Fundamentals of the pure spinor formalism

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DOI
10.5117/9789056296414

Publication date
2010

Citation for published version (APA):

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Chapter 5

Decoupling of unphysical states

Any theory whose spectrum is defined as the cohomology of a certain nilpotent fermionic operator must have the property that all amplitudes with an exact state vanish. Otherwise two operators representing the same physical state give rise to different scattering amplitudes and there is no way to prefer either of the two answers. In chapter 2 it was shown that decoupling of unphysical states is guaranteed if all insertions are $Q_S$ closed. For the non-minimal pure spinor formalism this was easy to show, cf. (2.61). The minimal formalism, however, contains constant spinors ($C_\alpha$) and constant tensors ($B_{mn}$) in its amplitude prescription. These constant tensors enter the theory via the picture changing operators. It was argued in [22] that amplitudes are independent of $C$ and $B$, because the Lorentz variation of the PCOs is $Q_S$ exact.

In this chapter it will be shown by explicit computations that the amplitudes do depend on the choice of the constant tensors and $Q_S$ exact states do not decouple. This happens already at tree level, but in this case one can show that there is a unique Lorentz invariant operator that can replace the PCOs in the tree-level amplitude prescription. With this replacement $Q_S$ exact terms do decouple and one can further show that this prescription is equivalent to the tree-level prescription obtained by integrating over $C$ [22], which correctly reproduces known tree-level amplitudes.

Amplitudes at one loop are discussed next. The main result will be that the PCOs, $Y_C$, are not $Q_S$ closed. Furthermore a no-go theorem will be proved which states that $Q_S$ closed Lorentz covariant PCOs lead to vanishing of all one-loop amplitudes. Hence if one wishes to replace the PCOs by $Q_S$ closed ones, manifest Lorentz invariance cannot be maintained.
Note that \( Q_S Y_C \neq 0 \) by itself does not imply that \( Q_S \) exact states do not decouple. It only implies that the standard argument for decoupling of unphysical states that involves integrating \( Q_S \) by parts does not automatically lead to decoupling. Hence one needs another argument. This new argument does not use integration of \( Q_S \) by parts. Rather it makes use of an invariance of the path integral measure and the fact the zero mode integrals act as projectors on a certain Lorentz scalar. Then one can show that the integrand that results from \( Q_S \) exact insertions does not contain this scalar, hence amplitudes that contain unphysical states vanish after integration.

Even though there is a proof of decoupling of unphysical states in the formulation with integration over \( C \) and \( B \), the fact that the PCOs are not \( Q_S \) closed is somewhat unsatisfactory. The technical origin of the problem is that the PCOs are \( Q_S \) closed only in a distributional sense and it turns out that the amplitudes are singular enough so that distributional identities do not hold. To understand why the amplitudes are singular, let us recall that the PCOs originate from gauge fixing zero mode invariances as discussed in chapter 4. The PCOs contain eleven delta functions of the form \( \delta(C^I_\alpha \lambda^\alpha) \), where \( C^I_\alpha \) are the constant spinors mentioned above. It turns out that for any choice of \( C^I \) that give an irreducible set of eleven constraints, the solution of \( C^I_\alpha \lambda^\alpha = 0 \) is given by \( \lambda^\alpha = 0 \), which is the tip of the cone that represents pure spinor space. As discussed in [21], the \( \lambda^\alpha = 0 \) locus should be removed from the pure spinor space. Thus this prescription corresponds to a singular gauge fixing condition and the problems with \( Q_S \) closedness of the PCOs reflect that fact. Furthermore the PCOs are not globally defined on pure spinor space. Ultimately one would like to use globally defined, \( Q_S \) closed PCOs that gauge fix the zero modes of \( \lambda \) to a nonzero value. Such an operator has not been found. Note however that this operator cannot be a Lorentz scalar, due to the no-go theorem.

There is one final point that deserves to be mentioned in this introduction. As stated in chapter 2 the most complicated loop amplitude computations have only been performed in the non-minimal pure spinor formalism. This suggests that the minimal loop computations are technically more involved. The analysis of this chapter, in particular the previously unnoted invariance of the path integral measure, might be used to simplify minimal loop computations.

### 5.1 Tree level

In the first part of this section, a number of tree-level amplitudes is computed in the formulation without an integral over the constant spinors \( C \). The conclusion will be that these amplitudes are not Lorentz invariant and unphysical states do not decouple. In the second part a manifestly Lorentz invariant prescription without constant spinors is presented. As will be shown this new prescription leads to decoupling
of unphysical states and is equivalent to the prescription with an integral over the constant spinors $C$.

5.1.1 No $C$ integration

This section presents two problems regarding the minimal amplitude prescription (2.22) when it is evaluated using the definition of the zero mode measure (2.25) and the usual definition of a delta function:

$$\int dx \delta(x)f(x) = f(0), \quad x\delta'(x) = -\delta(x). \quad (5.1)$$

The problems are

- $A$ is not Lorentz invariant or equivalently $A$ depends on the choice of $C$’s
- $Q_S$ exact states do not decouple.

Lorentz invariance

In section 2.1.2 it was argued the PCOs are Lorentz invariant inside correlators if they are $Q_S$ closed. The $Q_S$ variation is given in (2.29) and this seems to vanish but if one chooses $C_\alpha = \delta_\alpha^+$, the result is $Q_SY_C = \lambda^+\delta(\lambda^+)$. This is not zero because the measure contains $\frac{1}{(\lambda^+)^4}$. All one can use is $(\lambda^+)^4\delta(\lambda^+) = 0$. This problem is made even more explicit in the computation below. It will be shown that choosing particular $C$’s does not result into a Lorentz invariant answer.

Let us choose

$$C^1_\alpha = \delta_\alpha^+, \quad (C^2)^{a_1a_2} = \delta_1^{[a_1}\delta_2^{a_2]}, \ldots, (C^{11})^{a_1a_2} = \delta_4^{[a_1}\delta_5^{a_2]}, \quad \text{all other } C^I_\alpha = 0. \quad (5.2)$$

Note $C^I_\alpha$ has rank eleven for this choice, as it should. As is discussed in section 5.5 the lack of Lorentz invariance, which is shown below, would also be found, if any other choice was made, see footnote 9. The three-point tree-level function is given by

$$A = \langle \lambda^\alpha A_{1\alpha}(z_1)\lambda^\beta A_{2\beta}(z_2)\lambda^\gamma A_{3\gamma}(z_3)Y_{C_1}(\infty) \cdots Y_{C_{11}}(\infty) \rangle. \quad (5.3)$$

The PCOs operators are inserted at infinity, since this simplifies the computation. All OPE’s of the PCOs with the vertex operators vanish due to this choice. Therefore one can replace all fields in (5.3) by their zero modes:

$$A = \int [d\lambda]d^{16}\theta^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}(\theta)C^1_{\alpha_1} \theta^{\alpha_1} \cdots C^{11}_{\alpha_{11}} \theta^{\alpha_{11}} \delta(C^1_{\alpha_1}\lambda_{\alpha_1}) \cdots \delta(C^{11}_{\alpha_{11}}\lambda_{\alpha_{11}}) \quad (5.4)$$

$$= \int [d\lambda]d^{16}\theta^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}(\theta)\theta^{+\theta_12} \cdots \theta^{+\theta_{45}} \delta(\lambda^+)\delta(\lambda_{12}) \cdots \delta(\lambda_{45}) \quad (5.4)$$

1See section 3.2.1 for notational conventions.
\[ \int \frac{d\lambda^+ \wedge d\lambda_{12} \wedge \cdots \wedge d\lambda_{45}}{\lambda^{+3}} d^{16} \theta \lambda^{\alpha} \lambda^{\beta} \lambda^{\gamma} f_{\alpha \beta \gamma}(\theta) \theta^+ \theta_{12} \cdots \theta_{45} \delta(\lambda^+) \delta(\lambda_{12}) \cdots \delta(\lambda_{45}). \]

The only term that contributes is the one with \( \alpha \beta \gamma = + + + \), in all other cases there is an integral of the form \( \int d\lambda_{ab} \lambda_{ab} \delta(\lambda_{ab}) \) (no sum). There is a subtlety with these integrals, for instance

\[ \int [d\lambda](\lambda^+)^2 \lambda_{cd} \delta(\lambda^+) \delta(\lambda_{12}) \cdots \delta(\lambda_{45}) = \int d\lambda^+ d^{10} \lambda_{ab} \frac{\lambda_{cd}}{\lambda^+} \delta(\lambda^+) \delta(\lambda_{12}) \cdots \delta(\lambda_{45}) = \int d\lambda^+ \frac{1}{\lambda^+} \delta(\lambda^+) \int d\lambda_{cd} \lambda_{cd} \delta(\lambda_{cd}) = \infty. \] (5.5)

Note however that (5.5) has \( N \) charge one (cf. (3.50)). Since the outcome of the integral (maybe after some regularisation) must be a number, which does not transform under \( N \), the integral has to vanish. In other words only integrals with zero \( N \) charge, like \( \int [d\lambda](\lambda^+)^3 \delta(\lambda^+) \delta(\lambda_{12}) \cdots \delta(\lambda_{45}) \) can be non-vanishing. After the integration over the \( \lambda \) zero modes one is left with

\[ A = \int d^{16} \theta f_{+++} \theta^+ \theta_{12} \cdots \theta_{45}, \] (5.6)

where \( f_{+++} = A^1_+ A^2_+ A^3_+ \) and this can be evaluated with the help of the explicit expressions for the gamma matrices from section 3.2.4. If one chooses the external states to be two gauginos and one gauge boson the amplitude becomes:

\[ A = \int d^{16} \theta (\xi^a_{s_1} \theta_k a^k + \xi^1_{s_1} \theta^a \theta^k) (\xi^b_{s_2} \theta_l b^l + \xi^2_{s_2} \theta^b \theta^l) \theta^c a^3 c^3 + \theta_{12} \cdots \theta_{45} = \epsilon^{abde} \xi^1_{s_1} \epsilon^2_{s_2} a^3 c^3. \] (5.7)

This answer is not Lorentz invariant and different from the expected answer,

\[ \xi^1 \gamma^m \xi^2 a^3_m = 2(\xi^1 \xi^2 a_3^a + \xi^1 \xi^2 a_3^b a_3^c) - \frac{1}{4} \epsilon^{abde} \xi^1_{a_b} \xi^2_{c_d} a^3_c + \xi^1_{a_b} \xi^2_{b_d} a^3_c + \xi^1_{a_b} \xi^2_{c_d} a^3_c. \] (5.8)

where \( m \) is an \( SO(10) \) index and all Latin letters that come before \( m \) in the alphabet are \( SU(5) \) indices. In conclusion this shows that tree-level amplitudes do not yield Lorentz invariant answers when one does not integrate over \( C \).

**Dependence on \( C^I \)**

On top of the lack of Lorentz invariance amplitudes depend on the choice of constant spinors \( C^I \). In other words they are not invariant under \( C^I_\alpha \rightarrow C^I_\alpha + \delta C^I_\alpha \). This variation changes the \( I \)th PCO by a \( Q_S \) exact quantity. However when one computes a tree-level amplitude with the \( I \)th PCO replaced by this \( Q_S \) exact quantity, it does not vanish. Hence incidentally this computation demonstrates that not all \( Q_S \) exact states decouple. In the computation below the same \( C^I \)'s as in (5.2) are used and
\[ \delta C^{11} = \delta^{1}, \] where the 1 is an SU(5) vector index. The delta only has one non-vanishing component. This changes \( Y_{C_{11}} \) by

\[ \delta Y_{C_{11}} = \delta C_{11\alpha} \theta^{\alpha} \delta(C_{11} \lambda) + C_{11\alpha} \theta^{\alpha} \delta C_{11\beta} \lambda^{\beta} \delta'(C_{11} \lambda) \]  

(5.9)

Under this change in \( C_{\alpha}^{I} \) the tree-level three-point function changes by

\[ \delta\mathcal{A} = \langle V_{1}(z_{1})V_{2}(z_{2})V_{3}(z_{3})Y_{C_{1}}(\infty) \cdots Y_{C_{10}}(\infty)\delta Y_{C_{11}}(\infty) \rangle = \int d^{16}\theta d^{11}\lambda \alpha A_{\alpha}^{1} A_{1}^{2} A_{2}^{3} Q_{S}(Y_{C_{1}} \cdots Y_{C_{10}}) \theta^{1} \theta_{45} \delta'(\lambda_{45}) \]  

(5.10)

There is a total of four \( \lambda^{\alpha} \)'s in the numerator (one hidden in \( Q_{S} \)) one of them has to be \( \lambda_{45} \) and the other three have to be \( \lambda^{+} \) to give a non-vanishing answer. The term that contributes comes from \( Q_{S} \) hitting \( \theta^{+}\delta(\lambda^{+}) \), this \( \lambda^{+} \) then cancels against a \( \lambda^{+} \) in the denominator and the variation becomes

\[ \delta\mathcal{A} = \int d^{16}\theta d^{11}\lambda A_{\alpha}^{1} A_{\alpha}^{2} A_{\alpha}^{3} Q_{S}(Y_{C_{1}} \cdots Y_{C_{10}}) \theta^{1} \theta_{45} \delta'(\lambda_{45}) \]  

(5.11)

By choosing suitable polarisations it is not difficult to see this does not always vanish.

