Chapter 2

Pure spinor formalism

This chapter introduces the pure spinor formalism in a flat background. The worldsheet action is an educated guess originally written down by Berkovits. His starting point was not an analog of the Polyakov action, i.e. an action with $2d$ diff $\times$ Weyl invariance, instead he directly wrote down an analog of the worldsheet action in conformal gauge. This means that the action must have a conformal symmetry, zero central charge and a nilpotent fermionic operator that is used to define the spectrum, similar to the way a BRST operator defines a spectrum. Berkovits’ proposal satisfies these conditions and on top of that it exhibits manifest spacetime supersymmetry and the worldsheet fields are free. This chapter will discuss the explicit form of the action and some of its properties. Also the prescription for computing scattering amplitudes is provided. This chapter does not contain any explicit computations using this prescription. A good exposition of computations can be found in [20], section 5.1.2 of this thesis also contains some computations.

A number of years after the pure spinor formalism was introduced, Berkovits presented a different but similar formalism. To distinguish the two, the original one was renamed to minimal pure spinor formalism and the modification, the non-minimal pure spinor formalism. The latter was introduced to get rid of some awkward features of the former which will be discussed in due course. Both formalisms are described below. The most recent loop computations, which are also the more complicated ones, have only been performed in the non-minimal formalism, cf. section 2.3 for a precise overview.

This chapter utilises a lot of basic (mathematical) techniques that may or may not be familiar to the reader. In any case these techniques are explained in detail in the next chapter, which can serve either as a necessary addition for a reader new to the subject or as a useful reference for an expert.
2.1 Minimal pure spinor formalism

The worldsheet action in the minimal pure spinor formalism for the left movers in conformal gauge and flat target space is given by

\[ S = \int d^2 z \left( \frac{1}{2} \partial x^m \tilde{\partial} x_m + p_\alpha \tilde{\partial} \theta^\alpha - w_\alpha \tilde{\partial} \lambda^\alpha \right), \]  

(2.1)

with \( m = 0, \ldots, 9 \) and \( \alpha = 1, \ldots, 16 \). The fields \( p_\alpha \) and \( w_\alpha \) have conformal weight one and are Weyl spinors, \( \theta^\alpha \) and \( \lambda^\alpha \) have conformal weight zero and are Weyl spinor of opposite chirality. In addition \( \lambda^\alpha \) is a pure spinor, i.e. it satisfies

\[ \lambda^\alpha \gamma^{m}_{\alpha\beta} \lambda^\beta = 0, \]  

(2.2)

where \( \gamma^{m}_{\alpha\beta} \) are the ten dimensional Pauli matrices, which are defined in section 3.2. The decomposition of a Weyl spinor under the \( SU(5) \) subgroup, \( 16 \rightarrow 1 \oplus \bar{10} \oplus 5 \), which is used intensively throughout this work, is also discussed there. Since the worldsheet action consists of two \( \beta\gamma \) systems quantisation seems straightforward, but \( \lambda^\alpha \) is a pure spinor and therefore the \( \lambda w \) part is actually a curved \( \beta\gamma \) system \cite{21}. To deal with this we work on a patch in pure spinor space that is defined by \( \lambda^+ \neq 0 \). On this patch the pure spinor condition expresses \( \lambda^a \) in terms of \( \lambda_{ab} \) and \( \lambda^+ \), with \( a, b = 1, \ldots, 5 \). The solution is (in \( SU(5) \) covariant components)

\[ \lambda^a = \frac{1}{8} \frac{1}{\lambda^+} \varepsilon^{abcde} \lambda_{bc} \lambda_{de}. \]  

(2.3)

A constraint on fields in the action induces a gauge invariance on the conjugate fields. In this case the gauge transformations are given by

\[ \delta w_\alpha = \Lambda_m \gamma^{m}_{\alpha\beta} \lambda^\beta. \]  

(2.4)

In the original papers, e.g. \cite{22}, this gauge invariance is dealt with by only using gauge invariant quantities. This means \( w_\alpha \) can only appear in the Lorentz current \( N^{mn} \), the ghost number current \( J \) and the stress energy tensor \( T(\lambda w) \):

\[ N^{mn} = \frac{1}{2} w_\alpha (\gamma^{mn})^\alpha_\beta \lambda^\beta, \quad J = w_\alpha \lambda^\alpha, \quad T(\lambda w) = w_\alpha \partial \lambda^\alpha. \]  

(2.5)

Since the \( \lambda w \) part of the action is not free due to the pure spinor constraint it is not obvious what the OPE between \( w \) and \( \lambda \) will be. One way to proceed is by properly fixing the gauge invariance of (2.4). By making the gauge choice \( w_a = 0 \) and employing BRST methods, one can replace \( \int d^2 z w_\alpha \partial \lambda^\alpha \) by the free action,

\[ \int d^2 z (\omega_+ \tilde{\partial} \lambda^+ + \frac{1}{2} \omega^{ab} \tilde{\partial} \lambda_{ab}). \]  

(2.6)

The details can be found in section 3.3.2. One might have expected BRST ghosts associated to the gauge fixing of \( w_\alpha \). It turns out these can be integrated out. As a
check of the validity of this procedure the OPE of the Lorentz currents \(N_{mn}^{\alpha}|_{w=0}\) should give rise to the Lorentz algebra. Using (2.6) one finds

\[
N_{mn}(z)\lambda^\alpha(w) \sim \frac{1}{z-w} \frac{1}{2} (\gamma^{mn})^\lambda_\alpha, \quad J(z)\lambda^\alpha(w) \sim \frac{1}{z-w} \lambda^\alpha, \quad (2.7)
\]

\[
N_{mn}(z)N_{pq}(w) \sim -\frac{3}{(z-w)^2} (\eta^{[m} \eta_{q]}^{n}) + \frac{1}{z-w} (\eta^{[m} \eta N_{q]}^{n} - \eta^{m} \eta N_{[q]}^{n}),
\]

\[
J(z)J(w) \sim -\frac{4}{(z-w)^2}, \quad J(z)N_{mn}(w) \sim \text{regular},
\]

\[
N_{mn}(z)T(w) \sim \frac{1}{(z-w)^2} N_{mn}(w), \quad J(z)T(w) \sim \frac{-8}{(z-w)^3} + \frac{1}{(z-w)^2} J(w).
\]

The explicit computations can be found in appendix 3.3 and it should be noted there are subtleties regarding the double poles in the OPE. There is freedom to add conserved currents to the Lorentz currents without changing the single poles, which must have the above form for Lorentz currents. However if one demands that \(N_{mn}\) is a primary field with Lorentz level \(-3\), the above OPE’s follow unambiguously.

