Chapter 3

Basic techniques

This chapter contains the mathematical details of a lot of the arguments used in the previous chapter. The starting point will be the definition of a representation and all results will follow without the need for any further prerequisites\(^1\). Important results in this chapter include

- The Wick rotated Lorentz group, \(SO(10)\), has an \(SU(5)\) subgroup.
- A pure spinor has eleven independent components in ten dimensions.
- There exist unique Lorentz invariant measures for the zero modes of \(\lambda^\alpha\) and \(N^{mn}\).
- Proof of equation (2.75).

Furthermore this chapter contains results on representation theory and invariant tensors that will be useful in due course.

3.1 Invariant tensors

Before the definition of an invariant tensor is given it is necessary to recall how the vector and spinor representations of \(SO(N)\) are defined.

*Definition* A representation of \(SO(N)\) consists of an \(d\) dimensional vector space and a map

\[
    f : SO(N) \times \mathbb{C}^d \rightarrow \mathbb{C}^d, \quad (3.1)
\]

\[
    f(A, v) = g(A)v, \quad (3.2)
\]

\(^1\)Section 3.2.3 is an exception where knowledge ofDynkin labels is assumed. These are pedagogically introduced in [39].
where $g(A)$ is a linear map from $\mathbb{C}^d$ to itself for every $A \in SO(N)$. In addition $g$ must satisfy
\[ g(AB)v = g(A)g(B)v, \quad g(e)v = v, \quad (3.3) \]
where $e$ is the unit element of $SO(10)$.

The fundamental representation is given by $d = N$ and $g$ is the identity map ($g(A) = A$). In physics notation, which is used throughout this thesis, this representation would be denoted as
\[ v^a \rightarrow A^a_b v^b, \quad (3.4) \]
or even shorter
\[ v \rightarrow Av. \quad (3.5) \]
In order to see this is a representation note both sides of (3.3) reduce to $ABv$. A second representation of $SO(N)$ is given by
\[ v_a \rightarrow v_b (A^{-1})^b_a \text{ or } v \rightarrow (A^{-1T})v. \quad (3.6) \]
This also satisfies the defining condition for representations because
\[ A^{-1T} (B^{-1T}v) = (A^{-1T} B^{-1T})v = (AB)^{-1T}v. \quad (3.7) \]
In fact this can be generalised to construct a second representation from any given one. One just replaces $v \rightarrow g(A)v$ by
\[ v \rightarrow (g(A))^{-1T}v. \quad (3.8) \]
This is called the conjugate representation. Note the position of the indices on the conjugate representation is opposite to the original representation. This is very convenient because together with the rule that indices can only be summed over if one is up and one is down, tensors transform as indicated by their free indices. In particular combinations without free indices are invariant. For example for an arbitrary representation and its conjugate
\[ w_a v^a \rightarrow w_b ((g(A))^{-1})^b_a g(A)^a_c v^c = w_b \delta^b_c v^c = w_a v^a. \quad (3.9) \]
The first equality is a consequence of (3.3) with $B = A^{-1}$.

An invariant tensor is a tensor that transforms into itself under all elements of the group. For example $\delta^a_b$ is an invariant tensor for any representation. Note the range, that $a$ and $b$ run over, depends on the (dimension of the) representation. Its transformation is given by
\[ \delta^a_b \rightarrow g(A)^a_c \delta^c_d ((g(A))^{-1})^d_b = \delta^a_b. \quad (3.10) \]
For $SO(N)$ $\delta^{ab}$ is also an invariant tensor where $a, b$ denote the vector representation, hence they run from 1 to $N$. This tensor is invariant because
\[ \delta^{ab} \rightarrow A^a_c A^b_d \delta^{cd} = (AA^T)^{ab} = \delta^{ab}. \quad (3.11) \]
The last equality follows from the definition of $SO(N)$. For an arbitrary representation of $SO(N)$ of dimension $d$ with the property $\det(g(A)) = 1 \forall A \in SO(N)$, $\epsilon^{a_1 \cdots a_d}$ is an invariant tensor:

$$
\epsilon^{a_1 \cdots a_d} \rightarrow (g(A))^{a_1}_{b_1} \cdots (g(A))^{a_d}_{b_d} \epsilon^{b_1 \cdots b_d} = (\det(g(A)))\epsilon^{a_1 \cdots a_d}. \quad (3.12)
$$

Since the fundamental representation falls in this class, $\epsilon^{m_1 \cdots m_N}$ is an invariant tensor. Invariant tensors can be used to construct invariants from tensors. Objects that consist of (covariant) tensors and invariant tensors transform according to their free indices. In particular combinations without free indices are invariant. For example,

$$
v_a w_b \delta^{ab} \rightarrow v_c w_d (B^{-1})^c_a (B^{-1})^d_b \delta^{ab} = v_c w_d \delta^{cd}, \quad (3.13)
$$

where (3.11) with $A = B^{-1}$ was used in the last equality.

For the purposes of this thesis two representations, $v$ an $w$, are defined to be equivalent if they have the same dimension and $w$ can be contracted with invariant tensors such that the resulting index structure exactly matches the indices of $v$. For example the vector representation of $SO(N)$ is equivalent to its conjugate because $\delta^{ab} w_b$ has the same index structure as $v^a$ and therefore transforms as a fundamental vector.

A representation is reducible if the matrix $g(A)$ is blockdiagonal for all $A \in SO(N)$. In addition the same blocks must appear for all $A$’s and the number of blocks must be two or greater.

The complex conjugate of a representation, $g(A)$, is given by $g^*(A)$. One can check this always defines a representation if $g(A)$ did. If a representation is equivalent to its complex conjugate it is real. For $SU(N)$ the conjugate of the fundamental representation is equivalent to the complex conjugate because $A^{-1T} = A^*$.

### 3.2 Clifford algebra and pure spinors

The Clifford algebra in ten dimensions with Euclidian signature is given by

$$
\{\Gamma^m, \Gamma^n\}^a_b = 2\delta^{mn} \delta^a_b, \quad m, n = 0, \cdots, 9 \quad a, b = 1, \cdots, 32. \quad (3.14)
$$

These $\Gamma^m$’s can be used to construct a representation of the Lorentz algebra and by exponentiating also of the Lorentz group. The objects, $\Sigma^{mn} = 1/4[\Gamma^m, \Gamma^n]$, satisfy the Lorentz algebra.

**Definition** Let

$$
(J^{mn})^p_q = \delta^p[m] \delta^q_n, \quad (3.15)
$$

then $A^p_q = (\epsilon^{1/2} \omega_{mn}(J^{mn}))^p_q \in SO(10)$ and each element of $SO(10)$ is covered by an $\omega$. The spinor representation is defined by

$$
g(A(\omega))^a_b = (\epsilon^{-1/2} \omega_{mn}(\Sigma^{mn}))^a_b. \quad (3.16)
$$
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With this definition \((\Gamma^m)^a_b\) is an invariant tensor. Note that the notation implies that one has to transform the \(b\) index as a conjugate spinor. Let us show this by considering infinitesimal transformations:

\[
(\Gamma^m)^a_b \rightarrow (\delta^m_n + \frac{1}{2} \omega_{pq} (J^{pq})^m_n) (\delta^a_c - \frac{1}{2} \omega_{pq} (\Sigma^{pq})^a_c) (\Gamma^n)^c_d (\delta^d_b + \frac{1}{2} \omega_{pq} (\Sigma^{pq})^d_b). \tag{3.17}
\]

With spinor indices suppressed and only keeping terms to first order (plus one second order term) in \(\omega\) this becomes

\[
\Gamma^m \rightarrow \Gamma^m + (1 - \frac{1}{2} \omega_{pq} \Sigma^{pq}) \Gamma^m (1 + \frac{1}{2} \omega_{pq} \Sigma^{pq}) - (1 - \frac{1}{2} \omega_{pq} (J^{pq})^m_n) \Gamma^n. \tag{3.18}
\]

By using the definition of \(\Sigma\) and \(J\) the second and third term can be shown to be equal. This proves \((\Gamma^m)^a_b\) is an invariant tensor.

