Macroscopic charge quantization in single-electron devices

Burmistrov, I.; Pruisken, A.M.M.

Published in:
Physical Review B

DOI:
10.1103/PhysRevB.81.085428

PHYSICAL REVIEW B **81**, 085428 (2010)

Macroscopic charge quantization in single-electron devices

I. S. Burmistrov[1,2] and A. M. M. Pruisken[3]
[1]*L. D. Landau Institute for Theoretical Physics, Kosygina St. 2, 117940 Moscow, Russia*
[2]*Department of Theoretical Physics, Moscow Institute of Physics and Technology, 141700 Moscow, Russia*
[3]*Institute for Theoretical Physics, University of Amsterdam, Valckenierstraat 65, 1018XE Amsterdam, The Netherlands*
(Received 7 December 2009; published 22 February 2010)

In a recent paper by the authors [I. S. Burmistrov and A. M. M. Pruisken, Phys. Rev. Lett. **101**, 056801 (2008)] it was shown that single-electron devices (single-electron transistor or SET) display "macroscopic charge quantization" which is completely analogous to the quantum Hall effect observed on very different electron systems. In this investigation we present more detail on these findings. Based on the Ambegaokar-Eckern-Schön (AES) theory of the Coulomb blockade we introduce a general response theory that probes the sensitivity of SET to changes in the boundary conditions. This response theory defines a set of physical observables and we establish the contact with the standard results obtained from ordinary linear-response theory. The response parameters generally define the renormalization behavior of the SET in the entire regime from weak coupling with large values of the tunneling conductance all the way down to the strong-coupling phase where the system displays the Coulomb blockade. We introduce a general criterion for charge quantization that is analogous to the Thouless criterion for Anderson localization. We present the results of detailed computations on the weak-coupling side of the theory, i.e., both perturbative and nonperturbative (instantons). Based on an effective theory in terms of quantum spins we study the quantum critical behavior of the AES model on the strong-coupling side. Consequently, a unifying scaling diagram of the SET is obtained. This diagram displays all the super universal topological features of the θ angle concept that previously arose in the theory of the quantum Hall effect.

DOI: 10.1103/PhysRevB.81.085428 PACS number(s): 73.23.Hk, 73.43.Nq

I. INTRODUCTION

A. Coulomb blockade

The Coulomb blockade in nanostructures is one of the cornerstones of modern condensed-matter physics. The simplest approach to electron tunneling through quantum dots was proposed by Ambegaokar *et al.* in 1982.[1,2] Their model [in brief Ambegaokar-Eckern-Schön (AES) model] became the focus of a stream of experimental and theoretical papers[3–5] following the first experimental indications of "macroscopic charge quantization" in single-electron devices in 1991.[6]

To experimentally control the transport of electrons[7] one generally uses the so-called "single-electron transistor" or SET.[8] This is a mesoscopic metallic island that is capacitively coupled to a gate and connected to two metallic reservoirs through tunneling contacts with a total conductance g [see Fig. 1(a)].

The experimental conditions of the AES model are limited and extremely well known.[5,9–11] The model is nevertheless richly complex and much of the physical consequences have remained unknown. Over the years, however, it has slowly become more evident that the AES theory of the Coulomb blockade is in many ways similar to the theory of the quantum Hall effect.[12] For example, the AES model is asymptotically free in $0+1$ space-time dimension possesses instantons and has an instanton angle θ. This immediately raises the fundamental question whether the experimental phenomenon of "macroscopic charge quantization" in the SET is possibly related to the "robust quantization" of the Hall conductance observed on very different electronic systems.

The AES model has a number of very significant advantages as compared to the more conventional theories of the θ vacuum or instanton vacuum. For example, the winding numbers of the theory ("topological charge") are quantized at the outset of the problem. This is quite unlike the usual situation where the historical controversies in quantum field theory continue to haunt the subject. For example, it has been pointed out only very recently that the θ vacuum concept generally displays "massless chiral edge excitations" that are very different from those in the "bulk" of the system. Disentwining these different types of excitation is synonymous for separating the *fractional* topological sectors of the theory from the *integral* ones.

Remarkably, it turns out that the existence of "massless chiral edge excitations" in the problem automatically reveals the existence of the quantum Hall effect. This fundamental phenomenon previously remained concealed. However, it provides the resolution to longstanding problems such as the quantization of topological charge and the meaning of instantons and instanton gases.

The existence of "massless chiral edge excitations" has furthermore led to the idea of "super universality" which states that all the fundamental features of the instanton vacuum concept are precisely those of the quantum Hall effect.[12] These include not only the robust quantization of the Hall conductance but also the existence of "gapless excitations" at $\theta = \pi$ or, equivalently, "quantum criticality" of the quantum Hall plateau transition.

It is of interest to know whether these advances possibly also apply to the AES model. In this case, the microscopic origins of the integral and fractional topological sectors are far more obvious. For example, the integral sectors directly emerge from quantum statistics and they describe the quantum system (SET) in thermal equilibrium. On the other hand,

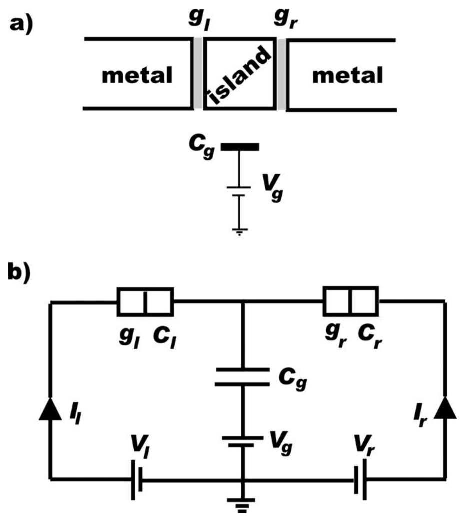

FIG. 1. (a) Sketch of the SET device. (b) Equivalent circuit of the SET.

the fractional topological sectors do not describe "edge" excitations but, rather, they have the meaning of perturbing external fields that take the SET out of thermal equilibrium. The great advantage of the AES model, however, is that the θ dependence can be studied on the strong-coupling side. The AES model is therefore an outstanding laboratory where the various different aspects of "super universality" can be explored and investigated in great detail.

It should be mentioned that the AES model in a different context is also known as the "circular brane model."[13] It is furthermore of direct physical interest in the theory of granular metals at intermediate temperatures.[14]

B. Charge quantization

The phrase "macroscopic charge quantization" usually refers to the charge of an isolated island that is disconnected from the reservoirs. It is given by the naive strong-coupling limit of the AES model where the tunneling conductance g is put equal to zero.

This naive approach leads to the electrostatic picture of the Coulomb blockade where the average charge (Q) on the island is *robustly* quantized in units of e as the temperature (T) goes to absolute zero. This quantization breaks down for very special values of the gate voltage

$$V_g^{(k)} = e(k+1/2)/C_g \tag{1}$$

where k is an integer and C_g denotes the gate capacitance. At these very special values of V_g a first-order *quantum phase transition* occurs separating two different phases with $Q=k$ and $k+1$, respectively.

The electrostatic picture of the Coulomb blockade gets fundamentally complicated when the tunneling conductance g is finite. It is well known, for example, that due to the strong charge fluctuations in the SET the averaged charge Q on the island is generally unquantized.[15] Despite the impressive list of existing theoretical work on both the strong-coupling side[5,16] ($g \ll 1$) and weak-coupling side[14] ($g \gg 1$) of the problem, it is not known what the electrostatic or semiclassical picture of the SET exactly stands for. This fundamental drawback clearly upsets the concept of "robust charge quantization" in single-electron devices.

C. Outline of this investigation

The main objective of this investigation is to show that the SET displays macroscopic charge quantization in much the same way as the two-dimensional electron gas displays the quantum Hall effect. We develop a complete quantum theory of the SET and introduce a unifying scaling diagram that spans the entire range from weak to strong coupling.

We benefit from the advances made over the years in the theory of the quantum Hall effect. We present, in particular, the "physical observables" of the AES theory that measure the sensitivity of the SET to changes in the boundary conditions. In the present context this means that the quantum system is taken out of thermal equilibrium by perturbing fields. The main problem to be solved is how to lay the bridge between the sensitivity to the boundary conditions on the one hand and the standard expressions for linear response obtained from the Kubo formalism on the other.

1. Electrostatic picture revisited

To start we briefly review the microscopic origins of the AES model and summarize the results known from previous work in Sec. II. To see the concept of "physical observables" at work we consider in Sec. III the trivial case of an isolated island at finite T obtained by putting the tunneling conductance g equal to zero. This simple but instructive example sets the stage for most of the analysis in the remainder of this paper.

We point out, first of all, that the averaged charge Q on the isolated island is a measure of the sensitivity of the system to changes in the boundary conditions. This notion immediately suggests a generalized Thouless criterion that relates the robust quantization of Q on the island to the appearance of an energy gap.

Second, we show how the renormalization behavior of Q at finite T provides a complete knowledge of the low-energy dynamics of the isolated island. This behavior involves two different kinds of fixed points, i.e., stable ones at $Q=k$ which describe the robust quantization of charge as T goes to zero, and unstable ones at $Q=k+1/2$ describing the transition between the states $Q=k$ and $k+1$ of the island.

2. Two sets of physical observables

Armed with the insights obtained from the isolated island we next embark on the general problem with finite g in Sec. IV. We introduce two slightly different but physically equivalent sets of response parameters g' and q'. The different expressions that we obtain stem from slightly different ways of handling the fractional topological sectors of the AES theory.

The first set is the simpler one which is a direct generalization of the results obtained for an isolated island. The second set is slightly more involved, but our findings permit a direct comparison with the expressions obtained from linear-response theory.

In both cases, however, one may think of g' and q' in terms of the sensitivity of the SET to changes in the boundary conditions. In both cases also one may think of g' in terms of the SET conductance. The quantity q' is new and in general very different from the conventionally studied averaged charge Q on the island. Within linear-response theory we express q' in terms of the antisymmetric current-current correlation function. We identify this new quantity with the previously unrecognized quasiparticle charge of the SET.

In complete analogy with the theory of the quantum Hall effect, we relate the conditions for "macroscopic charge quantization" $g'=0$ and $q'=k$ with integer k to the appearance of an energy gap in the SET. For $g=0$ these conditions are identically the same as those obtained from the electrostatic picture of the SET. For finite g, however, these conditions describe an entirely different physical state of the SET. They describe the macroscopic quantization of the *quasiparticle charge*, rather than the *averaged charge* on the island.

3. Explicit computations

This takes us to the second part of this investigation where we explicitly compute, in Secs. V and VI, the observable theory g' and q' in the various different regimes in g of interest. We benefit from having two different definitions of g' and q'. The different computational schemes provide a direct check on the universal and nonuniversal parts of the AES theory.

In Sec. V we consider the weak-coupling phase $g\to\infty$ of the AES model. We report the detailed results for the renormalization-group β functions based on ordinary perturbation theory as well as instantons. Even though this Section is self-contained, we refer the reader to the literature for a more detailed exposure to the instanton calculational technique.[12]

In Sec. VI we address the strong-coupling phase of the AES theory and study, in particular, the quantum critical behavior of the SET at finite g. For this purpose we first map the critical behavior of the AES model onto an effective theory of quantum spins. We employ Abrikosov's pseudo fermion technique and extract the β functions of the AES theory near the critical point.

The most important results of this investigation are encapsulated in the unifying scaling diagram in the g'-q' plane as illustrated in Fig. 10. The flow lines clearly indicate that the phenomenon of "macroscopic charge quantization" is a universal feature of single-electron devices that always appears in the limit where T goes to zero. Figure 10 furthermore displays all the super universal features of the θ angle concept that previously arose in the theory of the quantum Hall effect. We end the paper with a conclusion in Sec. VII.

II. AES MODEL

A. Action

It is well understood by now that the AES model of the Coulomb blockade is a limiting case of the so-called universal theory of zero dimensional electron systems.[17] To start we briefly review the microscopic origins of this model. The experimental design of the SET is illustrated in Fig. 1(a). The hamiltonian is split in three distinctly different parts

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_c+\sum_{s=l,r}\mathcal{H}_T^{(s)}. \tag{2}$$

The first part is the free-electron piece

$$\mathcal{H}_0=\sum_{k,s=l,r}\epsilon_k^{(s)}a_k^{(s)\dagger}a_k^{(s)}+\sum_\alpha\epsilon_\alpha d_\alpha^\dagger d_\alpha. \tag{3}$$

The index s runs over of the reservoirs on the left-hand side (l) and right-hand side (r) of the island respectively. The subscript k denotes the electronic states in the reservoirs and α those on the island. The $\epsilon_k^{(a)},\epsilon_\alpha$ are the energies relative to the Fermi level.

The second term in Eq. (2) is the result of the Coulomb interaction between the electrons on the island

$$\mathcal{H}_c=E_c\left(\sum_\alpha d_\alpha^\dagger d_\alpha-q\right)^2. \tag{4}$$

$E_c=e^2/[2(C_l+C_r+C_g)]$ stands for the charging energy and $q=C_gV_g/e$ represents the external charge on the island [see Fig. 1(b)].

The last part of Eq. (2) describes the tunneling of electrons between the reservoir and the island

$$\mathcal{H}_T^{(s)}=\sum_{k\alpha}t_{k\alpha}^{(s)}a_k^{(s)\dagger}d_\alpha+\text{H.c.} \tag{5}$$

The matrix $t_{k\alpha}^{(s)}$ contains the amplitudes for tunneling between the reservoirs and the island. To characterize this tunneling it is convenient to introduce the following Hermitean matrices:

$$\hat{g}_{kk'}^{(s)}=4\pi^2[\delta(\epsilon_k^{(s)})\delta(\epsilon_{k'}^{(s)})]^{1/2}\sum_\alpha t_{k\alpha}^{(s)}\delta(\epsilon_\alpha)t_{\alpha k'}^{(s)\dagger}, \tag{6}$$

$$\check{g}_{\alpha\alpha'}^{(s)}=4\pi^2[\delta(\epsilon_\alpha)\delta(\epsilon_{\alpha'})]^{1/2}\sum_k t_{\alpha k}^{(s)\dagger}\delta(\epsilon_k^{(s)})t_{k\alpha'}^{(s)}. \tag{7}$$

The first matrix acts in the Hilbert space of states of a single reservoir and the second one in the Hilbert space of states of the island. One should think of the delta functions in Eqs. (6) and (7) as being smoothed out over a scale δE such that $\max\{\delta,\delta^{(l,r)}\}\ll\delta E\ll T$. Here, δ and $\delta^{(l,r)}$ stand for mean level spacing of single-particle states on the island and reservoirs, respectively.