The OPE’s for the matter variables can be straightforwardly derived from (2.1):

\[
x_m(z)x^n(w) \sim -\eta^{mn} \log|z-w|^2, \quad p_\alpha(z)\theta^\beta(w) \sim \delta^\alpha_\beta \frac{1}{z-w}, \quad (2.8)
\]

The action (2.1) is invariant under a nilpotent fermionic symmetry generated by \(Q_S\):

\[
Q_S = \oint dz \lambda^\alpha d_\alpha, \quad (2.9)
\]

where

\[
d_\alpha = p_\alpha - \frac{1}{2} \gamma^m_{\alpha\beta} \gamma^\beta \partial x_m - \frac{1}{8} \gamma^m_{\alpha\beta} \gamma_m \gamma_\delta \theta^\beta \theta^\gamma \partial \theta^\delta. \quad (2.10)
\]

The transformations it generates are given by

\[
\delta x^m = \lambda \gamma^m, \quad \delta \theta^\alpha = \lambda^\alpha, \quad \delta \lambda^\alpha = 0, \quad \delta d_\alpha = -\Pi^m (\gamma_m \lambda_\alpha), \quad \delta w_\alpha = d_\alpha, \quad (2.11)
\]

where \(\Pi^m = \partial x^m + \frac{1}{2} \theta \gamma^m \partial \theta\) is the supersymmetric momentum and again we restrict to the left movers (so in particular, the full transformation for \(x^m\) contains a similar additive term with right moving fields).

It seems very natural to consider \(Q_S\) as a BRST operator that showed up after fixing a worldsheet symmetry, in particular diffeomorphism invariance. However to

\footnote{The unconventional subscript \(S\) is used to distinguish this operator from another nilpotent fermionic operator which will appear in chapter 4.}
date nobody has succeeded in substantiating this conjecture, although the authors of [23] describe how it is possible to obtain the pure spinor formalism as a twisted version of a gauge fixed string theory with diffeomorphism invariance. In chapter 4 the worldsheet action in conformal gauge will be derived by gauge fixing a worldsheet action with diffeomorphism symmetry. However the 2d coordinate invariant action is already invariant under $Q_S$, the gauge fixing of the diffeomorphisms gives rise to a second nilpotent fermionic operator. This is a different point of view where $Q_S$ is not a BRST operator of fixing 2d coordinate invariance.

The main motivation to introduce the pure spinor formalism is its manifest supersymmetry. This symmetry is generated by

$$q_\alpha = \oint dz (p_\alpha + \frac{1}{2} \gamma_{\alpha\beta} \theta^\beta \partial x_m + \frac{1}{24} \gamma_{\alpha \beta}^m (\gamma_m \gamma^\delta \theta^\gamma \theta^\delta)).$$

(2.12)

### 2.1.1 Spectrum

Physical states are defined as element of the cohomology of $Q_S$ with $J_{\lambda w}$ charge one and conformal weight zero. In theories derived from a worldsheet diffeomorphism invariant action, the conformal weight constraint follows from the condition that physical states must be annihilated by the BRST operator. In the case of the pure spinor action the operator $Q_S$ does not impose a constraint on the conformal weight and it has to be included by hand. In chapter 4 the origin of conformal weight constraint is explained from first principles in the case of the pure spinor formalism. The reason to look at ghost number one states is more subtle. At least one can say that the cohomology at this $J_{\lambda w}$ charge yields the super-Maxwell multiplet (for the open string).

Hence elements of the physical spectrum satisfy:

$$Q_S V(z) = 0, \quad V(z) \sim V(z) + Q_S \Omega(z).$$

(2.13)

Let us focus on the massless spectrum. The most general vertex operator (before imposing the above conditions) at $J_{\lambda w}$ charge one with conformal dimension zero and $k^2 = 0$ is given by

$$V(z) = e^{ik \cdot X(z, \bar{z})} \lambda^\alpha(z) A_\alpha(\theta(z)).$$

(2.14)

A number of comments are in order

- For the $X$ sector one uses the standard operators (1.116) and note that the weight is only non positive when no derivatives on $X$ are present.

- The weight of the $p, \theta$ and $w, \lambda$ sector is only non positive when $V$ only contains $\lambda$ and $\theta$. 

48
Chapter 2 - Pure spinor formalism

- The total weight of $V$ in the massless case can only be zero if it only consists of weight zero fields. This determines the form of (2.14) completely.

- Since there are no negative weight fields, there is no tachyon present in the spectrum. There is an infinite tower of massive states, but these will not be considered in this thesis.

After using the gauge invariance to set a number of components to zero the solution to (2.13) is given by [24]

$$V = \lambda^\alpha A_\alpha(x, \theta), \quad (2.15)$$

where

$$A_\alpha(x, \theta) = e^{ik\cdot x}(\frac{1}{2} a_m(\gamma^m \theta)_\alpha - \frac{1}{3}(\xi \gamma^m \theta)(\gamma^m \theta)_\alpha + \cdots), \quad (2.16)$$

where $a_m$ and $\xi^\alpha$ are the polarisations and $k^m$ is the momentum. They satisfy $k^2 = k^m a_m = k^m (\gamma_m \xi)_\alpha = 0$, there is a residual gauge invariance $a_m \to a_m + k_m \omega$ and the ellipsis contains products of $k^m$ with $a_m$ or $\xi^\alpha$. The operator $V(z)$ can be used as unintegrated vertex operator.