### 5.1.2 Including \( C \) integration

Obtaining amplitudes which are not Lorentz invariant is a serious problem and one might ask why the tree-level amplitude computations [22, 57] in the minimal pure spinor formalism gave Lorentz invariant answers and why \( Q_{S} \) exact states decoupled. Both these points are explained in the first part of this section. In the second part the tree-level amplitude prescription is reformulated in a way that does not contain any constant spinors.

Lorentz invariance is restored by integrating over all possible choices of \( C_{\alpha}^{I} \), and this also results in decoupling of \( Q_{S} \) exact states as will become apparent in this section. The manifestly Lorentz invariant tree-level amplitude in the minimal formalism is given by

\[ \mathcal{A} = \int [dC] \langle V_{1}(z_{1})V_{2}(z_{2})V_{3}(z_{3}) \rangle \int dz_{4}U_{4}(z_{4}) \cdots \int dz_{N}U_{N}(z_{N})Y_{C_{1}}(\infty) \cdots Y_{C_{11}}(\infty). \]  

(5.12)

After performing the OPE’s and replacing the fields by their zero modes this becomes

\[ \mathcal{A} = \int [dC] \int [d\lambda] d^{16}\theta \lambda^{\alpha} \lambda^{\beta} \lambda^{\gamma} f_{\alpha\beta\gamma}(\theta)(C^{1}\theta) \delta(C^{1}\lambda) \cdots (C^{11}\theta) \delta(C^{11}\lambda). \]  

(5.13)
Note the eleven PCOs, $Y_C$, have been replaced by a manifestly Lorentz invariant PCO which will be called $Y$:

$$ Y \equiv \int [dC](C^{1\lambda})\delta(C^{1\lambda}) \cdots (C^{11\lambda})\delta(C^{11\lambda}). $$ (5.14)

Now one uses

$$ \int [dC][d\lambda]\lambda^\alpha \lambda^\beta \lambda^\gamma C^1_{\beta_1} \cdots C^{11}_{\beta_{11}}\delta(C^{1\lambda}) \cdots \delta(C^{11\lambda}) = (\epsilon T)^{\alpha\beta\gamma}_{\beta_1 \cdots \beta_{11}}. $$ (5.15)

This is justified by Lorentz invariance, because the LHS is Lorentz invariant and the only invariant tensor with the appropriate symmetries is\(^2\) $(\epsilon T)$, as can be verified with [40]. Thus

$$ \mathcal{A} = (\epsilon T)^{\alpha\beta\gamma}_{\alpha_1 \cdots \alpha_{11}} \int d^{16}\theta f_{\alpha\beta\gamma}(\theta)\theta^{\alpha_1} \cdots \theta^{\alpha_{11}}. $$ (5.17)

The amplitude $\mathcal{A}$ is manifestly Lorentz invariant.

This prescription also ensures the decoupling of unphysical states. Amplitudes with unphysical states will be denoted by $\mathcal{B}$ throughout this chapter, while $\mathcal{A}$ is used for any amplitude, so at tree level with $V_1 = Q_S\Omega$,

$$ \mathcal{B} = \int [dC]\langle Q_S\Omega(z_1) V_2(z_2) V_3(z_3) \prod_{i=4}^N dz_i U(z_i) \rangle \sim \int [dC]\langle \lambda^\alpha(z_2) \lambda^\beta(z_3) g_{\alpha\beta}(d,\theta,N) Q_S(C^1_{\alpha_1} \theta^{\alpha_1} \cdots C^{11}_{\alpha_{11}} \theta^{\alpha_{11}}) \delta(C^{1\lambda}) \cdots \delta(C^{11\lambda}) \rangle. $$ (5.18)

This can be written in the following form:

$$ \mathcal{B} = \int [dC]\langle \lambda^\alpha(z_2) \lambda^\beta(z_3) g_{\alpha\beta}(d,\theta,N) Q_S(C^1_{\alpha_1} \theta^{\alpha_1} \cdots C^{11}_{\alpha_{11}} \theta^{\alpha_{11}}) \delta(C^{1\lambda}) \cdots \delta(C^{11\lambda}) \rangle \sim \int [dC]\langle \lambda^\alpha(z_2) \lambda^\beta(z_3) g_{\alpha\beta}(d,\theta,N) C^1_{\alpha_1} \lambda^{\alpha_1} \cdots C^{11}_{\alpha_{11}} \theta^{\alpha_{11}} \delta(C^{1\lambda}) \cdots \delta(C^{11\lambda}) \rangle. $$ (5.19)

where in going from the first to the second line an overall numerical factor of eleven was omitted. Such overall inconsequential factors will be neglected throughout this

\(^2\)Incidentally, the following related integral can also be computed using Lorentz invariance:

$$ \int [dC]d\lambda^{\alpha_1} \cdots d\lambda^{\alpha_{11}} C^1_{\beta_1} \cdots C^{11}_{\beta_{11}} \delta(C^{1\lambda}) \cdots \delta(C^{11\lambda}) = \ c_1 \delta^{[\alpha_1}_{\beta_1} \cdots \delta^{\alpha_{11}]}_{\beta_{11}} + c_2 \gamma^{[\alpha_1 \alpha_2 mnp}_{\beta_1 \beta_2 \beta_3} \delta_{\beta_{11}],} $$

where $c_1$ and $c_2$ are nonzero numerical constants. This structure follows from the fact $\text{Asym}^{11}_{16} \otimes \text{Asym}^{11}_{16}'$ contains two scalars (see section 3.2.2 for explanation about the notation and the argument). The constants can be computed using judicious choices of the indices. For example, the integral vanishes for the choice $\alpha_1 = \beta_1, \cdots, \alpha_{11} = \beta_{11} = +, 12, \ldots, 35, 5$, implying that one needs a nonzero constant $c_2$. Equation (5.16) corrects formula (3.25) of [22].
chapter. After using the OPE’s to integrate out the nonzero modes one gets:

\[
B = \int [dC] d^{16} \theta \prod_{\alpha} \lambda^\alpha \lambda^\beta f_{\alpha \beta}(\theta) C^1_{\alpha_1} \lambda^{\alpha_1} C^2_{\alpha_2} \theta^{\alpha_2} \cdots C^{11}_{\alpha_{11}} \theta^{\alpha_{11}} \delta(C^1 \lambda) \cdots \delta(C^{11} \lambda) = \\
\int d^{16} \theta f_{\alpha \beta}(\theta) (\epsilon T)^{\alpha_1 \beta_1 \cdots \alpha_{11} \beta_{11}} \frac{1}{\alpha \beta} \theta^{\alpha_1} \cdots \theta^{\alpha_{11}} = 0,
\]

(5.20)

where \( f_{\alpha \beta}(\theta) \) is some function of \( \theta \) zero modes and (5.15) was used in the second equality. The integral vanishes because \(^3\, 126 \otimes \text{Asym}^{10} \, 16\) does not contain a scalar (see section 3.2.2 for explanation about the notation and the argument), in other words

\[
(\epsilon T)^{\beta_1 \beta_2 \cdots \beta_{11}} = 0.
\]

(5.21)

In this case one can also write out \((\epsilon T)\) explicitly and check that its trace contains a contraction of an antisymmetric tensor \((\epsilon)\) and a symmetric one \((\gamma_\alpha^\beta)\).

Lorentz invariant tree-level prescription without constant spinors

There exists a replacement for the eleven PCOs that does not contain any constant spinors and is manifestly Lorentz covariant. The prescription that uses this replacement is equivalent to the one given in [22], when the integral over \( C \) in included. The prescription is given by

\[
A = \langle V_1(z_1)V_2(z_2)V_3(z_3) \int dz_4 U_4(z_4) \cdots \int dz_N U_N(z_N) \Lambda_{\alpha \beta \gamma}(\infty) \rangle
\]

(5.22)

\[
(\epsilon T)^{\beta_1 \beta_2 \cdots \beta_{11} \gamma} = \theta^{\beta_1}(\infty) \cdots \theta^{\beta_{11}}(\infty).
\]

The replacement of the eleven PCOs \( Y_C \) is called \( \Lambda_{\alpha \beta \gamma}(\infty) \). After integrating out the nonzero modes and replacing the fields by their zero modes \( A \) reduces to

\[
A = \int d^{16} \theta [d\lambda] \prod_{\alpha} \lambda^\alpha \lambda^\beta f_{\alpha \beta \gamma}(\theta) (\epsilon T)^{\beta_1 \beta_2 \cdots \beta_{11} \gamma_1} \theta^{\beta_1} \cdots \theta^{\beta_{11}} \Lambda_{\beta_1 \cdots \beta_{11} \gamma}.
\]

(5.23)

The tensor \( \Lambda_{\alpha \beta \gamma} \) is defined by

\[
\int [d\lambda] \lambda^\alpha \lambda^\beta \lambda^\gamma \Lambda_{\alpha \beta} = \delta(\alpha \beta \gamma) \theta^{\beta_1} \cdots \theta^{\beta_{11}} \Lambda_{\beta_1 \cdots \beta_{11} \gamma}.
\]

(5.24)

and is a function of the \( \lambda \)'s only. More accurately, all components contain eleven delta functions or derivatives thereof. The precise form of (5.24) follows from the fact that the integral must be an invariant tensor combined with the pure spinor constraint. Detailed arguments are provided in section 3.5. Explicit expressions of the components can be found by examining certain components of (5.24). In order

\(^3\)Note 126 denotes a gamma matrix traceless symmetric rank two tensor (recall that \( \lambda^\alpha \lambda^\beta \sim \lambda^{\gamma m npqr} \lambda^\alpha^\beta_{\gamma m npqr} \)).
to see what conditions (5.24) imposes on $\Lambda_{+++}$ note that choosing $\alpha\beta\gamma = +++$

\[ \int [d\lambda] \lambda^+^3 \Lambda_{+++} = 6. \quad (5.25) \]

Moreover this is the only condition because for all other choices the LHS of (5.24)

\[ \text{is not invariant under } M, \text{ the generator of a } U(1) \text{ subgroup of Lorentz group (see section 3.2.1 for the definition of } M). \]

Therefore the LHS is equal to zero. The solution is given by

\[ \Lambda_{+++} = 6\delta(\lambda^+)\delta(\lambda_{12})\cdots\delta(\lambda_{45}). \quad (5.26) \]

It is possible to verify this object is indeed part of a representation of the Lorentz group. In order to do so one needs to check the Lorentz algebra holds when acting on $\Lambda_{+++}$. First note

\[ (N^a_b)^a_b \Lambda_{+++} = N_{ab} \Lambda_{+++} = 0, \quad N\Lambda_{+++} = \frac{15}{4} \Lambda_{+++}, \quad (5.27) \]

$N^{mn}$ denote the realisation of Lorentz generators $M^{mn}$ in terms of pure spinors, see section 3.3 for the precise expressions. The nontrivial commutation relations that remain to be checked are

\[ [N_{ab}, N^{cd}] \Lambda_{+++} = -\frac{1}{2} \delta^{[c}_{[a} N^{d]}_{b]} \Lambda_{+++} = -\frac{1}{5} \delta_{[a} \delta_{b]} N \Lambda_{+++} = -\frac{3}{4} \delta_{[a} \delta_{b]} \Lambda_{+++}, (5.28) \]

\[ [N^a_b, N^{cd}] \Lambda_{+++} = \frac{1}{2} \delta^{[c}_{b} N^{d}]_{a} \Lambda_{+++}. \quad (5.29) \]

