Fundamentals of the pure spinor formalism
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Chapter 4

BRST quantisation of the pure spinor superstring

As mentioned throughout chapter 2 there are several unexplained aspects of the pure spinor formalism. These include

- the origin of the picture changing operators,
- the conformal weight constraint on the vertex operators,
- the $b$ ghost equation (2.34),
- the relation between integrated and unintegrated vertex operators (2.17).

In a theory derived from first principles, for example the bosonic string, the above aspects all follow from one starting point, namely

$$Z = \int \mathcal{D}g\mathcal{D}X \frac{1}{VolG} e^{-S_P}. \quad (4.1)$$

In addition to providing an explanation for the above aspects a first principles derivation of the pure spinor formalism could also help in the search of a simplified version. Furthermore in chapter 2 it was advertised that one can replace all fields by their zero modes in a correlator that only contains weight zero fields. This will also be proved in this chapter.

In this chapter a first principles derivation is provided. There have been many works in the past involving modifications and/or extensions of the pure spinor formalism with the same aim, see for example [47, 48, 49, 50, 23, 51, 52, 53]. The approach of this chapter is different and is guided by topological string constructions. Instead of searching for a model with a local symmetry which after gauge
fixing would lead to the pure spinor formalism with $Q_S$ and the pure spinors emerging as a BRST operator and ghost fields, the pure spinors $\lambda$ will be considered as “matter” fields as well and the worldsheet theory as a sigma model with a nilpotent symmetry $Q_S$ and target space the ten-dimensional superspace times the pure spinor space. To construct a string theory this theory will be coupled to two-dimensional gravity in a way that preserves the fermionic symmetry $Q_S$ and then BRST quantise the resulting theory in a conventional fashion.

4.1 Coupling to 2d gravity

To construct a string theory the pure spinor worldsheet action will be coupled to two-dimensional gravity in a way that preserves the $Q_S$ symmetry. Subsequently this system will be quantised using BRST methods. Since this model has zero central charge, one should couple it to topological gravity$^1$. This approach is thus similar to the construction of topological string theories, see [54] for a review. In that context one starts from a supersymmetric sigma model which upon topological twisting yields a topological sigma model. In this procedure one of the supersymmetry charges is identified with the BRST operator of the sigma model. The corresponding operator in the case at hand is the nilpotent operator $Q_S$. Note that the pure spinor sigma model has been obtained by twisting an $N = 2$ model in [23].

The first step in this procedure is thus to relax the conformal gauge in the action (2.1) (or (2.56) for the non-minimal version). The part that involves the $x^m$ is standard$^2$,

$$S_X = \int d^2\sigma \left( \frac{1}{4} \sqrt{-g} g^{ab} \partial_a x^m \partial_b x_m \right)$$

(4.2)

The rest of the action (2.1) (or (2.56) for the non-minimal version) is a sum of first order actions involving a field of dimension one and a field of dimension zero (with an overall sign that depends on whether the fields are bosonic or fermionic). The covariantisation of all these terms is the same, so it suffices to discuss one of them, say

$$S_{(p,\theta)} = \int d^2z p_\alpha \partial^\alpha \theta^\alpha.$$ 

(4.3)

The fields of dimension one are vectors on the worldsheet, so $p_\alpha$ is more accurately labeled as $p_{a\alpha}$. However, only the $z$-component participates in (4.3), as one can conclude by looking at the conformal weight of the various objects in (4.3). Similarly, only the $\bar{z}$ component of the right-moving momentum $\tilde{p}_{a\alpha}$ participates in the action.

$^1$By definition topological gravity does not change the central charge of the conformal field theory obtained after gauge fixing

$^2$The worldsheet has a Euclidean signature and the conventions are the same as in chapter 1, i.e. $z = \sigma^1 + i\sigma^2$, the flat metric is $g_{z\bar{z}} = 1/2$ etc.
To account for this, one can introduce projection operators

\[ P^{(\pm)}_a = \frac{1}{2}(\delta_a^b \mp iJ_a^b), \tag{4.4} \]

where \( J_a^b \) is the complex structure of the worldsheet, i.e. it satisfies

\[ J_a^b J_b^c = -\delta_c^a, \quad \nabla_c J_a^b = 0. \tag{4.5} \]

In terms of the worldsheet volume form and the worldsheet metric, it is given by

\[ J_a^b = -\epsilon_{ac} g_{cb}, \quad \epsilon_{ab} = \sqrt{\hat{g}} \hat{\epsilon}_{ab} \text{ and } \hat{\epsilon}_{01} = 1, \]

and holomorphic and anti-holomorphic functions on the worldsheet are defined by

\[ J_a^b \partial_b \phi = i \partial_a \phi \text{ and } J_a^b \partial_b \bar{\phi} = -i \partial_a \bar{\phi}, \]

respectively. Using (4.5) one shows that

\[ P^{(\pm)}_a P^{(\pm)}_b = P^{(\mp)}_c P^{(\mp)}_d = 0. \tag{4.6} \]

Notice also that

\[ g^{ab} P^{(\pm)}_b = g^{cb} P^{(\mp)}_a. \tag{4.7} \]

One can obtain vectors with only \( z \)-component by multiplying by \( P^{(+)}_a \) and vectors with only \( \bar{z} \)-component by multiplying by \( P^{(-)}_a \):

\[ \hat{p}_a = P^{(+)}_a p_b, \quad \hat{\bar{p}}_a = P^{(-)}_a \bar{p}_b. \tag{4.8} \]

In other words, the only nonzero component of \( P^{(+)}_a \) is \( P^{(+)}_z = 1 \) and the only nonzero component of \( P^{(-)}_a \) is \( P^{(-)}_{\bar{z}} = 1 \). More generally, these projection operators can be used to covariantise any tensor given in conformal gauge. The action (4.3) can then be covariantised as

\[ S_{(p,\theta)} = \int d^2 \sigma \sqrt{-g} g^{ab} \hat{p}_a \bar{\theta} \partial_b \theta^\alpha. \tag{4.9} \]

In summary the action of the minimal model coupled to gravity is given by

\[ S_{\sigma} = \int d^2 \sigma \sqrt{-g} \left( \frac{1}{4} \partial_a x^m \partial_b x^m + \hat{p}_a \partial_b \theta^\alpha - \bar{\hat{w}}_{\alpha \beta} \partial_b \lambda^\alpha \right) \tag{4.10} \]

with an obvious addition for the case of the non-minimal model. The stress energy tensor for the model can be obtained by varying w.r.t. the worldsheet metric,

\[ T_{ab} = \frac{2}{\sqrt{-g}} \frac{\delta S_\sigma}{\delta g^{ab}} = \frac{1}{2} (\partial_a x^m \partial_b x^m - \frac{1}{2} g_{ab} g^{cd} \partial_c x^m \partial_d x^m) \]

\[ + \ (p_{(a|\alpha}) \partial_b \theta^\alpha - \frac{1}{2} g_{ab} g^{cd} p_{ca} \partial_d \theta^\alpha) + T^{(\lambda \mu)}_{ab} \tag{4.11} \]

The contribution of the pure spinor part (and the non-minimal variables) is the same as the one for the \( (p,\theta) \) part with \( p \to w \) and \( \theta \to \lambda \) and an overall minus sign.
Chapter 4 - BRST quantisation of the pure spinor superstring

(with similar replacements for the non-minimal fields). This stress energy tensor is (manifestly) traceless and covariantly conserved, reflecting the fact that the action is invariant under diffeomorphisms and Weyl transforms,

$$\delta g_{ab} = \mathcal{L}_\epsilon g_{ab} + 2\phi(\sigma)g_{ab}$$  \hspace{1cm} (4.12)

$$\delta \Phi = -\epsilon^a \partial_a \Phi$$

$$\delta P_a = -\epsilon^a \partial_a P + \partial_a \epsilon^b P_b$$

where $\epsilon^a(\sigma), \phi(\sigma)$ are diffeomorphism and Weyl gauge parameters, $\mathcal{L}_\epsilon$ is the Lie derivative (cf. (1.65)), $\Phi = \{x^m, \theta^\alpha, \lambda^\alpha, \ldots\}$ collectively denotes all worldsheet scalars and $P_a = \{p_{a\alpha}, w_{a\alpha}, \ldots\}$ collectively denotes all worldsheet vectors.

