L(y)}{\eta} - \frac{\alpha - r}{L(y)} + \frac{\alpha - r}{\eta}\right) \\ &= v_0(y) + \frac{ay}{\alpha} \\ &\quad + \frac{\eta}{\delta\alpha}\left[(\alpha - r)\left(\frac{1}{\eta} - \frac{1}{L(y)}\right) - \ln\frac{L(y)}{\eta}\right] \end{aligned} \tag{5}$$

We can now use these equations to find upper bounds for the value function u_1 using the differential equation for L, which turns out to be very stable in the sense that it can be solved with high accuracy using standard numerical methods.

To calculate the associated strategies $c_1(x)$ and $\pi_1(x)$ for $x = \frac{\eta}{\delta y L(y)}$ we use the fact that

$$\begin{aligned} u_1'(x) &= y \\ u_1''(x) &= -1/\tfrac{\partial^2 v_1}{\partial y^2} = 1/\tfrac{\partial x(y)}{\partial y} \\ &= -\left(\frac{\eta(L(y) + yL'(y))}{\delta y^2 L^2(y)}\right)^{-1} \end{aligned}$$

so

$$
\begin{aligned}
c_1(x)/x &= \frac{1}{xu_1'(x)} = \frac{\delta L(y)}{\eta} \\
\pi_1(x) &= -\frac{\phi}{\sigma x}\frac{u_1'(x)}{u_1''(x)} = \frac{\phi \delta y L(y)}{\sigma \eta} y \frac{\eta(L(y)+yL'(y))}{\delta y^2 L^2(y)} \\
&= \frac{\phi}{\sigma}(1+\frac{yL'(y)}{L(y)})
\end{aligned}
$$

To obtain a more explicit expression for this investment strategy π_1 we can substitute the differential equation for $L(y)$ and we find that

$$
\begin{aligned}
\pi_1(x) &= \frac{\phi}{\sigma}\frac{L(y)-\alpha+r-(\frac{\alpha}{\eta}L(y)-\alpha-\frac{ay\delta}{\eta}L(y))}{L(y)-\alpha+r} \\
&= \frac{\phi}{\sigma}\frac{r+L(y)\frac{\eta-\alpha+ay\delta}{\eta}}{L(y)-\alpha+r} \\
&= \phi\frac{r\eta+\delta L(y)(ay-1)}{\eta\sigma(L(y)-\alpha+r)}
\end{aligned}
$$

Note that the results shows that we find the old proportional strategies for large values of x. This makes sense since for very large values of x the loss of income becomes relatively unimportant and we expect the optimal strategy to be close to a strategy in which income does not play any role at all.

We can check whether the upperbound that we derived actually equals the value function u_1 itself, by substituting it back into the HJB equations. It turns out[4] that this is the case only when $\phi = 0$. Apparently, our restriction to deterministic functions λ when transforming the probability measure with regard to the jump intensity of τ is too narrow when $\phi > 0$ and this creates what is known as a duality gap. The result suggest that in general the function λ should be made dependent on the Brownian paths as well. However, the resulting dual problem then becomes as hard to solve as the original problem so other methods may be preferable in that case.

However, the dual result for the case $\phi > 0$ is very useful to suggest good initial values for the numerical solution of the (primal) optimization problem and we give an example of this in the next section.

V. NUMERICAL RESULTS

We show numerical results for the following parameter values:

$$\mu = 0.15,\ \sigma = 0.20,\ \eta = 0.10$$

$$\delta = 0.30,\ r = 0.03,\ a = 0.10.$$

The HJB equation was solved by a numerical algorithm which uses the dual approximation obtained above to find good initial values and then searches for the value of $u_1(0)$ which, when substituted in the approximation derived in Theorem 2.1, generates solutions that are concave and satisfy the bounds on u_1, c_1 and π_1 that we derived.

[4]Calculations available from the authors upon request.

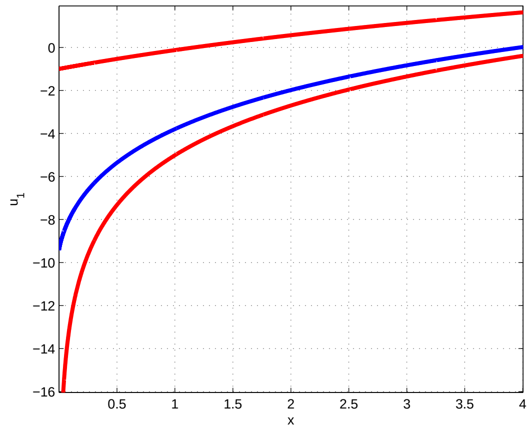

Fig 1. : Value function u_1.