The classical dimensionless conductance (in units e^2/h) of the junction between a reservoir and the island can be expressed as follows:[5]

$$g_s=\sum_k\hat{g}_{kk}^{(s)}\equiv\sum_\alpha\check{g}_{\alpha\alpha}^{(s)}. \tag{8}$$

Therefore, each nonzero eigenvalue of $\hat{g}^{(s)}$ or $\check{g}^{(s)}$ corresponds to the transmittance of some "transport" channel between a reservoir and the island.[18] The effective number of these "transport" channels $(N_{\text{ch}}^{(s)})$ is given by

$$N_{\rm ch}^{(s)} = \frac{\left(\sum_k \hat{g}_{kk}^{(s)}\right)^2}{\sum_{kk'} \hat{g}_{kk'}^{(s)} \hat{g}_{k'k}^{(s)}} \equiv \frac{\left(\sum_\alpha \check{g}_{\alpha\alpha}^{(s)}\right)^2}{\sum_{\alpha\alpha'} \hat{g}_{\alpha\alpha'}^{(s)} \hat{g}_{\alpha'\alpha}^{(s)}}. \tag{9}$$

The effective dimensionless conductance $g_{\rm ch}^{(s)}$ of a 'transport' channel can be written as follows:

$$g_{\rm ch}^{(s)} = \frac{\sum_{kk'} \hat{g}_{kk'}^{(s)} \hat{g}_{k'k}^{(s)}}{\sum_k \hat{g}_{kk}^{(s)}} \equiv \frac{\sum_{\alpha\alpha'} \hat{g}_{\alpha\alpha'}^{(s)} \hat{g}_{\alpha'\alpha}^{(s)}}{\sum_\alpha \check{g}_{\alpha\alpha}^{(s)}}. \tag{10}$$

The dimensionless conductance g_s then becomes

$$g_s = g_{\rm ch}^{(s)} N_{\rm ch}^{(s)}. \tag{11}$$

In what follows we will always assume

$$g_{\rm ch}^{(l,r)} \ll 1. \tag{12}$$

Notice that under these circumstances the conductances $g_{l,r}$ can still be large provided the effective number of channels $N_{\rm ch}^{(l,r)} \gg 1$ is sufficiently large.

We furthermore assume that the mean level spacing is negligible $\delta \ll T/\max\{1,g\}$,[10] and the charging energy is sufficiently large $E_c \gg \delta$ such that the effects of the exchange interaction can be ignored.[5,17]

1. Path integral representation

Given this sequence of limitations one can express the dynamics of the SET in terms of a single abelian phase $\Phi(\tau)$ with τ standing for the imaginary time.[1] This field generally describes the potential fluctuations on the island according to $V(\tau) = i\dot{\Phi}(\tau)$. The quantum-mechanical partition function Z can be written as a sum over winding numbers W according to

$$Z[q] = \sum_{W=-\infty}^{\infty} e^{2\pi i q W} Z_W \tag{13}$$

where Z_W is the integral over all paths $\Phi(\tau)$ that start with $\Phi(0)$ at $\tau=0$ and end with $\Phi(\beta)=\Phi(0)+2\pi W$ at $\tau=\beta$ with β the inverse temperature,

$$Z_W = \int_{\Phi(\beta)=\Phi(0)+2\pi W} \mathcal{D}\Phi(\tau) e^{-\mathcal{S}_d[\Phi]-\mathcal{S}_c[\Phi]}. \tag{14}$$

Here, the action $\mathcal{S}_d$ describes the tunneling between the island and the reservoirs

$$\mathcal{S}_d[\Phi] = \frac{g}{4}\int_0^\beta d\tau_1 d\tau_2 \alpha(\tau_{12}) e^{i\Phi(\tau_1)-i\Phi(\tau_2)} \tag{15}$$

where $g=g_l+g_r$ and $\tau_{12}=\tau_1-\tau_2$. The kernel $\alpha(\tau)=\alpha(\tau+\beta)$ in time and frequency representation is given by

$$\alpha(\tau) = -T^2 \operatorname{cosec}^2(\pi T\tau) = \frac{T}{\pi}\sum_{\omega_n} |\omega_n| e^{-i\omega_n\tau} \tag{16}$$

with $\omega_n = 2\pi T n$. The second term $\mathcal{S}_c$ corresponds to the charging energy due to the Coulomb interaction between the electrons on the island

$$\mathcal{S}_c[\Phi] = \frac{1}{4E_c}\int_0^\beta d\tau \dot{\Phi}^2. \tag{17}$$

Finally, the exponential factor containing the external charge q in Eq. (13) describes the coupling between the island and the gate of the SET.

2. Functional integral representation

An elegant formulation of the AES theory is obtained using the $O(2)$ field variable

$$\mathcal{Q}(\tau) = \begin{pmatrix} \cos\Phi & \sin\Phi \\ \sin\Phi & -\cos\Phi \end{pmatrix} \tag{18}$$

with $\mathcal{Q}^2(\tau)=\mathbf{1}$. The partition function can now be expressed as follows

$$Z[q] = \int_{\partial V} \mathcal{D}[\mathcal{Q}] e^{-\mathcal{S}[\mathcal{Q}]} \tag{19}$$

where the subscript ∂V indicates that the functional integral is performed with periodic boundary conditions

$$\mathcal{Q}(0) = \mathcal{Q}(\beta). \tag{20}$$

The action is given by

$$\mathcal{S}[\mathcal{Q}] = \int_0^\beta d\tau_1 d\tau_2 \gamma(\tau_{12}) \operatorname{tr} \mathcal{Q}(\tau_1)\mathcal{Q}(\tau_2) - \frac{q}{2}\int_0^\beta d\tau \operatorname{tr} \sigma_y \mathcal{Q}\partial_\tau \mathcal{Q} \tag{21}$$

with $\boldsymbol{\sigma}$ denoting the Pauli matrices. The kernel $\gamma(\tau_{12})$ in frequency representation can be written as follows:

$$\gamma(i\omega_n) = \frac{g}{4\pi}|\omega_n| + \frac{1}{8E_c}\omega_n^2. \tag{22}$$

Alternatively one may express the action in terms of the $O(2)$ vector field, $\mathcal{Q}(\tau)=\sigma_z(n_x(\tau)+i\sigma_y n_y(\tau))$, with $\boldsymbol{n}^2(\tau)=1$. The integer valued topological charge of the system can be expressed in three different ways

$$\begin{aligned} \mathcal{C}[\mathcal{Q}] &= -\frac{i}{4\pi}\int_0^\beta d\tau \operatorname{tr} \sigma_y \mathcal{Q}\partial_\tau \mathcal{Q} \\ &= -\frac{1}{2\pi}\int_0^\beta d\tau \epsilon_{\mu\nu} \boldsymbol{n}_\mu \cdot \partial_\tau \boldsymbol{n}_\nu \\ &= \frac{1}{2\pi}\int_0^\beta d\tau \dot{\Phi}. \end{aligned} \tag{23}$$

This quantity is nothing but the number of times (W) the $O(2)$ vector field $\boldsymbol{n}$ is winding around. It is important to emphasize that both the periodicity statement of Eq. (20) and the quantization of topological charge in Eq. (23) are fundamental features of the AES theory that inherently describe the SET in thermal equilibrium. This theory only depends on the external charge q modulo 2π. If, for example, one splits q into a fractional piece $-\pi < \theta(q) \le \pi$ and an integral piece $k(q)$ (see Fig. 2)

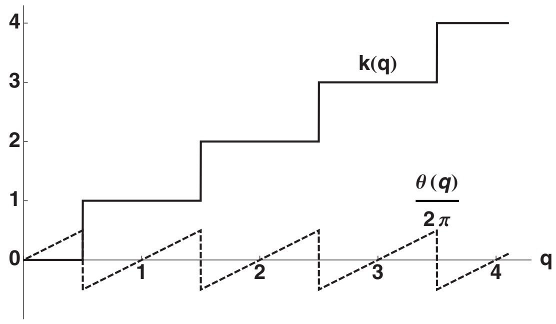

FIG. 2. Integer $k(q)$ and fractional $\theta(q)$ pieces of q.

$$q = \frac{\theta(q)}{2\pi} + k(q) \tag{24}$$

then the quantum-mechanical partition function only depends on the fractional piece $\theta(q)$,

$$Z[q] = Z[\theta(q)/2\pi]. \tag{25}$$

To extract the integral piece $k(q)$ from the AES theory one must in general consider perturbing fields that take the SET out of thermal equilibrium.

B. Instantons

One of the most impressive features of the tunneling term of Eq. (15) is that it possesses stable classical minima $\Phi_W(\tau)$ for each topological sector W. We term these classical solutions "instantons" since they are completely analogous to Yang-Mills instantons.[19] The general expression for $\Phi_W(\tau)$ is given by[20,21]

$$e^{i\Phi_W(\tau)} = \prod_{a=1}^{|W|} \frac{1 - z(\tau)z_a}{z(\tau) - z_a^*}. \tag{26}$$

with

$$z(\tau) = e^{-2\pi i T\tau}. \tag{27}$$

For instantons ($W>0$) the complex parameters z_a are all inside the unit circle and for anti-instantons ($W<0$) they are outside. The classical action

$$\mathcal{S}_d[\Phi_W] = \frac{g}{2}|W| \tag{28}$$

is finite and independent of the complex parameters z_a which are the $2|W|$ zero modes in the problem.

In the limit where $g \to \infty$ one may generally think in terms of a dilute gas of single instantons and anti-instantons. One identifies $\tau_0 = \arg z_1 / 2\pi T$ as the *position* (in time) of the single instanton whereas $\lambda = (1-|z_1|^2)\beta$ is the *scale size* or the duration of the potential pulse $i\dot{\Phi}_{\pm 1}(\tau)$. The thermodynamic potential $\Omega_{\text{inst}} = -T \ln Z$ of the dilute instanton gas[22] can be expressed in a standard manner as an integral over τ_0 and λ according to[12,23]

$$\beta\Omega_{\text{inst}} = -\int_0^\beta d\tau_0 \int_0^\beta \frac{d\lambda}{\lambda^2} g(\lambda) D e^{-g(\lambda)/2 + 2(T-2/\lambda)/E_c(\lambda)} \cos 2\pi q. \tag{29}$$

with $D = 2e^{-\gamma-1}$ and $\gamma \approx 0.577$ the Euler constant. Here, the quantities $g(\lambda)$ and $E_c(\lambda)$ have the same radiative corrections as those obtained from ordinary perturbation theory[24]

$$g(\lambda) = g - 2 \ln \lambda\Lambda, \quad E_c(\lambda) = E_c\left(1 - \frac{2}{g}\ln \lambda\Lambda\right) \tag{30}$$

with $\Lambda = gE_c/\pi^2 D$ standing for the frequency or energy scale. From Eq. (30) we obtain the renormalization-group equations which to order g^{-1} are given by

$$\beta_g = \frac{dg(\lambda)}{d \ln \lambda} = -2 - \frac{4}{g(\lambda)}, \quad \beta_c = \frac{d \ln E_c(\lambda)}{d \ln \lambda} = -\frac{2}{g(\lambda)}. \tag{31}$$

Here, we have included the perturbative contribution[25,26] of order $1/g(\lambda)$ into β_g. Based on perturbation theory alone one expects that the quantum system is a good "conductor" at high temperatures

$$g(T) = -2 \ln \beta/\xi \gg 1 \tag{32}$$

and an "insulator" at low temperatures

$$g(T) = \exp[-(\beta/\xi)^z] \ll 1. \tag{33}$$

Here, z is a dynamical exponent that is as of yet unknown and ξ denotes the dynamically generated correlation length in the time domain

$$\xi = \Lambda^{-1} g^{-1} e^{g/2}. \tag{34}$$

These standard ideas and expectations do not reveal much about the θ angle concept on the strong-coupling side, however. For example, there are the conflicting claims made by the semiclassical picture of the Coulomb blockade which say that the system displays a vanishing energy gap or a "quantum phase transition" when q passes through half-integral values (see Sec. III B). These conflicting scenarios raise fundamental questions about the exact meaning of the topological excitations in the problem and, in particular, the dilute gas of instantons written in Eq. (29).

III. PHYSICAL OBSERVABLES

As pointed out many times in our previous work, traditional instanton results such as Eq. (29) are of limited significance since they merely describe the regular or noncritical pieces of the theory which are of secondary interest. In order to be able to understand the low energy dynamics of the SET and, in particular, the phenomenon of charge quantization one must develop an entirely different approach to the AES model and reconsider the traditional renormalization-group ideas in quantum field theory all together.

Recall that conventionally one defines a renormalized theory by specifying how the ultraviolet singularity structure of the bare theory can be absorbed in counter terms. There are many ways of doing this and normally, in the theory of critical exponent values in ϵ expansions for example, one chooses a specific scheme based on computational advantages.

The infrared problems associated with the instanton angle θ dramatically alter the physical objectives of the renormalization group. The extensive list of studies on the AES model is in many ways a reflection of what started many years ago in quantum field theory. There are the perturbative weak coupling analyzes, the instanton investigations as well as the various different attempts toward the strong-coupling phase of the SET. Each of these distinctly different approaches to the AES model provide different pieces of knowledge in physics. They are completely disconnected, however, and have physically very little in common.

The basic idea pursued in the theory of the quantum Hall effect is to provide a unifying renormalization theory of the instanton angle θ based on the response of the system to infinitesimal changes in the boundary conditions. This idea is very close to the criterion of Anderson localization originally proposed by Thouless.[27] It is also very close to 't Hooft's idea on duality based on twisted boundary conditions[28] which states that gapless excitations must in general exist when θ passes through odd multiples of π. Unlike these well-known principles in physics, however, one now relates the sensitivity to boundary conditions to a set of "physical observables" that provide a very general definition of the renormalization behavior of the system. In the context of the quantum Hall effect these physical observables have previously been recognized as the macroscopic conductance parameters of the system.

The AES model is an interesting and highly nontrivial example where the theory of physical observables can be explored and investigated in great detail. In this Section we show that the problem of charge quantization in the SET is completely analogous to the robust quantization of the Hall conductance observed in the disordered electron gas in two dimensions. It turns out that the AES model is extremely interesting in and of itself because of the long ranged nature of the tunneling term or the nonlocal properties of the kernel $\gamma(\tau_{12})$ in Eq. (21). For the sake of simplicity we assume throughout the present section that $\gamma(\tau_{12})$ is local in time and postpone the refinements and extensions of the argument to Sec. IV.

As a trivial but very instructive example of our general definition of physical observables we study the isolated mesoscopic island in Sec. III B. This naive strong-coupling example reveals much the conceptual structure of the instanton angle θ and sets the stage for the remainder of this investigation.