Because of the symmetric form of $\Lambda_{+++}$ it suffices to check

\[ [N_{12}, N^{12}] \Lambda_{+++} = -\frac{3}{4} \Lambda_{+++}, \quad (5.30) \]

\[ [N_{12}, N^{13}] \Lambda_{+++} = 0, \quad (5.31) \]

\[ [N^{12}, N^{23}] \Lambda_{+++} = -\frac{1}{2} N^{13} \Lambda_{+++}. \quad (5.32) \]

Let us start with the LHS of (5.30)

\[ [N_{12}, N^{12}] \Lambda_{+++} = N_{12} N^{12} \Lambda_{+++} = N_{12} \left[ \frac{1}{2} 6\lambda^+ \delta(\lambda^+)\delta(\lambda_{12})\delta(\lambda_{13})\cdots\delta(\lambda_{45}) \right] = \]

\[ \frac{3}{2} \left( -w_+ \lambda_{12} - \frac{1}{2} \frac{1}{\lambda^+} w^{ab} \lambda_{ab} \lambda_{12} + \frac{1}{\lambda^+} w^{ab} \lambda_{1a} \lambda_{2b} \right) \left[ \lambda^+ \delta(\lambda^+)\delta'(\lambda_{12})\delta(\lambda_{13})\cdots\delta(\lambda_{45}) \right] = \]

\[ = (0 - \frac{9}{4} + \frac{3}{2}) \Lambda_{+++} = -\frac{3}{4} \Lambda_{+++}, \quad (5.33) \]

Note that $N_{12}$ does not contain factors of $(\lambda_{12})^2$ (possible such factors cancel out). This is useful when acting with $N_{12}$ in this second line. In going from the second to the last line $x\delta'(x) = -\delta(x)$ was used twice. The other two commutators, (5.31) and (5.32), follow along the same lines.
Chapter 5 - Decoupling of unphysical states

It is instructive to compute the next two levels (distinguished by \( N \) charge) of the components of \( \Lambda_{\alpha\beta\gamma} \). For the components on the second (\( N = \frac{11}{4} \)) level consider

\[
N^{a_1a_2} \Lambda_{++} = -\frac{1}{2} \Lambda^{a_1a_2}_{++} - \frac{1}{2} \Lambda_{++}^{a_1a_2} + \frac{1}{2} \Lambda_{++}^{a_1a_2} = -\frac{3}{2} \Lambda^{a_1a_2}_{++} \Rightarrow (5.34)
\]

The factor of \(-\frac{1}{2}\) is consistent with \( N^{ab}w_+ = -\frac{1}{2} w^{ab} \). Going to the next level (\( N = \frac{7}{4} \))

\[
N^{b_1b_2} \Lambda_{++}^{a_1a_2} = -\frac{1}{2} \epsilon^{a_1a_2b_1b_2} \epsilon \Lambda_{e++} - \frac{1}{2} \Lambda_{++}^{a_1a_2b_1b_2} + \frac{1}{2} \Lambda_{++}^{a_1a_2b_1b_2} = (5.35)
\]

This seems to leave freedom to define one of the two components, which would indeed be true if \( \Lambda_{\alpha\beta\gamma} \) was just a symmetric rank three tensor and nothing more. However \( \Lambda_{\alpha\beta\gamma} \) is gamma matrix traceless,

\[
\gamma_{\mu_1}^{\alpha\beta} \Lambda_{\alpha\beta\gamma} = 0. \qquad (5.36)
\]

This imposes one additional condition that relates components of equal \( N \) charge to each other. Consequently all components of \( \Lambda_{\alpha\beta\gamma} \) are uniquely fixed in terms of \( \Lambda_{++} \). Note that this is consistent with the discussion under (3.134), where Lorentz invariance arguments were used to come to the same conclusion.

Decoupling of \( Q_S \) exact states

The new insertion \( \Lambda_{\alpha\beta\gamma} \) was motivated by manifest Lorentz invariance, but it also results in a prescription in which \( Q_S \) exact states decouple. Indeed, the tree-level amplitude with one \( Q_S \) exact state,

\[
B = \langle Q_S \Omega(z_1)V_2(z_2)V_3(z_3) \prod_{i=4}^{N} \int dz_i U(z_i)(\epsilon T)_{\beta_1\ldots\beta_{11}}^{\delta_1\delta_2\delta_3}\theta^{\beta_1}\ldots\theta^{\beta_{11}}(\infty)\Lambda_{\delta_1\delta_2\delta_3}(\infty) \rangle,
\]

can be written in the following form:

\[
B = \langle \lambda^{\alpha}(z_2)\lambda^{\beta}(z_3) f_{\alpha\beta}(\theta)Q_S((\epsilon T)_{\beta_1\ldots\beta_{11}}^{\delta_1\delta_2\delta_3}\theta^{\beta_1}\ldots\theta^{\beta_{11}}\Lambda_{\delta_1\delta_2\delta_3}) \rangle = (5.37)
\]

\[
\langle \lambda^{\alpha}(z_2)\lambda^{\beta}(z_3) f_{\alpha\beta}(\theta)(\epsilon T)_{\beta_1\ldots\beta_{11}}^{\delta_1\delta_2\delta_3}\lambda^{\beta_1}\theta^{\beta_2}\ldots\theta^{\beta_{11}}\Lambda_{\delta_1\delta_2\delta_3} \rangle. (5.38)
\]

After using the OPE’s to integrate out the nonzero modes one gets:

\[
B = \int d^{16}\theta[d\lambda] \lambda^{\alpha}\lambda^{\beta} f_{\alpha\beta}(\theta)(\epsilon T)_{\beta_1\ldots\beta_{11}}^{\delta_1\delta_2\delta_3}\lambda^{\beta_1}\theta^{\beta_2}\ldots\theta^{\beta_{11}}\Lambda_{\delta_1\delta_2\delta_3} = (5.40)
\]

\[
\int d^{16}\theta f_{\alpha\beta}(\theta)(\epsilon T)_{\beta_1\ldots\beta_{11}}^{\alpha\beta_1}\theta^{\beta_2}\ldots\theta^{\beta_{11}} = 0.
\]

The last line vanishes because all traces of \( (\epsilon T) \) vanish (cf. (5.21)).
5.1.3 Global issues

The computations in section 5.1.1 showed that not all $Q_S$ exact states decouple. From this result it is tempting to conclude that the PCOs are not $Q_S$ closed. This is true as will be shown in section 5.3, but one cannot conclude it just yet. For the computations above involve integrations over the space of pure spinors, which is a manifold that cannot be covered by one coordinate patch. Therefore the computations in section 5.1.1 can best be viewed as evidence for the need of a globally well defined PCO. Note however that when one integrates over $C$, Lorentz invariance can be used and consequently any possible ambiguity goes away. Alternatively one can use $\Lambda$, globally defined in (5.24), which will be done in the one-loop computations in the next section.

5.2 One loop

In this section one-loop amplitudes with one unphysical state are considered both in the prescription with an integral over $B$ and without. Let us first consider the case in which there is no $B$ integration. All amplitudes, including those with an unphysical state, can be evaluated by first integrating out the nonzero modes. One is then left with a certain zero mode integral. At tree level one could show that these integrals vanish after the $\lambda$ integration is performed, cf. (5.40). This section contains the corresponding one-loop computation. The result is that the zero mode integrals do not vanish after the $\lambda, N$ integrations.

The analysis of amplitudes with an unphysical state when one includes an integral over $B$ is analogous to the tree-level case. After one has integrated out the nonzero modes the zero mode integral over $\lambda$ and $N$ can be performed by Lorentz invariance. Recall that decoupling of unphysical states at tree level followed from the vanishing of the trace of $\epsilon T$, cf. (5.40). This $\epsilon T$ showed up in the $\lambda$ zero mode integral (5.15). The analogous one-loop zero mode integral can be evaluated to give the one-loop analog of $\epsilon T$. Moreover one-loop amplitudes with an unphysical state are proportional to the trace of this one-loop invariant tensor. However this trace does not vanish. Therefore the question whether $Q_S$ exact states decouple remains unanswered in this section. The computation including the $B$ integral does show that the PCOs are not $Q_S$ closed. In section 5.4 it will be shown using a different argument that unphysical states decouple to all orders, when one integrates over $B$ and $C$.

5.2.1 No $B$ integration

A one-loop amplitude with one unphysical state is given by

$$B^{(N)} = \langle Q_S \Omega_1(z_1) \prod_{i=2}^{N} dz_i U_i(z_i) \int du \int \tilde{b}_{B_1}(u,w)(\lambda B^2 d)(y) \cdots (\lambda B^{10} d)(y)$$
$(\lambda d)(y)\delta(B^1 N(y)) \cdots \delta(B^{10} N(y))\delta(J(y))\Lambda_{\delta_1 \delta_2 \delta_3}(y)(\epsilon T)^{\delta_1 \delta_2 \delta_3}_{\beta_1 \cdots \beta_{11}} \theta^{\beta_1}(y) \cdots \theta^{\beta_{11}}(y)$,  

(5.41)

where $\lambda Bd = B_{mn} \lambda \gamma^{mn} d$. Note that the $Y_C$ insertions have been replaced by the Lorentz covariant insertion, $\Lambda_{\alpha\beta\gamma}$, as in the tree-level computation. This is equivalent with inserting $Y_C$ and integrating over $C$. On the torus one cannot insert the PCOs such that all their OPE’s would vanish. They are inserted at some arbitrary point $y$. For later convenience $\tilde{b}$ is inserted at a different point, $w$.

The next step is integrating $Q_S$ by parts. When $Q_S$ acts on $\tilde{b}$ one gets a total derivative in moduli space, as usual. If this total derivative is non-vanishing the theory has a BRST anomaly. These total derivative terms will be suppressed below because they are not important for our discussion. The terms that come from $Q_S$ hitting a picture raising operator, $Z_B$, vanish since the $Q_S$ variation vanishes without subtleties, cf. (2.32). The vertex operators are also $Q_S$ closed. The only non-vanishing terms come from $Q_S$ hitting a $\theta$. This results in a $\lambda^{\beta_1}$ contracted with $\Lambda_{\alpha\beta\gamma}(\epsilon T)^{\alpha\beta\gamma}_{\beta_1 \cdots \beta_{11}}$, very similar to tree-level amplitudes with an unphysical state. However the one-loop pure spinor zero mode integration also involves $N^{mn}$. As will be shown

$\lambda^{\beta_1} \Lambda_{\alpha\beta\gamma}(\epsilon T)^{\alpha\beta\gamma}_{\beta_1 \cdots \beta_{11}}$  

(5.42)

does not vanish after the one-loop pure spinor zero mode integrals have been performed.

After integrating $Q_S$ by parts the amplitude (5.41) becomes

$B^{(N)} = \langle \Omega_1(z_1) \prod_{i=2}^{N} dz_i U_i(z_i) \rangle \int du \mu(u) \tilde{b}_B(u, w)(\lambda B^2 d)(y) \cdots (\lambda B^{10} d)(y)(\lambda d)(y)$

(5.43)

$\delta(B^2 N(y)) \cdots \delta(B^{10} N(y))\delta(J(y))\Lambda_{\delta_1 \delta_2 \delta_3}(y)(\epsilon T)^{\delta_1 \delta_2 \delta_3}_{\beta_1 \cdots \beta_{11}}(y)\lambda^{\beta_1}(y)\theta^{\beta_2}(y) \cdots \theta^{\beta_{11}}(y))$.

In this subsection $B^{(N)}$ will be evaluated without integrating over $B$. The particular choice of $B$ used here is given by

$(B^1)_{ab} = \delta_a^{[1} \delta_b^{2]} \cdots \cdots, (B^{10})_{ab} = \delta_a^{[4} \delta_b^{5]} \cdots \cdots, (B^T)^{ab} = (B^T)^{a}_{b} = 0$.

(5.44)

The amplitude $B^{(N)}$ can be evaluated by first integrating out the nonzero modes and then evaluating the zero mode integrals. The nonzero mode integration is a little tedious since there is quite a number of $N\lambda$ OPE’s one has to consider. Therefore the nonzero mode integration is explained in detail after the subsection on the zero mode integrals. Once the nonzero mode integrals have been performed the amplitude $B^{(N)}$ can be written as a sum of terms that are all proportional to a certain $\lambda, N$ zero mode integral, $I_{\beta_2 \cdots \beta_{11}}$. This integral contains (5.42). In the next section it will be shown $I_{\beta_2 \cdots \beta_{11}}$ does not vanish. This non-vanishing does not prove that there exists a non-vanishing amplitude with a $Q_S$ exact state, because there may be additional cancellations when one performs the remaining integrals. It does show however that
the PCOs are not $Q_S$ closed, i.e. (5.42) does not vanish when integrated against an
arbitrary function.