The integrated vertex operators can again be obtained by an educated guess based on comparison with the bosonic string and/or the RNS string. In those theories the integrated vertex operator satisfies

$$Q_S U(z) = \partial V(z) \quad (2.17)$$

This equation also has a solution in the pure spinor formalism, which is given by

$$U = \partial \theta^\alpha A_\alpha(x, \theta) + \Pi^m A_m(x, \theta) + d_\alpha W^\alpha(x, \theta) + \frac{1}{2} N^{mn} \mathcal{F}_{mn}(x, \theta), \quad (2.18)$$

with

$$A_m = \frac{1}{8} D_\alpha \gamma^{\alpha\beta} A_\beta, \quad (2.19)$$

$$W^\beta = \frac{1}{10} \gamma^\alpha_m (D_\alpha A^m - \partial^m A_\alpha), \quad (2.20)$$

$$\mathcal{F}_{mn} = \frac{1}{8} D_\alpha (\gamma_{mn})^\alpha_\beta W^\beta, \quad (2.21)$$

where $D_\alpha = \frac{\partial}{\partial \theta_\alpha} + \frac{1}{2} \theta^\beta \gamma^{m}_{\alpha\beta} \partial_m$.

2.1.2 Tree-level prescription

Originally the amplitude prescription in the pure spinor formalism was motivated by analogy to the bosonic string. The guiding principles are given by

- There are three unintegrated vertex operators and $N - 3$ integrated ones to deal with the CKG.
The total $J_{\lambda w}$ charge must equal the charge anomaly (2.7). The $N$ point open string tree-level amplitude prescription presented in [22] satisfies the above guiding principles.

$$A = \langle V_1(z_1)V_2(z_2)V_3(z_3) \int dz_4U_4(z_4) \cdots \int dz_NU_N(z_N)Y_{C_1}(y_1) \cdots Y_{C_{11}}(y_{11}) \rangle =$$

$$\int [D^{10}\phi][D^{16}d][D^{16}\theta][D^{11}\lambda][D^{11}w]V_1(z_1)V_2(z_2)V_3(z_3) \int dz_4U_4(z_4) \cdots \int dz_NU_N(z_N) Y_{C_1}(y_1) \cdots Y_{C_{11}}(y_{11}) e^{-S},$$

(2.22)

where $[D\phi]$ denotes functional integration over the field $\phi$. The functional integration over $x^m$ have been studied in detail and the same correlation functions appear in the RNS formalism. This factor will be ignored when it is not relevant to the computation.

$Y_C$ are the picture changing operators (PCOs):

$$Y_C(y) = C_\alpha \theta^\alpha(y) \delta(C_\beta \lambda^\beta(y)),$$

(2.23)

where $C_\alpha$ is a constant spinor. The presence of the PCOs in the amplitude prescription is explained from first principles in chapter 4. In short, they come from fixing a gauge invariance due to the zero modes of the weight zero fields, $\lambda^\alpha, \theta^\alpha$. Note the weight one fields do not have zero modes at tree level. At higher loops there will also be PCOs for these fields. Since the PCOs are introduced as a gauge fixing term, amplitudes should be independent of the constant tensors $C_\alpha$. The name picture changing operator was also given to an operator in the RNS formalism (1.209). These operators change the (bosonic) ghost number of the vertex operators. The $\lambda, w$ sector can be seen as a ghost sector since it is not part of the ten dimensional superspace and they have the “wrong” spin-statistics relation. Since $Y_C$ change the $J_{\lambda w}$ charge by one, these operators were also named picture changing operators.

The functional integral (2.22) is evaluated by first using the OPE’s of (2.7) and (2.8). Note that this operation reduces the total conformal dimension of the worldsheet fields involved in the OPE. For example in the $p, \theta$ OPE, the conformal weight of $p_\beta(z)\theta^\alpha(w)$ is one and the conformal weight of $\delta_\beta^\alpha$ is zero. Thus in the end the correlator only contains worldsheet fields of weight zero. This can be evaluated by replacing the fields by their zero modes and performing the zero mode integrations. The justification for this step is given in section 4.6.

After integrating out the nonzero modes the amplitude reduces to

$$A = \int [d\lambda] d^{16}\theta \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(\theta)(C^1\theta)\delta(C^1\lambda) \cdots (C^{11}\theta)\delta(C^{11}\lambda),$$

(2.24)

2This can be inferred from the Riemann Roch theorem (1.74) and the fact that weight zero fields have precisely one zero mode at any genus.
where $f_{\alpha\beta\gamma}$ depends on all the polarisations and momenta. Note the functional integration of $x^m$ is omitted here as will be done in all computations in this thesis. A priori $f_{\alpha\beta\gamma}$ also depends on $z_1, z_2, z_3$. Of course we expect the final result to be independent of these coordinates. Also note all the fields are zero modes including those in the measure. $[d\lambda]$ is the unique Lorentz invariant measure of +8 ghost number on the space of pure spinors (cf. section 3.4). It is given by

\[ [d\lambda]\lambda^\alpha \lambda^\beta \lambda^\gamma = d\lambda^\alpha_1 \wedge \cdots \wedge d\lambda^\alpha_{11}(\epsilon T)^{\alpha\beta\gamma}_{\alpha_1 \cdots \alpha_{11}}, \]  

(2.25)

where

\[ (\epsilon T)^{\alpha\beta\gamma}_{\alpha_1 \cdots \alpha_{11}} = \epsilon_{\alpha_1 \cdots \alpha_{16}} \gamma^\alpha_1 \cdots \gamma^\alpha_{11} \gamma^{\alpha_1 \beta \alpha_2 \alpha_3 \alpha_4 \cdots \alpha_{16}}. \]  

(2.26)

Note no gamma trace is subtracted. This tensor is already gamma matrix traceless as explained in section 3.4.

**Lorentz invariance**

The PCOs contain constant spinors. Therefore the prescription is not manifestly Lorentz invariant and one has to check Lorentz invariance by hand. The Lorentz variation of one PCO is given by:

\[ M^{mn} Y_C = \frac{1}{2}(C\gamma^{mn}\theta)\delta(C\lambda) + \frac{1}{2}(C\theta)(C\gamma^{mn}\lambda)\delta'(C\lambda) = Q_S \left[ \frac{1}{2}(C\gamma^{mn}\theta)(C\theta)\delta\delta'(C\lambda) \right]. \]  

(2.27)

The last equality shows that the Lorentz variation of the PCO is $Q_S$ exact. This decouples if all other insertions are $Q_S$ closed and $\langle Q_S K \rangle$ vanishes for all $K$. The second condition is satisfied because after integrating out the non-zero modes $\langle Q_S X \rangle$ reduces to

\[ \int [d\lambda] d^{16}\theta \lambda^\alpha \lambda^\beta \lambda^\gamma D_\alpha f_{\beta\gamma}(\theta)C^1 \theta \delta(C^1 \lambda) \cdots C^{11} \theta \delta(C^{11} \lambda) = 0, \]  

(2.28)

because $\int d^{16}\theta D_\alpha g(\theta) = 0$ for any function $g$. The first condition is more subtle. The vertex operators are $Q_S$ closed, due to the physical state condition (2.13). In order to see whether the PCOs are closed consider

\[ Q_S Y_C = C_\alpha \lambda^\alpha \delta(C_\beta \lambda^\beta). \]  

(2.29)

This seems to be zero, but there are subtleties due to the presence of factors of $\lambda$ in the denominator form the measure (2.25). A detailed exposition of these subtleties can be found in chapter 5.