The Clifford algebra has a solution in which the 32 by 32 components \(\Gamma\) matrices are off diagonal:

\[
\Gamma^m = \begin{pmatrix} 0 & \gamma^m_{\alpha\beta} \\ \gamma^m_{\alpha\beta} & 0 \end{pmatrix}, \tag{3.19}
\]

where \(\alpha, \beta = 1, \cdots, 16\). The notation suggests that there is a sixteen dimensional representation. Moreover it suggests that the two \(\gamma\)'s are invariant tensors with respect to this new representation. To see this first of all note the Clifford algebra now reduces to

\[
\gamma^{(m\alpha\beta)} \gamma_{(n\beta\gamma)} = 2 \delta^{mn} \delta^\alpha. \tag{3.20}
\]

In particular \((\gamma^m)^{\alpha\beta}\) is the inverse of \((\gamma^m)_{\alpha\beta}\). The Lorentz generators \(\Sigma\) become

\[
\Sigma^{mn} = \frac{1}{4} \begin{pmatrix} (\gamma^{[m} \gamma^{n]} \gamma^\alpha)_{\beta} & 0 \\ 0 & (\gamma^{[m} \gamma^{n]} \gamma^\beta)_{\alpha} \end{pmatrix}. \tag{3.21}
\]

This implies the representation of the Lorentz group is reducible. An explicit solution to (3.20) is given in the next section after some explanation about how representations decompose under subgroups. From this explicit solution one can see the two representations are irreducible. The two blocks are the Weyl representation and its conjugate. The 32 dimensional spinor is called a Dirac spinor. To see \((\gamma^m)^{\alpha\beta}\) is an invariant tensor, note since \(\Sigma\) satisfies the Lorentz algebra so does \(\frac{1}{4} (\gamma^{[m} \gamma^{n]} \gamma^\alpha)_{\beta}\). These are the Lorentz generators in the Weyl representation. By a similar argument as for the \(\Gamma\)'s one sees \(\gamma\) is an invariant tensor.

3.2.1 The \(SU(N)\) subgroup of \(SO(2N)\)

This section is devoted to showing \(SO(2N)\) has an \(SU(N)\) subgroup. In addition it will be demonstrated how representations of \(SO(2N)\) decompose into representations
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of $SU(N)$. Part of this analysis is based on [39]. To start define for any $SO(2N)$ vector $v$:

$$v^a = \frac{1}{2}(v^{2a} - iv^{2a+1}), \quad v_a = \frac{1}{2}(v^{2a} + iv^{2a+1}), \quad a = 1, \ldots, N.$$  \hfill (3.22)

The Clifford algebra can now be written as

$$\{\Gamma_a, \Gamma^b\} = \delta^b_a,$$  \hfill (3.23)

with all other anticommutators zero. The $SU(N)$ subalgebra of $SO(2N)$ consists of the generators,

$$T_a = \Gamma_j \tilde{T}_a^j \Gamma^k, \quad a = 1, \ldots, N^2 - 1, \quad j, k = 1, \ldots, N,$$  \hfill (3.24)

where $\tilde{T}_a^j$ are Gell-Mann matrix elements for $SU(N)$, i.e. they satisfy $[\tilde{T}_a^j, \tilde{T}_b^l] = f_{abc} \tilde{T}_c^j$. By virtue of the Clifford algebra one can show the $T_a$ also satisfy the $SU(N)$ algebra:

$$[T_a, T_b] = [\Gamma_j \tilde{T}_a^j \Gamma^k, \Gamma_l \Gamma^m \tilde{T}_b^l \Gamma^m] =$$

$$\tilde{T}_a^j \tilde{T}_b^l \Gamma^m \Gamma^l \Gamma^m - \Gamma_j \Gamma_l \{\Gamma^k, \Gamma^m\} + \{\Gamma_j, \Gamma_l\} \Gamma^m \Gamma^k - \Gamma_l \{\Gamma_j, \Gamma^m\} \Gamma^k =$$

$$\Gamma_j ([\tilde{T}_a^j, \tilde{T}_b^l])^{j} \Gamma^k = f_{abc} \Gamma^l \Gamma^k = f_{abc} \tilde{T}_c.$$

Moreover the $T_a$ form a subalgebra of $SO(2N)$ since

$$\Gamma_j \Gamma^k = \frac{1}{2}\{\Gamma_j, \Gamma^k\} + \frac{1}{2}\{\Gamma_j, \Gamma^k\}$$

$$= \frac{1}{2}\delta^k_j - \frac{i}{2} \Sigma^{2j,2k} + \frac{1}{2} \Sigma^{2j+1,2k} - \frac{1}{2} \Sigma^{2j,2k+1} + \frac{i}{2} \Sigma^{2j+1,2k+1}. \hfill (3.26)$$

The $\delta^k_j$ does not contribute to $T_a$ because $\tilde{T}_a^j$ is traceless.

The $SO(2N)$ algebra is given by

$$[M^{mn}, M^{pq}] = -(\delta^{mp} M^{aq} - \delta^{aq} M^{mp}) m, n, p, q = 1, \ldots, 2N. \hfill (3.27)$$

One can also give this algebra with the components of the generators labelled by the indices from (3.22):

$$[M^{ab}, M^{cd}] = -\frac{1}{2}\delta^{[a}_{[c} M^{b]}_{d]}, \quad a, b, c, d = 1, \ldots, N, \hfill (3.28)$$

$$[M^a_b, M^c_d] = \frac{1}{2}(\delta^a_d M^c_b - \delta^c_b M^a_d), \hfill (3.29)$$

$$[M^a_b, M^{cd}] = \frac{1}{2} \delta^{[a}_{[c} M^{d]}_{b}], \quad [M^a_b, M_{cd}] = -\frac{1}{2} \delta^{[a}_{[c} M^{d]}_{b}], \hfill (3.30)$$

$$[M^{ab}, M^{cd}] = [M_{ab}, M_{cd}] = 0. \hfill (3.31)$$
These equalities can be proved by using (3.22), $M^a_b = -M^a_b$ and noting
\[
\begin{align*}
(\delta_{2N})^a_b &= \frac{1}{4}(\delta_{2N})^{2a,2b} - i(\delta_{2N})^{2a+1,2b} + i(\delta_{2N})^{2a,2b+1} + (\delta_{2N})^{2a+1,2b+1} \\
&= \frac{1}{2}(\delta_{2N})^a_b \equiv \frac{1}{2}\delta^a_b, \quad \text{(3.32)}
\end{align*}
\]
\[
\begin{align*}
(\delta_{2N})^{ab} &= \frac{1}{4}(\delta_{2N})^{2a,2b} - i(\delta_{2N})^{2a+1,2b} - i(\delta_{2N})^{2a,2b+1} - (\delta_{2N})^{2a+1,2b+1} \\
&= 0, \quad \text{(3.33)}
\end{align*}
\]

where $\delta_k$ is the $k$ dimensional Kronecker delta. From (3.29) one sees the $SO(2N)$ algebra has an $N^2$ dimensional subalgebra. This subalgebra contains a $U(1)$ generated by $M \equiv M^a_a$ and the other $N^2 - 1$ generators$^2$,
\[
(M_S)^a_b \equiv M^a_b - \frac{1}{5}\delta^a_b M^c_c, \quad \text{(3.34)}
\]

are traceless and generate an $SU(N)$:
\[
[(M_S)^a_b, (M_S)^c_d] = [M^a_b - \frac{1}{5}\delta^a_b M^c_e, M^c_d - \frac{1}{5}\delta^c_d M^f_f] = \]
\[
-\frac{1}{2}(\delta^a_d M^c_b - \frac{1}{5}\delta^a_d \delta^c_b M^e_e - \delta^c_b M^a_d - \frac{1}{5}\delta^c_b \delta^a_d M^f_f) = -\frac{1}{2}(\delta^a_d (M_S)^c_b - \delta^c_b (M_S)^a_d).
\]
The $U(1)$ charges of the generators are given by
\[
[M, M^{ab}] = -M^{ab}, \quad [M, M^a_b] = 0, \quad [M, M_{ab}] = M_{ab}. \quad \text{(3.36)}
\]

This concludes the proof of the existence of the $SU(5)$ subgroup. The next step is to examine how $SO(10)$ representations behave under $SU(5)$ transformations.

Every representation of $SO(2N)$ can be decomposed into representations of $SU(N)$. This means the vector space, that the tensors live in, can be written as a direct sum of subspaces and the subgroup does not mix the elements of the subspaces. The vector representation of $SO(2N)$ for example decomposes into the vector of $SU(N)$ and its conjugate, which for $SU(N)$ is equivalent to the complex conjugate. The subspace of $\mathbb{C}^{2N}$ that is invariant under $SU(N)$ is $V = \{v|v^a = 0; v \in \mathbb{C}^{2N}\}$. The variation of an $SO(2N)$ vector by an element of the $SU(5)$ subgroup is
\[
v^m \rightarrow (e^{\omega_a^b M^a_b})_m^nv^n = (A(\omega)v)^m, \quad \text{(3.37)}
\]

where $\omega^a_a = 0$. If $v \in V$, $(A(\omega)v)$ is also an element of $V$. To show this one needs to prove $(A(\omega)v)^c = 0$:
\[
(A(\omega)v)^a = 2(e^{\omega^d_c M^d_c})^a_b v^b + 2(e^{\omega^d_c M^d_c})^{ab} v_b = 0. \quad \text{(3.38)}
\]

$^2$The subscript $S$ on $M$ has no relation with the subscript on the nilpotent fermionic operators $Q_S$. 