The stress energy tensor (4.11) can be rewritten as

$$T_{ab} = P^{(+)}_a P^{(+)}_b T^{B}_{cd} + P^{(-)}_a p_{c\alpha} \left(P^{(-)}_b \partial_d \theta^\alpha\right) + \cdots$$  \hspace{1cm} (4.13)

where the ellipsis indicate the contribution from the pure spinor and non-minimal variables, which will be suppressed from now on since they are similar to the $(p, \theta)$ contribution. The anti-holomorphic contribution of $x^m$ is also suppressed. The first term in (4.13) is the covariantisation of the stress energy tensor appearing in Berkovits’ work,

$$T^{B}_{ab} = \frac{1}{2} \partial_a x_m \partial_b x^m + p_{a\alpha} \partial_b \theta^\alpha + \cdots$$  \hspace{1cm} (4.14)

while the second term is proportional to the $\theta^\alpha$ field equation. This additional term can be removed by modifying the transformation rule of $p_{a\alpha}$ in (4.12).

4.1.1 Topological gravity and $Q_S$ invariance

If one was to quantise the model just described one would find that it is anomalous, since the diffeomorphism ghosts would contribute $c = -26$ and the original sigma model had $c = 0$. This problem is avoided by extending the $Q_S$ symmetry to act on the worldsheet metric, so that the 2d gravity is topological. With this aim, the following transformation rule is introduced,

$$\delta g_{ab} = P^{(-)}_a P^{(-)}_b \psi_{cd} \equiv \hat{\psi}_{ab}, \quad \delta S \hat{\psi}_{cd} = 0.$$  \hspace{1cm} (4.15)

where $\psi_{ab}$ is a new field that has only one holomorphic component, $\psi_{zz}(z)$. (To extend this discussion to the anti-holomorphic sector one would also need to turn on $\tilde{\psi}_{zz}(\bar{z})$, i.e. the full transformation is $\delta_S g_{ab} = P^{(-)}_a P^{(-)}_b \psi_{cd} + P^{(+)}_a P^{(+)}_b \tilde{\psi}_{cd}$.)

Since the metric now transforms, the action is not invariant anymore and its $Q_S$ variation yields,

$$\delta_S S_\sigma = -\frac{1}{2} \int d^2 \sigma \sqrt{g} T^{ab} \delta_S g_{ab} = -\frac{1}{2} \int d^2 \sigma \sqrt{g} g^{ac} g^{bd} T^{B}_{ab} \hat{\psi}_{cd},$$  \hspace{1cm} (4.16)
where again only the holomorphic sector is discussed, and the second equality makes use of the fact that due to the projector operators the second term in (4.13) does not contribute. To construct an invariant action one now has to add a new term to the action,

$$S_\sigma \to S = S_\sigma + \frac{1}{2} \int d^2 \sigma \sqrt{g} g^{ac} g^{bd} G_{ab} \hat{\psi}_{cd}$$  \hspace{1cm} (4.17)$$

The new action is invariant under the condition there exists a $G_{ab}$ transforming as

$$\delta_S G_{ab} = T^B_{ab}.$$  \hspace{1cm} (4.18)$$

Note that because $\hat{\psi}_{ab}$ has only one fermionic component, the variation of the explicit worldsheet metrics in the new term does not contribute. Including both sectors one finds that for the discussion to go through $G_{ab}$ must be traceless. In conformal gauge and complex coordinates the (holomorphic part of) equation (4.18) becomes

$$T^B(z) = \{Q_S, G(z)\}.$$  \hspace{1cm} (4.19)$$

The $G$ currents generate a fermionic symmetry of the action in conformal gauge. In the language of [54] equation (4.19) defines the pure spinor action in conformal gauge to be a topological conformal theory.

Equation (4.18) for $G_{ab}$ is the equation for a composite “b-field”, cf. (2.34). Such a composite field has been constructed in conformal gauge in the non-minimal formalism. In the minimal case it was more difficult to solve equation (4.18). A detailed account of its solution will be given in section 4.4. Once the conformal gauge solution to (4.18) has been found, it can be covariantised to obtain a $Q_S$, diffeomorphism and Weyl invariant action.

### 4.2 Adding vertex operators

The vertex operators should be invariant under the symmetries of the theory, in this case: diffeomorphisms, Weyl transformations, $Q_S$ transformations and the transformations generated by $G_{ab}$. In order to preserve the $Q_S$ symmetry the vertex operators depend, in addition to the worldsheet coordinate $\sigma^a_i$, on its $Q_S$ partner $\zeta^a_i$,

$$\delta_S \sigma^a_i = \zeta^a_i, \quad \delta_S \zeta^a_i = 0,$$  \hspace{1cm} (4.20)$$

or in complex coordinates,

$$\delta_S z_i = \zeta_i, \quad \delta_S \bar{z}_i = \bar{\zeta}_i, \quad \delta_S \zeta_i = 0, \quad \delta_S \bar{\zeta}_i = 0.$$  \hspace{1cm} (4.21)$$

\footnote{It is not possible to choose $\delta_S \sigma^a_i = 0$, since the $\zeta$’s are needed to fix the residual gauge invariance of the symmetry generated by $G_{ab}$.}
Since $\zeta_i$ is a fermionic variable the $i$th vertex operator $V_i$ has the expansion (in complex basis)

$$V_i[\varphi](z_i, \zeta_i) = V_i^{(0)}[\varphi](z_i) + \zeta_i V_i^{(1)}[\varphi](z_i), \quad (4.22)$$

where only the holomorphic part of the vertex operator is given. The symmetry generated by $Q_S$ poses further constraints on the vertex operators:

$$\delta_S (V_i[\varphi](z_i, \zeta_i)) = 0. \quad (4.23)$$

The $Q_S$ transformation can act either on worldsheet fields $\varphi$ or on the positions $z_i$ and we obtain

$$\delta_S V_i[\varphi](z_i, \zeta_i) = (\delta_S V_i^{(0)})(z_i) + \zeta_i \left( \partial V_i^{(0)}(z_i) - (\delta_S V_i^{(1)})(z_i) \right) \quad (4.24)$$

which implies

$$\delta_S V_i^{(0)} = 0, \quad \delta_S V_i^{(1)} = \partial V_i^{(0)}, \quad (4.25)$$

where now $Q_S$ acts only on the fields. The equality is exactly the relation between integrated and unintegrated vertex operators in the pure spinor formalism postulated in (2.17). Moreover from (4.25) one finds that the integrated vertex operator

$$U_i = \int dz V_i^{(1)} \quad (4.26)$$

is $Q_S$ invariant.

The second transformation in (4.25) can be rewritten in a form that is useful to determine how $G$ acts on the superfield components

$$\delta_S V_i^{(1)} = \delta_S \{ G, V_i^{(0)} \}. \quad (4.27)$$

The partial derivative in (4.25) is generated by $T$ and this can be replaced by a $G$ transformation followed by a $Q_S$ transformation. The $G$ transformations of the components are given by

$$\{ G, V_i^{(0)} \} = V_i^{(1)}, \quad [ G, V_i^{(1)} ] = 0. \quad (4.28)$$

Hence in order to construct a vertex operator invariant under the $G$ symmetry one has to integrate over $\zeta$:

$$\int d\zeta_i V_i[\varphi](z_i, \zeta_i). \quad (4.29)$$

Finally invariance under diffeomorphisms is achieved in the same way as in the bosonic string (cf. (1.78)), namely by integrating over the worldsheet coordinate $z_i$. 