A. Background fields

Consider a fixed background matrix field $U_0(\tau)$ or $\mathcal{Q}_0(\tau) = U_0^{-1}\sigma_z U_0$ that varies slowly in time. We assume that $\mathcal{Q}_0$ satisfies the classical equations of motion and carries a small fractional topological charge, i.e., $\mathcal{Q}_0$ violates the boundary conditions of Eq. (20). The theory in the presence of the background field

$$Z[q;\mathcal{Q}_0] = \int_{\partial V} \mathcal{D}[\mathcal{Q}] e^{-\mathcal{S}[U_0 \mathcal{Q} U_0^{-1}]} \tag{35}$$

then provides all the important information on the quantum system at low energies. To relate the background field action the appearance of an energy gap in the SET one must separate the constant pieces in $\mathcal{Q}_0$ from the parts that couple to the matrix field variable $\mathcal{Q}$. Employing the split of Eq. (24) and keeping in mind that $\mathcal{C}[\mathcal{Q}]$ is quantized then one can write

$$\begin{aligned} \exp\{2\pi i q \mathcal{C}[U_0 \mathcal{Q} U_0^{-1}]\} = \exp\{&2\pi i k(q) \mathcal{C}[\mathcal{Q}_0] \\ &+ i\theta(q) \mathcal{C}[U_0 \mathcal{Q} U_0^{-1}]\}. \end{aligned} \tag{36}$$

Using this identity one can split the theory of Eq. (35) into pieces that are periodic and nonperiodic in the external charge q according to

$$Z[q;\mathcal{Q}_0] = e^{2\pi i k(q) \mathcal{C}[\mathcal{Q}_0]} Z[\theta(q)/2\pi;\mathcal{Q}_0]. \tag{37}$$

It is clear that only the periodic piece probes the sensitivity of the SET to changes in the boundary conditions. Provided the $\gamma(\tau_{12})$ is local in time one obtains the effective action in $\mathcal{Q}_0$ in terms of a derivative expansion. The result is of the same form as the AES action itself

$$\mathcal{S}_{\text{eff}}[\mathcal{Q}_0] = 2\pi i k(q) \mathcal{C}[\mathcal{Q}_0] + \mathcal{S}_\theta[\mathcal{Q}_0] \tag{38}$$

$$\begin{aligned} \mathcal{S}_\theta[\mathcal{Q}_0] &= \ln \frac{Z[\theta(q)/2\pi;\mathcal{Q}_0]}{Z[\theta(q)/2\pi]} \\ &= \int_0^\beta d\tau_1 d\tau_2 \gamma'(\tau_{12}) \text{tr}\, \mathcal{Q}_0(\tau_1)\mathcal{Q}_0(\tau_2) - i\theta' \mathcal{C}[\mathcal{Q}_0] \\ &\quad + \mathcal{O}(\mathcal{Q}_0^3). \end{aligned} \tag{39}$$

except that the bare quantities γ and $\theta(q)$ are replaced by the effective expressions γ' and θ', respectively. As a criterion for a mass gap or energy gap in the SET one can now state that $\mathcal{S}_\theta[\mathcal{Q}_0]$ must vanish order by order in an expansion in powers of the derivative acting on $\mathcal{Q}_0$. This means that not only the γ' and θ' are exponentially small in β but also the infinite series of higher order terms not written in Eq. (39). Under these circumstances the effective action is given by

$$\mathcal{S}_{\text{eff}}[\mathcal{Q}_0] = 2\pi i k(q) \mathcal{C}[\mathcal{Q}_0]. \tag{40}$$

In the context of the disordered electron gas one identifies this result as the action of "massless chiral edge excitations." The quantity $k(q)$ is recognized as the robustly quantized Hall conductance with sharp steps occurring at the center of the Landau bands (i.e., $q=m+1/2$ with integer m).

Presently, the background field $\mathcal{Q}_0$ merely stands for a perturbing field that takes the SET out of thermal equilibrium. The quantity $k(q)$, however, is identified as the robustly quantized quasiparticle charge of SET. This quantity, as we shall see in Sec. IV, is in general very different from the averaged charge Q on the island.

B. Isolated island

To see these general statements at work we go back to the path-integral representation of Sec. II A 1 and consider the simple problem with the tunneling conductance g equal to zero. The classical equation of motion of the Coulomb term of Eq. (17) is given by $\partial^2\Phi/\partial\tau^2=0$ which is simply solved by writing

$$\Phi(\tau) = 2\pi T(W+\phi)\tau. \tag{41}$$

The integer W generally stands for the integral topological sectors of the system and $-1/2<\phi<1/2$ denotes the perturbing background field with a fractional topological charge. We can write

$$\begin{aligned} Z[q;\phi] &= \sum_W \exp\left\{2\pi i q(W+\phi) - \frac{\pi^2}{\beta E_c}(W+\phi)^2\right\} \\ &= Z[q]e^{-\mathcal{S}_{\text{eff}}[\phi]} \end{aligned} \tag{42}$$

The "effective action" $\mathcal{S}_{\text{eff}}[\phi]$ in Eq. (42) has the same general form as the original AES theory (in the absence of tunneling)

$$\mathcal{S}_{\text{eff}}[\phi] = -2\pi i q'\phi - \frac{\pi^2}{\beta E_c'}\phi^2 + \mathcal{O}(\phi^3) \tag{43}$$

except that the bare parameters q and E_c are now replaced by the effective or "observable" ones q' and E_c' respectively. It is readily seen that

$$q' = q + \frac{1}{2\beta E_c}\frac{\partial \ln Z[q]}{\partial q}, \tag{44}$$

$$\frac{1}{E_c'} = \frac{1}{E_c}\left(1 + \frac{1}{2\beta E_c}\frac{\partial^2 \ln Z[q]}{\partial q^2}\right). \tag{45}$$

Similar expressions can be written down for the coefficients of the higher order terms in $\mathcal{S}_{\text{eff}}$ which in general are irrelevant.

1. Further evaluation

To investigate the criterion for charge quantization written in Eq. (40) we must evaluate the observable theory of Eqs. (44) and (45) in the limit $T=0$. Making use of the Poisson summation formula

$$\sum_W e^{2\pi i x W} = \frac{1}{2\pi}\sum_n \delta(x-n) \tag{46}$$

one can express the partition function of Eq. (42) as a rapidly converging sum over quantum numbers n according to

$$Z[q;\phi] = \sum_n \exp\{2\pi i n\phi - \beta E_c(n-q)^2\}. \tag{47}$$

We immediately recognize the grand partition function for Eq. (4) with the integer n now standing for the number of electrons on the island. The effective action can now be written as follows:

$$\mathcal{S}_{\text{eff}}[\phi] = 2\pi i\langle n\rangle\phi - 2\pi^2(\langle n^2\rangle - \langle n\rangle^2)\phi^2 + \mathcal{O}(\phi^3). \tag{48}$$

Comparison with Eq. (43) shows that q' is none other than the averaged charge $\langle n\rangle$ on the island and E_c' is related to the variance

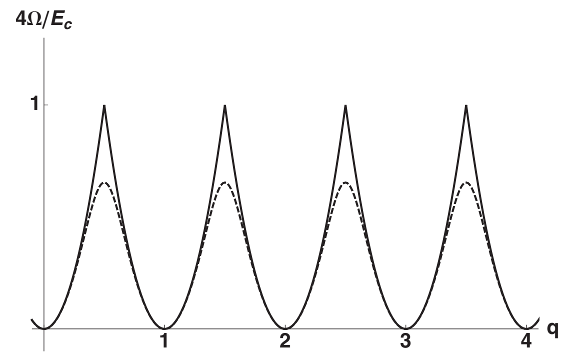

FIG. 3. The thermodynamic potential Ω of the isolated island at finite (dashed curve) and zero (solid curve) temperatures.

$$q' = \langle n\rangle, \quad \frac{1}{\beta E_c'} = 8(\langle n^2\rangle - \langle n\rangle^2). \tag{49}$$

To obtain explicit expressions for q' and E_c' we follow up on Eq. (37) and split Eq. (47) into periodic and nonperiodic parts in q according to

$$Z[q;\phi] = e^{2\pi i k(q)\phi} Z[\theta(q)/2\pi;\phi]$$

$$Z\left[\frac{\theta(q)}{2\pi};\phi\right] = \sum_{n'} \exp\left\{2\pi i n'\phi - \beta E_c\left(n' - \frac{\theta(q)}{2\pi}\right)^2\right\}. \tag{50}$$

It is immediately clear that $Z[\theta(q)/2\pi;\phi]$ in the limit $\beta \to\infty$ is independent of ϕ. In complete accordance with the general statement of Eq. (40) we conclude that the island for all values of $-\pi<\theta(q)<\pi$ develops an energy gap. The quantity q' or the averaged charge $\langle n\rangle$ on the island is quantized

$$q' = \langle n\rangle = k(q) \tag{51}$$

with sharp steps occurring at half-integral values of q where the energy gap vanishes. The thermodynamic potential

$$\beta\Omega[q] = \beta E_c\left(\frac{\theta(q)}{2\pi}\right)^2 \tag{52}$$

displays a "cusp" at half-integral values of q indicating that the transition is a first-order one (see Fig. 3).

2. Renormalization

This takes us to the most important part of this exercise which is to show that the physical observables generally define the renormalization behavior of the island at finite T. Notice that Eq. (50) is dominated by the terms with $n' = 0, \pm 1$. Write

$$Z[\theta(q)/2\pi;\phi] = Z[\theta(q)/2\pi]e^{-\mathcal{S}^0_{\text{eff}}[\phi]} \tag{53}$$

$$\mathcal{S}^0_{\text{eff}}[\phi] = -i\theta'\phi - \frac{\pi^2}{\beta E_c'}\phi^2 + \mathcal{O}(\phi^3) \tag{54}$$

then the explicit results for the thermodynamic potential $\Omega[q]$ and the physical observables θ' and E_c' can be written as follows:

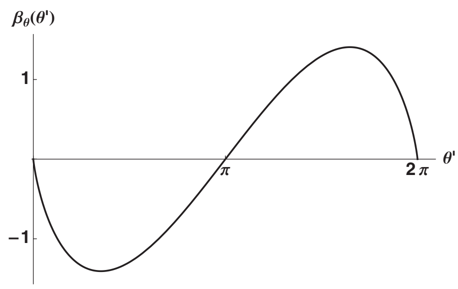

FIG. 4. β_θ function [Eq. (59)] for the isolated island.

$$\beta\Omega[q] = -\ln Z[\theta(q)/2\pi] = \frac{1}{4}(\beta E_c)(1-\Delta_0)^2 - \ln[1+e^{-(\beta E_c)\Delta_0}] \quad (55)$$

$$\theta' = \theta(q) - \frac{\pi}{E_c}\frac{\partial\Omega[q]}{\partial q} = \pm\frac{2\pi}{e^{(\beta E_c)\Delta_0}+1} \quad (56)$$

$$\frac{1}{\beta E_c'} = \frac{1}{\beta E_c}\left(1 - \frac{1}{2E_c}\frac{\partial^2\Omega[q]}{\partial q^2}\right) = \left|\frac{\theta'}{2\pi}\right|\left(1 - \left|\frac{\theta'}{2\pi}\right|\right). \quad (57)$$

Here, $\pm$ denotes the sign of $\theta(q)$ and Δ_0 is recognized as the dimensionless energy gap of the island which vanishes near the critical point according to

$$\Delta_0 = \left(1 - \left|\frac{\theta(q)}{\pi}\right|\right). \quad (58)$$

Finally, we express the response quantity θ' in differential form and obtain (see Fig. 4)

$$\beta_\theta(\theta') = \frac{d\theta'}{d\ln\beta} = \frac{\theta'}{2\pi}[2\pi - |\theta'|]\ln\left[\frac{|\theta'|}{2\pi - |\theta'|}\right]. \quad (59)$$

This result clearly translates the physics of the isolated island in the language of the renormalization group. Notice that the quantity E_c' in Eq. (57) does not lead to more complex renormalization behavior since it is expressed in terms of θ' alone. The same is true for the higher terms in Eq. (54).

We identify two different kinds of strong-coupling fixed points, a stable one at $\theta'=0$ and a critical one at $\theta'=\pm\pi$:

(1) near the critical fixed point $\theta'=\pm\pi$ we find

$$\beta_\theta(\theta') = \pm\pi + \theta' \quad (60)$$

which is a standard result for a first-order transition in one dimension. Equation (60) determines the energy-gap exponent ν according to

$$\frac{1}{\nu} = \left[\frac{\partial\beta_\theta}{\partial\theta'}\right]_{\theta'=\pm\pi} = 1. \quad (61)$$

Equations (48) and (57) tell us that near criticality the charge on the island is broadly distributed, i.e., the fluctuations are of the same order of magnitude as the averaged value θ'.

(2) Near the stable fixed point at $\theta'=0$ we find

$$\beta_\theta(\theta') = \theta' \ln|\theta'| \quad (62)$$

indicating that the averaged charge q' on the isolated island is robustly quantized with corrections that are exponentially small in β; i.e.,

$$q' = k(q) + \frac{\theta'}{2\pi} = k(q) \pm e^{-\beta E_c \Delta_0}. \quad (63)$$

Similarly, the root-mean-square fluctuations in θ' as well as the higher order moments all render exponentially small in β.

IV. GENERAL RESPONSE THEORY

A. Background fields

Armed with the insights obtained from the isolated island we next address the AES theory with finite values of g. To discuss the tunneling term $\mathcal{S}_d[\Phi]$ with varying boundary conditions on the Φ field one must generalize the expression for the kernel $\alpha(\tau_{12})$ in Eq. (16) which is periodic in time. Write

$$\alpha_\phi(\tau_{12}) = \frac{T}{\pi}\sum_n e^{-i(\omega_n+2\pi T\phi)\tau_{12}}|\omega_n + 2\pi T\phi| \quad (64)$$

with $-1/2<\phi<1/2$. The appropriate result for the tunneling term $\mathcal{S}_d$ in Eq. (16) is then obtained if one replaces $\alpha(\tau_{12})$ by the following expression:

$$\alpha(\tau_{12}) \to e^{i(2\pi T\phi)\tau_1}\alpha_\phi(\tau_{12})e^{-i(2\pi T\phi)\tau_2} = \alpha(\tau_{12}) + 2T^2|\phi| - 2iT^2\phi\cot(\pi T\tau_{12}). \quad (65)$$

Equation (65) essentially tells us that one cannot insert a background field $\mathcal{S}_d[\Phi]\to\mathcal{S}_d[\Phi+\Phi_0]$ with $\Phi_0=2\pi T\phi$ carrying a fractional topological charge unless one changes the kernel $\alpha(\tau_{12})$ into $\alpha_\phi(\tau_{12})$. Given Eq. (65) it is straightforward to discuss the effect of the more general background field

$$\Phi_0(\tau) = (\omega_m + 2\pi T\phi)\tau \quad (66)$$

and the result can be written as follows:

$$\mathcal{S}_d[\Phi+\Phi_0] = \frac{g}{4}\int_0^\beta d\tau_1 d\tau_2 e^{i\Phi(\tau_1)-i\Phi(\tau_2)+i\Phi_0(\tau_1)-i\Phi_0(\tau_2)}\alpha_\phi(\tau_{12}) = \frac{g}{4\pi}\int_0^\beta d\tau_1 d\tau_2 e^{i\Phi(\tau_1)-i\Phi(\tau_2)}T\sum_n e^{-i\omega_n\tau_{12}}|\omega_n + \dot{\Phi}_0|. \quad (67)$$

Notice that Eq. (66) now satisfies the classical equations of motion of the AES theory as a whole, i.e., not for only the isolated island as discussed in the previous Section but also for the theory in the presence of tunneling. We will next embark on the distinctly different ways of handling the background field methodology depending on the topological charge of the field Φ_0.