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The first term is zero because $v^a = 0$. The second term because

$$(M^c_d)_{ab} = 0.$$  
(3.39)

This follows from (3.15), (3.22) and (3.33). The other components are

$$(A(\omega)v)_a = (e^{\omega^d c M^d c}_a b v_b = v_a - v_b \omega^b_a + O(\omega^2).$$  
(3.40)

For a vector with $v_a = 0$ we get

$$(A(\omega)v)_a = 0,$$  
(3.41)

$$(A(\omega)v)^a = (e^{\omega^d c M^d c}_a b v_b = v^a + \omega^a_b v_b + O(\omega^2).$$  
(3.42)

From (3.40) and (3.42) one sees the two representations are each others conjugate. Since there are only two $N$ dimensional representations of $SU(N)$, namely the vector and its conjugate which is equivalent to the complex conjugate of the vector, one can conclude

$$2N \rightarrow N \oplus \bar{N}.$$  
(3.43)

As shown above $SO(10)$ has a sixteen dimensional spinor representation. This also decomposes under the $SU(5)$ subgroup. To find the precise decomposition note that any representation of the Clifford algebra is also a representation of $SO(10)$. Since the Clifford algebra in the form of (3.23) is just a set of raising and lowering operators, representations are easily constructed by choosing a vacuum $|0\rangle$ that satisfies

$$\Gamma_a |0\rangle = 0.$$  
(3.44)

32 states are created by acting with $\Gamma^a$:

$$e^{-} = |0\rangle, \quad e^{a_1 \cdots a_k} = \Gamma^{a_1} \cdots \Gamma^{a_k} |0\rangle, \quad k = 1, \ldots, 5.$$  
(3.45)

Note that all $e$’s are antisymmetric in their indices, so that there is indeed a total of 32 basis vectors. This representation is the Dirac spinor. These basis vectors can also be labelled with downstairs $SU(5)$ indices

$$e_+ = e^-, \quad e_{bcde} = \epsilon_{abcde} e^a, \quad e_{cde} = \frac{1}{2} \epsilon_{abcde} e^{ab},$$  
(3.46)

$$e_{de} = \frac{1}{6} \epsilon_{abcde} e^{abc}, \quad e_e = \frac{1}{24} \epsilon_{abcde} e^{abcd}, \quad e_- = \frac{1}{120} \epsilon_{abcde} e^{abde} = e^{+}.$$  

A generic spinor can be written as

$$\xi = \xi^+ e_+ + \xi_e e^a + \frac{1}{2} \xi_{ab} e^{ab} + \frac{1}{2} \xi_{ab} e_{ab} + \xi^a e_a + \xi^- e_-.$$  
(3.47)

The $M$ charges of the states are given by

$$Me^{-} = -\frac{5}{4} e^-, \quad Me^{a_1 \cdots a_k} = -\frac{1}{4} (5 - 2k) e^{a_1 \cdots a_k}.$$  
(3.48)
This can also be interpreted as $M$ charges of the components

$$M\xi^+ = -\frac{5}{4}\xi^+, \quad M\xi_{a_1...a_k} = -\frac{1}{4}(5-2k)\xi_{a_1...a_k}. \quad (3.49)$$

Because the difference of the number of $\Gamma^a$'s and $\Gamma_a$'s is always even in the $SO(10)$ generators, all $SO(10)$ transformations will change the $M$ charge by an integer. This shows the reducibility of the Dirac spinor into two Weyl spinors. Incidentally we can read off the decomposition under the $SU(5)$ subgroup:

$$16 \rightarrow 1_{\frac{5}{4}} \oplus 10_{\frac{1}{4}} \oplus \bar{5}_{\frac{3}{4}} \quad \lambda^a \rightarrow \lambda^+, \lambda_{a_1a_2}, \lambda^a, \quad (3.50)$$

$$16' \rightarrow 1_{\frac{5}{4}} \oplus 10_{\frac{1}{4}} \oplus \bar{5}_{\frac{3}{4}} \quad w_{\alpha} \rightarrow w^+ , w^{a_1a_2} , w_{a}. \quad (3.51)$$

where the subscripts are the $U(1)$ charges. For completeness the decomposition of the vector and the antisymmetric rank two tensor of $SO(10)$ are also specified:

$$10 \rightarrow 5_{\frac{-1}{2}} \oplus \bar{5}_{\frac{3}{2}} \quad v^m \rightarrow v^a , v_a, \quad (3.52)$$

$$45 \rightarrow 1_0 \oplus 24_0 \oplus 10_{-1} \oplus 10_1 \quad M^{mn} \rightarrow M^a_{a_1} , (M_S)^a_b , M^{ab} , M_{ab}. \quad (3.53)$$

### 3.2.2 Charge conservation and tensor products

In order to solve the pure spinor constraint (2.2) one needs an explicit representation of the gamma matrices. The $M$ charge conservation property of invariant tensors proves a large number of components of invariant tensors is zero, which is very useful if one is doing computations by using the explicit expressions of the tensors, in particular gamma matrices. An invariant tensor $T^{\alpha\beta}_{\gamma\delta}$ satisfies

$$0 = MT^{\alpha\beta}_{\gamma\delta} = (M^u(\alpha) + M^u(\beta) + M^d(\gamma) + M^d(\delta))T^{\alpha\beta}_{\gamma\delta}, \quad (3.54)$$

where $M^u(+) = -\frac{5}{4}, M^u(a_1a_2) = -\frac{1}{4}, M^u(a) = \frac{3}{4}, M^d(+) = \frac{5}{4}, M^d(a_1a_2) = \frac{1}{4}, M^d(a) = -\frac{3}{4}$. The $u$ is for up and the $d$ for down. This refers to the position of the Weyl index not the $SU(5)$ indices. So if the $M$ charges of the indices of a components do not sum up to zero the component vanishes. In this case one can for instance conclude $T^{+}_{b_1b_2,c,d} = 0$, because the $M$ charge of the components is $-\frac{1}{4}(5+1+3+3) \neq 0$.

In this thesis questions of the following type often arise: how many independent invariant tensors $T^{m\delta}_{(\alpha\beta\gamma)}$ exist? The upper index $\delta$ denotes the Weyl representation, the lower indices stand for the conjugate Weyl representation and $m$ is the ten dimensional vector. To answer this question first of all note that the space of all tensors with the index structure and symmetries of $T$ forms a representation of $SO(10)$. The question how many independent invariant tensors exist in that space now translates to what the dimension of the invariant subspace is. This number can be obtained by computing the number of scalars in the relevant tensor product.
This is one of the features of the computer algebra program LiE [40]. For the case of \( T \) one computes

\[
10 \otimes 16 \otimes \text{Sym}^3 16' = 1 \oplus 45 \oplus 45 \oplus 45 \oplus \cdots ,
\]

(3.55)

where the ellipsis denotes higher dimensional irreducible representations. The above decomposition shows that the space of invariant tensors with the symmetries of \( T \) is one dimensional. Based on this result we can for example conclude

\[
\gamma^m_{(\alpha \beta \gamma \delta)} \propto \gamma^n_{(\alpha \beta \gamma \delta)} \epsilon^e_{m} \epsilon^\delta_{n}.
\]

(3.56)

In order to find the constant of proportionality, computing a single component on both sides suffices. Alternatively one can contract both sides with a suitable invariant tensor.