**Zero mode integral**

After integrating out the nonzero modes, which is discussed in detail in the next
subsection, one-loop amplitudes (5.43) can be written as a sum of terms that are
proportional to the following zero mode integral,

$$I_{\beta_1, \beta_2 \ldots \beta_{11}}^{\alpha_1} \equiv$$

$$\int [d\lambda] [dN] \lambda^{\alpha_1} (\lambda \gamma^{13} d) \cdots (\lambda \gamma^{45} d) (\lambda d) \delta(N^{12}) \cdots \delta(N^{45}) \delta(J) \Lambda_{\alpha \beta \gamma} (\epsilon T)^{\alpha \beta \gamma}_{\beta_1 \cdots \beta_{11}}.$$

Note that there is one unintegrated vertex operator at one loop, which explains the
presence of $\lambda^{\alpha_1}$. The factors $\lambda^{\gamma_{ab}} \delta(N^{ab})$ originate from the picture raising operators
$Z_B$ and $\delta(N^{12})$ stems from the $b$ ghost. If one of the states is an unphysical state,
the amplitude can be written as a sum of terms proportional to the trace of $I_{\beta_1, \beta_2 \ldots \beta_{11}}^{\alpha_1}$
which is called $I_{\beta_2 \ldots \beta_{11}}^{\beta_2 \ldots \beta_{11}}$:

$$I_{\beta_2 \ldots \beta_{11}}^{\beta_2 \ldots \beta_{11}} \equiv I_{\alpha_1 \beta_2 \ldots \beta_{11}}^{\alpha_1} =$$

$$\int [d\lambda] [dN] \lambda^{\beta_1} (\lambda \gamma^{13} d) \cdots (\lambda \gamma^{45} d) (\lambda d) \delta(N^{12}) \cdots \delta(N^{45}) \delta(J) \Lambda_{\alpha \beta \gamma} (\epsilon T)^{\alpha \beta \gamma}_{\beta_1 \cdots \beta_{11}}.$$

This integral is the one-loop analog of (5.40) (or (5.20)). Therefore this integral must
vanish if the PCOs are $Q_S$ closed. Note that, in spite of the notation, $I_{\beta_1, \beta_2 \ldots \beta_{11}}^{\alpha_1}$ is
not manifestly Lorentz invariant. Whether it is Lorentz invariant remains to be seen.

Let us proceed by evaluating $I_{\beta_2 \ldots \beta_{11}}^{\beta_2 \ldots \beta_{11}}$.

After using expression (3.127) for $[dN]$ to evaluate the $N$ integral in $I_{\beta_2 \ldots \beta_{11}}^{\beta_2 \ldots \beta_{11}}$ one finds

$$I_{\beta_2 \ldots \beta_{11}} = \int [d\lambda] \frac{1}{(\lambda^+)^8} \lambda^{\beta_1} (\lambda \gamma^{13} d) \cdots (\lambda \gamma^{45} d) (\lambda d) \Lambda_{\alpha \beta \gamma} (\epsilon T)^{\alpha \beta \gamma}_{\beta_1 \cdots \beta_{11}}.$$

In this form it becomes apparent that the problems with factors of $\lambda^+$ in the denominator only increase at one loop. At this point one can only surmise this. To find a definitive answer one has to evaluate the $\lambda$ integral. This can be done by expanding the integrand by powers of $\lambda^+$, using the explicit gamma matrix expression from section 3.2.4:

$$\left(\frac{1}{(\lambda^+)^8} (\lambda \gamma^{13} d) \cdots (\lambda \gamma^{45} d) (\lambda d) =$$

$$(\lambda^+)^2 D_{12} d_+ + \frac{1}{2} \lambda^+ \lambda_{a_1 a_2} \left( D_{12} \delta_{a_1 a_2} + \frac{1}{2} \epsilon^{a b c a_1 a_2} d_c D_{12 a b} d_+ \right) +$$

$$\frac{1}{8} \lambda_{a_1 a_2} \lambda_{a_3 a_4} \left( D_{12} \delta_{a_1 a_2 a_3 a_4} d_a + \epsilon^{a b c a_1 a_2} d_c D_{12 a b} d_{a_3 a_4} +$$

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\[
\frac{1}{2} \epsilon^{a b_1 a_2 c_2 d_3 a_3 f d_e d_f D_{12 a b d e d_f}} + \sum_{k=3}^{6} \frac{1}{(\lambda^+)^{k-2}} \lambda_1 \lambda_2 \cdots \lambda_{2k-1} \lambda_{2k} Y^{a_1 \cdots a_{2k}}(d),
\]
where
\[
D = d_1^2 \cdots d_{10}^2, \quad D_{a_1 \cdots a_k} = \frac{\partial}{\partial d_{a_{k-1} a_k}} \cdots \frac{\partial}{\partial d_{a_1 a_2}} D. \tag{5.49}
\]
The \(Y(d)\)'s can be expressed in terms of the \(d\)'s similar to the first three terms. Note that the minimal number of \(d\)'s in \(Y^{a_1 \cdots a_{2k}}\) is \(k - 1\). This is the reason the series stops at \(k = 6\). The maximum number of \(d\)'s in \(Y^{a_1 \cdots a_{2k}}\) is \(k\). The \(\lambda\) integration of (5.48) can be evaluated term by term. \(I_{\beta_2 \cdots \beta_{11}}\) then becomes
\[
I_{\beta_2 \cdots \beta_{11}} = \sum_{k=0}^{6} (I_k)_{a_1 \cdots a_{2k}} \beta_2 \cdots \beta_{11} Y^{a_1 \cdots a_{2k}}. \tag{5.50}
\]
The integrals \(I_k\) are investigated order by order in the sequel of this subsection.

For \(k = 0, 1, 2\) one can use the definition of \(\Lambda_{\alpha \beta \gamma}\), (5.24), and the fact that the invariant tensor \((\epsilon T)\) is traceless, (5.21), to show the \(\lambda\) integrals vanish:
\[
(I_0)_{\beta_2 \cdots \beta_{11}} = \int [d\lambda] \lambda^{\alpha_1} (\lambda^+)^2 \Lambda_{\alpha_1 \alpha_2 \alpha_3} (\epsilon T)^{\delta_1, \delta_2, \delta_3} (\beta_1 \cdots \beta_{11}) = (\epsilon T)^{\alpha_1 \alpha_2 \alpha_3} (\beta_1 \cdots \beta_{11}) = 0, \tag{5.51}
\]
\[
(I_1)_{a_1 a_2 \beta_2 \cdots \beta_{11}} = \int [d\lambda] \lambda^{\alpha_1} \lambda^{\alpha_2} \Lambda_{\alpha_1 \alpha_2 \alpha_3} (\epsilon T)^{\delta_1, \delta_2, \delta_3} (\beta_1 \cdots \beta_{11}) = (\epsilon T)^{\alpha_1 \alpha_2 \alpha_3} (\beta_1 \cdots \beta_{11}) = 0,
\]
\[
(I_2)_{a_1 \cdots a_4 \beta_2 \cdots \beta_{11}} = \int [d\lambda] \lambda^{\alpha_1} \lambda^{\alpha_2} \lambda^{\alpha_3} \Lambda_{\alpha_1 \alpha_2 \alpha_3} (\epsilon T)^{\delta_1, \delta_2, \delta_3} (\beta_1 \cdots \beta_{11}) = (\epsilon T)^{\alpha_1 \alpha_2 \alpha_3} (\beta_1 \cdots \beta_{11}) = 0.
\]

If \(k > 2\), however, there are also factors of \(\lambda^+\) in the denominator. As shown in appendix A.1 the \(\lambda\) integrals do not vanish anymore. For example consider the integral \(I_3\). By \(M\) charge conservation all components of \(I_3\) vanish except when the indices are chosen to be
\[
\beta_2, \ldots, \beta_{11} = +, b_1 b_2, \ldots, b_9 b_{10}, c_1, c_2, c_3, c_4 \tag{5.52}
\]
or
\[
\beta_2, \ldots, \beta_{11} = b_1 b_2, \ldots, b_9 b_{10}, c_1, c_2, c_3, c_4.
\]
This is explained in detail in the first part of appendix A. Let us explicitly compute \(I_3\) for the first choice of indices. Since \(\text{Sym}^3_{10} \otimes \text{Asym}_{10}^3 \otimes \text{Asym}_{45}^4\) contains one scalar, one finds
\[
(I_3)_{a_1 \cdots a_6 + c_1 c_2 c_3 c_4} = \int [d\lambda] \frac{1}{\lambda^+} \lambda^{\alpha_1} \lambda^{a_2} \cdots \lambda^{a_5} \Lambda_{\alpha_1 \alpha_2 \alpha_3} (\epsilon T)^{\alpha \beta \gamma} (\beta_1 + c_1 c_2 c_3 c_4) b_1 \cdots b_{10} = \tag{5.53}
\]
\[
\int [d\lambda] \frac{1}{\lambda^+} \lambda^{\alpha_1} \lambda^{a_2} \cdots \lambda^{a_5} \Lambda_{\alpha_1 \alpha_2 \alpha_3} (\epsilon T)^{\alpha \beta \gamma} (\beta_1 + c_1 c_2 c_3 c_4) b_1 \cdots b_{10} =
\]

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where \((\epsilon_{10})^{b_1 \cdots b_20}\) is antisymmetric under both \(b_{2i-1} \leftrightarrow b_{2i}\) and \(b_{2i-1} b_{2i} \leftrightarrow b_{2j-1} b_{2j}\) and \((\epsilon_{10})^{12131415232425343545} = 1\). The two permutations add terms to make the RHS symmetric under \(a_{2i-1} a_{2i} \leftrightarrow a_{2j-1} a_{2j}\). The constant \(c_1\) is computed in appendix A.2 and is given by

\[
c_1 = \frac{129}{2}. \tag{5.54}
\]

The integral \(I_4\) can be computed similarly, but this computation will not be presented here. The next integral is \(I_5\). The only choice of \(\beta_2, \ldots, \beta_{11}\) that leads to a nonzero answer for \(I_5\) is

\[
(I_5)_{a_1 \cdots a_{10} b_1 \cdots b_{12}} = \int [d \lambda] \frac{1}{(\lambda^+)^2} \lambda^{\beta_1} \lambda_{a_1 a_2} \cdots \lambda_{a_9 a_{10}} \Lambda_{\delta_1 \delta_2 \delta_3} (\epsilon T)^{\delta_1 \delta_2 \delta_3} \beta_{b_1 \cdots b_{12}} =
\]

\[
- \frac{2}{5} \epsilon_{b_1 a_1 a_2 a_3 a_4} \epsilon_{b_1 a_5 a_6 a_7 a_8} \epsilon_{b_1 a_9 a_{10} b_2} (\epsilon_{10})^{b_1 \cdots b_{20}} \epsilon_{b_1 b_{14} b_{16} b_{18} b_{19} b_{20}} + 14 \text{ perms}. \tag{5.55}
\]

The details are given in appendix A.2. Finally \(I_6\) can be evaluated as:

\[
(I_6)_{a_1 \cdots a_{12} \beta_2 \cdots \beta_{11}} = \int [d \lambda] \frac{1}{(\lambda^+)^4} \lambda^{\beta_1} \lambda_{a_1 a_2} \cdots \lambda_{a_{11} a_{12}} \Lambda_{\alpha \beta \gamma} (\epsilon T)^{\alpha \beta \gamma} \beta_{b_1 \cdots b_{11}} =
\]

\[
\epsilon_{b_1 a_1 a_2 a_3 a_4} \epsilon_{b_2 a_5 a_6 a_7 a_8} \epsilon_{b_3 a_9 a_{10} a_{11} a_{12}} (\epsilon T)^{b_1 b_2 b_3} + \text{permutations} = 0. \tag{5.56}
\]

This vanishes because \((\epsilon T)^{b_1 b_2 b_3} = 0\) and that follows from the \(M\) charge conservation rule for invariant tensors. In other words it is not possible to choose \(\beta_2, \ldots, \beta_{11}\) such that the total \(M\) charge of the components is zero (cf. equation (3.54)). This concludes the computation of the pure spinor zero mode integrals that appear at one loop. It has been shown that the \(Q_S\) variation of the PCO as given in (5.42) does not vanish after the integration over the pure spinor sector in a typical one-loop zero mode integral. Therefore the PCOs are not \(Q_S\) closed.