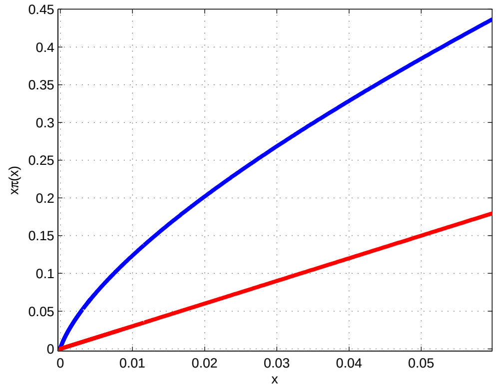

Fig 2. : Optimal policy π_1.

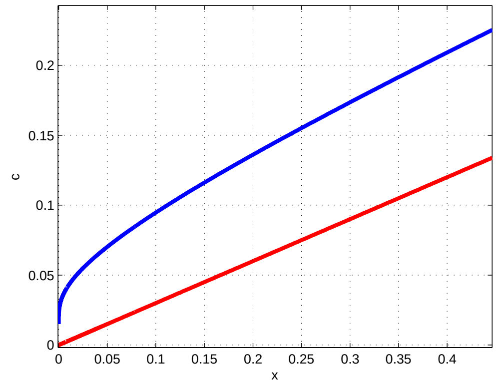

Fig 3. : Optimal policy c_1.

Figure 1 shows the value function for the case with certain income, terminating income and no income respectively, and figures 2 and 3 show the optimal investment and consumption policies for the terminating income and no income case.

VI. CONCLUSIONS

We have characterized the optimal policies for a problem of optimal investment and consumption of an income stream that terminates at a random time, under the assumption that a sufficiently smooth concave solution exists to the associated nonlinear Hamilton-Jacobi-Bellman equation. In future work we will address the question under which conditions such a solution can be guaranteed to exist.

We have also shown how dual methods can be used to find good numerical approximations for the value function and optimal policies. If the diffusion term is not switched off ($\phi > 0$) there is a duality gap, so the set of equivalent martingale measures that is used to define the dual problem needs to be extended in that case.

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APPENDIX

In this Appendix we derive a simplified expression for the functions involved in the dual problem. We have that

$$v_1(y) \quad = \quad y\,\mathbb{E}\int_0^\infty D_t a_t dt + \mathbb{E}\int_0^\infty e^{-\delta t} V(ye^{\delta t} D_t)dt$$

where

$$\begin{aligned} V(ye^{\delta t}D_t) &= -1 - \ln(ye^{\delta t}D_t) \\ &= -1 - \ln y - \delta t - \ln D_t \end{aligned}$$

and $D_t = Y_t/y$ equals

$$\begin{aligned} D_t &= \exp\Big[-rt + \tfrac{r-\mu}{\sigma}W_t - \tfrac{(r-\mu)^2}{2\sigma^2}t \\ &\qquad - \int_0^t (\lambda(s)-\eta)(1-N_s)ds + \int_0^t \ln\tfrac{\lambda(s)}{\eta}dN_s\Big] \\ &:= e^{-rt}\left(\tfrac{d\mathbb{Q}}{d\mathbb{P}}\right)_{\mathcal{F}_t} \end{aligned}$$

is the so-called *deflator* associated with a change to a new measure $\mathbb{Q}$. Since N_t has rate $\lambda(t)$ under $\mathbb{Q}$ we have

$$\begin{aligned} y\mathbb{E}\int_0^\infty D_t a_t dt &= ay\int_0^\infty e^{-rt}\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{t\le\tau} \mid \mathcal{F}_t]dt \\ &= ay\int_0^\infty e^{-rt-\int_0^t \lambda(s)ds}dt \end{aligned}$$

and

$$\mathbb{E}\int_0^\infty e^{-\delta t}(-1-\ln y - \delta t)dt \quad = \quad \tfrac{1}{\delta}(-2-\ln y)$$

while

$$\begin{aligned} &\mathbb{E}\int_0^\infty e^{-\delta t}(-\ln D_t)dt \\ &= -\mathbb{E}\int_0^\infty e^{-\delta t}\Big[-rt + \tfrac{r-\mu}{\sigma}W_t - \tfrac{(r-\mu)^2}{2\sigma^2}t \\ &\qquad - \int_0^t (\lambda(s)-\eta)(1-N_s)ds + \int_0^t \ln\tfrac{\lambda(s)}{\eta}dN_s\Big]. \end{aligned}$$