### 3.2.3 Dynkin labels and gamma matrix traceless tensors

Throughout this work irreducible representations are denoted by their dimensions. This is slightly ambiguous. A more precise label is the Dynkin label of the highest weight state of the representation [39].

\[
10 \leftrightarrow (1, 0, 0, 0, 0), \quad 16 \leftrightarrow (0, 0, 0, 1, 0), \quad 16' \leftrightarrow (0, 0, 0, 0, 1), \quad 45 \leftrightarrow (0, 1, 0, 0, 0).
\]

(3.57)

There is one further irreducible representation of interest, which is given by symmetric and gamma matrix traceless tensors:

\[
T^{((\alpha_1 \cdots \alpha_n))} \leftrightarrow (0, 0, 0, n, 0) \leftrightarrow \text{Gam}^n 16,
\]

(3.58)

where the Dynkin labels are specified. These representations are discussed in more detail in [41]. There are three gamma matrix traceless tensors that are of particular interest:

\[
(T_1)^{((\alpha_1 \alpha_2 \alpha_3))}_{[\beta_1 \cdots \beta_{11}]}, \quad (T_2)^{((\alpha_1 \cdots \alpha_8))}_{[[m_{11}, \cdots, [m_{10n10}]]}, \quad (T_3)^{((\alpha_1 \cdots \alpha_{11}))}_{[\beta_1 \cdots \beta_{11}][[m_{11}, \cdots, [m_{10n10}]]}.
\]

(3.59)

The first one has already appeared in chapter 2, the other two will play a role in one-loop computations. For the three tensors above the computer algebra program LiE can be used to conclude there is only one independent invariant tensor. Note this is consistent with the arguments in [33], where it is argued that a tensor which is symmetric and gamma matrix traceless, let us say in some indices \( \alpha_i \), is completely specified by the components where the \( \alpha \)'s are all \( + \). In order to see this implies there is only one independent invariant tensor of the form of \( T_1 \) note that for an invariant tensor the components

\[
(X_1)^{+++}_{\beta_1 \cdots \beta_{11}}
\]

(3.60)
are only nonvanishing if

\[ \beta_1 \cdots \beta_{11} = +, 12, 13, \ldots, 45, \]  

(3.61)

which follows from the charge conservation property of invariant tensors. By antisymmetry of the \( \beta \)'s there is only one independent component in (3.60). If one now uses the argument from [33] the entire invariant tensor is completely specified by a single component, therefore the space of invariant tensors of the form of \( T_1 \) is one dimensional. The above argument applies equally well to \( T_2 \) and \( T_3 \).

### 3.2.4 Explicit expression for gamma matrices

A solution to the Clifford algebra for the ten dimensional Pauli matrices (3.20) is given by

\[
(\gamma^k)_{\alpha\beta} = \left( \begin{array}{ccc} 0 & 0 & \delta^k_b \\ 0 & -\epsilon^{ka_1a_2b_1b_2} & \delta^k_a \\ \delta^k_a & 0 & 0 \end{array} \right), \quad (\gamma_k)_{\alpha\beta} = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & \delta^{[a_1}_b \delta_{a_2]}_k \\ 0 & \delta^{[a_1}_k \delta_{a_2]}_b & 0 \end{array} \right),
\]

(3.62)

where all Latin indices are \( SU(5) \) vector indices. The top left corner of the matrices is the +, + component, top middle is the +, \( b_1 b_2 \) component and top right is the +, \( b \) component etc. Note these matrices are skew diagonal, this is a consequence of the charge conservation property of invariant tensors.

In chapter 5 not only the gamma matrices itself will be important, but also their antisymmetrised products. In particular the three form gamma matrices. Their explicit expression is given by:

\[
(\gamma_{k_1k_2k_3})^{\alpha\beta} = -\frac{1}{6}(\gamma_{k_1} \gamma_{k_2} \gamma_{k_3})^{\alpha\beta} = \left( \begin{array}{ccc} 0 & \epsilon_{k_1k_2k_3b_1b_2} & 0 \\ \epsilon_{k_1k_2k_3a_1a_2} & 0 & 0 \\ 0 & 0 & 0 \end{array} \right)
\]

(3.64)

\[
(\gamma^{k_1}_{k_2k_3})^{\alpha\beta} = -\frac{1}{6}((\gamma^{k_1}_{k_2} \gamma_{k_3})^{\alpha\beta} - (\gamma^{k_2}_{k_1} \gamma_{k_3})^{\alpha\beta} + (\gamma^{k_3}_{k_1} \gamma_{k_2})^{\alpha\beta}) = \frac{1}{2} \left( \begin{array}{ccc} 0 & 0 & -\delta^{k_1}_{[a_1} \delta^{k_2}_{b_2]} \\ 0 & -\delta^{k_1}_{[a_2} \delta^{k_2}_{b_1]} \\ -\delta^{k_1}_{[a_1} \delta^{k_2}_{b_1]} & 0 & 0 \end{array} \right),
\]

(3.65)
\[
\left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & -\delta_{b_1}^{k_1} \delta_{b_2}^{k_2} \delta_{k_3}^a - \frac{1}{2} \delta_{b_1}^{a} \delta_{b_2}^{k_1} \delta_{k_3}^b \\
\delta_{[a_1}^{k_1} \delta_{a_2}^{k_2} \delta_{k_3}^b + \frac{1}{2} \delta_{[a_1}^{b} \delta_{a_2}^{k_1} \delta_{k_3}^b \\
0 & 0 & 0
\end{array} \right),
\tag{3.66}
\]

\[
(\gamma^{k_1} \gamma^{k_2} \gamma^{k_3})_{\alpha\beta} = \frac{1}{6} (\gamma^{[k_1} \gamma^{k_2} \gamma^{k_3]})_{\alpha\beta} = \left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -\epsilon^{k_1 k_2 k_3}_{ab}
\end{array} \right).
\tag{3.67}
\]

### 3.2.5 Pure spinors

A pure spinor is a Weyl spinor that satisfies

\[
\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0.
\tag{3.68}
\]

After plugging in the explicit expression for the gamma matrices this becomes

\[
2\lambda^+ \lambda^a - \frac{1}{4} \epsilon^{abced} \lambda_{bc} \lambda_{de} = 0,
\tag{3.69}
\]

\[
2\lambda^b \lambda_{ab} = 0.
\tag{3.70}
\]

These equations are solved by

\[
\lambda^a = \frac{1}{8} \frac{1}{\lambda^+} \epsilon^{abced} \lambda_{bc} \lambda_{de}.
\tag{3.71}
\]

This is clearly a solution to the first equation. For the second equation one makes use of the fact that a six component $SU(5)$ tensor vanishes when antisymmetrised over all indices:

\[
0 = \lambda_{[ab_1} \lambda_{b_2 b_3} \lambda_{b_4 b_5]} = 6 \lambda_{a[b_1} \lambda_{b_2 b_3} \lambda_{b_4 b_5]}.
\tag{3.72}
\]

The result (3.71) shows that a pure spinor has eleven independent components. A number of great significance in the pure spinor formalism, since it plays a crucial role in the vanishing of the central charge of the pure spinor formalism action in conformal gauge (2.1).

### 3.3 Pure spinor Lorentz generators

The goal of this section is deriving the Lorentz generator OPE’s as given in (2.7). This can be achieved by breaking manifest $SO(10)$ invariance to manifest $SU(5)$ invariance. As a warm up exercise the Lorentz currents for an unconstrained Weyl spinor are studied. Incidentally the results obtained in this exercise apply to the Lorentz generators of the $p, \theta$ sector of the worldsheet action (2.1). The Lorentz generators of an unconstrained bosonic spinor $\xi^\alpha$ and its conjugate variable $y_\beta$ are given by

\[
M^{mn} = \frac{1}{2} y_\alpha (\gamma^{mn})^\alpha_{\beta} \xi^\beta, \quad \gamma^{mn} = \frac{1}{2} (\gamma^m \gamma^n - \gamma^n \gamma^m).
\tag{3.73}
\]
In this subsection these components are given in terms of the $SU(5)$ components of $\xi$ and $y$. The $SU(5)$ components of (3.73) are given by

$$M_{kl} = \frac{1}{4} y_\alpha (\gamma^k)^{\alpha \beta} (\gamma_l)^{\beta \gamma} \delta \xi^\gamma,$$

$$M^k_l = \frac{1}{4} (y_\alpha (\gamma^k)^{\alpha \beta} (\gamma_l)^{\beta \gamma} - y_\alpha (\gamma_l)^{\alpha \beta} (\gamma^k)^{\beta \gamma}) \delta \xi^\gamma,$$

$$M^{kl} = \frac{1}{4} y_\alpha (\gamma^k)^{\alpha \beta} (\gamma^l)^{\beta \gamma} \delta \xi^\gamma.$$

In order to write these Lorentz generators in terms of the $SU(5)$ components of $y$ and $\xi$ one uses the explicit expressions of the gamma matrices (3.62) and (3.63).

$$M_{kl} = -\frac{1}{2} y_\alpha (\gamma^k)^{\alpha \beta} (\gamma_l)^{\beta \gamma} \xi^\gamma,$$

$$M^k_l = \frac{1}{2} y^k_l \xi^\gamma + \frac{1}{4} y_a \epsilon^{abc} \xi^c,$$

$$M^{kl} = -\frac{1}{4} \delta_l^k \xi^\gamma + \frac{1}{4} \delta^k_l y_\alpha (\gamma^k)^{\alpha \beta} (\gamma_l)^{\beta \gamma} \xi^\gamma + \frac{1}{2} y^k_a \xi^a,$$

$$M = M^k_k = -\frac{5}{4} y_\alpha (\gamma^k)^{\alpha \beta} (\gamma_l)^{\beta \gamma} \xi^\gamma + \frac{3}{4} y_a \xi^a,$$

$$M^S = M^k_k = -\frac{1}{10} \delta^k_l \delta^l_k y_\alpha (\gamma^k)^{\alpha \beta} (\gamma_l)^{\beta \gamma} \xi^\gamma.$$