**Nonzero mode integration**

It remains to demonstrate that all one-loop amplitudes with an unphysical state can be written as a sum of terms proportional to \(I_{\beta_2 \cdots \beta_{11}}\). After this proof the argument will be modified to prove that \(A^{(N)}\) can be written as a sum of terms proportional to \(I_{\beta_1 \cdots \beta_{11}}\). In general the amplitude, \(B^{(N)}\), becomes a sum of terms of the form

\[
B^{(N)}_{i_1 \cdots i_k} = \int [D\lambda][D\theta][D\theta][D\theta] (\prod_{i=2}^N \int dz_i) f_{m_{n_1} \cdots m_{n_k}} (z_1, \ldots, z_N) \tag{5.57}
\]

\[
N^{m_{n_1}} (z_{i_1}) \cdots N^{m_{n_k}} (z_{i_k}) (\lambda \gamma^{13} d)(y) \cdots (\lambda \gamma^{45} d)(y) (\lambda d)(y) \Lambda_{\alpha \beta \gamma} (y)(\epsilon T)^{\alpha \beta \gamma} \theta^{\beta_2}(y) \cdots \theta^{\beta_{11}}(y) \int d\mu(u) b_{B,1}(u, w) \delta(N^{13}(y)) \cdots \delta(N^{45}(y)) \delta(J(y)) e^{-S},
\]

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where the indices in the PCOs are $SU(5)$ indices, $i_j \in \{2, \ldots, N\}$ and $f_{m_1 \cdots n_k}$ does not contain any $\lambda$’s or $w$’s. The number $k$ indicates how many vertex operators provide an $N^{mn}$. The functional integrals over $\lambda$ and $N$ can be evaluated by performing the OPE’s to remove all fields of nonzero weight. Then one replaces the fields by their zero modes and performs the integration over these modes. In order to perform the OPE between $N^{mn}$ and $\delta(BN)$ one has to Taylor expand $\delta(BN)$, as discussed in [22],

$$\delta((BN(y)) = \delta(BN_0 \omega(y) + \hat{N}(y)) =$$

$$\delta(BN_0 \omega(y)) + (\hat{N}(y))\delta'(BN_0 \omega(y)) + \frac{1}{2}(\hat{N}(y))^2\delta''(BN_0 \omega(y)) + \cdots,$$

where $\hat{N}$ denotes $N$ after omission of the zero mode. The holomorphic one form $\omega(y)$ is constant on the torus:

$$\omega(y) = \frac{1}{4\pi^2 \tau_2},$$

where $\tau_2$ is the imaginary part of the modulus $\tau$. The $b$ ghost also contains $N^{mn}$’s which have to be taken into account if one is removing all fields of nonzero weight. Let us start with the first term, the local $b$ ghost, $b_\mu(u)$. The second term of $b(u, y)$, with the integration in it, will be dealt with later. After replacing $\tilde{b}(u, y)$ by $b_\mu(u)$ in the amplitude, $B^{(N)}_{i_1 \cdots i_k}$, becomes a sum over $n$, which counts the number of $N^{mn}$’s the local $b$ ghost provides, of the following objects:

$$B^{(N)}_{i_1 \cdots i_k, n} = \int [d\lambda][dN][Dd][D\theta](\prod_{i=2}^{N} \int dz_i) \int du \mu(u) \sum_{j=0}^{3} f_{jm_1 n_1 \cdots m_{k+n} n_{k+n}}(z, u, w)$$

$$N^{m_1 n_1}(z_{i_1}) \cdots N^{m_k n_k}(z_{i_k}) N^{m_{k+1} n_{k+1}}(w) \cdots N^{m_{k+n} n_{k+n}}(w)(\lambda \gamma_{13} d(y)) \cdots (\lambda \gamma_{45} d(y))$$

$$(\lambda d(y) \lambda \beta_1(y)(\epsilon T)^{\alpha\beta_1 \beta_{11}}_{\beta_{11}})_{\Lambda_{\alpha \beta} \gamma}(y) \theta^{\beta_2}(y) \cdots \theta^{\beta_{11}}(y)$$

$$\delta^{(j)}(N^{12}(w))\delta(N^{13}(y)) \cdots \delta(N^{45}(y)) \delta(J(y))e^{-S},$$

(5.60)

where $\delta^{(j)}$ denotes the $j$th derivative of the delta function and the sum runs from zero to three because $b$ does not contain $\delta^{(4)}(B^4 N)$ or higher derivatives.

The product of the eleven delta functions, including the one from $b$, becomes a sum of products of eleven $\delta^{(j)}(B^I N_0)$ after the Taylor expansion. Let us start with the first term in this sum, i.e. the one without $\hat{N}$’s and no derivatives on the delta functions. In this case the $N^{m_j n_j}(z_j)$’s from (5.57) have OPE’s with themselves and with the $\lambda$’s from the PCOs. Let us first concentrate on the term in which all $N^{mn}$’s get contracted with an explicit $\lambda$. That term is given by

$$C^{(N)}_{c_1 \cdots c_k} = \int [d\mu][dN][D^{16}d][D^{16}\theta](\prod_{i=2}^{N} \int dz_i) \int du f_{m_1 n_1 \cdots m_{k+n} n_{k+n}}(z_1, \cdots, z_N, u)$$

\footnote{Since the distinction between worldsheet fields and their zero modes plays a central role in the argument, zero modes are denoted in an explicit way, unlike in other parts of this work.}
\[ \prod_{l=1}^{k} F(z_{i_l}, y) F(w, y) N^{m_1 n_1} \cdots N^{m_{k+n} n_{k+n}} \lambda^2 \left( \lambda_0 \gamma^{13} d(y) \right) \cdots \left( \lambda_0 \gamma^{45} d(y) \right) \]

\[ (\lambda_0 d(y))(\Lambda_0)_{\alpha \beta \gamma}(eT)^{\alpha \beta \gamma}_{\beta_1 \cdots \beta_{11}} \theta^2 \cdots \theta^2 \delta(N_0^{12}) \cdots \delta(N_0^{45}) \delta(J_0) e^{S_{\nu \beta}} , \] (5.61)

where

\[ F(z, y) = \partial_z \log E(z, y) \] (5.62)

and \( E(z, y) \) is the holomorphic prime form, which goes like \( z - y \) when \( z \to y \) [3, 18]. \( N^{mn} \) are abstract Lorentz generators for the \( \lambda, w \) sector and they act to the right. They should not be thought of as containing (zero) modes of the \( \lambda \) or \( w \) worldsheet fields. The \( N^{mn} \) merely multiply every index on a \( \lambda \) or \( w \) they hit by a two form gamma matrix. Up to now only contractions between \( N^{mn} \) and the explicit \( \lambda \)'s have been considered, but if two or more \( N^{mn} \)'s contract with each other in \( B_{i_1 \cdots i_k, n}^{(N)} \) one gets a term of the form \( C_{i_1 \cdots i_k, m}^{(N)} \), with \( l + m < k + n \), where the poles in \( z_i - z_j \) are included in the unspecified function \( f \).

The last step of our argument is showing all terms with derivatives on the delta functions can also be written as a sum of terms of the form \( C_{i_1 \cdots i_k, n}^{(N)} \). To see this note that if a derivative acts on \( \delta(N^{ab}) \) one of the \( N^{mn} \) must provide this zero mode, otherwise the integral vanishes. This step just reduces the number of \( N^{mn} \)'s in \( B_{i_1 \cdots i_k, n}^{(N)} \) that must be contracted, so in fact it becomes of the form \( C_{i_1 \cdots i_k, m}^{(N)} \) where \( k + n - l - m \) is the number derivatives acting on the delta functions. Since the zero mode measures \([d\lambda]\) and \([dN]\) are Lorentz invariant one can pull the \( N \) out of these integrals. This concludes the main part of the argument that a one-loop amplitude can be written as a sum of terms proportional to \( I_{\beta_2 \cdots \beta_{11}} \).

One still needs to consider the second term in \( \bar{b}(u, w) \). This was not included in the above discussion because it contains \( \partial N^{mn}(v) \). This does not change the argument much, after the OPE's this part of the amplitude will also have the form of \( C_{i_1 \cdots i_k, n}^{(N)} \) where the effect of the \( v \) derivative and the integral over \( v \) are included in \( f \).

To see \( \mathcal{A}^{(N)} \) can be written as a sum of terms proportional to \( F_{\beta_1 \cdots \beta_{11}}^{(N)} \) one can use the above reasoning with a slight adjustment. This consists of replacing \( \lambda^{\beta_1}(y) \) by \( \lambda^{\alpha_1}(z_1) \) in (5.57) and adding an \( \alpha_1 \) index to \( f \). The only effect this has is the replacement of some \( F(z_1, y) \) by \( F(z_1, z_i) \) in (5.61), apart from the fact \( \alpha_1 \) and \( \beta_1 \) are not contracted anymore.

**Four point function**

The one-loop four-point function with an unphysical state in the formulation without an integral over \( B \) vanishes. This should come as a surprise after the result of the previous section, where it was shown that the \( Q_S \) variation of the PCOs does not vanish. The vanishing of the amplitude is instead achieved after the integral over the \( d \) zero modes has been performed.
The one-loop four-point amplitude is an example of an amplitude in which only the zero modes contribute (cf. [22]). It turns out only three terms have enough factors of \( d_a \) and \( N^{mn} \) to give a non-vanishing answer. This will become clear in equation (5.64) below. Thus one can immediately replace all the fields in (5.41) by their zero modes:

\[
B^{(4)} = \int [d\lambda][dN]d^{16}dd^{16}\theta\bar\Omega S\prod_{i=2}^4 U_i\bar{b}_B^i(\lambda B^2d)\cdots(\lambda B^{10}d)(\lambda d)
\]

(5.63)

\[
\delta(B^1N)\cdots\delta(B^{10}N)\delta(J)\Lambda_{a\beta\gamma}(\epsilon T)_{\beta_1\cdots\beta_{11}}^a\theta^{\beta_1}\cdots\theta^{\beta_{11}}.
\]

For the \( d \) integration to be non-vanishing there must be a total of sixteen \( d \) zero modes, therefore the only terms of \( b_B \) that contribute are the ones with four \( d \)'s and there are only three such terms:

\[
(b_B)|_{d^4} = -\frac{1}{1536}\gamma^{\alpha\beta}_{mnp}(d\gamma^{mnp}d)(Bd)\alpha(Bd)\beta\delta'(BN)
\]

(5.64)

\[
-\frac{1}{8}c^{\gamma\delta\alpha\rho}_1N^{mn}d_\rho(Bd)\alpha(Bd)\beta\gamma\delta''(BN)
\]

\[
-\frac{1}{16}c^{\delta\gamma\alpha\beta}_4N^{mn}N^{pq}(Bd)\alpha(Bd)\beta(Bd)\gamma(Bd)\delta''(BN),
\]

where the invariant tensors \( c_1 \) and \( c_4 \) can be read off from (2.39)-(2.42) and (3.139)-(3.148). Note the \( N \) integration will only be non-vanishing if the fourth vertex operator provides an \( N^{mn} \) zero mode. Moreover there are no terms in the \( b \) ghost with three \( d \)'s and no derivatives on \( \delta(BN) \). Such terms could have contributed here. The three terms above turn out to all be proportional to (for \( B_{ab} = \delta^1_{\alpha\beta}\delta^2_{\nu\mu}\), \( B^a_b = B^{ab} = 0\))

\[
d^{12}d_3d_4d_5\delta'(N^{12}).
\]

(5.65)

For the first term this follows from direct computation using the gamma matrices as listed in section 3.2.4. Actually, one could have predicted the fact that three of the four \( d_a \)'s are \( d_a \)'s and one is a \( d^{ab} \), by looking at the \( M \) charge of the full term. \( \delta'(N^{12}) \) has \( M \) charge two and since \( \gamma^{\alpha\beta}_{mnp}(d\gamma^{mnp}d)(Bd)\alpha(Bd)\beta\delta'(BN) \) has \( M \) charge zero, the \( d \) part must have \( M \) charge minus two. The only way four \( d \)'s can give \( M \) charge minus two is when three of them are a \( d_a \) (\( M \) charge \(-\frac{3}{4}\)) and the fourth is a \( d^{ab} \) (\( M \) charge \(-\frac{1}{4}\)).