90
4.3 BRST quantisation

The action $S$ in equation (4.17) constructed in the previous section is invariant under diffeomorphisms and local Weyl transformations. This theory can be quantised using the BRST methods developed in chapter 1. Recall that BRST quantisation amounts to adding a term $Q_V \Psi$ to the action, where $\Psi$ is the gauge fixing fermion and $Q_V$ is the BRST operator (cf. (1.51)). However in the case at hand there is a second nilpotent fermionic symmetry, generated by $Q_S$. In order to preserve both symmetries the gauge fixing term is of the following type

$$S \rightarrow S + \delta_V \delta_S \Psi. \quad (4.30)$$

In fact the order of the symmetries does not matter, since as shown in [54] $Q_V$ and $Q_S$ anticommute:

$$\{Q_V, Q_S\} = 0. \quad (4.31)$$

A proper gauge fixing condition for the group of diffeomorphism and Weyl transformation has been discussed at length in section 1.2.1 and the $Q_S$ variation in (4.30) ensures that the $G$ symmetry is also gauge fixed. The gauge fixing fermion has two terms in general, one that involves the metric ($L_1$) and one that involves vertex operators positions ($L_2$). The latter is only necessary on the sphere and on the torus, since only in those cases $L_1$ leaves residual gauge invariance, which can be fixed by imposing a condition on the vertex operator positions. The two gauge fixing terms are given by

$$L_1 = \delta_V \delta_S (\hat{\beta}^{ab} [g_{ab} - \hat{g}_{ab}(\tau)]), \quad L_2 = \delta_V \delta_S \left( \sum_{j=1}^{\kappa/2} \hat{\beta}^a_j (\sigma^a_\tau - \hat{\sigma}^a_\tau) \right), \quad (4.32)$$

where $\kappa$ is the number of conformal killing vectors, $\hat{g}$ is the reference metric and $\hat{\sigma}$ are some chosen worldsheet positions. The $\beta$’s are bosonic fields which can be concluded from the fact that the $L$’s must be bosonic. Furthermore $\hat{\beta}^{ab}$ is a tensor density such that $L_1$ is coordinate invariant. The object $\hat{\beta}^a_j$ does not depend on the worldsheet coordinates, it is similar to the $B^i_a$’s from gauge fixing the residual gauge invariance in the bosonic string (cf.(1.80)), the only difference is the statistics.

The next step is performing the $Q_S$ and $Q_V$ transformations in the gauge fixing terms (4.32). To this end it is useful to have an overview of the transformations. The diffeomorphism and Weyl ghosts, $c^a$ and $C_\omega$ have $Q_S$ partners,

$$\delta_S c^a = \gamma^a, \quad \delta_S C_\omega = \gamma_\omega, \quad (4.33)$$

which are bosonic BRST ghosts for the fermionic symmetry generated by $G$. Note that due to the nilpotency of both charges and the fact that they anticommute, the fields will appear in quartets. These quartets are given in figure 4.1.
Using the transformations in figure 4.1 one can process the gauge fixing terms in (4.32). Let us start with $L_1$:

$$L_1 = \delta_V (\tilde{b}^{ab}_{\hat{a}}[g_{ab} - \hat{g}_{ab}(\tau)] + \tilde{\beta}^{ab}_{\hat{a}}[\hat{\psi}_{ab} - \hat{\tau}^k \partial \hat{g}_{ab}(\tau)])$$

$$= B^{ab}_{\hat{a}}[g_{ab} - \hat{g}_{ab}(\tau)] - \tilde{b}^{ab}_{\hat{a}}[2C_\omega g_{ab} + \mathcal{L}_c g_{ab} - \xi^k \partial \hat{k} \hat{g}_{ab}(\tau)] - p^{ab}_{\hat{a}}[\hat{\psi}_{ab} - \hat{\tau}^k \partial \hat{k} \hat{g}_{ab}(\tau)]$$

$$+ \tilde{\beta}^{ab}_{\hat{a}}[\mathcal{L}_c \hat{\psi}_{ab} - \mathcal{L}_c \gamma ab] - 2\gamma \omega g_{ab} + 2C_\omega \hat{\psi}_{ab} ,$$

where $\partial \hat{k} \hat{g}_{ab}(\tau) = \partial \hat{g}_{ab}(\tau)/\partial \tau^k$ is a derivative of the reference metric w.r.t. the moduli and $\hat{\psi}_{ab}$ is defined in (4.15). This gauge fixing action contains the usual gauge fixing terms for the metric and the ghost actions for $b, c$ and $\tilde{\beta}, \gamma$. The gauge fixing term for the residual gauge invariance can also be processed:

$$L_2 = \delta_V \left( \sum_{j=1}^{\kappa/2} \beta^{a}_{\hat{a}}(\sigma^a_j - \hat{\sigma}^a_j) + \tilde{\beta}^{a}_{\hat{a}}(\hat{\sigma}^a_j) \right)$$

$$= \sum_{j=1}^{\kappa/2} B^{a}_{\hat{a}}(\sigma^a_j - \hat{\sigma}^a_j) - \tilde{b}^{a}_{\hat{a}}c^a(\sigma^a_j) - p^{a}_{\hat{a}}\hat{\sigma}^a_j + \tilde{\beta}^{a}_{\hat{a}}\gamma^a(\sigma^a_j) .$$

At this point all gauge symmetries have been treated, except the ones associated with zero modes of the original fields $X, p, \theta, w, \lambda$. These will be discussed in the next section.

To summarise, a general scattering amplitude is given by

$$Z = \int d\mu \prod_{i=1}^{N} V_i[\varphi](\sigma_i, \zeta_i) \exp (-S - L_1 - L_2) ,$$

where $\varphi$ is a superfield and $S, L_1, L_2$ are the action, the gauge fixing term, and the residual gauge fixing term, respectively.
where $S, L_1$ and $L_2$ are given in (4.17), (4.34) and (4.35), $d\mu_\sigma$ is the measure factor associated with $X, p, \theta, w, \lambda$ (and non-minimal variables) that will be discussed in the next section and $d\mu$ is the measure that follows from the analysis of this section, i.e.

$$d\mu = \prod_i d^2\sigma_i \sqrt{g(\sigma_i)} d^2\zeta_i \prod_{k=1}^{\mu} d\tau^k d\xi^k d\zeta^k \prod_{j=1}^{\kappa/2} db^\jmath ab dp^j_a d\beta^\jmath ab dB^j_a \times \mathcal{D}\psi_{ab} \mathcal{D}g_{ab} \mathcal{D}c^a \mathcal{D}\gamma^a \mathcal{D}C_\omega \mathcal{D}\gamma_\omega \mathcal{D}p^{ab} \mathcal{D}\beta^{ab} \mathcal{D}B^{ab} \mathcal{D}\tilde{h}^{ab} \tag{4.37}$$

The first line contains the integration over all constant “fields” while the fields in the second line are functionally integrated over. The integration over most of these variables can be done exactly.

As in previous sections only the holomorphic sector is discussed. Firstly, integrating over $B^{ab}$ and $g_{ab}$ sets the worldsheet metric equal to the reference metric $\hat{g}_{ab}$ in all expressions. Integrating over $B^\jmath ab_a, p^j_a$, leads to delta functions $\delta(z_j - \hat{z}_j) \delta(\zeta_j)$ which can be used to integrate over $z_j, \zeta_j$. So $\kappa/2$ insertions will involve $V^{(0)}_i(\hat{z}_j)$ while the remaining $(N - \kappa/2)$ vertex operators will involve $V^{(1)}_i(z_i)$ and will be integrated. Furthermore integrating out $b^\jmath, b^\G$ leads to the insertion $c(\hat{z}_j) \delta(\gamma(\hat{z}_j))$.