The current $J$ can also be written in terms of the $SU(5)$ components of its constituents:

$$J = y_\alpha \xi^\alpha = y_\alpha (\gamma^k)^{\alpha \beta} (\gamma_l)^{\beta \gamma} \xi^\gamma + \frac{1}{2} y_\alpha (\gamma^k)^{\alpha \beta} (\gamma^l)^{\beta \gamma} \xi^\gamma + \frac{1}{2} y_\alpha (\gamma^k)^{\alpha \beta} (\gamma^l)^{\beta \gamma} \xi^\gamma.$$

### 3.3.1 Lorentz current OPE’s with unconstrained spinors

For unconstrained spinors there is no need to break to $SU(5)$ in order to derive the OPE of the Lorentz currents. It can be derived by the $SO(10)$ covariant OPE of the bosonic spinors $\xi_\alpha$ and $y^\beta$:

$$y_\alpha (z) \xi^\beta (w) \sim \delta^\beta_\alpha \frac{1}{z - w}.$$

The OPE of the pure spinor Lorentz current with itself is given by

$$M^{m_1 m_2} (z) M^{n_1 n_2} (w) \sim \frac{1}{4} \frac{1}{z - w} (-y_\alpha (z) \gamma^{m_1 m_2} \gamma^{n_1 n_2} \xi^\gamma (w) +$$

$$y_\alpha (w) \gamma^{n_1 n_2} \gamma^{m_1 m_2} \xi^\gamma (z)) + \frac{1}{4} \text{Tr}(\gamma^{m_1 m_2} \gamma^{n_1 n_2}) (z - w)^2.$$
The following two identities can be used
\[
\left[ \frac{1}{2} \gamma^{m_1 m_2}, \frac{1}{2} \gamma^{n_1 n_2} \right] = \frac{1}{2} (\eta^{n_1 [m_2} \gamma^{m_1]n_2} - \eta^{n_2 [m_2} \gamma^{m_1]n_1}),
\] (3.79)
\[
\text{Tr}(\gamma^{m_1 m_2} \gamma^{n_1 n_2}) = -16 \eta^{m_1 [n_1} \eta^{n_2]m_2}.
\] (3.80)

The MM OPE now reduces to
\[
M^{m_1 m_2}(z) M^{n_1 n_2}(w) \sim \frac{-(\eta^{n_1 [m_2} M^{m_1]n_2} - \eta^{n_2 [m_2} M^{m_1]n_1})}{z - w}
- 4 \eta^{m_1 n_2} \eta^{m_2 n_1} - \eta^{m_1 n_1} \eta^{m_2 n_2}
\] (z - w)^2.
\] (3.81)

One can read off the algebra of the Lorentz charges from the single pole in the OPE
\[
[M^{m_1 m_2}, M^{n_1 n_2}] = -(\eta^{n_1 [m_2} M^{m_1]n_2} - \eta^{n_2 [m_2} M^{m_1]n_1}).
\] (3.82)

In case the worldsheet fields are fermionic, the OPE remains the same:
\[
p_\alpha(z) \theta^\beta(w) \sim \delta^\beta_\alpha \frac{1}{z - w}.
\] (3.83)

The Lorentz generator for the fermionic variables has a minus sign:
\[
M^{mn} = -p\gamma^{mn}\theta.
\] (3.84)

This sign is necessary to reproduce the commutation relation (3.82). As a consequence the sign in the double pole in the OPE changes from -4 to +4. This coefficient is called the level. One would like the Lorentz current of the combined \(p, \theta, \lambda, w\) sector to have level one, since this is the level of the \(\psi\) sector in the RNS formalism. This implies the \(N_{\lambda w}\) generators must have level \(-3\). The next two subsections contain an explanation how such currents can be obtained from the pure spinor action after gauge fixing.

### 3.3.2 Gauge fixing \(w_\alpha\) invariance

As mentioned before \(\lambda w\) part of the pure spinor action 2.1 has a gauge invariance. To deal with this one can start by relaxing the pure spinor condition on \(\lambda^\alpha\) and introducing a Lagrange multiplier \(l_m\) to impose it in the path integral. The \((w, \lambda)\) part of the action (2.1) thus now reads
\[
S_{(w, \lambda)} = \int d^2 z \left(w_\alpha \bar{\partial} \lambda^\alpha + l_m (\lambda \gamma^m \lambda) \right).
\] (3.85)

where \(\lambda^\alpha\) is now an unconstrained bosonic Weyl spinor. This action has a gauge invariance\(^3\),
\[
\delta w_\alpha = \Lambda^a (\gamma_a \lambda)_\alpha, \quad \delta l^\alpha = \frac{1}{2} \bar{\partial} \Lambda^a.
\] (3.86)

\(^3\)There is also a gauge invariance associated with \(l_a\), but since the constraint from which this is derived is completely solved by the \(l^a\) constraint, the \(l_a\) gauge invariance will not be present anymore after gauge fixing the \(l^a\) invariance.
This gauge transformation has rank five, so one can gauge fix it by requiring

$$w^a = 0.$$  \hfill (3.87)

Following the steps of BRST quantisation (and expressing the gamma matrices in the \(U(5)\) basis) one finds that corresponding ghost action is given by

$$\int d^2z \left( \bar{C}_b (\gamma_a)^b_\beta \lambda^\beta C^a + w^a \pi_a \right) = \int d^2z \left( \bar{C}_a \lambda^+ C^a + w^a \pi_a \right),$$ \hfill (3.88)

where \(\bar{C}_b, C^a, \pi_a\) are the corresponding antighost, ghost and auxiliary fields. Integrating them out sets \(w^a = 0\) and inserts the factor \((\lambda^+)^5\) in the path integral measure. Furthermore, integrating out \(l^a\) leads to the delta function \(\delta(2\lambda_a \lambda^+ + \frac{1}{4} \epsilon_{abcde} \lambda^b \lambda^{de})\) which can be used to integrate out \(\lambda_a\) (so we are left with the eleven independent components \(\lambda^+, \lambda^{ab}\)) and also results in the insertion \((\lambda^+)^{-5}\) in the path integral measure, which cancels the factor \((\lambda^+)^5\) from the ghosts. Finally integrating out \(l^a\) sets \(\lambda \gamma_m \lambda\) to zero and hence removes \(l^a\) from the action. Therefore \(l^a\) is pure gauge and since it does not appear in the action anymore, BRST quantisation amounts to removing the measure factor associated to \(\lambda_a\) from the path integral measure. The end result is that the action (3.85) becomes the free action

$$\int d^2z (w + \bar{\partial} \lambda^+ + \frac{1}{2} w_{ab} \bar{\partial} \lambda^{ab}),$$ \hfill (3.89)

with all factors coming from eliminating the 5 and gauge fixing the gauge invariance canceling out.

The gauge fixed action (3.89) is no longer invariant under \(Q_S \neq \oint dz \lambda^\alpha d_\alpha\), but it is invariant under \(\hat{Q}_S\) defined by

$$\hat{Q}_S w_\alpha = d_\alpha - \frac{d_\alpha}{\lambda^+} (\gamma^a \lambda)_\alpha.$$ \hfill (3.90)

On all other fields \(\hat{Q}_S\) acts the same as \(Q_S\). Note the second term in (3.90) is a gauge transformation with \(\Lambda_a = \frac{d_\alpha}{\lambda^+}, \Lambda^a = 0\). This implies that when acting on gauge invariant quantities \(Q_S = \hat{Q}_S\). Moreover \(\hat{Q}_S w_\alpha = 0\). So that for instance

$$\hat{Q}_s N^{mn}|_{w_a=0} = Q_S N^{mn} = \frac{1}{2} \lambda \gamma^{mn} d.$$ \hfill (3.91)

\(\hat{Q}_S\) also satisfies

$$\hat{Q}_S^2 = 0,$$ \hfill (3.92)

on all fields including \(w\), unlike \(Q_S\).
3.3.3 Currents containing pure spinors

As argued before one would like to find Lorentz currents constructed from the fields in the gauge fixed action (3.89) that (by definition) satisfy the Lorentz algebra and have level minus three. Let us start by looking what one gets by just imposing the gauge condition (3.87) on the Lorentz generators of (3.75):

\[ N_{kl} = -\frac{1}{2} w^{-} \lambda_{kl} + \frac{1}{4} w^{ab} \lambda_{kl} \lambda_{ab} + \frac{1}{2} w^{ab} \lambda_{ka} \lambda_{lb}, \]  
\[ N^{kl} = \frac{1}{2} w^{kl} \lambda^{+}, \]  
\[ N = -\frac{5}{4} w^{-} \lambda^{+} - \frac{1}{4} w^{ab} \lambda_{ab}, \]  
\[ (N_S)^k_l = \frac{1}{2} (-\frac{1}{5} \delta^k_l w^{ab} \lambda_{ab} + w^{ak} \lambda_{at}). \]

The number current in the gauge \( w_a = 0 \) becomes

\[ J = w^{-} \lambda^{+} + \frac{1}{2} w^{ab} \lambda_{ab}. \]

One might expect that imposing \( w_a = 0 \) in all (gauge invariant) operators depending on \( w_\alpha \) does not break Lorentz covariance. For the OPE's of \( N \) and \( J \) with \( \lambda \) Lorentz covariance is indeed not lost, as will be shown below. The \( NN \) OPE is not Lorentz covariant anymore after imposing the gauge condition. The single pole is the same as in (3.81), the level of the OPE, however, depends on which \( SU(5) \) components one chooses. This spoils Lorentz invariance, but it can be cured as demonstrated below.