The second term can be reduced as follows:

\[
(c_1)\gamma^{\alpha\beta\rho}_{mnp}N^{mn}d_\rho(Bd)\alpha(Bd)\beta\delta''(BN) =
\]

(5.66)

\[
(c_1)_{12\ a_1\cdots a_8}d^{a_1a_2a_3}d_a\frac{1}{2}\epsilon^{a_3a_4}d_{12}a_2b\frac{1}{2}\epsilon^{a_5a_6}d_{12}c\delta'(N^{12}),
\]

where the \( M \) charge conservation property of invariant tensors was used together with \((Bd)\alpha = 0\). After observing that \((c_1)_{aba_1\cdots a_8}\) is an \( SU(5) \) invariant tensor that
is antisymmetric in the middle three pairs of indices \( (a_1a_2, a_3a_4, a_5a_6) \) and there is only one invariant tensor with these symmetries \([40]\), namely \( \epsilon_{aba_1a_2|a_3|a_5a_6a_7a_8} + 5 \text{ perms} \), one finds that the second term in the \( b \) ghost is proportional to

\[
(c_1)_{12345678}d^{a_7a_8}d_3d_4d_5\delta' (N^{12}) = d^{12}_3d_4d_5\delta' (N^{12}).
\] (5.67)

The same logic can be applied to the third term although this case is slightly simpler. \( \alpha, \beta, \gamma, \delta \) has to be \( +, ab, cd, ef \) and since \( (Bd)_+ = d^{12} \) one automatically gets this factor.

The third integrated vertex operator must provide an \( N^{12} \) zero mode. It then follows that \( \mathcal{B}^{(4)} \) is proportional to \( I_{\beta_2 \cdots \beta_{11}} \). This integral can be written as a sum over \( k \) just as in (5.48). In this sum the \( k = 0, 1, 2, 6 \) terms vanish because of the \( \lambda \) integration and the \( k = 4, 5 \) terms vanish due to the \( d \) integration (note that \( b_{B^1} \) contains three \( d_a \)’s and \( Y_4, Y_5 \) contain at least three \( d_a \)’s). The \( k = 3 \) term is given by

\[
(b_{B^1})|d^4(I_3)_{a_1 \cdots a_6\beta_2 \cdots \beta_{11}(Y_3)^{a_1 \cdots a_6}} = \frac{1}{32}d^{12}_3d_4d_5\left( \epsilon_{aba_1a_2c}d_cD_{12ab}\epsilon_{a_3a_4a_5a_6}d_d + \right.
\]

\[
\epsilon_{aba_1a_2c}d_c\epsilon_{dea_3a_4}d_fd_fD_{12abdec}d_5a_5a_6 + \frac{1}{2}\epsilon_{aba_1a_2c}\epsilon_{dea_3a_4f}\epsilon_{gha_5a_6j}d_c\epsilon_{d_fd_fd_j}D_{12abdecgh}d_5. \]
\] (5.68)

\[
\int [d\lambda] - d^{12}_3d_4d_5 \epsilon_{aba_1a_2c}d_c\epsilon_{dea_3a_4}d_fd_fD_{12abdec}d_5a_5a_6 + \frac{1}{2}\epsilon_{aba_1a_2c}\epsilon_{dea_3a_4f}\epsilon_{gha_5a_6j}d_c\epsilon_{d_fd_fd_j}D_{12abdecgh}d_5.
\]

\[
\int [d\lambda] - d^{12}_3d_4d_5 \epsilon_{aba_1a_2c}d_c\epsilon_{dea_3a_4}d_fd_fD_{12abdec}d_5a_5a_6 + \frac{1}{2}\epsilon_{aba_1a_2c}\epsilon_{dea_3a_4f}\epsilon_{gha_5a_6j}d_c\epsilon_{d_fd_fd_j}D_{12abdecgh}d_5.
\]

where the following identity was used

\[
D_{12abdec}d^ef = - \delta_{c}^{[e} \delta_d^{f]} D_{12ab} - \delta_1^{[c} \delta_2^{f]} D_{abcd} - \delta_3^{[c} \delta_6^{f]} D_{cd12} \] (5.69)

and the integral vanishes because \( \epsilon T \) is traceless.

Thus, for the four-point one-loop amplitudes with a \( Q_S \) exact state the terms that do not vanish after the \( \lambda, N \) integral now vanish because they contain a square of fermionic quantity, namely \( d_a d_a \) (no sum). Decoupling of unphysical states in higher point function is much more tedious to check since the nonzero mode integrations are non-trivial and the lack of manifest Lorentz invariance.

### 5.2.2 Including \( B \) integration

At tree level decoupling of unphysical states was restored after integrating over the constant spinors \( C \). In this section manifest Lorentz invariance for one-loop amplitudes is restored by including the \( B \) integration. Whether this leads to decoupling
of unphysical is the subject of this section. Similar to the tree-level case one can show that all amplitudes are proportional to a certain invariant tensor (at tree level this was \((\epsilon T)\)) and amplitudes with \(Q_S\) exact states are proportional to the trace of this invariant tensor. However, at one loop the trace of this tensor does not vanish.

Following the same steps as in the previous subsection (section 5.3 contains details of these steps), one can show that all amplitudes can be written as a sum of terms proportional to the following zero mode integral

\[
X_{\alpha_1 \cdots \alpha_{11}}^{\beta_1 \cdots \beta_{11}} m_1 n_1 \cdots m_{10} n_{10} \equiv \int \frac{dB}{[dC][d\lambda][dN]} \lambda^{\alpha_1} \cdots \lambda^{\alpha_{11}}
\]

(5.70)

\[
B_{m_1 n_1}^1 \cdots B_{m_{10} n_{10}}^{10} C_{\beta_1}^1 \cdots C_{\beta_{11}}^{11} \delta(C^1 \lambda) \cdots \delta(C^{11} \lambda) \delta(B^1 N) \cdots \delta(B^{10} N) \delta(J).
\]

Proportional here means in the sense of tensor multiplication: in the terms that appear after contractions, the tensor \(X\) is multiplied by gamma matrices. Evaluating the integrals in (5.70) is much easier than one might have anticipated, because \(X\) must be an invariant tensor, that is symmetric and gamma matrix traceless in the \(\alpha\)'s, antisymmetric in the \(\beta\)'s and antisymmetric in both \(m_i \leftrightarrow n_i\) and \(m_i n_i \leftrightarrow m_j n_j\).

To find out how many independent invariant tensors with these properties exist, one has to compute the number of scalars in the relevant tensor product, which is one (see also section 3.2.3). The relevant invariant tensor has already appeared in the one-loop prescription in chapter 2:

\[
(\epsilon TR)_{\alpha_1 \cdots \alpha_{11}}^{\beta_1 \cdots \beta_{11} m_1 n_1 \cdots m_{10} n_{10}} \equiv (\epsilon T)_{\beta_1 \cdots \beta_{11}}^{(\alpha_1 \alpha_2 \alpha_3 R_{m_1 n_1 \cdots m_{10} n_{10}}^{\alpha_4 \cdots \alpha_{11}})},
\]

(5.71)

where the double brackets denote gamma matrix traceless, cf. section 3.5. Lorentz invariance has completely fixed \(X\), there is no freedom remaining.

Starting from a correlator with an unphysical state and integrating \(Q_S\) by parts, it will hit a \(\theta\) from a PCO (where the total derivative in moduli space obtained when \(Q_S\) acts on \(\tilde{b}\) is again suppressed, this derivative does not play a role here). This means all amplitudes with an unphysical state can be written as a sum of terms proportional to the trace of \((\epsilon TR)\):

\[
\int \frac{dB}{[dC][d\lambda][dN]} \lambda^{\alpha_2} \cdots \lambda^{\alpha_{11}} B_{m_1 n_1}^1 \cdots B_{m_{10} n_{10}}^{10} \delta(C^{11} \lambda) \delta(B^1 N) \cdots \delta(B^{10} N) \delta(J) = (\epsilon TR)_{\alpha_1 \beta_2 \cdots \beta_{11} m_1 n_1 \cdots m_{10} n_{10}}^{\alpha_2 \cdots \alpha_{11}}.
\]

(5.72)

There are two independent invariant tensors with indices and symmetries of the trace of \((\epsilon TR)\), so one expects a non-vanishing trace. Indeed, it is proved in section 5.2.3 that this trace does not vanish, which provides another proof for the fact that the PCO is not \(Q_S\) closed. The non-vanishing of the trace implies the proof of decoupling of unphysical states at tree level does not generalise to one loop and one needs a new argument. Such a new argument is presented in section 5.4, where it is shown that unphysical states decouple to all loop order.
Comparison to non-minimal formalism

In this subsection a brief comparison with the non-minimal formalism [25] is made. In this case all insertions are $Q_S$ closed and decoupling of unphysical states follows straightforwardly.

In the non-minimal formalism the PCOs are replaced by

$$N = e^{-(\lambda \bar{\lambda} + r \theta + \frac{1}{2} N_{mn} N^{mn} + \frac{1}{2} S_{mn} \lambda \gamma^{mn} d + J \bar{J} + \frac{1}{2} S \lambda d)}, \quad (5.73)$$

This is invariant under $Q_S$:

$$Q_S N = (\lambda r - \lambda r + \bar{N}^{mn} \frac{1}{2} \lambda \gamma_{mn} d - \bar{N}^{mn} \frac{1}{2} \lambda \gamma_{mn} d + \bar{J}(\lambda d) - \bar{J}(\lambda d)) N = 0. \quad (5.74)$$

Thus, all problematic terms of the minimal formalism are manifestly absent here and $Q_S$ exact states decouple. In other words, these amplitudes vanish because two equal terms are subtracted.

5.2.3 Non-vanishing of the trace of $(\epsilon TR)$

In this subsection the trace $\text{Tr} (\epsilon TR)$ of the tensor $(\epsilon TR)$ is computed. To show that this trace does not vanish it is convenient to define a tensor $Y$ and an operator $\hat{X}$:

$$Y_{m_1 \cdots n_{10}} \equiv \bar{\lambda}_{\alpha_4} \cdots \bar{\lambda}_{\alpha_{11}} R^{\alpha_4 \cdots \alpha_{11}}_{m_1 \cdots n_{10}}, \quad (5.75)$$

$$\hat{X} \equiv \psi_{\beta_1 2} \cdots \psi_{\beta_{16}} \bar{\lambda}_{\alpha_1} \cdots \bar{\lambda}_{\alpha_3} T_{\beta_1 2 \cdots \beta_{16}, \alpha_1 \alpha_2 \alpha_3} \psi_{\bar{\alpha}} \frac{\partial}{\partial \bar{\lambda}_{\alpha}}, \quad (5.76)$$

where $\psi_{\alpha}$ is a fermionic Weyl spinor and $\bar{\lambda}_{\alpha}$ is a pure spinor of opposite chirality to $\lambda^{\alpha}$. Note that, because $\bar{\lambda}_{\alpha}$ is a constrained spinor, $\partial/\partial \bar{\lambda}_{\alpha}$ is only defined up to a gauge transformation:

$$\delta \frac{\partial}{\partial \bar{\lambda}_{\alpha}} = A^m (\gamma_m \bar{\lambda})^{\alpha}. \quad (5.77)$$

The operator $\hat{X}$, however, is well defined, since it is gauge invariant. This follows from

$$\bar{\lambda}^{\beta} \gamma^{q} \psi_{\beta_{12}} \cdots \psi_{\beta_{16}} \bar{\lambda}_{\alpha_1} \cdots \bar{\lambda}_{\alpha_3} T_{\beta_1 2 \cdots \beta_{16}, \alpha_1 \alpha_2 \alpha_3} = 0. \quad (5.78)$$

That can be shown by noting there are no scalars in Asym$^6 16'$ $\otimes$ 10 $\otimes$ Gam$^4 16'$, where Gam means the symmetric and gamma matrix traceless tensor product. Note one can use

$$\frac{\partial}{\partial \bar{\lambda}_{\alpha}} \bar{\lambda}_{\beta} = \delta_{\beta}^{\alpha} \quad (5.79)$$

when $\partial/\partial \bar{\lambda}_{\alpha}$ is part of a gauge invariant quantity, $S_{\alpha} (\partial/\partial \bar{\lambda}_{\alpha})$, because

$$S_{\alpha} \frac{\partial}{\partial \bar{\lambda}_{\alpha}} \bar{\lambda}^{\gamma} \gamma^{m} \bar{\lambda} = S_{\gamma}^{m} \gamma^{m} \bar{\lambda} = 0, \quad (5.80)$$
the last equality is a consequence of gauge invariance.