Note that the $V^{(0)}_i$ and $V^{(1)}_i$ do not depend on the ghost fields\footnote{The ghost fields are consistently denoted as $c/\gamma$ and lowercase $b/\beta$. In this case there are two $b/\beta$ ghosts and four $c/\gamma$ ghosts.}, so the path integral factorises into a part that only depends on the ghosts and the rest. One might anticipate that the ghost contributions will cancel each other since $c^a, C_\omega$ and the $\gamma^a, \gamma_\omega$ are related by the $Q_S$ symmetry. So to simplify the presentation the ghosts are set to zero. The complete computation including the ghosts is given in section 4.5. The scattering amplitudes thus take the form

$$\langle V_1 \cdots V_n \rangle = \int d\mu_\sigma e^{-S_\sigma} d\tilde{\mu} e^{-\tilde{S}} \prod_{j=1}^{\kappa/2} V^{(0)}_j(\hat{z}_j) \prod_{i=\kappa/2+1}^N dz_i V^{(1)}_i(z_i), \tag{4.38}$$

where

$$d\tilde{\mu} e^{-\tilde{S}} = \prod_{k=1}^\mu d\tau^k d\bar{\tau}^k D\psi_{ab} Dp^{ab} \exp \left( \int d^2\sigma (\sqrt{\hat{g}}^{-1} G^{ab}\hat{\psi}_{ab} + p^{ab}[\hat{\psi}_{ab} - \hat{\tau}^k \partial_k \hat{g}_{ab}(\tau)]) \right) \tag{4.39}$$

Integrating out $p^{ab}$ gives a delta function that sets $\hat{\psi}_{ab} = \hat{\tau}^k \partial_k \hat{g}_{ab}(\tau)$. Finally integrating out $\hat{\tau}^k$ leads to $(6g - 6)$ (of which $(3g - 3)$ are holomorphic) insertions of $G^{ab}$,

$$\langle V_1 \cdots V_n \rangle = \int d\mu_\sigma e^{-S_\sigma} \prod_{k} d\tau^k (G, \partial_k \hat{g}) \prod_{j=1}^{\kappa/2} V^{(0)}_j(\hat{z}_j) \prod_{i=\kappa/2+1}^N dz_i V^{(1)}_i(z_i) \tag{4.40}$$
where \((G, \partial_k \hat{g}) = \int_\Sigma d^2 \sigma \sqrt{\hat{g}} G^{ab} \partial_k \hat{g}_{ab}\).

### 4.3.1 Summary

Let us summarise the results so far. The starting point was a theory with a fermionic nilpotent symmetry \(Q_S\) and zero central charge. This theory was coupled to topological gravity in a way that preserves the \(Q_S\) symmetry. Quantising this system using BRST methods leads to the formula (4.40) for the scattering amplitudes. In this formula the position of \(\kappa/2\) of the vertex operators \(V_i^{(0)}\) is fixed while the remaining ones, \(V_i^{(1)}\), are integrated. These vertex operators satisfy (in the holomorphic sector),

\[
\delta S V_i^{(0)} = 0, \quad \delta S V_i^{(1)} = \partial V_i^{(0)}.
\]

Furthermore, one needs \((6g - 6)\) insertions ((3g - 3) holomorphic ones) of the field \(G_{ab}\) defined by

\[
\delta S G_{ab} = T_{ab}
\]

where \(T_{ab}\) is the stress energy tensor of the worldsheets theory. This composite field is the analogue of the \(b\) ghost in the scattering prescription of bosonic string theory. One may have anticipated these results based on the scattering amplitude prescription for the bosonic string and studies of topological strings. Indeed this is precisely the prescription used in the literature. The novelty here is its derivation from a first principles BRST quantisation. Notice that these results hold irrespectively of what the original sigma model is.

### 4.4 Pure spinor measure

Let us now return to the pure spinor sigma model. Two aspects deserve further attention. The first is finding an explicit form of the current \(G_{ab}\). The second is determining whether the sigma model path integral measure \(d\mu_\sigma\) contains gauge directions, i.e. whether evaluating the functional integral would lead to divergences. Let us start with the second one.

It turns out that there are gauge directions in \(d\mu_\sigma\) and they are given by the zero modes of the sigma model (or matter) fields. The zero modes are gauge directions because by definition a zero mode is annihilated by the kinetic operator in the action and therefore zero modes do not appear in the action. For fermionic zero modes this does not present a problem; the vertex operators can provide the appropriate number of fermionic zero modes so that the final expressions are non-vanishing. Non-compact bosonic zero modes however are a problem, even in the presence of vertex operators. The action \(S_\sigma\) does not contain a convergence factor because of the zero mode gauge invariance. This can be remedied by gauge fixing the bosonic zero
mode gauge invariances, as is discussed in this section. Due to the $Q_S$ invariance, part of the invariance related to the fermionic zero modes is also fixed.

On a genus $g$ surface, a worldsheet scalar $\Phi$ has one zero mode $\Phi_0$ and a worldsheet vector $P$ has $g$ zero modes, $P_0(z) = \sum_{I=1}^g P^I I(z)$, where $\omega_I(z)$ are the $g$ holomorphic Abelian differentials of first kind satisfying $\int_{\mathcal{A}_I} dz \omega_J = \delta_{IJ}$ and the contour integral is around the $g$ non-trivial A-cycles of a genus $g$ surface. Note that $\Phi_0$ and $P_0$ are constants. In the minimal pure spinor formalism there are ten zero modes $x^m_0$, sixteen zero modes $\theta^\alpha_0$ and eleven zero modes $\lambda^\alpha_0$ from the worldsheet scalars and $16g$ zero modes $d^I_{\alpha}, I = 1, \ldots, g$, and $11g$ zero modes $w^I_{\alpha}$ from the worldsheet vectors. Of these $x^m_0, \lambda^\alpha_0$ and $w^I_{\alpha}$ are bosonic. The treatment of the zero modes of $x^m$ is standard and will not be discussed here. Furthermore $w_{\alpha}$, which transforms under the gauge transformation (2.4), will be traded for the gauge invariant variables,

$$N_{mn} = \frac{1}{2} w_{\alpha} (\gamma_{mn})^{\alpha \beta} \lambda^\beta, \quad J = w_{\alpha} \lambda^\alpha,$$

where $N_{mn}$ is the (contribution of the pure spinors to the) Lorentz current and $J$ is the ghost generator. As discussed in [27], the pure spinor condition implies enough relations between $N_{mn}$ and $J$ so that one can express the eleven independent components of $w_{\alpha}$ in terms of $J$ and ten component of $N_{mn}$. In what follows the $11g$ zero modes of $N_{mn}, J$ will be denoted by $N^I_{mn}, J^I$.

The zero mode gauge invariances cause divergences in the functional integral. Hence one can apply BRST quantisation to obtain a finite result. The BRST transformations corresponding to the zero mode gauge invariance are given by

$$\delta V \lambda^\alpha_0 = c^\alpha, \quad \delta V \theta^\alpha_0 = \gamma^\alpha, \quad \delta V d^I_{\alpha} = \gamma^I_{\alpha}, \quad \delta V w^I_{\alpha} = c^I_{\alpha},$$

where $c^\alpha, c^I_{\alpha}$ are constant fermionic ghosts and $\gamma^\alpha, \gamma^I_{\alpha}$ are constant bosonic ghosts. The transformations for $\lambda^\alpha_0, w^I_{\alpha}$ require some explanation, since $\lambda^\alpha$ satisfy a quadratic constraint and $w_{\alpha}$ has a gauge invariance. These zero modes are most easily described in $U(5)$ variables since the system in terms of $\lambda^+, \lambda^{ab}, w_+, w_{ab}$ is unconstrained and has no gauge invariance (see section 3.3.2). The BRST transformation is then given by shifting these variables by their zero modes. Reversing the steps in section 3.3.2 one may express $c^\alpha$ in terms of the eleven zero modes of $\lambda^+, \lambda^{ab}$ and $c^I_{\alpha}$ in terms of the $11g$ zero modes of $w_+, w_{ab}$. The arbitrariness due to the gauge invariance (2.4) is then eliminated by passing to the gauge invariant variables $N^I_{mn}, J^I$.