The OPE of \( J \) and \( N^{mn} \) with \( \lambda \) are given by

\[ J(z) \lambda^\alpha(w) \sim \frac{1}{z-w} \lambda^\alpha(w), \quad N^{mn}(z) \lambda^\alpha(w) \sim \frac{1}{z-w} \frac{1}{2} \gamma^{mn} \lambda^\beta(w). \]

In order to check these OPE's we set \( w_a = 0 \) and use the free field OPE's

\[ w^{-}(z) \lambda^{+}(w) \sim \frac{1}{z-w}, \quad w^{ab}(z) \lambda_{cd}(w) \sim \frac{1}{z-w} \delta^{[a}_{c} \delta^{b]}_{d}. \]

Let us start with \( J \):

\[ J(z) \lambda^{+}(w) = w^{-}(z) \lambda^{+}(w) \sim \frac{1}{z-w} \lambda^{+}(w) \]

and similarly for \( \lambda_{ab} \). \( \lambda^a \) is more involved. By using

\[ w^{-}(z) \frac{1}{\lambda^{+}(w)} \sim \frac{1}{z-w} \frac{-1}{(\lambda^{+})^2(w)}, \]

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one can reproduce the Lorentz invariant answer:

\[ J(z)\lambda^a(w) = (w^- \lambda^+ - \frac{1}{2} w_{ab} \lambda_{ab})(z) \frac{\epsilon^{abcde} \lambda_{bc} \lambda_{de}}{8\lambda^+}(w) \sim \frac{1}{z-w} \frac{1}{8\lambda^+} \epsilon^{abcde} \lambda_{bc} \lambda_{de}(w). \]

(3.102)

Let us continue with the trace of \( N_{mn} \). In terms of unconstrained spinors it is given by

\[ N = -\frac{5}{2} \lambda^+ w^- - \frac{1}{2} w_{ab} \lambda^{ab} + \frac{3}{2} w^a \lambda_a. \]

(3.103)

From here one can see that the expected charge of \( \lambda^a \) is \( -\frac{3}{2} \). The OPE of \( N \) with \( \lambda^+ \) or \( \lambda_{ab} \) trivially reproduces the Lorentz invariant result, the OPE of \( N \) with \( \lambda^a \) is

\[ N(z)\lambda^a(w) = (-\frac{5}{2} \lambda^+ w^- - \frac{1}{2} w_{ab} \lambda^{ab})(z) \frac{\epsilon^{abcde} \lambda_{bc} \lambda_{de}}{8\lambda^+}(w) \sim \frac{1}{z-w} (\frac{3}{2} - \frac{1}{2} - \frac{1}{2}) \epsilon^{abcde} \lambda_{bc} \lambda_{de}(w). \]

(3.104)

All other components of the \( N \) OPE can be checked along the same lines.

The \( N_{mn} \) OPE is a different story. The single pole always leads to the correct Lorentz algebra, but the coefficient of the double pole depends on which \( SU(5) \) components we choose to take. For instance

\[ N(z)N(w) \sim -\frac{35}{16} \frac{1}{(z-w)^2} = -\frac{7}{4} \eta^i \eta^l \frac{1}{(z-w)^2} \]

(3.105)

\[ N^{12}(z)N_{12}(w) \sim \frac{1}{4} \frac{1}{(z-w)^2} + \frac{1}{z-w} \frac{1}{2} (N^1(w) + N^2(w)) = -1 \frac{1}{(z-w)^2} (-\eta^1 \eta^2) + \frac{1}{z-w} \frac{1}{2} (N^1(w) + N^2(w)). \]

(3.106)

The first OPE would imply a Lorentz current level of \(-\frac{7}{4}\) and the second one \(-1\). It will be shown below that it is possible to deform the currents in equations (3.93)-(3.96) by conserved quantities such that the level of the \( N_{mn}N_{pq} \) OPE is minus three [42]. There is not only a freedom to add conserved quantities to \( N_{mn} \), also \( J \) and the stress energy tensor \( T_{\lambda w} \) are subject to this freedom. However now that the Lorentz current is completely fixed by the level -3 constraint, the form of the deformation of the number current \( J \) is unique determined by demanding that the OPE of \( J \) and \( N \) does not contain any poles (2.7). Similarly by demanding that the Lorentz currents are primary field the (pure spinor part of) the stress energy tensor is completely determined. If one now computes the \( JT \) OPE, a \( J_{\lambda w} \) number anomaly value of minus eight follows. This cannot be adjusted.

The deformations are most easily given after bosonization of \( \lambda \) and \( w \), which is given by

\[ \lambda^+ \cong e^{x-\phi}, \quad w^- \cong e^{-x+\phi} \partial x, \quad \lambda^+ w^- \cong \partial \phi, \]

(3.107)
where $\phi, \chi$ are chiral bosons satisfying
\[ \phi(z)\phi(0) \sim -\ln z, \quad \chi(z)\chi(0) \sim \ln z. \] (3.108)

Now define
\[ s = \chi - \phi, \quad 2t = \phi + \chi \leftrightarrow \phi = \frac{1}{2}(2t - s), \quad \chi = \frac{1}{2}(s + 2t) \] (3.109)

The OPE’s for these new variables are
\[ s(z)s(0) \sim \text{regular}, \quad t(z)t(0) \sim \text{regular} \quad t(z)s(0) \sim \ln z. \] (3.110)

The original worldsheet fields $\lambda$ and $w$ can be expressed in terms of $s,t$ as
\[ \lambda^+ \cong e^s, \quad w^- \cong \frac{1}{2}e^{-s}(\partial s + 2\partial t), \quad \lambda^+ w^- \cong \frac{1}{2}(2\partial t - \partial s). \] (3.111)

The Lorentz currents of (3.93)-(3.96) in bosonised form are given by\footnote{In [42] the Lorentz currents, denoted $(N^B)_{mn}$ here, have a different normalisation. The relation with ours is given by}
\[ N = -\frac{5}{8}(2\partial t - \partial s) - \frac{1}{8}w^{ab}\lambda_{ab}, \] (3.113)
\[ N^{ab} = \frac{1}{2}e^s w^{ab}, \] (3.114)
\[ (N_S)^a_b = \frac{1}{2}(w^{ac}\lambda_{bc} - \frac{1}{5}\delta^a_b w^{cd}\lambda_{cd}), \] (3.115)
\[ N_{ab} = e^{-s}\left[-\frac{1}{2}(\frac{1}{2}\partial s\lambda_{ab} + \partial t\lambda_{ab}) - \frac{1}{4}w^{cd}\lambda_{ab}\lambda_{cd} + \frac{1}{2}w^{cd}\lambda_{ac}\lambda_{bd}\right]. \] (3.116)

The deformations one should add to (3.93)-(3.96) to make the $NN$ OPE Lorentz invariant are given by [42]:
\[ \Delta N = -\frac{5}{8}\partial s, \] (3.117)
\[ \Delta N^{ab} = 0, \] (3.118)
\[ \Delta (N_S)^a_b = 0, \] (3.119)
\[ \Delta N_{ab} = e^{-s}\left[-\frac{3}{4}\partial s\lambda_{ab} + \partial t\lambda_{ab}\right] = \partial(e^{-s}\lambda_{ab}) - \frac{1}{4}(\partial e^{-s})\lambda_{ab}. \] (3.120)

Note that the field equations imply the $\bar{\partial}$ operator annihilates these deformations. Hence the deformed charges are still conserved. Furthermore the deformations do not modify the $N\lambda$ OPE, which is manifest in the $s,t$ variables.

\[ N = -\frac{\sqrt{5}}{2}N^B, \quad N^{ab} = \frac{1}{2}(N^B)^{ab}, \quad (N_S)^a_b = \frac{1}{2}(N^B_S)^a_b, \quad N_{ab} = \frac{1}{2}(N^B)_{ab}. \] (3.112)
3.4 Lorentz invariant measures

The Lorentz invariant measures for both the weight zero field, $\lambda^\alpha$, and the weight one field, $N^{mn}$, are discussed below. Both these measures were first introduced in [22] and the $\lambda$ zero mode measure is also discussed in [43].