The first step of the argument is showing that \( \hat{X}Y \neq 0 \). The second and last step is proving this implies the trace of \((\epsilon T R)\) does not vanish. Consider the following component of \( \hat{X}Y \) in a Lorentz frame in which the only nonzero component of \( \lambda \) is \( \lambda_+ \):

\[
\hat{X}Y_{a_1 b_1 a_2 b_2 \ldots a_{10} b_{10}} = (\lambda_\gamma m \psi)(\lambda_\gamma n \psi)(\psi_\gamma^{mn} \psi)
\]

\[
[2(\lambda_\gamma a_1 b_1 a_2 a_3 a_4 \hat{\lambda})(\lambda_\gamma a_5 b_5 a_2 a_6 a_7 \hat{\lambda})(\lambda_\gamma a_8 b_8 a_3 b_9 \hat{\lambda})(\lambda_\gamma a_{10} b_{10} b_2 b_7 b_9 \hat{\lambda})
\]

\[
2(\lambda_\gamma a_1 b_1 a_2 a_3 a_4 \hat{\lambda})(\lambda_\gamma a_5 b_5 a_2 a_6 a_7 \hat{\lambda})(\lambda_\gamma a_8 b_8 b_3 b_9 \hat{\lambda})(\lambda_\gamma a_{10} b_{10} b_4 b_7 b_9 \hat{\lambda})
\]

\[
2(\lambda_\gamma a_1 b_1 a_2 a_3 a_4 \hat{\lambda})(\lambda_\gamma a_5 b_5 a_2 a_6 a_7 \hat{\lambda})(\lambda_\gamma a_8 b_8 b_3 a_9 \hat{\lambda})(\lambda_\gamma a_{10} b_{10} b_4 b_7 b_9 \hat{\lambda})
\]

\[
\text{+ permutations },
\]

where the permutations make the RHS antisymmetric in \( a_i b_i \leftrightarrow a_j b_j \). This reduces, up to an overall constant which is not zero\(^5\), to

\[
\hat{X}Y_{a_1 b_1 a_2 b_2 \ldots a_{10} b_{10}} = \epsilon^{c_1 \cdots c_5} \psi_{c_1} \cdots \psi_{c_5} (\lambda_+)^{10} \psi_+ + \epsilon_{a_1 b_1 a_2 a_3 a_4} \epsilon_{a_5 b_5 a_2 a_6 a_7} + \text{permutations} = \epsilon^{c_1 \cdots c_5} \psi_{c_1} \cdots \psi_{c_5} (\lambda_+)^{10} \psi_+ (\epsilon_{10})_{a_1 \ldots b_{10}} \neq 0.
\]

What remains is to show the non-vanishing of this tensor implies the non-vanishing of the trace of \((\epsilon T R)\).

\[
\hat{X}Y_{m_1 n_1 \ldots m_{10} n_{10}} = \epsilon^{\beta_1 \ldots \beta_{10}} [(\epsilon T)^{((\alpha_1 \alpha_2 \alpha_3 \psi_{a_1} \psi_{a_2} \ldots \psi_{a_{10}}) R_{m_1 n_1 \ldots m_{10} n_{10}} \lambda_{a_1} \ldots \lambda_{a_{10}})].
\]

For the term in the square brackets one can move the \( \alpha_{11} \) to \( (\epsilon T) \) by using

\[
0 = (\epsilon T)^{\alpha_{11} \alpha_2 \alpha_3} \psi_{\beta_1} \cdots \psi_{\beta_{16}} \psi_{\alpha_{11}} =
\]

\[
6(\epsilon T)^{\alpha_{11} \alpha_2 \alpha_3} \psi_{\beta_1} \cdots \psi_{\beta_{16}} \psi_{\alpha_{11}} + 11(\epsilon T)^{\alpha_{11} \alpha_2 \alpha_3} \psi_{\beta_{11}} \cdots \psi_{\beta_{16}}.
\]

The first line is zero because it contains an antisymmetrisation of seventeen indices that only take sixteen values.

\[
\hat{X}Y_{m_1 n_1 \ldots m_{10} n_{10}} = \epsilon^{\beta_1 \cdots \beta_{16}} [(\epsilon T)^{((\alpha_1 \alpha_2 \alpha_3 \psi_{a_1} \psi_{a_2} \ldots \psi_{a_{10}}) R_{m_1 n_1 \ldots m_{10} n_{10}} \lambda_{a_1} \ldots \lambda_{a_{10}})].
\]

Since \((\epsilon T R)^{\alpha_{11} \alpha_{12} \cdots} \) is fully antisymmetric in \( \beta_2 \cdots \beta_{11} \) and symmetric and gamma matrix traceless in \( \alpha_1 \cdots \alpha_{10} \), one can conclude from the non-vanishing of \( \hat{X}Y \) that

\[
(\epsilon T)^{((\alpha_1 \alpha_2 \alpha_3 \ R_{m_1 n_1 \ldots m_{10} n_{10}} \neq 0).
\]

\(^5\)Constants were omitted in the following two relations:

\[
(\lambda_\gamma m \psi)(\lambda_\gamma n \psi)(\lambda_\gamma p \psi)(\psi_\gamma^{mn} \psi) \propto \epsilon^{c_1 \cdots c_5} \psi_{c_1} \cdots \psi_{c_5} (\lambda_+)^3,
\]

\[
(\gamma_{a_1 b_1 a_2 a_3 a_4} \gamma_{a_5 b_5 a_2 a_6 a_7} \gamma_{a_8 b_8 b_3 b_9} \gamma_{a_{10} b_{10} b_4 b_7 b_9} \text{ + permutations} \propto (\epsilon_{10})_{a_1 b_1 \ldots a_{10} b_{10}}.
\]
5.3 No-go theorem for $Q_S$ closed, Lorentz invariant PCOs

In the previous section it was proved that the Lorentz invariant PCO was not $Q_S$ closed. A logical next step is to give a modified prescription in which the PCO is $Q_S$ closed. However the non-vanishing of the trace of the invariant tensor $\epsilon TR$, which played an important role in the previous section, places severe restrictions on a $Q_S$ closed PCO. It turns out that any Lorentz invariant $Q_S$ closed PCO leads to vanishing of all one loop amplitudes.

A Lorentz invariant $Q_S$ closed PCO is defined as an operator $Y$ that satisfies

- $Y = f_{\beta_1 \ldots \beta_{11}}(\lambda) \theta^{\beta_1} \ldots \theta^{\beta_{11}}$,
- $f_{\beta_1 \ldots \beta_{11}}(\lambda)$ has $J_{\lambda w}$ charge $-11$,
- $f_{\beta_1 \ldots \beta_{11}}(\lambda)$ is a Lorentz tensor,
- $Q_S Y = 0$.

The original proposal in [22] is the special case where the function $f$ is given by

\[ f_{\beta_1 \ldots \beta_{11}} = \int [dC] C^1_{[\beta_1} \ldots C^{11}_{\beta_{11}]} \delta(C^1 \lambda) \ldots \delta(C^{11} \lambda). \tag{5.90} \]

This satisfies the first three conditions, but although $Q_S Y \sim \lambda \delta(\lambda)$ the fourth bullet does not hold for (5.90).

Using the fact that $f$ is a Lorentz tensor one finds,

\[ \int [dB][d\lambda][dN] \lambda^{\alpha_1} \ldots \lambda^{\alpha_{11}} B^1_{m_1 n_1} \ldots B^{10}_{m_{10} n_{10}} f_{\beta_1 \ldots \beta_{11}}(\lambda) \delta(B^1 N) \ldots \delta(B^{10} N) \delta(J) = c_1 (\epsilon TR)^{\alpha_1 \ldots \alpha_{11}}_{\beta_1 \ldots \beta_{11} m_1 n_1 \ldots m_{10} n_{10}}, \tag{5.91} \]

for some $c_1$. This follows from the fact that $(\epsilon TR)$ is the unique Lorentz tensor with the indicated tensor structure. Now the crucial observation is that for functions $f$ such that $Q_S Y = 0$ the integral (5.91) must be equal to zero. Indeed, using

\[ 0 = Q_S Y = f_{\beta_1 \ldots \beta_{11}}(\lambda) \lambda^{\beta_1} \theta^{\beta_2} \ldots \theta^{\beta_{11}}. \tag{5.92} \]

leads to

\[ 0 = \int [dB][d\lambda][dN] \lambda^{\alpha_2} \ldots \lambda^{\alpha_{11}} B^1_{m_1 n_1} \ldots B^{10}_{m_{10} n_{10}} (f_{\beta_1 \ldots \beta_{11}}(\lambda) \lambda^{\beta_1} \theta^{\beta_2} \ldots \theta^{\beta_{11}}). \]

\[ f_{\beta_1 \ldots \beta_{11}} = (\epsilon T)^{\alpha \beta \gamma}_{\beta_1 \ldots \beta_{11}} \Lambda_{\alpha \beta \gamma}. \tag{5.89} \]
\[ \delta(B^{1}N) \cdots \delta(B^{10}N)\delta(J) = c_{1}(\epsilon TR)^{\alpha_{1}\cdots \alpha_{11}}_{\beta_{1} \cdots \beta_{11}m_{1}n_{1}\cdots m_{10}n_{10}} \theta^{\beta_{2}} \cdots \theta^{\beta_{11}}. \] (5.93)

The trace of \((\epsilon TR)\) does not vanish as shown in section 5.2.3. Hence one can conclude that
\[ c_{1} = 0. \] (5.94)

To prove vanishing of all one-loop amplitudes the above result is not enough, because there are also zero mode integrals with derivatives on the delta functions and \(N\) insertions. After the nonzero mode integration is performed, an arbitrary amplitude is reduced to a sum of zero mode integrals, all of which are of the form
\[
\mathcal{E}_{\beta_{1} \cdots \beta_{11}m_{1}n_{1}\cdots m_{10}n_{10}r_{1}s_{1}\cdots r_{L}s_{L}} = L \int [dB][dN][d\lambda] \prod_{j=1}^{L} N^{P_{j}/q_{j}} \prod_{i_{1}=1}^{L_{1}} B_{r_{1}s_{1}}^{1} \prod_{i_{2}=L_{1}+1}^{L+L_{2}} B_{r_{2}s_{2}}^{2} \cdots \prod_{i_{10}=L_{1}+\cdots+L_{9}+1}^{L} B_{r_{10}s_{10}}^{10},
\]

\[
\lambda^{\alpha_{1}} \cdots \lambda^{\alpha_{11}} f_{\beta_{1} \cdots \beta_{11}}(\lambda) B_{m_{1}n_{1}}^{1} \cdots B_{m_{10}n_{10}}^{10} \delta^{(L_{1})}(B^{1}N) \cdots \delta^{(L_{10})}(B^{10}N)\delta(J),
\] where all the fields are zero modes and \(L = \sum_{P=1}^{10} L_{P}\) and \(\delta^{(m)}(x)\) denotes the \(m\)-th derivative of \(\delta(x)\). All zero mode integrands have to be of the form (2.48), (2.49) for a non-vanishing answer. In order to write down the above zero mode integrand one starts from the general functions \(f_{B}, h_{B}\) from chapter 2 and uses the following four arguments.

- For each \(P\) the total number of \(B^{P}\)’s outside the delta functions is equal to the number of derivatives on \(\delta(B^{P}N)\) plus one. This can be inferred from the explicit form of the \(b\) ghost, (2.38), and the Taylor expansion of the delta functions. This is reflected in (5.95) because \(L_{P}\) appears in two places.

- For a nonzero answer the total number of \(N\) zero modes must equal the total number of derivatives on the delta functions. This gives the restriction \(L = \sum L_{P}\).

- One might have expected derivatives on \(\delta(J)\) as well, but for a non-vanishing answer there must also be enough \(J\) zero modes, so one can always reduce the amplitude to contain only \(\delta(J)\).

- Compared to (2.48) the \(\lambda\) dependence is less general. It is possible to restrict to this class of integrands because \(f_{\beta_{1} \cdots \beta_{11}}(\lambda)\) is a Lorentz tensor. To see this note the OPE’s of \(N\) and \(J\) with \(f\) do not introduce derivatives:

\[
N^{mn}(z)f_{\beta_{1} \cdots \beta_{11}}(\lambda(w)) \sim \sum_{i=1}^{11} (\gamma^{mn})^{\alpha}_{\beta_{i}} f_{\beta_{1} \cdots \alpha \cdots \beta_{11}}(\lambda(w)) \frac{1}{z-w},
\] (5.96)

\[
J(z)f_{\beta_{1} \cdots \beta_{11}}(\lambda(w)) \sim -11 f_{\beta_{1} \cdots \beta_{11}}(\lambda(w)) \frac{1}{z-w},
\] (5.97)

where the \(\alpha\) index is in the \(i\)th position.