1. Φ_0 with fractional topological charge

By taking $\Phi_0=2\pi T\phi\tau$ or $\omega_m=0$ then Eq. (66) can directly be used to probe the sensitivity of the SET to changes in the boundary conditions. Introducing the two-point correlation function

$$D(i\omega_n) = T\int_0^\beta \int_0^\beta d\tau_1 d\tau_2 e^{i\omega_n \tau_{12}} \langle e^{-i\Phi(\tau_1)+i\Phi(\tau_2)}\rangle. \quad (68)$$

then to lowest orders in the ϕ we obtain the total effective action

$$\mathcal{S}_{\rm tot}[\phi] = \mathcal{S}_{\rm eff}[\phi] + \frac{g}{4\pi}\sum_n D(i\omega_n)(|\omega_n + 2\pi T\phi| - |\omega_n|) \quad (69)$$

with $\mathcal{S}_{\rm eff}[\phi]$ given by Eq. (48) and below. Keeping in mind that $-1/2<\phi<1/2$ we split the sum in Eq. (69) in $n=0$ and $n\neq 0$ parts and we immediately obtain

$$\mathcal{S}_{\rm tot}[\phi] = \frac{g'}{2}|\phi| - 2\pi i q'\phi + \delta\mathcal{S}_{\rm tot}[\phi] \quad (70)$$

where $\delta\mathcal{S}_{\rm tot}$ stands for all the higher order terms in ϕ. The physical observables g' and q' are given as follows:

$$g' = gTD(i0), \quad (71)$$

$$q' = Q - \frac{g}{2\pi}T\sum_{n>0} \mathrm{Im}\, D(i\omega_n). \quad (72)$$

Here we have introduced the quantity

$$Q = q + \frac{i\langle\dot{\Phi}\rangle}{2E_c} = q - \frac{1}{2E_c}\frac{\partial\Omega[q]}{\partial q} \quad (73)$$

which generally stands for the averaged charge on the island. We see that in the presence of tunneling the averaged charge Q is different from q' which we now identify with the quasiparticle charge of the SET. We expect that the new quantity q' is quantized and, along with that, the quantity g' as well as all the higher dimensional terms in $\delta\mathcal{S}_{\rm tot}$ render exponentially small in the limit where β goes to infinity.

2. Higher dimensional terms

To obtain the leading-order corrections in $\delta\mathcal{S}_{\rm tot}$ one needs the four-point correlation function

$$D(i\omega_n, i\omega_m) = T^2\int_{12}\int_{34} e^{i\omega_n\tau_{12}+i\omega_n\tau_{34}} \langle e^{-i\Phi(\tau_1)+i\Phi(\tau_2)} e^{-i\Phi(\tau_3)+i\Phi(\tau_4)}\rangle_{cum}. \quad (74)$$

The effective action $\delta\mathcal{S}_{\rm tot}$ up to the third order in ϕ can be written as follows:

$$\delta\mathcal{S}_{\rm tot}[\phi] = \frac{\pi^2}{\beta E_c'}\phi^2 + \frac{2\pi i}{\beta F_c'}\phi|\phi| + \mathcal{O}(\phi^3) \quad (75)$$

where

$$\frac{1}{\beta E_c'} = \frac{1}{\beta E_c}\frac{\partial}{\partial q}(2q' - Q) + \frac{g^2T^2}{8\pi^2}D(i0,i0) - \frac{g^2T^2}{8\pi^2}\sum_{n,m\neq 0}\mathrm{sgn}(\omega_n\omega_m)D(i\omega_n, i\omega_m), \quad (76)$$

$$\frac{1}{\beta F_c'} = \frac{1}{4\beta E_c}\frac{\partial g'}{\partial q} + \frac{ig^2T^2}{8\pi}\sum_{n\neq 0}\mathrm{sgn}\ \omega_n D(i\omega_n, i0). \quad (77)$$

A more detailed discussion of the higher order terms will be presented elsewhere.[29]

3. Φ_0 with integral topological charge

A less obvious way of probing the energy gap in the SET is obtained by putting $\phi=0$ and, instead, we consider background fields with an integral topological charge only; i.e., $\Phi_0=\omega_m\tau$. Notice that this choice of Φ_0 is a special case of the instanton solution of Eq. (26) with $W=m$ but with all the parameters z_α put equal to zero. Even though this background field Φ_0 can formally be absorbed in a redefinition of the Φ field one can nevertheless proceed and define the effective action $\mathcal{S}_{\rm tot}[\Phi_0]$ by expanding in powers of Φ_0 or ω_m. Provided one finds a way to analytically continue the discrete Matsubara frequencies to fractional or infinitesimal values the final results are again a measure for the sensitivity of the SET to changes in the boundary conditions.

To start we consider the effective action at a tree level

$$\mathcal{S}[\Phi_0] = \frac{g}{2}|m| - 2\pi i q m + \frac{\pi^2}{\beta E_c}m^2. \quad (78)$$

We expect that the exact result retains the general form of Eq. (78) except that the bare parameters g, q and E_c are replaced by effective or observable ones. To lowest orders in m one can write

$$\mathcal{S}_{\rm tot}[m] = \mathcal{S}_{\rm eff}[m] + K(i\omega_m) - K(i0) \quad (79)$$

where $\mathcal{S}_{\rm eff}[m]$ is the same as Eq. (48) with ϕ replaced by m. We have introduced the quantity

$$K(i\omega_n) = -\frac{g}{4\beta}\int_0^\beta d\tau_1 d\tau_2 e^{i\omega_n\tau_{12}}\alpha(\tau_{12})\langle e^{i\Phi(\tau_1)-i\Phi(\tau_2)}\rangle. \quad (80)$$

To expand this theory in terms of a series in powers of ω_m we make use of the analytic properties of response functions. Specifically, following the standard prescription $i\omega_m \to \omega + i0^+$ we analytically continue the discrete set of imaginary frequencies $i\omega_m$ in Eq. (80) to real ones ω and subsequently we can take the limit $\omega\to 0$ (see Sec. IV A). The following total result is obtained for the effective action up to order m^3:

$$\mathcal{S}_{\rm tot}[m] = -\frac{g'}{2}|m| - 2\pi i q' m + \frac{\pi^2}{\beta E_c'}m^2 + \frac{2\pi i}{\beta F_c'}m|m|. \quad (81)$$

Here, the quantities q' and g' are given in terms of Kubo-like expressions as follows:

$$g' = 4\pi\ \mathrm{Im}\frac{\partial K^R(\omega)}{\partial\omega} \quad (82)$$

$$q' = Q + \mathrm{Re}\frac{\partial K^R(\omega)}{\partial \omega}. \quad (83)$$

$K^R(\omega)$ denotes the analytic continuation of $K(i\omega_m)$ and the limit $\omega \to 0$ is understood. As before the Q denotes the averaged charge on the island [see Eq. (73)]. Similar results can be written down for E_c' as well as F_c' but the expressions are considerably more complex than those of Eqs. (76) and (77) and the details will be presented elsewhere.

Even though the physical observables of this Section are formally different from those in the preceding Section they should nevertheless define the same renormalization behavior of the SET. In particular, in the presence of an energy gap the physical observables in both Eqs. (70) and (81) should all scale to zero as β goes to infinity except for the quantity q' that can take on arbitrary integral values. The main advantage of Eqs. (81)–(83), however, is that they directly lay the bridge between the background field methodology on the one hand, and results obtained from ordinary linear-response theory on the other (see Sec. IV B).

B. More about response functions

The function $K^R(\omega)$ can elegantly be expressed in terms of the retarded propagator $D^R(\omega)$ which is the analytic continuation of $D(i\omega_n)$ in Eq. (68). In Appendix A we derive the following relation:

$$K^R(\omega) = g\int_{-\infty}^{\infty} \frac{d\epsilon_1 d\epsilon_2}{4\pi^3}\epsilon_2 \frac{n_b(\epsilon_2) - n_b(\epsilon_1)}{\epsilon_1 - \epsilon_2 + \omega + i0^+}\mathrm{Im}\, D^R(\epsilon_1) \quad (84)$$

where $n_b(\epsilon)=[\exp(\beta\epsilon)-1]^{-1}$ denotes the Bose-Einstein distribution. A detailed computation of $K^R(\omega)$ in the weak and strong-coupling regimes is presented in Appendix D.

Given the function $K^R(\omega)$ one can obtain the response parameters g' and q' from Eqs. (82) and (83). However, it is more convenient to express these quantities directly in terms of $D^R(\omega)$ according to

$$g' = g\int_{-\infty}^{\infty} \frac{d\epsilon}{\pi}\epsilon \frac{\partial n_b}{\partial \epsilon}\mathrm{Im}\, D^R(\epsilon), \quad (85)$$

$$q' = Q + g\int_{-\infty}^{\infty} \frac{d\epsilon}{4\pi^2}\frac{\partial \epsilon n_b}{\partial \epsilon}\mathrm{Re}\, D^R(\epsilon). \quad (86)$$

C. Linear response

In this Section we establish the contact between the response quantities g' and q' and the well-known expressions for the SET conductance G and the nonsymmetrized current noise S_I.

1. SET conductance

If one applies a voltage difference $V=V_r-V_l$ between the reservoirs then the tunneling part $\mathcal{H}_T^{(l,r)}$ of Eq. (2) becomes time dependent[30]

$$\mathcal{H}_T^{(s)} = X^{(s)}e^{-ieV_s t} + X^{(s)\dagger}e^{ieV_s t},$$

$$X_s = \sum_{k\alpha} t_{k\alpha}^{(s)} a_k^{(s)\dagger} d_\alpha. \quad (87)$$

The operator for the current I_s that flows from a reservoir to the island can be expressed as follows:

$$\mathcal{I}_s = e\frac{d}{dt}\sum_k a_k^{(s)\dagger}a_k^{(s)} = -ieX^{(s)}e^{-ieV_s t} + h.c. \quad (88)$$

To the lowest order in $1/N_{ch}^{(s)}$ we find

$$I_s = -i\int_{-\infty}^{t} dt'\langle[\mathcal{I}^{(\alpha)}(t),\mathcal{H}_T^{(s)}(t')]\rangle = -2e\,\mathrm{Im}\, K_s^R(-eV_s). \quad (89)$$

The retarded correlation function is given by

$$K_s^R(\omega) = i\int_0^\infty dt e^{i\omega t}\langle[X^{(s)}(t),X^{(s)\dagger}(0)]\rangle \quad (90)$$

and the corresponding Matsubara correlation function by

$$K_s(i\omega_n) = \int_0^\beta d\tau e^{i\omega_n \tau}\langle T_\tau X^{(s)}(\tau)X^{(s)\dagger}(0)\rangle. \quad (91)$$

Repeating the same steps that led to the AES action starting from the Hamiltonian of Eq. (2) we obtain

$$K_s(i\omega_n) = -\frac{g_s}{4\beta}\int_0^\beta d\tau_1 d\tau_2 e^{i\omega_n \tau_{12}}\alpha(\tau_{12})D(\tau_{21}). \quad (92)$$

Comparison with Eq. (80) yields $K_s(i\omega_n)=(g_s/g)K(i\omega_n)$ or, equivalently,

$$K_s^R(\omega) = (g_s/g)K^R(\omega). \quad (93)$$

Based on the continuity equations for the current $I=I_l=-I_r=GV$ we finally find the SET conductance G in units of $[e^2/h]$ according to[16,31,32]

$$G = \frac{g_l g_r}{(g_l+g_r)^2}g' \quad (94)$$

with g' given by Eq. (82) or Eq. (85). Therefore, except for the constant $g_l g_r/(g_l+g_r)^2$ the conductance G is none other than the observable g' that measures the sensitivity of the SET to changes in the boundary conditions.

2. Quantum current noise

Similar to Eq. (89) we obtain the real part of the retarded correlation function as follows:

$$\mathrm{Re}\, K_s^R(-eV_\alpha) = \frac{i}{2e^2}\int_{-\infty}^{t} dt'\langle[\mathcal{I}_s(t),\mathcal{I}_s(t')]\rangle. \quad (95)$$

The quantity q' in Eq. (83) or Eq. (86) can therefore be expressed in terms of the current-current correlation function[33]

$$q' = Q - i\frac{(g_l+g_r)^2}{2g_l g_r}\frac{\partial}{\partial V}\int_{-\infty}^{0} dt\langle[\mathcal{I}(0),\mathcal{I}(t)]\rangle \tag{96}$$

in the limit where V goes to zero. We have thus found a novel interpretation of the so-called antisymmetric current-current correlation function that in different physical contexts has attracted a considerable amount of interest over the years.[34] Introducing the nonsymmetrized current noise[35]

$$S_I(\omega,V) = \int_{-\infty}^{\infty} dt e^{-i\omega t}\langle\mathcal{I}(t)\mathcal{I}(0)\rangle \tag{97}$$

then one can also write

$$q' = Q + \frac{(g_l+g_r)^2}{g_l g_r}PV\int\frac{d\omega}{2\pi}\frac{1}{\omega}\frac{\partial S_I(\omega,V)}{\partial V}. \tag{98}$$

Here, PV denotes the principal value and the limit $V\to 0$ is understood. Equations (94) and (98) are among the most significant results of this investigation.

V. WEAK-COUPLING REGIME, $g' \gg 1$

A. Perturbation theory

At a Gaussian level the AES action in frequency representation is given by

$$\mathcal{S}_0 = g\sum_{n>0}\left(n + \frac{2\pi^2 T}{gE_c}n^2\right)\Phi_n\Phi_{-n}. \tag{99}$$

To lowest order in an expansion in g the following result for $D(i\omega_n)$ is obtained

$$D(i\omega_n) = \beta\left[1 - \frac{2}{g}\sum_{s>0}\frac{1}{s+2\pi^2 Ts^2/(gE_c)}\right]\delta_{n,0} + \frac{2\pi i}{g}(1-\delta_{n,0}) \times\left(\frac{1}{i|\omega_n|} - \frac{1}{i|\omega_n| + igE_c/\pi}\right). \tag{100}$$

Using the representation $\delta_{n,0} = \lim_{\eta\to 0}\eta(i\omega_n+\eta)^{-1}$ one can perform analytic continuation to real frequencies and the retarded correlation function becomes

$$D^R(\omega) = \beta\left[1 - \frac{2}{g}\ln\frac{gE_c e^{\gamma}}{2\pi^2 T}\right]\lim_{\eta\to 0}\frac{\eta}{\omega+\eta+i0^+} + \frac{2\pi i}{g}\left(\frac{1}{\omega+i0^+} - \frac{1}{\omega + igE_c/\pi}\right). \tag{101}$$

Having carried out integration in Eqs. (85) and (86) with the help of the identity

$$\int_0^{\infty}\frac{dx}{x^2+\pi^2 z^2}\frac{x^2}{\sinh^2 x} = \frac{1}{2|z|} - 1 + |z|\psi'(1+|z|), \tag{102}$$

where $\psi(z)$ denotes the Euler di-gamma function, we obtain

$$g'(T) = g - 2\ln\frac{gE_c e^{\gamma+1}}{2\pi^2 T}, \quad q'(T) = q. \tag{103}$$

Here, $\gamma = -\psi(1) \approx 0.577$ denotes the Euler constant. The result for g' was originally obtained in Ref. 24 more than two decades ago. The quantity q', on the other hand, is unaffected by the quantum fluctuations to any order in an expansion in powers of $1/g$. To establish the renormalization of q' (θ renormalization) it is necessary to include the nonperturbative effects of instantons.