It is possible to restore manifest Lorentz invariance by integrating over all possible choices for $C$. This guarantees the prescription is Lorentz invariant. However it does not guarantee that $Q_S$ exact states will decouple. After including the $C$ integral (2.24) becomes

\[ A = \int [dC][d\lambda] d^{16}\theta \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(\theta)(C^1 \theta)\delta(C^1 \lambda) \cdots (C^{11} \theta)\delta(C^{11} \lambda). \]  

(2.30)
2.1.3 One and higher-loop prescription

Let us start with giving the one-loop amplitude prescription. Compared to a tree-level function a one-loop function exhibits three new features:

- PCOs for the weight one worldsheet fields $p, w$,
- zero mode integrals over $p, w$,
- a composite $b$ ghost constructed out of the worldsheet fields from (2.1).

The first two points are direct consequences of the presence of a zero mode of weight one fields on the torus. The new PCOs are given in terms of the gauge invariant quantities $N^{mn}$ and $J$:

$$Z_B(z) = \frac{1}{2} B_{mn} \lambda(z) \gamma^{mn} d(z) \delta(B_{mn} N^{mn}(z)), \quad Z_J(z) = \lambda^a(z) d^a(z) \delta(J(z)). \quad (2.31)$$

Note that the picture raising\(^3\) operators, $Z_B$ and $Z_J$, are $Q_S$-closed without subtleties:

$$Q_S Z_B = \frac{1}{4} B_{mn} \lambda \gamma^{mn} dB_{m'n'} \lambda \gamma^{m'n'} d \delta(B_{pq} N^{pq}) = \frac{1}{4} (B_{mn} \lambda \gamma^{mn} d)^2 \delta(B_{pq} N^{pq}) = 0. \quad (2.32)$$

This vanishes because it contains the square of a fermionic quantity. Let us also record the Lorentz variation of $Z_B$,

$$M^{mn} Z_B = Q_S [2 \eta^{[m} [\delta^e_r] B_{pq} N^{qr} \delta(BN)], \quad (2.33)$$

which is $Q_S$ exact.

All string theory amplitude prescriptions at one loop contain a $b$ ghost which satisfies

$$\{Q_S, b(z)\} = T(z). \quad (2.34)$$

In the RNS formalism this field appears as one of the two reparametrisation Faddeev Popov ghosts and note that at one loop there should be one (holomorphic) $b$ ghost insertion to absorb the zero mode (cf. table 1.1). In the bosonic string amplitude prescription, which the pure spinor amplitude prescription is analogous to, this $b$ insertion enters through $(b, \partial_\tau g)$, where the brackets have been defined in (1.69) and $\tau$ is the modulus of a genus one surface. While the full derivation of the form of this insertion will be given in chapter 4, it is possible to show this insertion in consistent with BRST invariance, since its variation equals a total derivative in moduli space which vanishes upon integrating over the moduli:

$$Q_S(b, \partial_\tau g) e^{-S} = (T, \partial_\tau g) e^{-S} = \frac{\partial S}{\partial g_{ab}} \frac{\partial g_{ab}}{\partial \tau} e^{-S} = -\frac{\partial}{\partial \tau} e^{-S}. \quad (2.35)$$

\(^3\)These are called raising operators because they have $+1$ $J_{\lambda w}$ charge, hence they raise the picture in the language of (1.209).
The derivative of the metric with respect to the modulus is called a Beltrami differential, \( \mu \), and on higher genus surfaces the Beltrami differential has an index that runs over the number of moduli, \( \mu_k \).

In the pure spinor formalism, however, the \( b \) ghost is constructed out of the worldsheet fields from (2.1) as explained from first principles in chapter 4. It is not possible to solve equation (2.34) in the minimal pure spinor formalism, because of ghost number \( (J \text{ charge}) \) conservation combined with gauge invariance of objects containing \( w_\alpha \). The former implies \( b \) must have ghost number minus one and since there are no gauge invariant quantities with negative ghost number the latter rules out any solution. A resolution to this problem is combining the (composite) \( b \) field with a PCO, \( Z_B \), such that

\[
\{Q_S, \tilde{b}_B(u,z)\} = T(u)Z_B(z).
\]

This equation ensures the \( Q_S \) variation of the \( b \) ghost vanishes after integrating over moduli space. The solution is given by

\[
\tilde{b}_B(u,z) = b_B(u) + T(u) \int_u^z dvB_{pq}\partial N^{pq}(v)\delta(BN(v)).
\]

The local \( b \) ghost, \( b_B(u) \), is a composite operator, constructed out of the worldsheet fields:

\[
b_B(z) = b_{B0}(z)\delta(BN(z)) + b_{B1}(z)\delta'(BN(z)) + b_{B2}(z)\delta''(BN(z)) + b_{B3}(z)\delta'''(BN(z)),
\]

where the primes denote derivatives, \( BN \equiv B_{mn}N^{mn} \) and

\[
b_{B0} = \frac{1}{2} G_{\gamma mn}dB_{mn} - \frac{1}{2} H_{\alpha\beta}(\gamma^p_{\gamma mn})_{\alpha\beta}\Pi_pB_{mn} + \frac{1}{2} K_{\alpha\beta\gamma}(\gamma^p_{\gamma mn})_{\beta\gamma}(\gamma_p\partial\theta)_{\alpha}B_{mn} + \frac{1}{2} S_{\alpha\beta\gamma}(\gamma^p_{\gamma mn})_{\beta\gamma}(\gamma_p\partial\lambda)_{\alpha}B_{mn},
\]