To maintain $Q_S$ invariance one must further require

$$\delta S \gamma^\alpha = c^\alpha, \quad \delta S c^I_{\alpha} = \gamma^I_{\alpha}.$$ 

\[\text{In the language of section 4.6, which contains a detailed account of what a zero mode really is, this Abelian differential is a realisation of $G_{0\alpha}$.} \]
To gauge fix the bosonic invariances one needs constant fermionic and bosonic ghost fields, \( b_\alpha, \tilde{b}_\alpha \) each containing eleven independent components, \( b^{mnI}, \tilde{b}^{mnI} \), each containing 10g independent components and \( b^I, \tilde{b}^I \), each containing g components and corresponding auxiliary fields. The \( Q_V \) and \( Q_S \) transformations of these fields are given in figure 4.2.

\[
\begin{align*}
  b_\alpha & \xrightarrow{V} \pi_\alpha \\
  b^{mnI} & \xrightarrow{V} \pi^{mnI} \\
  b^I & \xrightarrow{V} \pi^I \\
  \tilde{b}_\alpha & \xrightarrow{-V} \tilde{\pi}_\alpha \\
  \tilde{b}^{mnI} & \xrightarrow{-V} \tilde{\pi}^{mnI} \\
  \tilde{b}^I & \xrightarrow{-V} \tilde{\pi}^I \\

\text{Figure 4.2: } Q_V \text{ and } Q_S \text{ transformations of the auxiliary and } b \text{ ghost fields that play a role in gauge fixing zero mode invariances.}
\end{align*}
\]

The gauge fixing of the zero mode gauge invariances can be performed by introducing the following gauge fixing Lagrangian:

\[
L_3 = \delta_V \delta_S \left( b_\alpha \theta_0^\alpha + \sum_{l=1}^{g} (b^{mnI} N_{mn}^I + b^I J^I) \right) \tag{4.46}
\]

\[
= \delta_V \left( -b_\alpha \lambda_0^\alpha + \tilde{b}_\alpha \theta_0^\alpha + \sum_{l=1}^{g} \left( \frac{1}{2} b^{mnI} (d^l \gamma_{mn} \lambda_0) + \tilde{b}^{mnI} N_{mn}^I + b^I (d^l \lambda_0) + \tilde{b}^I (w^l \lambda_0) \right) \right)
\]

\[
= -\pi_\alpha \lambda_0^\alpha - \tilde{\pi}_\alpha \theta_0^\alpha + \sum_{l=1}^{g} \left( \pi^{mnI} \frac{1}{2} d^l \gamma_{mn} \lambda_0 - \pi^{mnI} N_{mn}^I + \pi^I d_\alpha^\alpha \lambda_0^\alpha - \pi^I J^I \right)
\]

\[
+ b^\alpha c^\alpha + \tilde{b}^\alpha \gamma^\alpha + \sum_{l=1}^{g} \left( \frac{1}{2} b^{mnI} (\gamma^l \gamma_{mn} \lambda_0 - d^l \gamma_{mn} c) - \frac{1}{2} \tilde{b}^{mnI} (c^l \gamma_{mn} \lambda_0 - w^l \gamma_{mn} c) \right)
\]

\[
+ b^I (\gamma^I \lambda_0 - d^l \gamma^I c) - \tilde{b}^I (\pi^I \lambda_0 - w^l c) \right).
\]

Integrating over \( b^\alpha \) and \( \tilde{b}^\alpha \) leads to delta functions for \( c^\alpha \) and \( \gamma^\alpha \), which can be used to integrate out \( c^\alpha, \gamma^\alpha \). This sets \( c^\alpha \) and \( \gamma^\alpha \) to zero, in particular four of the eight terms in the sum in the last two lines of (4.46) disappear. Both the bosonic variables \( b^{mnI}, b^I, \gamma^I \) and the fermionic variables \( \tilde{b}^{mnI}, \tilde{b}^I, c^I_\alpha \) only appear in the sum in the last lines of (4.46). Integration over these six variables leads to a factor of one, because the integration over the bosonic variables leads to a factor that is the inverse of the integral over the fermionic variables. More explicitly

\[
\int [db^I][d\gamma^I] e^{\sum_l b^I \gamma^I \lambda_0} = \left( \int [d\tilde{b}^I][dc^I] e^{\sum_l \tilde{b}^I c^I \lambda_0} \right)^{-1}, \tag{4.47}
\]

96
where the integration over $b^{mnI}$ is suppressed to avoid cluttering of the equation. So the zero mode measure now becomes

$$[d\mu_{\sigma}]_{z.m.} = [d^{16}\theta_0][d^{11}\pi_0][d^{11}\pi] \prod_{I=1}^g [d^{11}\pi_I][d^{11}\bar{\pi}_I][d^{11}N_I] \times (4.48)$$

$$\exp \left( -\pi_\alpha \lambda_0^\alpha - \bar{\pi}_\alpha \theta_0^\alpha + \sum_{I=1}^g \left( \pi^{mnI} \frac{1}{2} d^I \gamma_{mn} \lambda_0 - \bar{\pi}^{mnI} N_{mn}^I + \pi^I d_\alpha \lambda_0^\alpha - \bar{\pi}^I J^I \right) \right) ,$$

where $[d^{11}\lambda_0]$ and $\prod_I [d^{11}N_I]$ are the Lorentz invariant zero mode integration measures discussed in section 3.4, whose explicit form is not needed here. The auxiliary fields $\pi$ seem to have too many components. For instance $\pi_\alpha$ has sixteen components whereas only eleven are needed to gauge fix the zero modes of $\lambda$. Similarly $\pi_{mn}$ has 45 components while only ten are needed for the gauge fixing. This paradox can be resolved by realising that the exponent in (4.48) is invariant under a number of symmetries that render the “unwanted” components of $\pi$ pure gauge. The symmetry for $\pi_\alpha$ is similar to the gauge invariance for $w_\alpha$ (cf. (2.4)):

$$\delta \pi_\alpha = f^m (\gamma_m \lambda)_\alpha .$$

This can be used to remove five components of $\pi_\alpha$ and since $\bar{\pi}_\alpha = Q_S \pi_\alpha$ this propagates to $\bar{\pi}_\alpha$. The symmetry for the higher loop auxiliary fields is given by

$$\delta \pi^I_{mn} = (\lambda \gamma_{[m}) f^I_{\alpha n]} , \quad \delta \pi^I = - (\lambda \gamma^n f^I_n)$$

$$\delta \bar{\pi}^I_{mn} = (\lambda \gamma_{[m}) \bar{f}^I_{\alpha n]} , \quad \delta \bar{\pi}^I = - (\lambda \gamma^n \bar{f}^I_n)$$

This symmetry can be used to eliminate 35 out of the 45 components of each $\pi^I_{mn}$ and $\bar{\pi}^I_{mn}$, which is as expected since the number of BRST auxiliary fields should be equal to the number of gauge fixing conditions.

The next step is actually integrating out $\pi, \bar{\pi}, \pi^I, \bar{\pi}^I$. This can be done in multiple ways, one leads to the minimal formalism and another to the non-minimal formalism.

### 4.4.1 Minimal formulation

The fields $\pi_\alpha$ and $\bar{\pi}_\alpha$ have eleven independent components each. One way to parametrise them is to write

$$\pi_\alpha = p_i C^i_\alpha , \quad \bar{\pi}_\alpha = \bar{p}_i C^i_\alpha , \quad i = 1, \ldots , 11$$

where $p_i, \bar{p}_i$ are the independent components and $C^i_\alpha$ is a constant matrix of rank eleven. Then $[d^{11}\pi][d^{11}\bar{\pi}] = \prod_i d p_i d \bar{p}_i$ and integrating over $p^i$ yields $\prod_i \delta(C^i_\alpha \lambda_0^\alpha)$, while integrating over $\bar{p}^i$ yields $\prod_i C^i_\alpha \theta_0^\alpha$. Putting it differently, one may have started with ghosts and auxiliary fields $b^i, \bar{b}^i, p^i, \bar{p}^i$ and gauge fixing condition $C^i_\alpha \lambda_0^\alpha = 0,$
for the invariance due to the eleven zero modes of $\lambda^\alpha$ and gauge fixing condition $C_\alpha^I_0 \theta^0_0 = 0$ for the invariance due to eleven of the sixteen zero modes of $\theta$. Note that the insertions can be combined into eleven insertions of the picture lowering operator

$$Y_C = C_\alpha \theta^0_0 \delta(C_\alpha \lambda^\alpha_0).$$

(4.53)