B. Instantons

Since the infrared of the dilute instanton gas is well defined one can proceed and evaluate the integrals in Eq. (29). This leads to the much simpler expression[22,26,36]

$$\beta\Omega_{\text{inst}} = -\frac{g^2}{\pi^2}\beta E_c e^{-g/2}\ln\frac{\beta E_c}{2\pi^2 e^{\gamma}}\cos 2\pi q. \tag{104}$$

With the help of Eq. (73) we immediately find the temperature dependence of the average charge on the island and the result is

$$Q(T) = q - \frac{g^2}{\pi}e^{-g/2}\ln\frac{E_c}{2\pi^2 e^{\gamma}T}\sin 2\pi q. \tag{105}$$

To find the quantities q' and g', however, we still have to evaluate the instanton contribution to the correlation function $D(i\omega_n)$. For this purpose we first consider the expectation of an arbitrary operator $\mathcal{O}$ which can be expanded to lowest order in the topological sectors $W = \pm 1$ according to

$$\langle\mathcal{O}\rangle \approx \frac{1}{Z[q]}(\mathcal{O}_0 + e^{2\pi iq}\mathcal{O}_1 + e^{-2\pi iq}\mathcal{O}_{-1}) \tag{106}$$

where

$$\mathcal{O}_W = \int_{\Phi(\beta)=\Phi(0)+2\pi W}\mathcal{D}\Phi(\tau)\mathcal{O}(\Phi)e^{-\mathcal{S}_d[\Phi]-\mathcal{S}_c[\Phi]}. \tag{107}$$

Similarly, we expand the partition function according to

$$Z[q] \approx Z_0\left(1 + e^{2\pi iq}\frac{Z_1}{Z_0} + e^{-2\pi iq}\frac{Z_{-1}}{Z_0}\right) \tag{108}$$

Equation (106) can therefore be split in a $W=0$ part and an instanton part

$$\langle\mathcal{O}\rangle \approx \langle\mathcal{O}\rangle_0 + \langle\mathcal{O}\rangle_{\text{inst}}. \tag{109}$$

Here, $\langle\mathcal{O}\rangle_0 = \mathcal{O}_0/Z_0$ and

$$\langle\mathcal{O}\rangle_{\text{inst}} = e^{2\pi iq}\frac{\mathcal{O}_1 - \langle\mathcal{O}\rangle_0 Z_1}{Z_0} + e^{-2\pi iq}\frac{\mathcal{O}_{-1} - \langle\mathcal{O}\rangle_0 Z_{-1}}{Z_0}. \tag{110}$$

In the semiclassical evaluation of Eq. (110) it suffices to replace the operator $\mathcal{O}[\Phi]$ in the integrand of Eq. (107) by its classical value $\mathcal{O}[\Phi_W]$. The result for Eq. (110) can then be written in the typical instanton form

$$\langle \mathcal{O}\rangle_{\rm inst} = \sum_{W=\pm 1}\int_0^\beta d\tau_0 \int_0^\beta \frac{d\lambda}{\lambda^2}[\mathcal{O}[\Phi_W]-\langle\mathcal{O}\rangle_0] g(\lambda) D \exp \times\left\{-\frac{1}{2}g(\lambda)+\frac{2}{\beta E_c(\lambda)}\left(1-\frac{2\beta}{\lambda}\right)+2\pi i q W\right\} \tag{111}$$

where $\mathcal{O}[\Phi_{\pm 1}]$ generally depends on the position τ_0 and scale size λ of the instanton/anti-instanton.

We next apply these general results to the correlation function $D(i\omega_n)$. The operator specific parts of Eq. (29) are computed to be, to the leading order in $1/g$,

$$\int_0^\beta d\tau_0[\mathcal{O}[\Phi_W]-\langle\mathcal{O}\rangle_0] = \beta\left\{-\left(\frac{\lambda}{\beta}\right)\delta_{n,0}+\left(1-\frac{\lambda}{\beta}\right)^{(|n|-1)} \times\left(\frac{\lambda}{\beta}\right)^2\Theta(nW)\right\} \tag{112}$$

with Θ denoting the Heaviside step function. Inserting this result in Eq. (29) and performing the integral over λ we find the following result for the instanton part:

$$D_{\rm inst}(i\omega_n) = -\frac{g^2 E_c}{\pi^2 T^2}e^{-g/2}\left[\delta_{n,0}\cos 2\pi q - \pi i T e^{2\pi i q\,\mathrm{sgn}\,n} \times(1-\delta_{n,0})\left(\frac{1}{i|\omega_n|}-\frac{1}{i|\omega_n|+2\pi i T}\right)\right]. \tag{113}$$

Performing analytic continuation to real frequencies we obtain

$$D^R_{\rm inst}(\omega) = -\frac{g^2 E_c}{\pi^2 T^2}e^{-g/2}\left[\cos 2\pi q \lim_{\eta\to 0}\frac{\eta}{\omega+\eta+i0^+} - \pi i T e^{i2\pi q}\left(\frac{1}{\omega+i0^+}-\frac{1}{\omega+i2\pi T}\right)\right]. \tag{114}$$

Using Eqs. (85) and (86) we find the following nonperturbative corrections to g' and q':

$$g'_{\rm inst} = -\frac{g^3 E_c}{6T}e^{-g/2}\cos 2\pi q, \tag{115}$$

$$q'_{\rm inst} = Q(T) - \frac{g^3 E_c}{24\pi T}e^{-g/2}\sin 2\pi q. \tag{116}$$

Here, the expression for $Q(T)$ is given by Eq. (105).

Combining the perturbative and nonperturbative contributions of Eqs. (103), (105), (115), and (116) we obtain the final total result for the temperature dependence of g' and q',

$$g'(T) = g - 2\ln\frac{gE_c e^{\gamma+1}}{2\pi^2 T} - \frac{g^3 E_c}{6T}e^{-g/2}\cos 2\pi q, \tag{117}$$

$$q'(T) = q - \frac{g^3 E_c}{24\pi T}\left[1+\frac{24T}{gE_c}\ln\frac{E_c}{2\pi^2 e^\gamma T}\right]e^{-g/2}\sin 2\pi q. \tag{118}$$

Several remarks are in order. First of all, we notice that the amplitude of the oscillations in q' with varying q are much larger than those in the averaged charge $Q(T)$. Equations (117) and (118) are generally valid in the weak-coupling phase of the SET $g \ll 1$ such that $T \gg g^3 E_c e^{-g/2}$. The results are completely analogous to the instanton corrections to the conductances σ'_{xx} and σ'_{xy} in the theory of the quantum Hall effect[37,38] that have recently been investigated experimentally.[39] It should be mentioned that Eq. (115) coincides with the earlier computations reported in Ref. 14.

C. θ renormalization

To leading order in $1/g$ one can express Eqs. (117) and (118) in the following manner:

$$g'(T) = g(T) - Dg^2(T)e^{-g(T)/2}\cos 2\pi q, \tag{119}$$

$$q'(T) = q - \frac{D}{4\pi}g^2(T)e^{-g(T)/2}\sin 2\pi q. \tag{120}$$

Here, $D=(\pi^2/3)e^{-\gamma-1}\approx 0.68$ is a numerical constant and

$$g(T) = g - 2\ln\frac{gE_c}{6DT} \tag{121}$$

contains the perturbative quantum corrections to leading order in $1/g$. It is important to emphasize that same results of Eqs. (119) and (120) are obtained if one employs the much simpler expressions for g' and q' defined in Eqs. (71) and (72). The only difference is the numerical value of D which now equals $D=2\exp(-\gamma)$. At the same time, the charging energy E_c in Eq. (121) is replaced by $(6/\pi^2)E_c$.

Expressing Eqs. (119) and (120) in differential form

$$\beta_g = \frac{dg'}{d\ln\beta} = -2 - \frac{4}{g'} - Dg'^2 e^{-g'/2}\cos 2\pi q' \tag{122}$$

$$\beta_q = \frac{dq'}{d\ln\beta} = -\frac{D}{4\pi}g'^2 e^{-g'/2}\sin 2\pi q' \tag{123}$$

we obtain the renormalization-group functions $\beta_{g,q} = \beta_{g,q}(g',q')$ of the AES theory on the weak-coupling side. We have included the two loop correction[25] in the perturbative part of Eq. (122).

Equations (122) and (123) are among the most important results of this investigation. The results clearly demonstrate that instantons are the fundamental topological objects of the AES theory that describe the crossover behavior of the SET between the conducting phase at high temperatures and the Coulomb blockade phase that generally appears at much lower temperatures only.

VI. STRONG-COUPLING PROBLEM, $g' \ll 1$

A. Effective action for $\theta \approx \pi$

For small values of the tunneling conductance g we can simplify the Hamiltonian of Eq. (2) near the degeneracy point $\theta=\pi$ or $q=1/2$ by employing a projection onto the states with $Q=k(q)$ and $Q=k(q)+1$ of the isolated island.[15] The projected Hamiltonian can be written as follows:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_c + \mathcal{H}_T^{(l)} + \mathcal{H}_T^{(r)} \tag{124}$$

where

$$\mathcal{H}_c = E_c(k-q)^2 + \frac{\Delta}{2} - \Delta S^z, \tag{125}$$

$$\mathcal{H}_T^{(s)} = \sum_{k\alpha} t_{k\alpha}^{(s)} a_k^{(s)\dagger} d_\alpha S^+ + \text{h.c.} \tag{126}$$

Here, $\Delta = E_c(1-\theta/\pi) \geq 0$, $\boldsymbol{S}$ denotes the spin $s=1/2$ operators and $S^\pm = S^x \pm iS^y$.

A convenient representation is obtained by using Abrikosov's two-component pseudofermion fields $\bar{\psi}$ and ψ.[40,41] After integration over the electronic degrees of freedom one arrives at the following effective action to leading order in $1/N_{\text{ch}}^{(s)}$,

$$\mathcal{S} = \beta E_c(k-q)^2 + \beta\frac{\Delta}{2} + \int_0^\beta d\tau \bar{\psi}\left(\partial_\tau - \eta + \frac{\Delta}{2}\sigma_z\right)\psi + \frac{g}{4}\int_0^\beta d\tau_1 d\tau_2 \alpha(\tau_{12})[\bar{\psi}(\tau_1)\sigma_-\psi(\tau_1)][\bar{\psi}(\tau_2)\sigma_+\psi(\tau_2)]. \tag{127}$$

Here, σ_j with $j=x,y,z$ stand for the Pauli matrices and $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$. We have introduced the chemical potential η such that the limit $\eta \to -\infty$ is taken at the end of all calculations. This procedure ensures that only the physical states with pseudo fermion number $N_{pf}=1$ contribute to the quantities of physical interest. Following the prescription[40,41]

$$Z = \lim_{\eta\to-\infty} \frac{\partial}{\partial e^{\beta\eta}} Z_{pf} \tag{128}$$

we obtain the physical partition function Z from the pseudo fermion theory Z_{pf}. Similarly, we extract the physical expectation $\langle \mathcal{O} \rangle$ according to

$$\langle \mathcal{O} \rangle = \lim_{\eta\to-\infty}\left[\frac{Z_{pf}}{Z}\frac{\partial}{\partial e^{\beta\eta}}\langle \mathcal{O}\rangle_{pf} + \langle \mathcal{O}\rangle_{pf}\right]. \tag{129}$$

The brackets $\langle \ldots \rangle_{pf}$ denote the average with respect to the theory of Eq. (127).

In what follows we employ the effective action of Eq. (127) to investigate the phenomenon of macroscopic charge quantization as well as the renormalization behavior of the SET on the strong-coupling side. Equation (127) is similar to the XY case of the Bose-Kondo model for spin $s=1/2$.[43–45] Notice that the spin operators $\bar{\psi}(\tau)\sigma_\pm\psi(\tau)$ in Eq. (127) are the same as the AES operators $\exp[\pm i\Phi(\tau)]$ projected onto the states $Q=k$ and $Q=k+1$ of the isolated island. The projection onto the Hamiltonian of Eq. (124) is justified as long as $g \ll 1$, $|q-k-1/2| \ll 1$ and $\beta E_c \gg 1$.[42]

B. Leading logarithmic approximation

In what follows we limit the analysis of Eq. (127) to the so-called leading logarithmic approximation. This corresponds to the one-loop renormalization-group procedure of Refs. 44 and 45.

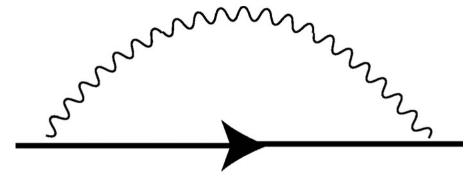

FIG. 5. The pseudofermion self-energy: solid line denotes $G(i\epsilon_n)$ whereas wavy line stands for the interaction $\alpha(i\omega_n)$ (see text).