\[
b_{B1} = \frac{1}{4} H^{\alpha\beta}(Bd)_{\alpha}(Bd)_{\beta} + \frac{1}{4} K_{\alpha\beta\gamma}(\gamma^p_{\gamma mn})_{\beta\gamma}(Bd)_{\alpha}\Pi_pB_{mn} + \frac{1}{4} K_{\alpha\beta\gamma}(\gamma^p_{\gamma mn})_{\alpha(\beta}(Bd)_{\gamma)\Pi_pB_{mn} + \frac{1}{4} L^{\alpha\beta\gamma\delta}((\gamma^p_{\gamma mn})_{\gamma\delta}(Bd)_{\alpha}(\gamma_p\partial\theta)_{\beta]) - (\gamma^p_{\gamma mn})_{\beta(\gamma_p\partial\theta)_{\alpha]}B_{mn} - ((\gamma^p_{\gamma mn})_{\alpha(\beta}(Bd)_{\gamma)\Pi_pB_{mn}\Pi_qB_{qr}),
\]

\[
b_{B2} = -\frac{1}{8} K_{\alpha\beta\gamma}(Bd)_{\alpha}(Bd)_{\beta}(Bd)_{\gamma} - \frac{1}{8} L^{\alpha\beta\gamma\delta}((\gamma^p_{\gamma mn})_{\gamma\delta}(Bd)_{\alpha}(Bd)_{\beta}) + \frac{1}{2} (\gamma^p_{\gamma mn})_{\alpha(\beta}(Bd)_{\gamma)\Pi_pB_{mn},
\]

\[
b_{B3} = -\frac{1}{16} L^{\alpha\beta\gamma\delta}(Bd)_{\alpha}(Bd)_{\beta}(Bd)_{\gamma}(Bd)_{\delta},
\]

where \( (Bd)_{\alpha} \equiv B_{mn}(\gamma^mn)_{\alpha} \) and \( G, H, K, L \) are given in appendix 3.6.
The one-loop amplitude prescription in the minimal pure spinor formalism is given by

$$A^{(N)} = \int d^2 \tau \langle \int d^2 u \mu(u) \tilde{b}_B(u, z_1) \prod_{P=2}^{10} Z_{BP}(z_P) Z_J(z_{11}) \prod_{I=1}^{11} Y_{CI}(y_I) \rangle^2$$

$$V_1(t_1) \prod_{T=2}^{N} \int d^2 t_T U_T(t_T),$$

where $\mu(u)$ is the $zz$ component of the Beltrami differential.

Above one loop there are no conformal killing vectors anymore, so that there is no unintegrated vertex operator. The number of metric moduli at genus $g$ is given by $6g - 6$ (cf. table 1.1) and all conformal weight one fields have $g$ zero modes each. This leads to the multiloop amplitude prescription of the minimal pure spinor formalism:

$$A^{(N)} = \int d^2 \tau_1 \cdots d^2 \tau_{3g-3} \prod_{P=1}^{3g-3} \int d^2 u_P \mu_P(u_P) \tilde{b}_B(u_P, z_P)$$

$$\prod_{P=3g-2}^{10g} Z_{BP}(z_P) \prod_{R=1}^{g} Z_J(v_R) \prod_{I=1}^{11} Y_{CI}(y_I) \prod_{T=1}^{N} \int d^2 t_T U_T(t_T).$$

As described in [22], the amplitudes (2.43) and (2.44) are evaluated by first using the OPE’s to remove all fields of nonzero weight. After this step all fields have weight zero. This can be evaluated by replacing the fields by their zero modes and performing the zero mode integrations. Therefore one needs to know how to integrate over the zero modes. For the $d, \theta, x$ variables this is standard, so only the integration over $\lambda, N, B, C$ is discussed.

A typical integral one encounters is given by [22]:

$$A = \int [d\lambda][dB][dC] \prod_{R=1}^{g} [dN_R] f(\lambda, N_R, J_R, C, B),$$

where $[dN]$ is the zero mode measure for (each zero mode of) $N_{mn}$ (cf. section 3.4.2). It must have $J_{\omega w}$ charge -8, since the $JT$ OPE in (2.7) implies only correlators with a total $J_{\omega w}$ charge of $8 - 8g$ can be non vanishing at $g$ loops [8]. It is given by

$$[dN] \lambda^{\alpha_1} \cdots \lambda^{\alpha_8} = dN^{m_1 n_1} \wedge \cdots \wedge N^{m_{10} n_{10}} \wedge dJ R^{\alpha_1 \cdots \alpha_8}_{m_1 n_1 \cdots m_{10} n_{10}},$$

with

$$R^{\alpha_1 \cdots \alpha_8}_{m_1 n_1 \cdots m_{10} n_{10}} \equiv \gamma_{m_1 n_1 m_2 n_2 m_3 n_3 m_4 n_4} \gamma_{m_5 n_5 m_6 n_6 m_7 n_7} \gamma_{m_8 n_8 n_3 n_6 m_9 n_9} \gamma_{m_{10} n_{10} m_4 n_4 n_7 n_9} + \text{permutations}.$$
The permutations make $R$ antisymmetric under exchange in both $m_i \leftrightarrow n_i$ and $m_in_i \leftrightarrow m_j n_j$, and the double brackets denote subtraction of the gamma trace. The zero mode integral (2.45) is only nonzero if the function $f$ depends on $(\lambda, N, J, C, B)$ as

$$f(\lambda, N, J, C, B) = h(\lambda, N, J, C, B) \prod_{R=1}^{g} \delta^{MR}(J) \prod_{P=1}^{10} \prod_{R=1}^{g} \delta^{LP,R}(B^P N_R) \prod_{I=1}^{11} \delta^{K_i}(C^I \lambda),$$

where the polynomial $h$ assumes the form

$$(\lambda)^{8g-8+\sum_{I=1}^{11}(K_i+1)} \prod_{R=1}^{g} (J_R)^M_R (N_R)^{L_P,R} \prod_{P=1}^{10} (B^P)^{L_P,R+1} \prod_{I=1}^{11} (C^I)^{K_i+1}.$$  (2.49)

The integration over the zero modes of the pure spinor variables and the constant tensors is defined in [22] as

$$A^{(N)} = c \prod_{I=1}^{11} \left( \frac{\partial}{\partial \lambda} \delta \frac{\partial}{\partial C^I} \right) K_I \prod_{P=1}^{10} \prod_{R=1}^{g} \left( \frac{\partial}{\partial B^P_{pq}} \delta \frac{\partial}{\partial N^R_{pq}} \right) \prod_{R=1}^{g} \left( \frac{\partial}{\partial J_R} \right)^M_R \left[ R_{m_1 n_1 \cdots m_{10} n_{10}} \frac{\partial}{\partial \lambda \alpha_4} \cdots \frac{\partial}{\partial \lambda \alpha_{11}} \frac{\partial}{\partial B^1_{m_1 n_1}} \cdots \frac{\partial}{\partial B^0_{m_{10} n_{10}}} \right] \phi(\lambda, N_R, J_R, C, B),$$  (2.50)

for some proportionality constant $c$.