Similarly, one may parametrise the $10g$ independent components of $\pi^{mnI}$ and of $\tilde{\pi}^{mnI}$ as

$$\pi^{mnI} = p^I_{jI} B^{mn}_{jI}, \quad \tilde{\pi}^{mnI} = \tilde{p}^I_{jI} B^{mn}_{jI}, \quad j = 1, \ldots, 10$$

(4.54)

where $p^I_{jI}, \tilde{p}^I_{jI}$ are the $10g$ independent components and $B^{mn}_{jI}$ are constants. Integrating over $p^I_{jI}, \tilde{p}^I_{jI}$ and $\pi^I, \tilde{\pi}^I$ leads to the insertions

$$\prod_{I=1}^{10} \left( (d^I_{\alpha} \lambda^\alpha_0) \delta(J^I) \prod_{j=1}^{10} \frac{1}{2} B^{mn}_{jI} (d^I_{\gamma_{mn}} \lambda_0) \delta(B^{mn}_{jI} N^I_{mn}) \right) = \prod_{R=1}^{g} Z_J(z_R) \prod_{P=1}^{10g} Z_{B^P}(w_p),$$

(4.55)

where the insertions have been reassembled in terms of the picture raising operators, $Z_B = \frac{1}{2} B^{mnI} d^I_{\gamma_{mn}} \lambda_0 \delta(B^{mnI} N^I_{mn})$, $Z_J = (\lambda^\alpha_0 d^I_{\alpha}) \delta(J^I)$, (4.56)

inserted at positions $z_R, w_R$. These insertions correspond to gauge fixing conditions $B^{mn}_{jI} N^I_{mn} = 0$, $J^I = 0$, for the gauge invariance due to the $11g$ $w_\alpha$ zero modes and $B^{mn}_{jI}(d^I_{\gamma_{mn}} \lambda_0) = 0$, $d^I_{\alpha} \lambda^\alpha_0 = 0$ for the gauge invariance due to $11g$ of the $16g$ zero modes of $d_\alpha$. Note that the constants $C^I_\alpha, B^{mn}_{jI}$ enter through a gauge fixing term and there is a formal argument, presented below (1.51), that says physical predictions do not depend on the gauge fixing term and therefore not on $B$ and $C$. However decoupling of $Q_S$ exact states in the pure spinor formalism is a non trivial subject which is discussed in the next chapter. The precise statements about Lorentz invariance and dependence on $B$ and $C$ are specified there.

What is left is to discuss $G_{ab}$. By definition, $G_{ab}$ should satisfy (now in complex coordinates and dropping the indices)

$$\delta_S G = T, \quad T = \frac{1}{2} \Pi^m \Pi_m + d_\alpha \partial \theta^\alpha - w_\alpha \partial \lambda^\alpha.$$

(4.57)

Since $\delta_S$ is nilpotent, this equation defines a cohomology class $[G]$, i.e. solutions $G$ up to $\delta_S$ exact terms. A solution of (4.57) is given by [55]

$$G_0 = \frac{C^I_\alpha G^\alpha_0}{C^I_\alpha \lambda^\alpha_0}, \quad G^\alpha = \frac{1}{2} \Pi^m (\gamma_m d)^\alpha - \frac{1}{4} N_{mn} (\gamma_{mn} \partial \theta)^\alpha - \frac{1}{4} J \partial \theta^\alpha - \frac{1}{4} \partial^2 \theta^\alpha,$$

(4.58)

for a constant spinor $C_\alpha$. This expression also appeared in [23] as a twisted worldsheet supersymmetry current. This solution is however not acceptable because it
contains a factor of \((C_\alpha \lambda^\alpha)^{-1}\). Allowing such operators renders the \(Q_S\) cohomology trivial. Indeed, consider the field \(\xi\)

\[
\xi = \frac{C_\alpha \theta^\alpha}{C_\alpha \lambda^\alpha}, \quad \delta_S \xi = 1. \tag{4.59}
\]

Then any closed operator \(V\) is also exact since

\[
\delta_S V = 0 \implies V = \delta_S (\xi V). \tag{4.60}
\]

A related issue is that the positions of the poles of \(G_0\) are also the positions of the zeros of the path integral insertions thus making the expressions ill-defined.

One might hope to arrive at well-defined expression by finding a different representative of the cohomology class \([G]\) such that the poles in the new \(G\) cancel against zeros in other path integration insertions. Indeed, such a representative \(G_1\) exists and it is given by \(G_1 = b_B/Z_B\), where \(Z_B\) is the picture raising operator in (4.56) and \(b_B\) is the picture-raised \(b\) ghost originally constructed in [22] by solving the equation,

\[
\delta_S b_B = Z_B T. \tag{4.61}
\]

It was shown in [56] that \(G_1\) is in the same cohomology class as \(G_0\) and the poles of \(G_1\) indeed cancel against zeros coming from the picture raising operators.

After the BRST quantisation the end result is that a multi-loop amplitude in the minimal pure spinor formalism should include \(3g - 3\) insertions of \(b_B\), \(10g - (3g - 3)\) insertions of \(Z_B\), \(g\) insertions of \(Z_J\) and eleven insertions of \(Y_C\). This is precisely the prescription proposed in chapter 2.

### 4.4.2 Non-minimal formulation

Let us now return to (4.48) and recall that \(\pi_\alpha\) and \(\bar{\pi}^\alpha\) are \(Q_S\) partners, \(\delta_S \pi_\alpha = \bar{\pi}^\alpha\), see figure 4.2, and each has eleven independent components. These are precisely the properties of the non-minimal variables \(\bar{\lambda}_\alpha\) and \(r_\alpha\), see section 2, so one may identify

\[
\pi_\alpha = \bar{\lambda}_\alpha^0, \quad \bar{\pi}^\alpha = r_\alpha^0, \tag{4.62}
\]

where \(\bar{\lambda}_\alpha^0, r_\alpha^0\) are the zero modes of \(\bar{\lambda}_\alpha\) and \(r_\alpha\). (Actually since the non-minimal variables are cohomologically trivial their nonzero modes do not contribute to any observable and one may only keep their zero modes). Recall also that the non-minimal sector has a gauge invariance similar to (2.4) (whose explicit form is not needed here) and the following combinations are gauge invariant [25]:

\[
\bar{N}_{mn} = \frac{1}{2}(\bar{w} r_{\gamma mn \lambda} - s r_{\gamma mn r}), \quad \bar{J} = \bar{w}^\alpha \bar{\lambda}_\alpha - s^\alpha r_\alpha, \tag{4.63}
\]

\[
S_{mn} = \frac{1}{2} s_{\gamma mn \bar{\lambda}}, \quad S = s^\alpha \bar{\lambda}_\alpha.
\]
The canonical momenta $\bar{w}^{\alpha}$ and $s^{\alpha}$ have 11$g$ zero modes each, as in the discussion of the minimal variables, can be traded for 10$g$ zero modes of $\bar{N}_{mn}^{I}$ and $S_{mn}^{I}$ and $g$ zero modes of $\bar{J}^{I}$ and $S^{I}$. Using the $Q_{S}$ transformations in (2.58) one finds
\[
\delta_{S}S_{mn}^{I} = \bar{N}_{mn}^{I}, \quad \delta_{S}S^{I} = \bar{J}^{I}.
\] (4.64)
Thus the fields $\bar{N}_{mn}^{I}, S_{mn}^{I}, S^{I}, \bar{J}^{I}$ have the same degrees of the freedom and the same $Q_{S}$ transformations as $\pi^{mnI}, \bar{\pi}^{mnI}, \pi^{I}, \bar{\pi}^{I}$. Therefore it is natural to identify them,
\[
\pi^{mnI} = \bar{N}_{mn}^{I}, \quad \bar{\pi}^{mnI} = S_{mn}^{I}, \quad \pi^{I} = S^{I}, \quad \bar{\pi}^{I} = \bar{J}^{I}.
\] (4.65)
With these identifications the exponential factor in (4.48) is precisely the regularisation factor $N$ of equation (2.67) (up to inconsequential numerical factors).