3.4.1 Measure for the zero modes of $\lambda$

From the $J_{\lambda^w}$ number anomaly in the $JT$ OPE (2.7) one can deduce a tree level correlator can only be nonzero if the $J_{\lambda^w}$ charge of the insertions is -8 (cf. section 1.3.1). Since there are no $w$ (or $N^{mn}$) zero modes at tree level, the measure for the $\lambda$ zero modes must have ghost number +8. In addition the measure must be Lorentz invariant. This results in

$$[d\lambda]\lambda^\alpha \lambda^\beta \lambda^\gamma = X_{\beta_1 \cdots \beta_{11}}^{\alpha \beta \gamma} d\lambda_{\beta_1} \wedge \cdots \wedge d\lambda_{\beta_{11}}$$

(3.121)

for some invariant tensor $X$. The number of independent invariant (3,11) tensors with spinor indices that are symmetric in the upper indices and antisymmetric in lower ones is one [40]. In other words there is only one possibility for $X$ which is $(\epsilon T)$, cf. (2.26). Because the LHS of (3.121) is zero when contracted with $\gamma^m_{\alpha \beta \gamma}$, the RHS should vanish too. It does because there are no scalars in $10 \otimes 16 \otimes \text{Asym}^{11} 16'$. Thus

$$\gamma^m_{\alpha \beta \gamma}(\epsilon T)^{\alpha \beta \gamma}_{\beta_1 \cdots \beta_{11}} = 0.$$ 

(3.122)

In equation (3.121) one is free to choose $\alpha \beta \gamma$. Different choices lead to different guises of the measure. In [21] it was shown all these are related to each other by a coordinate transformation in pure spinor space. A choice for $\alpha \beta \gamma$ that results in a convenient form of the measure is $\alpha \beta \gamma = +++$. This gives $[d\lambda]$ as

$$[d\lambda] = \frac{d\lambda^+ \wedge d\lambda_{12} \wedge \cdots \wedge d\lambda_{45}}{(\lambda^+)^3}.$$ 

(3.123)

The charge conservation property was used to conclude that $(\epsilon T)^{+++-}_{\beta_1 \cdots \beta_{11}}$ is only nonzero if $\beta_1, \cdots, \beta_{11} = +, b_1 b_2, b_3 b_4, \cdots, b_{19} b_{20}$. In the form (3.123) one explicitly sees factors of $\lambda^+$ in the denominator. These are the reason that the $Q_S$ variation of the PCO for $\lambda$, which is of the form $\lambda \delta(\lambda)$, does not vanish inside correlators as discussed in chapter 5.

3.4.2 Measure for the zero modes of $N^{mn}$

The $J_{\lambda^w}$ number anomaly and Lorentz invariance imply the measure for the zero modes of $N$ must be of the form

$$[dN]\lambda_{\alpha_1} \cdots \lambda_{\alpha_8} = X_{m_1 n_1 \cdots m_{10} n_{10}}^{\alpha_1 \cdots \alpha_8} dN_{m_1 n_1}^{m_{10} n_{10}} \cdots dN_{m_{10} n_{10}}^{m_{10} n_{10}} \wedge dJ.$$ 

(3.124)
There exists only one independent invariant tensor of this kind (cf. 3.2.3) and since (2.47) is an example:

\[
\left[dN\right]^{\lambda_1 \cdots \lambda_8} = R^{\alpha_1 \cdots \alpha_8}_{\alpha_1 \cdots \alpha_8} dN^{m_1 n_1} \wedge \cdots \wedge dN^{m_{10} n_{10}} \wedge dJ. \quad (3.125)
\]

A more explicit form of \([dN]\) is obtained by choosing all \(\alpha\)'s equal to +. The relevant gamma matrix components are

\[
\gamma^{++}_{\alpha_1 \cdots \alpha_5} = \epsilon_{\alpha_1 \cdots \alpha_5}, \quad \gamma^{a_1 \cdots a_5}_{++} = \epsilon^{a_1 \cdots a_5}, \quad (3.126)
\]

all other components of \(\gamma^{++}_{mnpqr}\) vanish. Using these one sees \([dN]\) can be written out as

\[
\left[dN\right]^{++}_{\lambda_1 \cdots \lambda_8} = \epsilon_{a_1 b_1 a_2 b_2 a_3 b_3 \cdots a_8 b_8} \epsilon^{a_1 b_1 a_2 b_2 a_3 b_3 \cdots a_8 b_8} dN^{a_1 b_1} \wedge \cdots \wedge dN^{a_{10} b_{10}} \wedge dJ = dN^{12} \wedge \cdots \wedge dN^{45} \wedge dJ = \lambda^{+11} d^{10} w^{ab} d w_+ \Rightarrow [dN] = (\lambda^+)^3 d w_+ d^{10} w^{ab}, \quad (3.127)
\]

where the gauge condition \(w_a = 0\) is imposed in the first equality of the second line.

## 3.5 Gamma matrix traceless projectors

In general the space of symmetric tensors forms an invariant subspace in tensor spaces that are direct products of a certain representation. For example the tensor space \(\otimes^k \mathbf{16}\) is given by the tensors \(T^{\alpha_1 \cdots \alpha_k}\). The subspace of symmetric tensors is given by \(T^{(\alpha_1 \cdots \alpha_k)}\). Since invariant subspaces are linear subspaces one can define a projection onto this subspace. In the case the space of symmetric tensors the projector is given by

\[
P^{\alpha_1 \cdots \alpha_k}_{\alpha_1' \cdots \alpha_k'} = \delta^{(\alpha_1'}_{\alpha_1} \cdots \delta^{\alpha_k')}_{\alpha_k}. \quad (3.128)
\]

Note \(P\) satisfies \(P^2 = 0\) and \(P\) is surjective. In the pure spinor formalism one is often interested in projections on the subspace of symmetric and gamma matrix traceless tensors, since the bilinear \(\lambda^\alpha \lambda^\beta\) has these properties. A tensor \(T^{\alpha_1 \cdots \alpha_k}\) is gamma matrix traceless when it satisfies

\[
T^{\alpha_1 \cdots \alpha_k} \gamma^m_{\alpha_i \alpha_j} = 0 \quad 1 \leq i, j \leq k \quad (3.129)
\]

for all choices of \(i\) and \(j\). Note this condition also defines a linear subspace. Also note that the above condition is preserved by Lorentz transformations. This is a consequence of the fact that \(\gamma^m_{\alpha \beta}\) is an invariant tensor. Hence gamma matrix traceless tensor form an invariant linear subspace in the space of all tensors \(T^{\alpha_1 \cdots \alpha_k}\). The explicit form of the projectors onto gamma matrix traceless tensors for arbitrary \(k\) will be specified in this section. The projection on symmetric tensors is already given in (3.128), therefore one only needs the projection of symmetric tensors onto gamma matrix traceless tensors.
Let us start with the case of three indices. Note that the required projector is an invariant tensor with three symmetrised upper spinor indices and three symmetrised lower spinor indices. The $SO(10)$ invariant tensors of the form $T^{(\alpha' \beta' \gamma')}_{(\alpha' \beta' \gamma')}$ form a vector space which is two dimensional as can be computed by counting the number of scalars in $\text{Sym}^3 \mathbf{16} \otimes \text{Sym}^3 \mathbf{16}'$ [40]. A basis of this vector space is given by

$$\{ \delta^{(\alpha' \beta' \delta' \gamma')}_{\alpha' \beta'}, \gamma_m^{(\alpha' \beta' \delta' \gamma')} \}. \quad (3.130)$$

Thus an arbitrary invariant tensor is given by

$$c_1 \delta^{(\alpha' \beta' \delta' \gamma')}_{\alpha' \beta'} + c_2 \gamma_m^{(\alpha' \beta' \delta' \gamma')} . \quad (3.131)$$