### 2.1.4 Decoupling of $Q_S$ exact states and PCO positions

The amplitude prescriptions, (2.22) and (2.43), put the PCOs at arbitrary points on the worldsheet. Of course the final result cannot depend on these positions, since they do not contain any physical significance. To study the dependence on the insertions point one looks at the worldsheet derivatives of the PCOs:

$$\partial Y_C(y) = Q_S[(C\partial \theta(y))(C\theta(y))\delta'(C\lambda(y))],$$  (2.51)

$$\partial Z_B(z) = Q_S[-B^q_p \partial N^p_q(z)\delta(BN(z))], \quad \partial Z_J(z) = Q_S[-\partial J(z)\delta(J(z))].$$  (2.52)

These are $Q_S$ exact, like the Lorentz variation of the PCOs. Hence the amplitude is only guaranteed to be independent of these insertions points if $Q_S$ exact states decouple. Due to the subtleties with $Q_S$ closedness of $Y_C$ this is non-trivial. In chapter 5 the problem is completely solved and a proof of decoupling of $Q_S$ exact states is given, hence also proving Lorentz invariance and independence of PCO positions.
Chapter 2 - Pure spinor formalism

2.2 Non-minimal pure spinor formalism

The minimal pure spinor formalism has the desired property of manifest spacetime supersymmetry. However, manifest Lorentz invariance is not present, due to the appearance of the constant spinors/tensors $C$ and $B$. Furthermore the $b$ ghost equation (2.34) could not be solved. These two problems are resolved in the non-minimal pure spinor formalism.

The non-minimal version of the formalism [25] (see [26] for a review) amounts to introducing a set of non-minimal variables, the complex conjugate $\bar{\lambda}_\alpha^\alpha$ of $\lambda^\alpha$, a fermionic constrained spinor $r^\alpha$ satisfying

$$\bar{\lambda}_\alpha^\alpha \gamma^m_m \bar{\lambda}_\beta^\beta = 0, \quad \bar{\lambda}_\alpha^\alpha \gamma^m_m r^\beta = 0$$

and their conjugate momenta, $\bar{w}^\alpha$ and $s^\alpha$. Analogous to the minimal formalism these conditions induce a gauge invariance:

$$\delta \bar{w}_\alpha^\alpha = \bar{\Lambda}^m (\gamma^m_m \bar{\lambda})^\alpha - \phi^m (\gamma^m_m r)^\alpha, \quad \delta s^\alpha = \phi^m (\gamma^m_m \bar{\lambda})^\alpha.$$  \hfill (2.54)

This implies $\bar{w}^\alpha$ and $s^\alpha$ can only appear in the gauge invariant quantities

$$\bar{N}^{mn} = \frac{1}{2} (\bar{\lambda} \gamma^m_m \bar{w} - s \gamma^m_m r), \quad \bar{J} = \bar{\lambda} \bar{w} - sr, \quad T_{\bar{\lambda} \bar{w}} = \bar{w}^\alpha \bar{\partial} \bar{\lambda}_\alpha - s^\alpha \partial r^\alpha,$$

$$S_{mn} = \frac{1}{2} s \gamma^m_m \bar{\lambda}, \quad S = s \bar{\lambda}.$$  \hfill (2.55)

The action (2.1) is modified by the addition of the term $S_{nm}$:

$$S \rightarrow S + S_{nm}, \quad S_{nm} = \int d^2z \left( -\bar{w}^\alpha \bar{\partial} \bar{\lambda}_\alpha + s^\alpha \partial r^\alpha \right)$$

and the generator $Q_S$ by

$$Q_S \rightarrow Q_S + \oint dz \bar{w}^\alpha r^\alpha.$$  \hfill (2.57)

This acts on the non-minimal variables as follows

$$\delta \bar{\lambda}_\alpha = r^\alpha, \quad \delta r^\alpha = 0, \quad \delta s^\alpha = \bar{w}^\alpha, \quad \delta \bar{w}^\alpha = 0.$$  \hfill (2.58)

These transformation rules imply that the cohomology is independent of the non-minimal variables. In other words the vertex operators can always be chosen such that they do not include these variables. A more natural point of view, which will be adopted in chapter 4, is to consider the non-minimal variables as fields that appear in the BRST treatment of gauge freedom due to shifts of the zero modes of the worldsheet fields. This also explains why vertex operators do not depend on the non-minimal fields and why only the zero modes of these fields appear in the path integral. Furthermore the OPE’s given in section 2.1 still comprise a complete
list, since the new fields do not have non zero modes. The tree-level amplitude prescription is given by

\[ \mathcal{A} = \langle V_1(z_1)V_2(z_2)V_3(z_3) \prod_{i=4}^{N} \int dz_i U_i(z_i)e^{-(\lambda(y)\bar{\lambda}(y)+r(y)\theta(y))} \rangle. \]  

(2.59)

Compared to the minimal case the PCOs have been replaced by

\[ \mathcal{N}(y) \equiv e^{(\{Q_S, -\bar{\lambda}(y)\theta(y)\})} = e^{-(\lambda(y)\bar{\lambda}(y)+r(y)\theta(y))}. \]  

(2.60)

Originally this factor was postulated by Berkovits, but it can also be derived from first principles. This will be done in chapter 4. Unlike the PCOs \( \mathcal{N} \) is \( Q_S \) closed without subtleties:

\[ Q_S e^{-(\lambda\bar{\lambda}+r\theta)} = Q_S [- (\lambda\bar{\lambda} + r\theta)] e^{-(\lambda\bar{\lambda}+r\theta)} = -(\lambda r - r\lambda)e^{-(\lambda\bar{\lambda}+r\theta)} = 0. \]  

(2.61)

Furthermore amplitude will not depend on the insertion point \( y \) since \( y \) only appears in a \( Q_S \) exact term. More precisely \( \mathcal{N} \) can be written as \( 1 + Q_S \Omega \) for some \( \Omega \) and all \( y \) dependence is in that \( \Omega \).