It remains to discuss $G_{ab}$. This field was constructed in [25] (with an elegant interpretation of the construction in terms of Čech cohomology given in [27])
\[
G_{B} = \frac{\bar{\lambda}_{\alpha}G^{\alpha}}{(\bar{\lambda}\bar{\lambda})} + \frac{\bar{\lambda}_{\alpha}r_{\beta}H^{[\alpha\beta]}}{(\bar{\lambda}\bar{\lambda})^{2}} - \frac{\bar{\lambda}_{\alpha}r_{\beta}r_{\gamma}K^{[\alpha\beta\gamma]}}{(\bar{\lambda}\bar{\lambda})^{3}} - \frac{\bar{\lambda}_{\alpha}r_{\beta}r_{\gamma}r_{\delta}L^{[\alpha\beta\gamma\delta]}}{(\bar{\lambda}\bar{\lambda})^{4}},
\] (4.66)
where $G^{\alpha}$ is given in (4.58) and $H^{\alpha\beta}, K^{\alpha\beta\gamma}, L^{\alpha\beta\gamma\delta}$ are specified in section 3.6. Note also that this field is cohomologically equivalent to $G_{0}$ [44]. Hence after a careful treatment of the zero mode invariances and finding the solution for $G$ in the non-minimal formalism, the functional integral derived from first principles (4.40) reduces to the amplitude prescription advocated in section 2.2.

Notice that $G_{B}$ field has poles as $\bar{\lambda}\bar{\lambda} \to 0$ so one might wonder whether this prescription suffers from the same problems as the one using $G_{0}$. Indeed, there is a non-minimal version of the argument around (4.59)-(4.60). The corresponding non-minimal $\xi$ field is [25]
\[
\xi_{nm} = \frac{\bar{\lambda}_{\alpha}\theta^{\alpha}}{\lambda_{\beta}\lambda^{\beta} + r_{\beta}\theta^{\beta}}.
\] (4.67)
This diverges as $(\bar{\lambda}\bar{\lambda})^{-11}$ so one must ensure that no operators which diverge with this rate are allowed. A related issue is that the path integral with the insertions just discussed will diverge if the insertions diverge as fast as $(\bar{\lambda}\bar{\lambda})^{-11}$. As discussed in [25, 27] this can only happen for genus $g > 2$ (since the pure spinor measure converges as $(\bar{\lambda}\bar{\lambda})^{11}$ and $G_{B}$ diverges as $(\bar{\lambda}\bar{\lambda})^{-3}$). One way to deal with this issue is look for a different representative $G_{(B,\epsilon)}$ of the $Q_{S}$ cohomology class of $[G]$ which is less singular than $G_{B}$ as $\bar{\lambda}\bar{\lambda} \to 0$. A construction of such a $G_{(B,\epsilon)}$ is presented in [27]. Using this $G_{(B,\epsilon)}$ field one then arrives at a prescription that in principle works to all orders. See also [29] for more recent work.

This solves the problem in principle. The actual construction of $G_{(B,\epsilon)}$ however is very complicated. Given that the issues with singularities are related to the $\bar{\lambda}\bar{\lambda} \to 0$ limit, a different approach would be to modify the gauge fixing condition for the
pure spinor zero modes such that they are fixed to a nonzero value. It would be interesting to investigate if such gauge fixing can be implemented and whether it would lead to a simpler scattering amplitude prescription. Moreover chapter 5 will provide further motivation to look for a different gauge fixing condition.

### 4.5 Ghost contribution

There remains one loose end that needs to be tied up. In equation (4.38) the ghost fields were set to zero without sound motivation. This is provided in this section. Without setting the ghost fields to zero (4.38) is given by

\[
\langle V_1 \cdots V_n \rangle = \int d\mu e^{-S_{\sigma}} d\mu_{gh} e^{-S_{gh}} d\tilde{\mu} e^{-\tilde{S}} \prod_{j=1}^{\kappa/2} V_j^{(0)}(\tilde{z}_j) \prod_{i=\kappa/2+1}^{N} \int dz_i V_i^{(1)}(z_i),
\]

where \(d\tilde{\mu} e^{-\tilde{S}}\) is given in (4.39),

\[
d\mu_{gh} = D\tilde{\beta}^{ab}D\tilde{b}^{ab}Dc^aD\gamma^aDC_{\omega}D\gamma_\omega DC_\kappa D\xi^kD\tilde{\xi}^k \prod_{j=1}^{\kappa} c^a(\tilde{\sigma}_j)\delta^a(\tilde{\sigma}_j)
\]

and

\[
S_{gh} = \int_\Sigma \left( 2\gamma_\omega \tilde{\beta}^{ab} \tilde{g}_{ab}(\tau) - 2C_{\omega}(\tilde{\beta}^{ab} \tilde{g}_{ab}(\tau) - \tilde{\beta}^{ab} \tilde{\psi}_{ab}) 
\]

\[
+ \tilde{\beta}^{ab}[\hat{\nabla}_a c_b + \hat{\nabla}_b c_a] + \tilde{\beta}^{ab}[\hat{\nabla}_a \gamma_b + \hat{\nabla}_b \gamma_a] + \tilde{\beta}^{ab} \xi^k \partial_k \tilde{g}_{ab}(\tau) 
\]

\[
- \tilde{\psi}_{ab}[\partial_c(\tilde{\beta}^{ab} c^c) - 2\tilde{\beta}^{c[b} \partial_c c^{a]} - \tilde{\beta}^{ab}[\hat{\xi}^k \partial_k \tilde{g}_{ab}(\tau) - \hat{\tau}^k \xi^l \partial_k \partial_l \tilde{g}_{ab}(\tau)] \right).
\]

where \(\hat{\nabla}_a\) is the covariant derivative associated with \(\tilde{g}_{ab}\). The goal is to show that the “BRST factor” in (4.68), let us call it \(X_{BRST}\), can be manipulated to give the result of section 4.3:

\[
X_{BRST} = \int d\mu_{gh} e^{-S_{gh}} d\tilde{\mu} e^{-\tilde{S}} \prod_{k=1}^{6g-6} d\tau^k(G, \partial_k \tilde{g}(\tau)).
\]

The first step is integrating out \(\gamma_\omega\) and \(\beta(\tau) \equiv \tilde{g}_{ab}(\tau) \tilde{\beta}^{ab}\). This sets the trace of \(\tilde{\beta}^{ab}\) equal to zero. The traceless part of \(\tilde{\beta}^{ab}\) will be denoted by \(\beta^{ab}\). Integrating out \(\tilde{\xi}^k\) introduces \((6g - 6)\) insertions of the \(\beta^{ab}\) zero modes, while integrating over \(p^{ab}, \tilde{\psi}_{ab}\) and \(\hat{\tau}^k\) leads to insertions of the zero mode of \(G\),

\[
\langle \tilde{G}, \partial_k \tilde{g} \rangle \equiv \int_\Sigma d^2 \sigma \left( \partial_c(\beta^{ab} c^c) - 2\beta^{c[b} \partial_c c^{a]} + 2\beta^{ab} C_\omega + \sqrt{g} G^{ab} + \beta^{ab} \xi^l \partial_l \right) \partial_k \tilde{g}_{ab}(\tau).
\]
After these integrations one is left with

\[ X_{\text{BRST}} = \int d\mu_{\beta\gamma} d\tilde{\mu}_{gh} e^{-\tilde{S}_{gh}}, \]  

(4.73)

where

\[ \tilde{S}_{gh} = \int d^2 \sigma \left( \beta^{ab}(\tilde{\nabla}_a \gamma_b + \tilde{\nabla}_b \gamma_a) + \tilde{b}^{ab}(2C_\omega \tilde{g}_{ab} + \tilde{\nabla}_a \tilde{c}_b + \tilde{\nabla}_b \tilde{c}_a) + \tilde{b}^{ab} \xi^k \partial_k \tilde{g}_{ab}(\tau) \right) \]  