We calculate the different parts in this integral separately:

$$-\mathbb{E}\int_0^\infty e^{-\delta t}[-rt + \tfrac{r-\mu}{\sigma}W_t - \tfrac{(r-\mu)^2}{2\sigma^2}t]dt \quad = \quad \tfrac{1}{\delta^2}(r + \tfrac{1}{2}\phi^2)$$

and

$$\begin{aligned} &-\mathbb{E}\int_0^\infty e^{-\delta t}[\int_0^t \ln\tfrac{\lambda(s)}{\eta}dN_s]dt \\ &= -\mathbb{E}\int_0^\infty e^{-\delta t}[\ln\tfrac{\lambda(\tau)}{\eta}\mathbf{1}_{t>\tau}]dt \\ &= -\int_0^\infty e^{-\delta t}[\int_0^t \eta e^{-\eta s}\ln\tfrac{\lambda(s)}{\eta}ds]dt \\ &= \int_0^\infty \frac{e^{-\delta t}}{-\delta}\eta e^{-\eta t}\ln\tfrac{\lambda(t)}{\eta}dt \\ &= -\frac{\eta}{\delta}\int_0^t e^{-(\eta+\delta)t}\ln\tfrac{\lambda(t)}{\eta}dt \end{aligned}$$

and, since $\tau = \inf\{t \ge 0\,; N_t = 1\}$

$$\begin{aligned} &\mathbb{E}\int_0^\infty e^{-\delta t}\left(\int_0^t (\lambda(s)-\eta)(1-N_s)ds\right)dt \\ &= \mathbb{E}\int_0^\infty e^{-\delta t}\left(\int_0^{\min(t,\tau)}(\lambda(s)-\eta)ds\right)dt. \end{aligned}$$

But using partial integration

$$\begin{aligned} &\mathbb{E}\int_0^{\min(t,\tau)}(\lambda(s)-\eta)ds \\ &= \int_0^t \eta e^{-\eta v}\int_0^v (\lambda(u)-\eta)dudv \\ &\quad + \int_t^\infty \eta e^{-\eta v}\int_0^t(\lambda(u)-\eta)dudv \\ &= \left[-e^{-\eta v}\int_0^v(\lambda(u)-\eta)du\right]_{v=0}^{v=t} \\ &\quad - \int_0^t -e^{-\eta v}(\lambda(v)-\eta)dv \\ &\quad + e^{-\eta t}\int_0^t(\lambda(u)-\eta)du \\ &= \int_0^t e^{-\eta v}(\lambda(v)-\eta)dv, \end{aligned}$$

and substituting this gives after another partial integration

$$
\begin{aligned}
&\mathbb{E}\int_0^\infty e^{-\delta t}\left(\int_0^{\min(t,\tau)}(\lambda(s)-\eta)ds\right)dt \\
&= \mathbb{E}\int_0^\infty e^{-\delta t}\int_0^t e^{-\eta u}(\lambda(u)-\eta)dudt \\
&= \tfrac{1}{\delta}\int_0^\infty e^{-(\delta+\eta)t}(\lambda(t)-\eta)dt.
\end{aligned}
$$

Collecting terms gives

$$
\begin{aligned}
v_1(y) &= \frac{r-2\delta+\frac{1}{2}\phi^2}{\delta^2}+\frac{-\ln y}{\delta} \\
&\quad +ay\int_0^\infty e^{-rt-\int_0^t\lambda(s)ds}dt \\
&\quad +\tfrac{\eta}{\delta}\int_0^\infty e^{-(\delta+\eta)t}\left(\tfrac{\lambda(t)-\eta}{\eta}-\ln\tfrac{\lambda(t)}{\eta}\right)dt \\
&= v_0(y)+\tilde{v}(y) \qquad (6)
\end{aligned}
$$

which is the expression used in section 3.