1. Pseudofermion Green function renormalization

Using Eq. (127) we find the following expression for the pseudofermion Green's function for $g=0$:

$$G_{0\pm}^{-1}(i\epsilon_n) = i\epsilon_n + \eta \mp \frac{\Delta}{2}, \tag{130}$$

where $\epsilon_n = \pi T(2n+1)$. The pseudofermion Green's function can be expressed in terms of the self-energy $\Sigma_\pm$,

$$G_\pm^{-1}(i\epsilon_n) = i\epsilon_n + \eta \mp \frac{\Delta}{2} - \Sigma_\pm(i\epsilon_n). \tag{131}$$

It is convenient to parameterize the self-energy as follows:

$$\Sigma_\pm(i\epsilon_n) = (i\epsilon_n + \eta)[1-\gamma(i\epsilon_n)] \mp [1-\gamma_s(i\epsilon_n)]\frac{\Delta}{2}. \tag{132}$$

The pseudofermion Green's function now becomes

$$G_\pm^{-1}(i\epsilon_n) = (i\epsilon_n + \eta)\gamma(i\epsilon_n) \mp \gamma_s(i\epsilon_n)\frac{\Delta}{2}. \tag{133}$$

The leading logarithmic approximation corresponds to the simplest diagram for the self-energy shown in Fig. 5. This leads to the following equation:

$$\Sigma_\pm(i\epsilon_n) = -\frac{gT}{4\pi}\sum_{\omega_l}\frac{|\omega_l|}{i\omega_l + i\epsilon_n + \eta \pm \frac{\Delta}{2} - \Sigma_\mp(i\omega_l + i\epsilon_n)} \tag{134}$$

which has to be solved self consistently. Recall that there is no renormalization of the interaction line [see Fig. 5] because of the absence of closed fermion loops in the pseudofermion diagrammatic technique, i.e., their contribution vanishes in the limit $\eta \to -\infty$.[40,41]

With logarithmic accuracy we see that both γ and γ_s depend on the single variable $x = \ln \Lambda/\max\{\Delta\gamma_s/\gamma, |i\epsilon_n + \eta|\}$ where Λ is an arbitrary high energy cutoff. Then from Eq. (134) we obtain

$$\gamma(x) = 1 + \frac{g}{4\pi^2}\int_0^x \frac{dy}{\gamma(y)}, \tag{135}$$

$$\gamma_s(x) = 1 - \frac{g}{4\pi^2}\int_0^x \frac{dy\,\gamma_s(y)}{\gamma^2(y)} \tag{136}$$

and, hence,

$$\gamma(x) = \gamma_s^{-1}(x) = \left(1 + \frac{g}{2\pi^2}x\right)^{1/2}. \tag{137}$$

2. Partition function and the average charge Q

Using Eq. (128) we find

$$Z = e^{\beta E_c(k-q)^2} e^{\beta\Delta/2} \lim_{\eta\to-\infty} e^{-\beta\eta} \sum_{\epsilon_n,\sigma=\pm} e^{i\epsilon_n 0^+} G_\sigma(i\epsilon_n). \tag{138}$$

Given the vertex functions $\gamma(x)$ and $\gamma_s(x)$ it is now trivial to evaluate the partition function

$$Z = 2e^{\beta E_c(k-q)^2} e^{\beta\Delta/2} \cosh\frac{\beta\Delta'}{2}. \tag{139}$$

Here,

$$\gamma = \left(1 + \frac{g}{2\pi^2}\ln\frac{\Lambda}{\max\{\Delta',T\}}\right)^{1/2}, \tag{140}$$

and $\Delta' = \Delta/\gamma^2$ stands for the renormalized energy gap between the ground state and first-excited state.

The average charge on the island is expressed in terms of the magnetization $M = \langle S_z \rangle$ of our spin model

$$Q(T) = k + \frac{1}{2} - M \tag{141}$$

where

$$M(T) = \frac{1}{2\gamma^2}\tanh\frac{\beta\Delta'}{2}. \tag{142}$$

Equation (142) has originally been obtained in Ref. 16 using slightly different techniques. Evaluating the result at $T=0$ we find

$$Q(T=0) = k + \frac{1}{2} - \frac{1}{1 + \frac{g}{2\pi^2}\ln\frac{\Lambda}{\Delta'}} \tag{143}$$

which is the familiar result of Matveev.[15] It does not resemble the simple expression for the averaged charge on an isolated island. This charge is, in fact, no longer quantized when the tunneling conductance g is finite (see Sec. VI C 1).

3. Correlation function $D^R(\omega)$

The diagram for the two-point correlation function $D(i\omega_n)$ is shown in Fig. 6. Because of the peculiar form of the pseudofermion interaction which couples the σ_- and σ_+ it is readily seen that the lowest-order contribution to the vertex function $\Gamma(i\epsilon_n)$ is proportional to $g^2 \ln(\Lambda/\max\{T,\Delta'\})$ (see Fig. 7). Within the leading logarithmic approximation one can therefore put $\Gamma(i\epsilon_n) = 1$ and, hence,

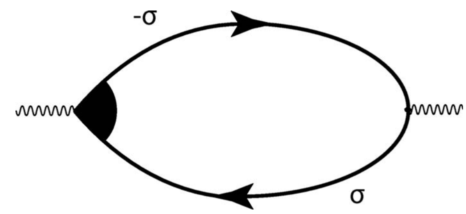

FIG. 6. The two-point correlation function: solid line denotes $G(i\epsilon_n)$ whereas black triangle denotes the vertex function $\Gamma(i\epsilon_n)$ (see text).

$$\begin{aligned} D(i\omega_n) &= -\frac{1}{2\cosh\beta\Delta'/2}\lim_{\eta\to-\infty}\frac{\partial}{\partial e^{\beta\eta}} \\ &\quad\times T\sum_{\epsilon_m} G_-(i\epsilon_m)G_+(i\epsilon_m + i\omega_n) \\ &= -\frac{\tanh\beta\Delta'/2}{\gamma^2}\frac{1}{i\omega_n - \Delta'}. \end{aligned} \tag{144}$$

After analytic continuation to real frequencies we obtain

$$D^R(\omega) = -\frac{\tanh\beta\Delta'/2}{\gamma^2}\frac{1}{\omega - \Delta' + i0^+}. \tag{145}$$

C. Physical observables

Given Eq. (145), it is possible to evaluate integrals in Eqs. (71) and (72) (definitions I) and in Eqs. (85) and (86) (definitions II) and compute the response functions in the strong-coupling phase. For both definitions (I) and (II) we obtain the same expression for the quasiparticle charge q',

$$q' = Q + \frac{1-\gamma^2}{2\gamma^2}\tanh\frac{\beta\Delta'}{2} \tag{146}$$

$$= k(q) + \frac{1}{e^{\beta\Delta'} + 1}. \tag{147}$$

However, different expressions for the response quantity g' are obtained and the result is

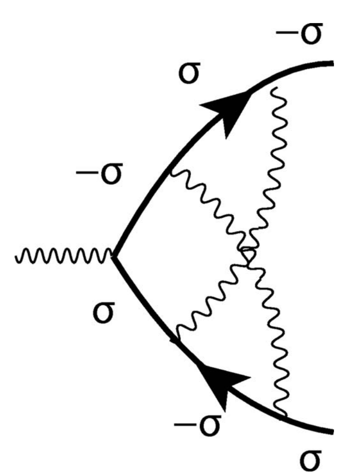

FIG. 7. The first nontrivial contribution to the vertex function $\Gamma(i\epsilon_n)$ (see text).

$$g'_I = \frac{g}{2\gamma^2}\frac{\tanh \beta\Delta'/2}{\beta\Delta'/2}, \tag{148}$$

$$g'_{II} = \frac{g}{2\gamma^2}\frac{\beta\Delta'}{\sinh \beta\Delta'}. \tag{149}$$

Equation (149) coincides with the result found in Ref. 16 using a somewhat different approach. It furthermore corresponds to the sequential tunneling approximation of Ref. 46.

There are several interesting conclusions that one can draw from these findings. First of all, we see that as T approaches absolute zero Eqs. (146)–(149) are independent of g and precisely coincide with the results obtained for the isolated island. Unlike Eq. (143), for example, we find that the new quantity q' is robustly quantized with infinitely sharp steps occurring when the external charge q passes through $k+1/2$.

However, Eqs. (148) and (149) for the response quantity g' do not unequivocally predict an exponential dependence on T when $\beta\Delta' \gg 1$. Moreover, as shown in Appendix C, the corrections to the quantities g'_{II} and q'_I to second order in g do not demonstrate an exponential dependence on T when T vanishes.

This means that the strong-coupling expansion in powers of g generally does not provide access to the Coulomb blockade phase where the SET develops an energy gap. The validity of the leading logarithmic approximation is therefore limited to the quantum critical phase $\beta\Delta' \lesssim 1$ which for our purposes is the most significant regime of the SET.

This takes us to the most important part of this exercise which is to employ Eqs. (146)–(149) in order to extract the scaling behavior of the SET on the strong-coupling side. Since our physical observables are essentially defined for finite-size systems (i.e., finite β) they should in general be distinguished from the ordinary thermodynamic quantities of the quantum spin system that are normally being considered. Emerging from Eqs. (146)–(149) there are two distinctly different renormalization-group schemes, to be discussed further below, that provide complementary information on the quantum system at zero temperature and finite temperatures, respectively.

1. RG at zero temperature

Equations (146)–(149) clearly show that two renormalizations are in general necessary to absorb the ultraviolet or high energy singularity structure of our spin system, i.e., one renormalization associated with the coupling constant (tunneling conductance) g and one associated with the "magnetic field" (energy gap) Δ. From the expressions at zero T

$$\frac{g}{\gamma^2} = \frac{g}{1+\frac{g}{2\pi^2}\ln\frac{\Lambda}{\Delta'}}, \quad \Delta' = \frac{\Delta}{\gamma^2} = \frac{\Delta}{1+\frac{g}{2\pi^2}\ln\frac{\Lambda}{\Delta'}} \tag{150}$$

we obtain the following renormalization-group β and γ functions to one-loop order

$$\beta_g = \frac{dg}{d\ln\Lambda} = \frac{g^2}{2\pi^2}, \quad \gamma_\Delta = \frac{d\ln\Delta}{d\ln\Lambda} = \frac{g}{2\pi^2}. \tag{151}$$

Employing the method of characteristics one can cast the thermodynamic quantities of the quantum spin system at $T=0$ in a general scaling form. For example, the magnetization M with varying "magnetic field" Δ can be expressed as follows:

$$M(\Delta) = M_0 f(\Delta M_0 \xi). \tag{152}$$

Here, the functions M_0 and ξ with varying g are determined by the β and γ functions according to

$$\left(\Lambda\frac{\partial}{\partial\Lambda} + \beta_g\frac{\partial}{\partial g}\right)\xi = 0 \tag{153}$$

$$\left(\gamma_\Delta + \beta_g\frac{\partial}{\partial g}\right)M_0 = 0. \tag{154}$$

One finds, for example, that the characteristic time scale ξ of the SET is given by

$$\xi = \Lambda^{-1} e^{-2\pi^2/g} \tag{155}$$

which has the same meaning as the weak-coupling expression of Eq. (34). The quantity M_0 and the scaling function $f(X)$ within the one-loop approximation are given by

$$M_0 = 1/2g, \quad f(X) = 2\pi^2 \ln^{-1} X^{-1}. \tag{156}$$

The result essentially tells us that the spontaneous magnetization only exists for the theory with $g=0$ but it vanishes for any finite value of g. In terms of the AES model this means that the averaged charge Q on the island is no longer quantized when finite values of the tunneling conductance are taken into account.

2. RG at finite temperatures

We next specialize to the physical observables at finite temperature. Since the expressions of Eqs. (146)–(149) are universal for $\beta\Delta' \ll 1$

$$g'(T) \simeq \frac{g}{2}\left(1+\frac{g}{2\pi^2}\ln\beta\Lambda\right)^{-1}, \tag{157}$$

$$q'(T) \simeq k+\frac{1}{2}-\frac{\beta\Delta}{4}\left(1+\frac{g}{2\pi^2}\ln\beta\Lambda\right)^{-1} \tag{158}$$

we immediately obtain the finite temperature β functions along the critical lines $q'=k+1/2$ for $g' \ll 1$ according to

$$\beta_g = \frac{dg'}{d\ln\beta} = -\frac{g'^2}{\pi^2} \tag{159}$$

$$\beta_q = \frac{dq'}{d\ln\beta} = \left(q'-k-\frac{1}{2}\right)\left(1-\frac{g'}{\pi^2}\right). \tag{160}$$

These results should be compared with Eq. (59) obtained for an isolated island. We see that the critical fixed point of an isolated island is the critical fixed point of the AES theory as

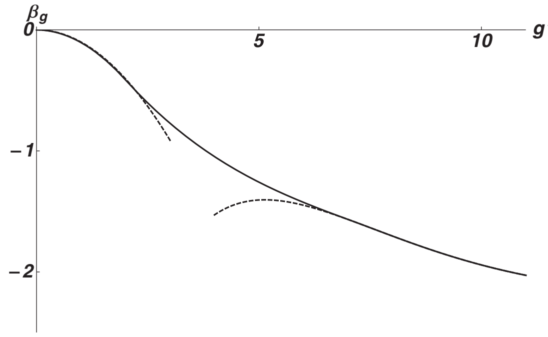

FIG. 8. The function β_g with varying g' along the critical line $q'=k+1/2$. An interpolation between the weak and strong-coupling branches has been drawn as a guide for the eyes (see text).

a whole with the SET conductance g' now playing the role of a *marginally irrelevant* scaling variable.

Next we compare Eqs. (159) and (160) with the weak-coupling results of Eqs. (122) and (123). In Figs. 8 and 9 we plot the functions β_g and $\partial\beta_q/\partial q$ along the critical line $q'=k+1/2$. A simple interpolation between the weak and strong-coupling branches indicates that both these functions decrease monotonically as g' increases.

Finally, it is not difficult to understand why the Coulomb blockade phase of the SET is beyond the scope of the present investigation. For example, given the fact that the theory with $q'\approx k$ and $g'\ll 1$ develops an energy gap then one generally expects [see also Eq. (62)]

$$\beta_g = g' \ln g' \tag{161}$$

$$\beta_q = (q'-k)\ln|q'-k| \tag{162}$$

which cannot be obtained using ordinary perturbation theory in g'.

VII. SUMMARY AND CONCLUSIONS

To summarize the results of this investigation we have sketched, in Fig. 10, a unifying scaling diagram of the SET in the g'-q' plane. This diagram is based on the strong-coupling results of Eqs. (59), (159), and (160) and the weak-coupling results of Eqs. (122) and (123).

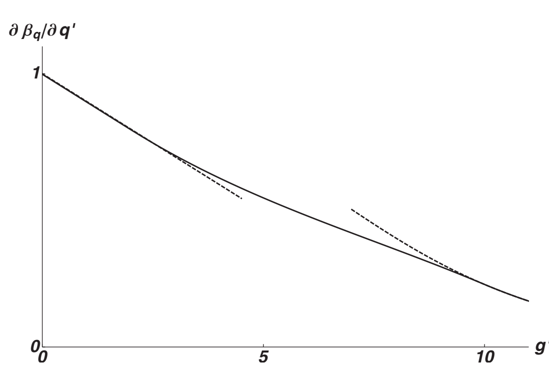

FIG. 9. The function $\partial\beta_q/\partial q'$ with varying g' along the critical line $q'=k+1/2$. An interpolation between the weak and strong-coupling branches has been drawn as a guide for the eye (see text).

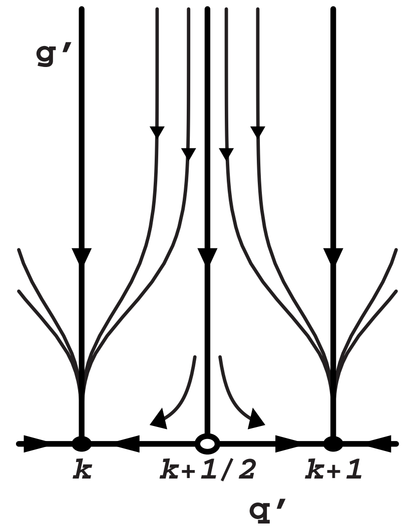

FIG. 10. Unified scaling diagram of the Coulomb blockade in terms of the SET conductance g' and the q'. The arrows indicate the scaling toward $T=0$ (see text).