After performing the OPE's between the vertex operators, which results in exactly the same function \( f_{\alpha\beta\gamma} \) as the minimal manipulations, all fields can be replaced by their zero modes:

\[ \mathcal{A} = \int d^{16}\theta f_{\alpha\beta\gamma}(\theta) \int [d\lambda][d\bar{\lambda}][dr] \lambda^{\alpha} \bar{\lambda}^{\beta} \lambda^{\gamma} e^{-(\lambda\bar{\lambda}+r\theta)}, \]  

(2.62)

where \([d\bar{\lambda}]\) and \([dr]\) are Lorentz invariant measures:

\[ [d\bar{\lambda}]\bar{\lambda}_{\alpha}\bar{\lambda}_{\beta}\bar{\lambda}_{\gamma} = (\epsilon T)^{\alpha_1 \cdots \alpha_{11}}_{\alpha\beta\gamma} d\bar{\lambda}_{\alpha_1} \cdots d\bar{\lambda}_{\alpha_{11}} \]  

(2.63)

and

\[ [dr] = (\epsilon T)^{\alpha\beta\gamma}_{\alpha_1 \cdots \alpha_{11}} \bar{\lambda}_{\alpha}\bar{\lambda}_{\beta}\bar{\lambda}_{\gamma} \frac{\partial}{\partial r_{\alpha_1}} \cdots \frac{\partial}{\partial r_{\alpha_{11}}}. \]  

(2.64)

The invariant tensor \((\epsilon T)\) with indices in the opposite positions compared to (2.26) is defined by

\[ (\epsilon T)^{\alpha_1 \cdots \alpha_{11}}_{\alpha\beta\gamma} = \epsilon^{\alpha_1 \cdots \alpha_{11}}_{\alpha\alpha_{12}} \gamma^{m}_{\alpha_{12}} \gamma^{n}_{\beta\alpha_{13}} \gamma^{p}_{\gamma\alpha_{14}} (\gamma_{mnp})_{\alpha_{15} \alpha_{16}}. \]  

(2.65)

We know \( \int [d\lambda][d\bar{\lambda}][dr] \lambda^{\alpha} \bar{\lambda}^{\beta} \lambda^{\gamma} e^{-(\lambda\bar{\lambda}+r\theta)} \) must be a Lorentz tensor with three spinor indices and it must also contain eleven \( \theta \)'s, because the \( r \) integration requires eleven \( r \)'s to be non vanishing and all the terms with eleven \( r \)'s also contain eleven \( \theta \)'s. There is only one invariant tensor, up to scaling, with these symmetries which is \((\epsilon T)\):

\[ \int d^{16}\theta f_{\alpha\beta\gamma}(\theta)(\epsilon T)^{\alpha\beta\gamma}_{\beta_1 \cdots \beta_{11}} \theta^{\beta_1} \cdots \theta^{\beta_{11}}. \]  

(2.66)
At higher loops two new issues arise, (1) appearance of the \( b \) ghost which is a composite field constructed from the worldsheet fields, including the non-minimal variables, (2) the weight one fields have zero modes. To deal with the second issue, \( \mathcal{N} \) will also include zero modes of weight one fields. For the one-loop case weight one fields have one zero mode, this results in

\[
\mathcal{N}(y) = e^{-(\lambda(y)\lambda(y)+r(y)\theta(y))} - \frac{1}{2}N_{mn}^0 N_0^{mn} + \frac{1}{2}S_{mn}^0 f_A dz \gamma^{mn} d + J^0 J^0 + S f_A dz \lambda d).
\]  

(2.67)

This is invariant under \( Q_S \):

\[
Q_S \mathcal{N}(y) = (\lambda r(y) - \lambda r(y) + N^{mn} (\frac{1}{2} \lambda \gamma_{mn} d - N^{mn} (\frac{1}{2} \lambda \gamma_{mn} d + J(\lambda d) - J(\lambda d))) \mathcal{N}(y) = 0.
\]  

(2.68)

The non-minimal \( b \) ghost satisfies

\[
\{Q_S, b_{nm}(z)\} = T_{nm}(z) \equiv T_{min}(z) + T_{\lambda \theta}(z).
\]  

(2.69)

This equation can be solved in the non-minimal formalism and its solution is given by

\[
b_{nm} = s^\alpha \partial \lambda^\alpha + \frac{\lambda^\alpha (2\Pi^m(\gamma_m d)^\alpha - N_{mn}(\gamma^{mn} \partial \theta)^\alpha - J \partial \theta^\alpha \frac{1}{4} \partial^2 \theta^\alpha)}{4\lambda}.
\]  

(2.70)

The one-loop amplitude prescription in the non-minimal pure spinor formalism is given by

\[
\mathcal{A}^{(N)} = \langle V_1(z_1) \prod_{i=2}^{N} \int dz_i U_i(z_i) \int dw \mu(w) b_{nm}(w) \mathcal{N}(y)\rangle,
\]  

(2.71)

where \( \mathcal{N} \) is given is (2.67). After integrating out the non zero modes by using the OPE’s a typical one-loop amplitude in the non-minimal formalism becomes

\[
\mathcal{A}^{(N)} = \int d^{16}d^{16}\theta \int [d\lambda][d\bar{\lambda}][dN][d\bar{N}][ds][dr] f(\lambda, \bar{\lambda}, \theta) \mathcal{N}^0,
\]  

(2.72)

where the Lorentz invariant measures are defined by

\[
[dN] \bar{\lambda}_{\alpha_1} \cdots \bar{\lambda}_{\alpha_8} = R_{\alpha_1 \cdots \alpha_8}^{m_1 \cdots m_1 n_1 \cdots n_1} dN_{m_1 n_1} \cdots dN_{m_1 n_1} dJ
\]  

(2.73)

and

\[
[ds] = R_{m_1 n_1 \cdots m_1 n_1}^{\alpha_1 \cdots \alpha_8} \bar{\lambda}_{\alpha_1} \cdots \bar{\lambda}_{\alpha_8} \partial S_{m_1 n_1} \cdots \partial S_{m_1 n_1} \partial S
\]  