One can determine the coefficients up to an overall normalisation by imposing vanishing of the gamma trace:

$$0 = c_1 \gamma_m^{(\alpha' \beta' \delta' \gamma')} + c_2 \gamma_m^{(\alpha' \beta' \delta' \gamma')} \gamma_n^{(\alpha' \beta' \delta' \gamma')} \quad (3.132)$$

$$= (c_1 + 40c_2) \delta^{(\alpha' \beta' \delta' \gamma')}_{\alpha' \beta' \delta' \gamma'},$$

where the following identity was used (cf. (3.56))

$$\gamma_n^{(\alpha' \beta' \delta' \gamma')} = 2\delta_n^{(\alpha' \beta' \delta' \gamma')} . \quad (3.133)$$

One could have anticipated ending up with one equation for $c_1, c_2$ because $\mathbf{10} \otimes \mathbf{16} \otimes \text{Sym}^3 \mathbf{16}'$ contains one scalar. In conclusion the projector is given by

$$P_{\alpha' \beta' \gamma'}^{\alpha \beta \gamma} = \delta^{(\alpha' \beta' \delta' \gamma')}_{\alpha' \beta'} \equiv \delta^{(\alpha \beta \delta \gamma)}_{\alpha' \beta'} - \frac{1}{40} \gamma^{(\alpha \beta \delta \gamma)}_{(\alpha' \beta' \delta' \gamma')} . \quad (3.134)$$

In summary the number of scalars in $\text{Sym}^3 \mathbf{16} \otimes \text{Sym}^3 \mathbf{16}'$ determined the number of degrees of freedom ($c_1$) and the number of scalars in $\mathbf{10} \otimes \mathbf{16} \otimes \text{Sym}^3 \mathbf{16}'$ determined the number of relations between them.

### 3.5.1 Arbitrary rank

The tensor in equation (3.134) is unique because the number of scalars in $\text{Gam}^3 \mathbf{16} \otimes (\mathbf{16}')^3$ is one (cf. (3.58) for the meaning of Gam). In fact there is one scalar in $\text{Gam}^n \mathbf{16} \otimes (\mathbf{16}')^n$ for any $n$. In order to write an explicit expression for $\delta^{(\alpha_1 \ldots \alpha_n)}_{\beta_1 \ldots \beta_n}$ for any $n$ one looks for a basis of rank $(n, n)$ invariant tensors that are symmetric in both their upper and lower indices. For even $n$ the number of scalars in $\text{Gam}^n \mathbf{16} \otimes \text{Gam}^n \mathbf{16}'$ is $\frac{n}{2} + 1$. For odd $n$ the number of scalars in $\text{Gam}^n \mathbf{16} \otimes \text{Gam}^n \mathbf{16}'$ is $\frac{n - 1}{2} + 1$. Since odd $n$ is of more relevance to this work the basis for odd $n$ is explicitly given. The $\frac{n - 1}{2} + 1$ basis elements are given by

$$T_1 = \delta^{(\alpha_1 \ldots \alpha_n)}_{\beta_1 \ldots \beta_n}, \quad T_2 = \gamma_m^{(\alpha_1 \alpha_2 \ldots \alpha_n \beta_1 \beta_2 \ldots \beta_n)} \gamma_n^{(\alpha_1 \alpha_2 \beta_1 \beta_2 \ldots \beta_n)} . \quad (3.135)$$
up to

\[ T_{k+1} = \gamma_{m_1}^{\alpha_1 \alpha_2 m_1} \cdots \gamma_{m_k}^{\alpha_{n-2} \alpha_{n-1} m_k} \delta_{\alpha_n}^{\beta_{n-2} \beta_{n-1}} \]  

(3.136)

where \( k = \frac{n-1}{2} \). In order to see these tensors are independent compute the following components:

\[ T_{+ \cdots +}, T_{b_1 + \cdots +}, \ldots, T_{a_1 \cdots a_k + \cdots +} \]  

(3.137)

One can conclude

\[ \delta_{\beta_1}^{(\alpha_1 \cdots \alpha_n)} = c_1 T_1 + \cdots + c_k T_k, \]

(3.138)

for some coefficients \( c_i \), which can be explicitly computed as was done for the \( n = 3 \) case. Note the above is for odd \( n \). Even \( n \) works very much in the same way, the only difference is the last \( \delta \) in all the \( T \)'s. If one removes this, the \( T \)'s form a basis for the even case.

### 3.6 Chain of operators for \( b \) ghost

This section is only for reference purposes. It does not contain any results or derivations. The following chain of operators plays an important role in the \( b \) ghost:

\[
\begin{align*}
Q S G^\alpha &= \lambda^\alpha T, \\
Q S H^{\alpha \beta} &= \lambda^\alpha G^\beta + g^{(\alpha \beta)}, \\
Q S K^{\alpha \beta \gamma} &= \lambda^\alpha H^{\beta \gamma} + h_1^{((\alpha \beta)) \gamma} + h_2^{\alpha((\beta \gamma))}, \\
Q S L^{\alpha \beta \gamma \delta} &= \lambda^\alpha K^{\beta \gamma \delta} + h_1^{((\alpha \beta)) \gamma \delta} + k_2^{\alpha((\beta \gamma)) \delta} + k_3^{\alpha \beta((\gamma \delta))}, \\
0 &= \lambda^\alpha L^{\beta \gamma \delta \rho} + l_1^{((\alpha \beta)) \gamma \delta \rho} + l_2^{\alpha((\beta \gamma)) \delta \rho} + l_3^{\alpha \beta((\gamma \delta)) \rho} + l_4^{\alpha \beta \gamma((\delta \rho))}.
\end{align*}
\]

(3.139) \( \quad \) (3.140) \( \quad \) (3.141) \( \quad \) (3.142)

The last equation implies there exists an \( S^{\alpha \beta \gamma} \) such that

\[ L^{\alpha \beta \gamma \delta} = \lambda^\alpha S^{\beta \gamma \delta} + s_1^{((\alpha \beta)) \gamma \delta} + s_2^{\alpha((\beta \gamma)) \delta} + s_3^{\alpha \beta((\gamma \delta))}. \]

(3.143) \( \quad \) (3.144)
The text below is essentially a summary of section 3 of [44]. The primary fields of weight two that solve the above equations are given by

\[
G^\alpha = \frac{1}{2} \Pi^m (\gamma_m d)^\alpha - \frac{1}{4} N_{mn} (\gamma^{mn} \partial \theta)^\alpha - \frac{1}{4} J \partial \theta^\alpha + \frac{7}{2} \partial^2 \theta, \quad (3.145)
\]

\[
H^{\alpha \beta} = \frac{1}{16} \gamma^{\alpha \beta m} (N^{mn} \Pi_n - \frac{1}{2} J \Pi^m + 2 \partial \Pi^m)
\]

\[
+ \frac{1}{96} \gamma^{\alpha \beta \gamma mnp} (\frac{1}{4} d \gamma^{mnp} d + 6 N^{mn} \Pi^c), \quad (3.146)
\]

\[
K^{\alpha \beta \gamma} = -\frac{1}{48} \gamma^\alpha_m (\gamma_n d)^\gamma N^{mn} - \frac{1}{192} \gamma^{\alpha \beta}_{mnp} (\gamma^m d) \gamma N^{np}
\]

\[
+ \frac{1}{192} \gamma^m_m [ (\gamma_n d)^\alpha N^{mn} + \frac{3}{2} (\gamma^m d)^\alpha J + 6 (\gamma^m \partial d)^\alpha ]
\]

\[
- \frac{1}{192} \gamma^m_{mnp} (\gamma^m d)^\alpha N^{np}, \quad (3.147)
\]

\[
L^{[\alpha \beta \gamma \delta]} = -\frac{1}{3072} (\gamma_{mnp}[^{\alpha \beta} (\gamma^m q r) \gamma^\delta]) N^{np} N_{qr}. \quad (3.148)
\]

NB1: Only the antisymmetric part of \( L^{\alpha \beta \gamma \delta} \) is given because in [44] the full \( L^{\alpha \beta \gamma \delta} \) is not given in terms of gauge invariant objects. An explicit expression is known within the \( Y \) formalism [44, 45, 46] and it is also proved all \( Y \) dependence from \( L^{\alpha \beta \gamma \delta} \) disappears when contracted with \( Z^{\alpha \beta \gamma \delta} \). In [22] \( L^{\alpha \beta \gamma \delta} \) is given as

\[
L^{\alpha \beta \gamma \delta} = c_{4 mnpq} N^{mn} N^{pq} + c_{5 mn} \gamma^m \gamma^n + c_6 \gamma^m \gamma^n J + c_7 \gamma^m \gamma^n J + c_8 \gamma^m \gamma^n J, \quad (3.149)
\]

with unknown coefficients.

NB2: the coefficients of the total derivative terms depend on the normal ordering prescription and the ones above are only consistent with the prescription of [44].