The universal features of this diagram are the quantum critical fixed points located at $q'=k+1/2$, $g'=0$, and the stable fixed points at $q'=k$, $g'=0$ that describe the "macroscopic charge quantization" of the SET. The results are in accordance with the concept of super universality that has previously been proposed in the context of the quantum Hall effect.[12]

We have established the relation between the quantity g' and the ordinary SET conductance G that one normally obtains from linear-response theory. The quantity q' is new and can similarly be expressed in terms of the antisymmetric current-current correlation function.

The quantization of q' is an interesting and important challenge for experimental research on single-electron devices. There are, however, other ways of experimentally probing the quasiparticle charge q' of the SET. In Sec. VII A below we will summarize the quantum critical properties of q' and point out how they are directly measurable in the experiment.

We conclude this paper with Sec. VII B below where we discuss in some detail the physical mechanism that is responsible for changing the quasiparticle charge q' of the SET as q passes through the critical point.

A. Quantum criticality

Equation (157) is the maximum value of $g'(T)$ as one varies the value of q. This maximum value vanishes logarithmically in T according to

$$g'_{\max}(T) = \frac{g}{2\gamma^2} = \frac{g}{2}\left(1+\frac{g}{2\pi^2}\ln\beta\Lambda\right)^{-1} = \pi^2\ln^{-1}(\beta/\xi) \ll 1. \tag{163}$$

Similarly, Eq. (158) determines the maximum slope of the quasiparticle charge $q'(T)$ with varying q. This slope diverges according to

$$\left[\frac{\partial q'(T)}{\partial q}\right]_{\max} = \frac{\beta E_c}{2}\left(1 + \frac{g}{2\pi^2}\ln \beta\Lambda\right)^{-1} = \frac{\pi^2}{g}(\beta E_c)\ln^{-1}(\beta/\xi). \tag{164}$$

The inverse of this quantity is a measure for the width Δq of the transition. This width vanishes as T goes to zero.

Notice that Eq. (164) is completely analogous to what happens at the plateau transitions in the quantum Hall regime. In that case we have

$$\left[\frac{\partial R_H}{\partial B}\right]_{\max} \propto \beta^{\kappa}, \tag{165}$$

i.e., the maximum slope of the Hall conductance R_H with varying magnetic field B diverges algebraically in β. The best experimental value of the critical exponent κ equals 0.42. Similar to the experiments on the quantum Hall effect, one may consider higher order derivatives of g' and q' with respect to q.[47] These quantities diverge even faster with higher powers of β.

The quantity $\partial q'/\partial q$ can directly be measured in the experiment since it determines the *renormalized* gate capacitance of the SET $C'_g = C_g \partial q'/\partial q$. As was shown recently, the *rate of energy dissipation* (P) in the SET due to a low-frequency gate voltage $V_g(t) = V_g + U_\omega \cos \omega t$ is given by[48]

$$P \propto \omega^2 |U_\omega|^2 \frac{C'^2_g}{g'}. \tag{166}$$

Therefore, the maximum in P with varying values of V_g diverges according to

$$P_{\max} \propto \left[\frac{C'^2_g}{g'}\right]_{\max} = \frac{\pi^2 C_g^2}{g^2}(\beta E_c)^2 \ln^{-1}(\beta/\xi). \tag{167}$$

Finally, it should be mentioned that the critical behavior of the SET is likely to change when the effective number of channels $N_{\mathrm{ch}}^{(l,r)}$ between the island and the reservoirs are finite rather than infinite.[14,49] Even though we expect that our theory of physical observables remains unchanged, Matveev[15] has argued that the critical behavior of the SET can be mapped onto the N-channel Kondo model.[50] This would mean that the transition at $q' = k + 1/2$ becomes a second-order one with a finite critical value of g' thus closely resembling the more complicated physics of the quantum Hall effect.[12] Progress along these lines will be reported elsewhere.[29]

B. Quantization of q'

We have seen that the Thouless criterion for the Coulomb blockade breaks down at points $q = k + 1/2$ where the energy gap Δ' vanishes. To understand how the critical features of the SET permit a change in q' one must think in terms of a dynamical process where a unit of external charge is added to the system at (imaginary) time 0 and removed at $\tau > 0$. This process is described by the two-point correlation function $D(\tau)$ given by

$$D(\tau) = \langle e^{i\Phi(0) - i\Phi(\tau)} \rangle. \tag{168}$$

Following Eq. (71), the tunneling through the SET involves the sum over all processes $D(\tau)$ according to

$$g' = gT \int_0^\beta d\tau D(\tau). \tag{169}$$

From Eq. (144) we obtain the following expression valid at $T = 0$ when q approaches $k + 1/2$ from below:

$$D(\tau) = \gamma^{-2} \Theta(\tau) e^{-\tau \Delta'}. \tag{170}$$

This general result includes the isolated island except that the AES operators are now renormalized $(\gamma \neq 1)$ and the energy gap Δ is replaced by the renormalized value Δ'.

Let us first assume that $D(\tau)$ denotes the correlation of an isolated island. Equation (169) then stands for a semiclassical picture of the SET where the island and reservoirs are essentially disconnected. Since the expectation value $\langle \tau \rangle$ is finite for $q < k + 1/2$,

$$\langle \tau \rangle = \frac{\int_0^\infty d\tau \tau D(\tau)}{\int_0^\infty d\tau D(\tau)} = \frac{1}{\Delta} \tag{171}$$

it is impossible that the tunneling processes described by Eq. (169) alter the static charge Q on the island. However, as one approaches the critical point then the expectation $\langle \tau \rangle$ diverges. It is thus possible that when q passes through $k + 1/2$, a unit of charge stays behind on the island. This extra charge is precisely what lowers the energy of the island; i.e., it permits the energy to jump from one parabolic branch $E_c(q-k)^2$ to the next $E_c(q-k-1)^2$ (see Fig. 3).

From the expression for q' in Eq. (72) it is clear that this semiclassical picture of the SET gets dramatically complicated when the tunneling conductance g is finite. In particular, the second term proportional to g is Eq. (72) clearly indicates that the quantization of q' goes hand in hand with strong charge fluctuations between the island and the reservoirs. Nevertheless, the *mechanism* for changing the quasiparticle charge q' of the SET remains essentially the same. This mechanism solely involves a vanishing energy gap Δ'. The only difference with the semiclassical picture is that the AES operators $e^{\pm i\Phi}$ in Eq. (168) generally stand for the *quasiparticle* operators of the SET, rather than those of ordinary electrons in an isolated island.

Let us next consider the tunneling process in some more detail. We are interested, first of all, in the energy difference δE between the states $|q+1\rangle$ and $|q\rangle$ of the SET. Here, $|q+1\rangle$ is formally defined as follows:

$$|q+1\rangle = \lim_{\tau_0 \to \beta} |q(\tau)\rangle \tag{172}$$

where $q(\tau) = q + 1$ for $0 < \tau < \tau_0$ and $q(\tau) = q$ for $\tau_0 < \tau < \beta$. After elementary algebra we obtain[29]

$$\delta E = \frac{\Delta'}{1 + e^{-\beta \Delta'}} + T \frac{1}{\gamma^2} \frac{d\gamma^2}{d \ln T}. \tag{173}$$

Since $\delta E \sim \Delta'$ at low temperatures $(T \leq \Delta')$ we conclude that the transition from $|q\rangle$ to $|q+1\rangle$ is energetically unfavorable.

Next, the rates for electron tunneling from reservoir to island (Γ_{01}) and backward (Γ_{10}) are computed to be[46,48]

$$\Gamma_{01/10} = \frac{g\Delta'}{4\pi\gamma^2}\left(\coth\frac{\Delta'}{2T} \pm 1\right). \tag{174}$$

As long as the energy gap Δ' is finite, the energy difference δE in Eq. (173) and the tunneling rates $\Gamma_{01/10}$ are not related to one another in any obvious manner. However, at the critical point $\Delta=0$ we find

$$\delta E = \frac{1}{\beta \ln \beta/\xi}, \quad \Gamma_{01} = \Gamma_{10} = \pi\delta E. \tag{175}$$

Hence, the energy difference between the states $|q+1\rangle$ and $|q\rangle$ determines the time the electron resides on the island. It is therefore possible that the tunneling processes alter the static charge q' of the SET as q passes through $k+1/2$.

ACKNOWLEDGMENTS

The authors are grateful to A. Abanov, O. Astafiev, A. Finkelstein, A. Ioselevich, A. Lebedev, and Yu. Makhlin for stimulating discussions. The research was funded in part by the Dutch National Science Foundations NWO and FOM, the Russian Ministry of Education and Science, U.S. CRDF, the Council for Grant of the President of Russian Federation (Grant No. MK-125.2009.2), RFBR (Grants No. 09-02-92474-MHKC and No. 07-02-00998), the Dynasty Foundation and RAS Programs “Quantum Physics of Condensed Matter” and “Fundamentals of nanotechnology and nanomaterials.”

APPENDIX A: ANALYTIC CONTINUATION OF $K(i\omega_n)$

In this appendix we perform the various steps that take us from $K(i\omega_n)$, Eq. (80), to the expression for $K^R(\omega)$ in Eq. (84). Starting from the correlation function

$$K(i\omega_n) = -\frac{g}{4\pi}T\sum_{\omega_m}|\omega_m+\omega_n|D(i\omega_m) \tag{A1}$$

we employ the following relations:

$$|\omega_m| = \int \frac{d\epsilon}{\pi}\frac{i\omega_m}{\epsilon + i\omega_m}, \tag{A2}$$

$$D(i\omega_m) = \int \frac{d\epsilon}{\pi}\frac{\mathrm{Im}\, D^R(\epsilon)}{\epsilon - i\omega_m}. \tag{A3}$$

and obtain the following expression:

$$K(i\omega_n) = \frac{g}{4\pi}\int \frac{d\epsilon_1 d\epsilon_2}{\pi^2}\frac{\epsilon_2\, \mathrm{Im}\, D^R(\epsilon_1)}{\epsilon_1+\epsilon_2+i\omega_n}T\sum_{\omega_m}\times\left[\frac{1}{\epsilon_1 - i\omega_m} + \frac{1}{\epsilon_2 + i\omega_m + i\omega_n}\right]. \tag{A4}$$

Evaluating the sum over ω_m we find

$$K(i\omega_n) = \frac{g}{4\pi}\int \frac{d\epsilon_1 d\epsilon_2}{\pi^2}\frac{\epsilon_2\, \mathrm{Im}\, D^R(\epsilon_1)}{\epsilon_1-\epsilon_2+i\omega_n}[n_b(\epsilon_2) - n_b(\epsilon_1)]. \tag{A5}$$

The analytic continuation to real frequencies is now trivial and we directly obtain Eq. (84).

APPENDIX B: USEFUL IDENTITIES

The identities obtained in this Section will be of use in Appendix C. First, we consider the derivative of the average charge Q with respect to g which can be obtained as follows:

$$\frac{\partial Q}{\partial g} = \frac{\partial^2 \ln Z_{pf}}{\partial g\, \partial \Delta} = \frac{1}{4\pi}\frac{\partial}{\partial \Delta}T\sum_{\omega_n}|\omega_n|D(i\omega_n). \tag{B1}$$

With the help of Eq. (A3), it is convenient to rewrite Eq. (B1) in the following manner:

$$\frac{\partial Q}{\partial g} = \frac{\partial}{\partial \Delta}\int \frac{d\epsilon}{2\pi^2}Y(\epsilon)\mathrm{Im}\, D^R(\epsilon), \tag{B2}$$

where

$$Y(\epsilon) = T\sum_{\omega_n>0}\frac{\omega_n\epsilon}{\omega_n^2+\epsilon^2}. \tag{B3}$$

A second useful identity for the expression appearing in Eq. (83) is given by

$$\mathrm{Re}\frac{\partial K^R(\omega)}{\partial \omega} = -\frac{g}{2\pi^2}\int d\epsilon\, \mathrm{Im}\, D^R(\epsilon)\frac{\partial Y(\epsilon)}{\partial \epsilon} \tag{B4}$$

where the limit $\omega \to 0$ is understood.

APPENDIX C: EVALUATION OF q' TO SECOND ORDER IN g

Based on Eqs. (B2) and (B4) we evaluate, in this Appendix, the expression for q' in Eq. (83) to second order in g. We start from the two-point correlation function $D(i\omega_n)$ which to first order in g is given by

$$D(i\omega_n) = -\frac{\tanh(\beta\Delta/2)}{i\omega_n - \Delta}\left[1 - \frac{g}{\pi}\frac{\beta Y(\Delta)}{\sinh(\beta\Delta)}\right] + \frac{g}{\pi}\tanh(\beta\Delta/2)\frac{\partial}{\partial\Delta}\left(\frac{Y(\Delta)}{i\omega_n - \Delta}\right) - \frac{g}{4\pi}\frac{|\omega_n|}{(i\omega_n - \Delta)^2}. \tag{C1}$$

This result can be written in the following form:

$$D(i\omega_n) = -\frac{\tanh(\beta\Delta_1'/2)}{\gamma_1^2}\frac{1}{i\omega_n - \Delta_1'} - \frac{g}{4\pi}\frac{|\omega_n|}{(i\omega_n-\Delta)^2}. \tag{C2}$$

where the renormalized energy gap and the renormalization factor are given as

$$\Delta_1' = \Delta - \frac{g}{\pi}Y(\Delta), \quad \frac{1}{\gamma_1^2} = \frac{\partial \Delta_1'}{\partial \Delta}. \tag{C3}$$

Equation (C2) implies the following expression for the retarded function:

$$D^R(\epsilon) = -\frac{\tanh(\beta\Delta_1'/2)}{\gamma_1^2}\frac{1}{\epsilon-\Delta_1'+i0^+} + \frac{g}{4\pi}\frac{i\epsilon}{(\epsilon-\Delta+i0^+)^2}. \tag{C4}$$

By inserting result (C4) for $D^R(\epsilon)$ in Eq. (B4) we find after elementary algebra

$$\mathrm{Re}\frac{\partial K_1^R(\omega)}{\partial\omega} = -\frac{g}{2\pi}\frac{\tanh(\beta\Delta_1'/2)}{\gamma_1^2}\frac{\partial Y(\Delta_1)}{\partial\Delta_1} + \frac{g^2}{32\pi^2}\frac{\partial}{\partial\Delta}\left(\Delta\frac{\partial}{\partial\Delta}\Delta\coth\frac{\beta\Delta}{2}\right) \tag{C5}$$

where the limit $\omega\to 0$ is understood. Next, by expanding Eq. (C5) to the second order in g we finally obtain

$$\mathrm{Re}\frac{\partial K_1^R(\omega)}{\partial\omega} = -\frac{g}{2\pi}\tanh\frac{\beta\Delta}{2}\left[\frac{\partial Y}{\partial\Delta} - \frac{g}{2\pi}\frac{\partial^2 Y^2}{\partial\Delta^2} - \frac{g}{2\pi}\frac{\beta}{\sinh\beta\Delta}\frac{\partial Y^2}{\partial\Delta}\right] + \frac{g^2}{32\pi^2}\frac{\partial}{\partial\Delta}\left(\Delta\frac{\partial}{\partial\Delta}\Delta\coth\frac{\beta\Delta}{2}\right). \tag{C6}$$