(4.74)

and

\[ d\mu_{\beta\gamma} = [d\beta^{ab}][d\gamma^a] \prod_{k=1}^{6g-6} \delta(\beta, \partial_k \tilde{g}) \prod_{j=1}^{\kappa} \delta(\gamma^a(\hat{\sigma}_j)) \]  

(4.75)

\[ d\tilde{\mu}_{gh} = [d\tilde{b}^{ab}][dc^a][dC_\omega] \prod_{k=1}^{6g-6} d\tau^k(G, \partial_k \tilde{g}(\tau)) \prod_{j=1}^{\kappa} c^a(\hat{\sigma}_j) \]  

(4.76)

The \( \beta\gamma \) system is now a standard CFT with a \( U(1) \) charge conservation and the path integral measure contains all appropriate zero mode insertions. It follows that the \( \beta \)-dependent part of (4.72) drops out of (4.73) since it is charged w.r.t. the \( \beta\gamma \) \( U(1) \). Integrating out \( C_\omega \) sets the trace of \( \tilde{b}^{ab} \) to zero; the traceless part will be denoted by \( b^{ab} \), and integrating out \( \xi^k \) leads to \((6g-6)\) insertions of the \( b^{ab} \) zero modes. The BRST factor is now given by

\[ X_{\text{BRST}} = \int d\mu_{\tau} d\mu_{\beta\gamma} d\mu_{bc} e^{-\int d^2 \sigma \left( \beta^{ab}(\tilde{\nabla}_a \gamma_b + \tilde{\nabla}_b \gamma_a) + b^{ab}(\tilde{\nabla}_a \tilde{c}_b + \tilde{\nabla}_b \tilde{c}_a) \right)}, \]  

(4.77)

with \( d\mu_{\beta\gamma} \) as in (4.75) and

\[ d\mu_{bc} = [db^{ab}][dc^a] \prod_{k=1}^{6g-6} (b, \partial_k \tilde{g}(\tau)) \prod_{j=1}^{\kappa} c^a(\hat{\sigma}_j), \quad d\mu_{\tau} = \prod_{k=1}^{6g-6} d\tau^k(G, \partial_k \tilde{g}(\tau)). \]  

(4.78)

It is now manifest that the integration over \((b^{ab}, c^a)\) cancels against the integration over \((\beta^{ab}, \gamma^a)\) and after integrating out \( b, c, \beta, \gamma \) one finds:

\[ X_{\text{BRST}} = \prod_{k=1}^{6g-6} d\tau^k(G, \partial_k \tilde{g}(\tau)). \]  

(4.79)

### 4.6 Replacing worldsheet fields by zero modes

In chapter 2 the amplitude prescription for the pure spinor formalism was presented. In order to evaluate the correlators it was stated that one should first remove all fields of nonzero conformal weight by using the OPE’s and thereafter replace the
remaining fields, which all have weight zero, by their zero modes. The first principles derivation of this chapter has set the stage for justifying that step.

First it is useful to clarify what a zero mode really is. In general a zero mode is an eigenstate of some (differential) operator. In the case of $\beta\gamma$ systems it is not clear from the action in conformal gauge, i.e. for a special choice of coordinates, what this operator is, since such an action is only defined on one coordinate patch. Consider the $p\theta$ action, which is a fermionic $\beta\gamma$ system. Its action in conformal gauge is given by

$$S(p\theta) = \int d^2z p_\alpha \tilde{\theta}^\alpha,$$

where the weights of $p$ and $\theta$ are one and zero respectively. In order to write this action in a coordinate free form, one uses the differential operators $P_0$ and $P^0_T$ defined in (1.66). The action (4.80) can now be written in either of the two following forms

$$S(p\theta) = (\hat{p}_\alpha, P_0 \theta^\alpha) = (P^0_T \hat{p}_\alpha, \theta^\alpha),$$

where $(\hat{p}_\alpha)_a = P^b_\alpha P^a_b$ is a projection operator defined in (4.4), not to be confused with the differential operator $P_n$, it depends on the complex structure of the worldsheet and in conformal gauge its only nonzero component is $P^{(+)z}_z = 1$.

The operators $P_n$ and $P^0_T_n$ do not have eigenstates because they change the rank of the tensors they act on, therefore zero modes cannot be eigenstates of one of these operators. Operators that can be diagonalised are $P^0_T P_n$ and $P_n P^0_T$:

$$P_n P^0_T F^{a_1 \cdots a_{n+1}}(\sigma) = v^2 J F^{a_1 \cdots a_{n+1}}(\sigma), \quad P^0_T P_n G^{a_1 \cdots a_n}(\sigma) = v^2 K G^{a_1 \cdots a_n}(\sigma),$$

where $F_J$ and $G_K$ are symmetric traceless tensors of respectively rank $n + 1$ and $n$. Any traceless symmetric worldsheet field can be expanded in a basis of eigenfunctions of $P^0_T P_n$ for some $n$. In addition it can expanded in eigenfunctions of $P_{n+1} P^0_T$.

This basis can be chosen to be orthonormal with respect to (1.69):

$$(F_J, F_{J'}) = \delta_{JJ'}, \quad (G_K, G_{K'}) = \delta_{KK'}.$$  

$\hat{p}_\alpha$ and $\theta^\alpha$ can be expanded as

$$\hat{p}^\alpha(\sigma) = \sum_J (p_\alpha)_J \hat{F}^a_J(\sigma), \quad \theta^\alpha(\sigma) = \sum_K \theta^\alpha_K G^a_K(\sigma).$$

There is a one to one correspondence between the nonzero modes of $P^0_T P_n$ and $P_n P^0_T$. The number of zero modes can differ. This follows from

$$(P_n P^0_T) P_n G_J(\sigma) = P_n (P^0_T P_n) G_J(\sigma) = (v^2 J P_n G_J(\sigma).$$

Thus $P_n G_J$ is an eigenfunction of $(P_n P^0_T)$. Along the same lines it follows that $P^0_T F_J$ is an eigenfunction of $(P^0_T P_n)$. This shows a one to one correspondence between the
modes that satisfy $P_n G_J \neq 0$ and $P_n^T F_J \neq 0$, these are precisely the nonzero modes. These can be separated in positive ($J, K > 0$) and negative modes ($J, K < 0$). The zero modes are denoted as $F_{0J}, G_{0k}$ where $j = 1, \cdots, \mu$ and $k = 1, \cdots, \kappa$. The values of $\mu$ and $\kappa$ depend on both the genus of the worldsheet and the value of $n$. Canonical quantisation amounts to imposing the commutation relations

$$[F_J, G_K] = \delta_{J,-K}, \quad [F_J, F_{J'}] = [G_K, G_{K'}] = 0. \quad (4.86)$$

A vacuum can be defined by

$$F_J |0\rangle = G_K |0\rangle = 0, \quad J, K > 0. \quad (4.87)$$

Consider a number of $\beta \gamma$ systems of weight one, which is the relevant one for the pure spinor formalism. In the Hilbert space language a correlator, that only contains weight zero fields,

$$\mathcal{A} = \langle \gamma^1(z_1) \cdots \gamma^N(z_N) \rangle, \quad (4.88)$$

can be expanded as

$$\mathcal{A} = \langle 0| \gamma^1(z_1) \cdots \gamma^N(z_N)|0\rangle = \langle 0| \sum_{j=1}^{\kappa} (\gamma^1_0)_j G_{0j}(z_1) \cdots \sum_{j'=1}^{\kappa} (\gamma^N_0)_{j'} G_{0j'}(z_N)|0\rangle, \quad (4.89)$$

because the positive modes vanish against the vacuum on the right and the negative ones against the vacuum on the left. Also note there are only non vanishing (anti)commutators between a positive $\gamma$ mode and a negative $\beta$ mode or vice versa, so one can (anti)commute all $\gamma$ modes through each other. This justifies replacing the fields in a correlator by their zero modes if all these fields are of weight zero.