(2.74)

---

4The zero mode of a holomorphic field \( \phi(z) \) is given by: \( \phi^0 \equiv \int_A dz \phi(z) \). \( A \) is the non-trivial A-cycle that satisfies \( \int_A \omega(z) = 1 \), where \( w(z) \) is the holomorphic one-form on the torus.
Chapter 2 - Pure spinor formalism

Note $b_{nm}$ has poles in $\lambda \bar{\lambda}$ which can cause the zero mode integrals over $\lambda$ and $\bar{\lambda}$ to diverge. At one loop this will not cause any problems because the measure $[d\lambda][d\bar{\lambda}][dN][d\bar{N}][dr][ds]$ goes like $(\lambda)^{11}(\bar{\lambda})^{11}$ and the $b$ field like $\bar{\lambda}/(\lambda \bar{\lambda})^4$ when $\lambda \to 0$. At three loops and higher the number of $b$ fields is high enough to cause divergences. They have originally been regularised in [27] and more recently in [28], but this method has not been applied to actual computations. (See however [29] where this regularisation method is reviewed and applied to the one-loop four-point amplitude with four integrated vertex operators. This requires a modification of the amplitude prescription that will not be discussed in this thesis.)

2.3 Results from the pure spinor formalism

In this chapter two new string theory formalisms have been introduced. Although it is not been proved rigorously, there is a lot of evidence that the minimal pure spinor formalism, the non-minimal pure spinor formalism and the RNS formalism are equivalent to each other.

Let us start with the equivalence between RNS and the minimal pure spinor formalism. The spectra of these two were shown to coincide in\textsuperscript{5} [24]. The most direct approach to show equivalence is to compare the amplitude computations. In [30] the equivalence was proved for $N$-point massless tree-level amplitudes with four or fewer Ramond states. For massless four-point one-loop amplitudes the amplitudes were shown to be identical in [31]. The four-point massless two-loop amplitude has been computed in the pure spinor formalism in [32]. This computation includes all possible choices (Neveu-Schwarz or Ramond) of the external states. The analogous computation in the RNS formalism is extremely complicated due to the sum over spin structures (cf. (1.213)) and is only successfully performed in the case of four NS states [17]. For this choice of external states the pure spinor result agrees with the RNS result. In conclusion one can say that the pure spinor formalism agrees with all known results of the RNS formalism. On top of this the pure spinor formalism produces more results, especially involving RR states, due to its manifest spacetime supersymmetry. All pure spinor computations referred to in this paragraph were performed in the manifestly Lorentz invariant version of the minimal formalism. This means including integrals over the constant tensors/spinors $C$ and $B$ (cf. (2.30)).

Equivalence of the minimal and non-minimal at tree-level is not difficult to show when one utilises the manifestly Lorentz invariant version of the minimal formalism.

\textsuperscript{5}This reference shows coincidence of the spectrum of the minimal pure spinor formalism and the spectrum of the Green-Schwarz superstring, yet another superstring formalism. However as explained in [10] the GS string is equivalent to the RNS string.
The $\lambda$ and $C$ integrals in (2.30) can be evaluated by Lorentz invariance:
\[
\int [dC][d\lambda] \lambda^\alpha \lambda^\beta \lambda^\gamma C^1_{\alpha_1} \cdots C^{11}_{\alpha_{11}} \delta(C^1 \lambda) \cdots \delta(C^{11} \lambda) = (\epsilon T)^{\alpha_1 \cdots \alpha_{11}}. \tag{2.75}
\]
Using this result (2.30) becomes
\[
\mathcal{A} = \int d^{16} \theta f_{\alpha \beta \gamma} (\theta)(\epsilon T)^{\alpha_1 \cdots \alpha_{11}} \theta^{\alpha_1} \cdots \theta^{\alpha_{11}}, \tag{2.76}
\]
which coincides with the non-minimal result (2.66). At higher loops there does not exist such a general proof, but in [33] the non-minimal one- and two-loop four-point functions are shown to coincide with their minimal counterparts. The most recent computation, the five-point one-loop amplitude, has only been computed in the non-minimal formalism [34]. In chapter 4 formal equivalence between the minimal and non-minimal formalism will be proved by providing a first principles derivation from the same starting point for both minimal and non-minimal.

The power of the pure spinor formalism is not only illustrated by the fact that the complexity of all the amplitudes mentioned in the previous paragraph does not depend on the number of external fermions (unlike RNS). In addition there exists a number of non-renormalisation theorems that have been proved in the pure spinor formalism and not in RNS. Four theorems are listed below in chronological order. It is also indicated which formalism is used in the reference.

- The $p$-loop four graviton function vanishes above one loop [22] (minimal). In other words the $R^4$ term in the low energy effective action does not receive perturbative corrections above one loop. This is a consequence of a conjectured selfduality of type IIB string theory, S-duality. In the RNS formalism the conjecture was verified only at two loops after much effort [11].

- The massless $N$-point multiloop ($g \geq 2$) function vanishes whenever $N < 4$ [22] (minimal). This result is the main ingredient of the proof of perturbative finiteness of string theory. As explained in [22] the only other possible obstruction to proving perturbative finiteness is the existence of unphysical divergences in the interior of moduli space. Such divergences are not expected in the pure spinor formalism. Within the RNS formalism there are no results beyond two loops.

- In [35] (non-minimal) two more conjectures based on string dualities are presented and subsequently proved. The first theorem states that when $0 < n < 12$, $\partial^n R^4$ terms do not receive perturbative corrections above $n/2$ loops. The second theorem states that when $n \leq 8$, perturbative corrections to $\partial^n R^4$ terms in the IIA and IIB effective actions coincide.
The analysis of the previous reference was extended to the open string in [36] (non-minimal). In this case it has been shown that the so-called double trace term, $\partial^2 t_8 (tr F^2)^2$, does not receive corrections above two loops, whereas no such restriction holds for the single trace term, $\partial^2 t_8 (tr F^4)$.

Furthermore the (non-minimal) pure spinor formalism has also caught up and overtaken the RNS formalism in the area of overall coefficients. The normalisation of the one-loop four-point in the non-minimal pure spinor formalism was computed in [37]. The tree-level and two-loop computations were performed in [38]. This reference also shows that the results from the pure spinor formalism are in agreement with predictions from S-duality. Moreover the results are consistent with factorisation.