We proceed by inserting the result for $D^R(\epsilon)$ in the expression of Eq. (B2) and find

$$\frac{\partial Q}{\partial g} = \frac{1}{2\pi}\frac{\partial}{\partial\Delta}\left[\frac{\tanh(\beta\Delta_1'/2)}{\gamma_1^2}Y(\Delta_1')\right] \tag{C7}$$

$$-\frac{g}{32\pi^2}\frac{\partial^2}{\partial\Delta^2}\left(\Delta^2\coth\frac{\beta\Delta}{2}\right). \tag{C8}$$

Up to second in g the expression for the averaged charge Q therefore becomes

$$Q = k(q) + \frac{1}{1+e^{\beta\Delta}} + \frac{g}{2\pi}\frac{\partial}{\partial\Delta}\left[\tanh\frac{\beta\Delta}{2}\left(Y - \frac{g}{2\pi}\frac{\partial Y^2}{\partial\Delta} - \frac{g}{2\pi}\frac{\beta Y^2}{\sinh\beta\Delta}\right)\right] - \frac{g^2}{64\pi^2}\frac{\partial^2}{\partial\Delta^2}\left(\Delta^2\coth\frac{\beta\Delta}{2}\right). \tag{C9}$$

Finally, collecting Eqs. (C6) and (C9) together we find the total result for q' as follows:

$$q' = k(q) + \frac{1}{1+e^{\beta\Delta}} + \frac{g}{2\pi}\left(Y\frac{\partial}{\partial\Delta} - \frac{g}{2\pi}\frac{\partial Y^2}{\partial\Delta}\frac{\partial}{\partial\Delta} - \frac{g}{2\pi}Y^2\frac{\partial^2}{\partial\Delta^2}\right)\tanh\frac{\beta\Delta}{2} + \frac{g^2}{64\pi^2}\Delta\frac{\partial^2}{\partial\Delta^2}\left(\Delta\coth\frac{\beta\Delta}{2}\right). \tag{C10}$$

The result can be written in a slightly more compact fashion according to

$$q' = k(q) + \frac{1}{1+e^{\beta\Delta_2'}} + \frac{g^2}{64\pi^2}\Delta\frac{\partial^2}{\partial\Delta^2}\left(\Delta\coth\frac{\beta\Delta}{2}\right). \tag{C11}$$

Here,

$$\Delta_2' = \Delta - (g/\pi)Y(\Delta_1') \tag{C12}$$

represents the second order in g expression for the renormalized gap.

Similarly, with the help of Eqs. (C1) and (C4) from Eqs. (82), (71), and (72) one can compute the other response parameters to the second order in g. The results can be summarized as follows:

$$g_I' = \frac{g}{2\gamma_1^2}\frac{\tanh\beta\Delta_1'/2}{\beta\Delta_1'/2}, \tag{C13}$$

$$q_I' = k(q) + \frac{1}{1+e^{\beta\Delta_2'}} - \frac{g^2\Delta}{64\pi^2}\frac{\partial^2}{\partial\Delta^2}\left(\Delta\coth\frac{\beta\Delta}{2}\right) + \frac{g^2}{8\pi^3}\beta\Delta_1'\tanh\frac{\beta\Delta_1'}{2}\mathrm{Im}\,\psi'\left(1+\frac{i\beta\Delta}{2\pi}\right), \tag{C14}$$

and

$$g_{II}' = \frac{g}{2\gamma_1^2}\frac{\beta\Delta_1'}{\sinh\beta\Delta_1'} - \frac{g^2}{4\pi^2}\left[1 + \frac{\partial}{\partial\Delta}\frac{\beta\Delta^2}{2\pi}\mathrm{Im}\,\psi'\left(1+\frac{i\beta\Delta}{2\pi}\right)\right], \tag{C15}$$

$$q_{II}' = k(q) + \frac{1}{1+e^{\beta\Delta_2'}} + \frac{g^2\Delta}{64\pi^2}\frac{\partial^2}{\partial\Delta^2}\left(\Delta\coth\frac{\beta\Delta}{2}\right). \tag{C16}$$

These results are different from those obtained in the leading logarithmic approximation. In some cases (i.e., g_{II}' and q_I') the corrections in g no longer predict an exponential dependence on T in the limit where T goes to zero. This clearly shows that the expansion to lowest orders in g does not provide access to the Coulomb blockade phase of the SET.

APPENDIX D: EVALUATION OF $K^R(\omega)$

In this appendix we present the results of explicit computations of the response function $K^R(\omega)$. These results can be used, first of all, as an independent check on the results of Eqs. (103), (115), (116), (147), and (149). Second, they show how the analytic continuation of $K(i\omega_n)$ to real frequencies works in explicit computations.

1. Weak-coupling regime $g' \gg 1$

Based on Eqs. (101) and (113) we obtain the following expression from Eq. (84),

$$K^R(\omega) = \frac{i\omega g}{4\pi}\left[1 - \frac{2}{g}\ln\frac{egE_c}{2\pi^2 T} + \frac{2}{g}\psi\left(1-\frac{i\omega}{2\pi T}\right)\right] - \sum_{n=1}^{n_{\max}}\frac{\frac{gE_c}{2\pi^2}}{n+\frac{gE_c}{2\pi^2 T}} - \frac{g^3E_c}{2\pi^2}e^{-g/2} \times\left\{e^{i2\pi q}\left[\psi(1) - \psi\left(1-\frac{i\omega}{2\pi T}\right)\right] + \cos 2\pi q\sum_{n=2}^{n_{\max}}\frac{1}{n}\right\}. \tag{D1}$$

Here, the cut off $n_{\max}$ appears due to the fact that, as usual, we use the low-frequency part of the kernel $\alpha(\tau)$ only. If one takes the proper expression for it into account then one finds $n_{\max} \sim E_F/T$ where E_F denotes the Fermi energy.

2. Strong-coupling regime $g' \ll 1$

Given Eq. (145) we can write

$$K^R(\omega) = -\frac{g \tanh\frac{\beta\Delta'}{2}}{4\pi^2\gamma^2}\int d\epsilon\epsilon\frac{n_b(\epsilon)-n_b(\Delta')}{\epsilon-\Delta'-\omega-i0^+}. \tag{D2}$$

Hence,

$$\operatorname{Im} K^R(\omega) = \frac{g \tanh\frac{\beta\Delta'}{2}}{4\pi\gamma^2}(\omega+\Delta')[n_b(\Delta')-n_b(\omega+\Delta')] \tag{D3}$$

$$\operatorname{Re} K^R(\omega) = \frac{g \tanh\frac{\beta\Delta'}{2}}{4\pi^2\gamma^2}(\Delta'+\omega)\operatorname{Re}\left[\psi\left(1-\frac{i(\Delta'+\omega)}{2\pi T}\right) - \psi\left(1-\frac{i\Delta'}{2\pi T}\right) - \frac{2\pi}{\Delta'}Y(\Delta')\right] \tag{D4}$$

By using the following representation of Bose-Einstein function in the sum over Matsubara frequencies:

$$n_b(\omega) = \frac{1}{\pi}\operatorname{Im}\,\psi\left(1+i\frac{\omega}{2\pi T}\right) - \frac{1}{2} + \frac{T}{\omega} \tag{D5}$$

we finally obtain

$$K^R(\omega) = \frac{g \tanh\frac{\beta\Delta'}{2}}{4\pi^2\gamma^2}(\omega+\Delta')\left[\psi\left(1-i\frac{\omega+\Delta'}{2\pi T}\right) - \psi\left(1-i\frac{\Delta'}{2\pi T}\right) - \frac{2\pi}{\Delta'}Y(\Delta') + \frac{iT\omega}{\Delta'(\omega+\Delta')}\right]. \tag{D6}$$

[1] V. Ambegaokar, U. Eckern, and G. Schön, Phys. Rev. Lett. **48**, 1745 (1982).
[2] For a review, see G. Schön and A. D. Zaikin, Phys. Rep. **198**, 237 (1990).
[3] For a review, see *Single Charge Tunneling*, edited by H. Grabert and M. H. Devoret (Plenum, New York, 1992).
[4] For a review, see H. Grabert and H. Horner, in the special issue on single charge tunneling, Z. Phys. B **85**, 317 (1991).
[5] For a review, see L. I. Glazman and M. Pustilnik, in *New Directions in Mesoscopic Physics, Towards to Nanoscience*, edited by R. Fazio, G. F. Gantmakher, and Y. Imry (Kluwer, Dordrecht, 2003).
[6] P. Lafarge, H. Pothier, E. R. Williams, D. Esteve, C. Urbina, and M. H. Devoret, Z. Phys. B **85**, 327 (1991).
[7] L. J. Geerligs, V. F. Anderegg, P. A. M. Holweg, J. E. Mooij, H. Pothier, D. Esteve, C. Urbina, and M. H. Devoret, Phys. Rev. Lett. **64**, 2691 (1990); H. Pothier, P. Lafarge, P. F. Orfila, C. Urbina, D. Esteve, and M. H. Devoret, Physica B **169**, 573 (1991).
[8] T. A. Fulton and G. J. Dolan, Phys. Rev. Lett. **59**, 109 (1987).
[9] G. Falci, G. Schon, and G. T. Zimanyi, Phys. Rev. Lett. **74**, 3257 (1995); Physica B **203**, 409 (1994).
[10] I. S. Beloborodov, K. B. Efetov, A. Altland, and F. W. J. Hekking, Phys. Rev. B **63**, 115109 (2001).
[11] K. B. Efetov and A. Tschersich, Phys. Rev. B **67**, 174205 (2003).
[12] For a review, see A. M. M. Pruisken and I. S. Burmistrov, Ann. Phys. (N.Y.) **322**, 1265 (2007); Ann. Phys. (N.Y.) **316**, 285 (2005), and references therein.
[13] S. L. Lukyanov, A. M. Tsvelik, and A. B. Zamolodchikov, Nucl. Phys. B **719**, 103 (2005).
[14] A. Altland, L. I. Glazman, A. Kamenev, and J. S. Meyer, Ann. Phys. (N.Y.) **321**, 2566 (2006), and references therein.
[15] K. A. Matveev, Sov. Phys. JETP **72**, 892 (1991).
[16] H. Schoeller and G. Schön, Phys. Rev. B **50**, 18436 (1994), and references therein.
[17] I. L. Kurland, I. L. Aleiner, and B. L. Altshuler, Phys. Rev. B **62**, 14886 (2000); I. L. Aleiner, P. W. Brouwer, and L. I. Glazman, Phys. Rep. **358**, 309 (2002).
[18] R. Landauer, IBM J. Res. Dev. **1**, 223 (1957).
[19] A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Yu. S. Tyupkin, Phys. Lett. B **59**, 85 (1975).
[20] S. E. Korshunov, JETP Lett. **45**, 434 (1987).
[21] S. A. Bulgadaev, Phys. Lett. A **125**, 299 (1987).
[22] X. Wang and H. Grabert, Phys. Rev. B **53**, 12621 (1996).
[23] See, e.g., R. Rajaraman, *Instantons and Solitons*, (Amsterdam, North-Holland, 1982); A. M. Polyakov, *Gauge Fields and Strings*, (Harwood Academic Publishers, Shur, 1987).
[24] F. Guinea and G. Schön, Europhys. Lett. **1**, 585 (1986); S. A. Bulgadaev, JETP Lett. **45**, 622 (1987).
[25] W. Hofstetter and W. Zwerger, Phys. Rev. Lett. **78**, 3737 (1997).
[26] I. S. Beloborodov, A. V. Andreev, and A. I. Larkin, Phys. Rev. B **68**, 024204 (2003).
[27] D. J. Thouless, in *Ill-Condensed Matter*, edited by R. Balian, R. Maynard, and G. Toulouse (North-Holland/World Scientific, Singapore, 1978), p. 1.
[28] G. 't Hooft, Nucl. Phys. B **138**, 1 (1978); **153**, 141 (1979); **190**, 455 (1981).
[29] I. S. Burmistrov and A. M. M. Pruisken (unpublished).
[30] G. D. Mahan, *Many-Particle Physics* (Plenum Press, New York, 1990).
[31] E. Ben-Jacob, E. Mottola, and G. Schön, Phys. Rev. Lett. **51**, 2064 (1983).
[32] C. Wallisser, B. Limbach, P. vom Stein, R. Schäfer, C. Theis, G.

Göppert, and H. Grabert, Phys. Rev. B **66**, 125314 (2002).

[33] I. S. Burmistrov and A. M. M. Pruisken, Phys. Rev. Lett. **101**, 056801 (2008).

[34] Y. Blanter, arXiv:cond-mat/0511478 (unpublished).

[35] Y. Imry, *Introduction to Mesoscopic Physics* (Oxford University, New York, 1997).

[36] S. V. Panyukov and A. D. Zaikin, Phys. Rev. Lett. **67**, 3168 (1991).

[37] A. M. M. Pruisken, Nucl. Phys. B **285**, 719 (1987); Nucl. Phys. B **290**, 61 (1987).

[38] A. M. M. Pruisken and I. S. Burmistrov, Phys. Rev. Lett. **95**, 189701 (2005).

[39] S. S. Murzin, A. G. M. Jansen, and I. Claus, Phys. Rev. Lett. **92**, 016802 (2004); **95**, 189702 (2005).

[40] A. A. Abrikosov, Physics **2**, 21 (1965).

[41] See, e.g., Yu. A. Izyumov and Yu. N. Skryabin, *Statistical Mechanics of Magneto-Ordered Systems* (Moskva, Nauka, 1987), in Russian.

[42] G. Göppert and H. Grabert, Phys. Rev. B **63**, 125307 (2001).

[43] A. I. Larkin and V. I. Melnikov, Sov. Phys. JETP **34**, 656 (1972) [Zh. Éksp. Teor. Fiz. **61**, 1232 (1971)].

[44] L. Zhu and Q. Si, Phys. Rev. B **66**, 024426 (2002).

[45] G. Zaránd and E. Demler, Phys. Rev. B **66**, 024427 (2002).

[46] I. O. Kulik and R. I. Shekhter, Zh. Eksp. Teor. Fiz. **68**, 623 (1975) [Sov. Phys. JETP **41**, 308 (1975)].

[47] H. P. Wei, D. C. Tsui, M. A. Paalanen, and A. M. M. Pruisken, Phys. Rev. Lett. **61**, 1294 (1988).

[48] Ya. I. Rodionov, I. S. Burmistrov, and A. S. Ioselevich, Phys. Rev. B **80**, 035332 (2009).

[49] Yu. V. Nazarov, Phys. Rev. Lett. **82**, 1245 (1999).

[50] P. Nozières and A. Blandin, J. Phys. (Paris) **41**, 193 (1980); A. M. Tsvelik and P. B. Wiegmann, Z. Phys. B: Condens. Matter **54**, 201 (1984); N. Andrei and C. Destri, Phys. Rev. Lett. **52**, 364 (1984).