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Note that the free indices on $E$ can be either contracted among each other or with $d$ or $\theta$ zero modes. The integral in (5.95) can be evaluated by using the definition of $B$ integration in (2.50). Let us call the integrand of (5.95) $g$ and write it as

$$g(\lambda, N, J, B^P) = \lambda^{\alpha_1} \cdots \lambda^{\alpha_{11}} h_{\alpha_1 \cdots \alpha_{11}}^{\beta_1 \cdots \beta_{11}} (N, J, B^P) \prod_{P=1}^{10} \delta^{(L_P)} (B^P N) f_{\beta_1 \cdots \beta_{11}} (\lambda), \quad (5.98)$$

where $h$ is a polynomial depending on $(N, J, B)$ as

$$(N)^L \prod_{P=1}^{10} (B^P)^{L_P+1}. \quad (5.99)$$

It also contains other fields (e.g. $\theta, d$) but these are suppressed.

The integrations can be performed using (2.50):

$$\int [dB][d\lambda][dN] g(\lambda, N, J, B^I) \equiv \frac{\partial}{\partial \lambda^{\alpha_1}} \cdots \frac{\partial}{\partial \lambda^{\alpha_{11}}} (\epsilon TR)^{\alpha_1 \cdots \alpha_{11}}_{\beta_1 \cdots \beta_{11} m_1 n_1 \cdots m_{10} n_{10}} \prod_{P=1}^{10} \frac{\partial}{\partial B^{P}_{m_1 n_1}} \frac{\partial}{\partial B^{P}_{m_{10} n_{10}}} \lambda^{\gamma_1} \cdots \lambda^{\gamma_{11}} h_{\alpha_1 \cdots \alpha_{11}}^{\beta_1 \cdots \beta_{11}} (\lambda, N, J, B^P) =$$

$$(\epsilon TR)^{\alpha_1 \cdots \alpha_{11}}_{\beta_1 \cdots \beta_{11} m_1 n_1 \cdots m_{10} n_{10}} \prod_{P=1}^{10} \frac{\partial}{\partial B^{P}_{m_1 n_1}} \frac{\partial}{\partial B^{P}_{m_{10} n_{10}}} h_{\alpha_1 \cdots \alpha_{11}}^{\beta_1 \cdots \beta_{11}} (\lambda, N, J, B^P) \equiv 0$$

This reduces to (2.50) with $K_I = 0$ if one chooses $f_{\beta_1 \cdots \beta_{11}} (\lambda)$ as in (5.90) and uses

$$h_{\alpha_1 \cdots \alpha_{11}}^{\beta_1 \cdots \beta_{11}} = \frac{\partial}{\partial C_{\beta_1}^{\alpha_1}} \cdots \frac{\partial}{\partial C_{\beta_{11}}^{\alpha_{11}}} (h_B)_{\alpha_1 \cdots \alpha_{11}}. \quad (5.101)$$

Using the above definition the integral in (5.95) can be evaluated as

$$E_{\alpha_1 \cdots \alpha_{11} p_1 q_1 \cdots p_L q_L}^{\beta_1 \cdots \beta_{11} m_1 n_1 \cdots m_{10} n_{10} r_1 s_1 \cdots r_L s_L} =$$

$$c_{L_1 \cdots L_{10}} \delta^{[p_1 q_1]} \cdots \delta^{[p_L q_L]} (\epsilon TR)^{\alpha_1 \cdots \alpha_{11}}_{\beta_1 \cdots \beta_{11} m_1 n_1 \cdots m_{10} n_{10}} + \text{symmetrisation in} [r_{L_{P-1}+1}, s_{L_{P-1}+1}], \ldots, [r_{L_P}, s_{L_P}], [m_P n_P], \quad (5.102)$$

for some constant $c_{L_1 \cdots L_{10}}$. Note the round brackets denote symmetrisation in

$$[p_1 q_1], \ldots, [p_L q_L]. \quad (5.103)$$

The second line above includes ten symmetrisations, one for each $P$. $E$ is symmetric in these indices because they all appear on $B^I$. (Note that by definition $L_0 = 0$).
To get some insight how to obtain (5.102) consider the case $L_1 = L = 1$. In that case the RHS of (5.95) is given by

$$
(\epsilon TR)_{m'_1 n'_1 \cdots m'_{10} n'_{10}} \frac{\partial}{\partial B_{p' q'}} \frac{\partial}{\partial N_{p' q'}} \frac{\partial}{\partial B_{m'_1 n'_1}^{m'_1}} \cdots \frac{\partial}{\partial B_{m'_{10} n'_{10}}^{m'_{10}}} N^{pq} B_{r_1 s_1}^1 B_{m_1 n_1}^1 \cdots B_{m_{10} n_{10}}^{10},
$$

where the spinor indices on $(\epsilon TR)$ are suppressed. The last nine $B$ differentiations are trivial resulting in:

$$
(\epsilon TR)_{m'_1 n'_1 m_{2n} \cdots m_{10} n_{10}} \frac{\partial}{\partial B_{p' q'}} \frac{\partial}{\partial N_{p' q'}} \frac{\partial}{\partial B_{m'_1 n'_1}^{m'_1}} N^{pq} B_{r_1 s_1}^1 B_{m_1 n_1}^1.
$$

Let us first perform the $N$ differentiation followed by the last two $B$ differentiations:

$$
(\epsilon TR)_{m'_1 n'_1 m_{2n} \cdots m_{10} n_{10}} \frac{\partial}{\partial B_{pq}^1} \frac{\partial}{\partial B_{m'_1 n'_1}^{m'_1}} B_{r_1 s_1}^1 B_{m_1 n_1}^1 =
$$

$$(\epsilon TR)_{m'_1 n'_1 m_{2n} \cdots m_{10} n_{10}} \delta_{r_1 s_1}^{[pq]} \delta_{m_1 n_1}^{m'_1 n'_1} = \delta_{r_1 s_1}^{[pq]} (\epsilon TR)_{m_1 n_1 \cdots m_{10} n_{10}} + (r_1 s_1 \leftrightarrow m_1 n_1),$$

which agrees with (5.102). The above computation clarifies the appearance of the Kronecker delta’s. It is a consequence of the fact $\frac{\partial}{\partial B_{pq}^1}$ and $\frac{\partial}{\partial N_{pq}}$ appear contracted. The symmetrisations in (5.102) follows from the product rule of differentiation.

With these preliminaries it is possible to prove that if $Q_SY = 0$ then all one-loop amplitudes vanish:

**No go theorem**

$$Q_SY = 0 \implies c_{D_1 \cdots D_{10}} = 0,$$

$$c_{D_1 \cdots D_{10}} = 0 \implies \text{all one loop amplitudes vanish},$$

**Proof of (5.107).** In terms of $f$ the condition on the LHS of (5.107) reads

$$0 = Q_SY = f_{\beta_1 \cdots \beta_{11}} (\lambda) \lambda^{\beta_1} \theta^{\beta_2} \cdots \theta^{\beta_{11}}.
$$

This implies

$$0 = E_{\alpha_1 \cdots \alpha_{11} | p_1 q_1 \cdots p_L q_L} c_{L_1 L_{10}} \delta_{r_1 s_1}^{[p_1 q_1]} \cdots \delta_{r_L s_L}^{[p_L q_L]} (\epsilon TR)_{\alpha_1 \cdots \alpha_{11} | \beta_1 \cdots \beta_{11} m_{10} n_{10}} + \text{symmetrisation in } ([r_{L_p - 1 + 1}, s_{L_p - 1 + 1}], \ldots, [r_{L_p}, s_{L_p}], [m_p n_p]),$$

Since the trace of $(\epsilon TR)$ does not vanish, the invariant tensor $\text{Tr} (\epsilon TR)$ has at least one non-vanishing component. Let us denote this index choice by hats. If one chooses

$$r_i s_i = \hat{m} \hat{p} \hat{n} \hat{p}, \quad i = L_{p - 1} + 1, \ldots, L_p,$$

$$p_i q_i = \hat{m} \hat{p} \hat{n} \hat{p}, \quad i = L_{p - 1} + 1, \ldots, L_p,$$

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Chapter 5 - Decoupling of unphysical states

the tensor on the RHS of (5.110) is non-vanishing. Therefore

\[ c_{L_1 \cdots L_{10}} = 0. \quad (5.113) \]

**Proof of (5.108).** As explained around (5.95) all amplitudes can be written as a sum of terms, where all terms contain a \( c_{L_1 \cdots L_{10}} \).

This no-go theorem can be used to prove that the \( Y_C \) as defined in (2.23) are not \( Q_S \) closed. In order to see this note that computations performed in the minimal pure spinor formalism including an integration over \( C \) have led to non-vanishing answers, see e.g. \([22]\). In fact even agreement with the RNS formalism has been show where possible. From the no-go theorem one can conclude \( Y \) is not \( Q_S \) closed:

\[ Q_S \int [dC](C^1 \theta) \cdots (C^{11} \theta) \delta(C^1 \lambda) \cdots \delta(C^{11} \lambda) \neq 0. \quad (5.114) \]

This implies that the individual factors, \( Y_C \), cannot be \( Q_S \) closed either:

\[ Q_S(C \theta)\delta(C \lambda) \neq 0. \quad (5.115) \]

5.4 Proof of decoupling of unphysical states

The PCO \( Y \) is not \( Q_S \) closed, hence the standard argument for decoupling of unphysical states does not apply. However that does not mean unphysical states do not decouple. One just has to use other arguments. A proof of decoupling of unphysical states in the minimal pure spinor formalism including integrals over \( C \) and \( B \) is presented in this section. Firstly the tree-level argument is reviewed in a form that generalises to the higher loops and it is shown that \( Q_S \) exact states decouple to all orders. Secondly a new symmetry of the insertions is exposed. This symmetry follows from the particular form of the picture raising operators, \( Z_B \) and it plays a crucial role in the proof. Finally this symmetry is combined with arguments based on Lorentz invariance to prove decoupling of unphysical states at every genus.

5.4.1 Tree-level amplitudes

After integrating out the nonzero modes every tree-level amplitude assumes the form

\[ \mathcal{A} = \int [d\lambda][dC]d^{16} \theta \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha \beta \gamma}(\theta, a, k) \theta^{\beta_1} \cdots \theta^{\beta_{11}} C^1_{\beta_1} \cdots C^{11}_{\beta_{11}} \delta(C^1 \lambda) \cdots \delta(C^{11} \lambda), \quad (5.116) \]

where \( a \) denotes all polarisations and \( k \) denotes all momenta. Note that the integration over the nonzero modes does not affect the factor of \( Y_{C_1} \cdots Y_{C^{11}} \). This can be justified either by writing \( Y_C \) as a function of only zero modes or by inserting the factor of \((Y_C)^{11}\) at \( z = \infty \) on the worldsheet. The three factors of \( \lambda \) originate from
the three unintegrated vertex operators and the factors of \( \theta, C \) and \( \delta(C\lambda) \) from the eleven picture changing operators \( Y_C \). In order to evaluate (5.116) first note that only terms with five \( \theta \)’s can contribute:

\[
A = \int [d\lambda][dC] d^{16}\theta \lambda^\alpha \lambda^\beta \lambda^\gamma f^{(5)}_{\alpha\beta\gamma\beta_1\ldots\beta_{16}}(a,k) \tag{5.117}
\]

\[
\theta^{\beta_1} \ldots \theta^{\beta_{16}} C^1_{\beta_1} \ldots C^1_{\beta_{11}} \delta(C^1\lambda) \ldots \delta(C^{11}\lambda).
\]

The next step is showing that the integration is a projection on the scalar in \( f^{(5)} \). To this end the tensor product \( (\lambda)^3(\theta)^5 \) is written in terms of its irreducible representations:

\[
\lambda^\alpha \lambda^\beta \lambda^\gamma \theta^{\beta_1} \ldots \theta^{\beta_{16}} = T^{\alpha\beta\gamma\beta_1\ldots\beta_{16}} T^{\alpha'\beta'\gamma'\beta_1'\ldots\beta_{16}'} \lambda^{\alpha'} \lambda^{\beta'} \lambda^{\gamma'} \theta^{\beta_1'} \ldots \theta^{\beta_{16}'} + \tag{5.118}
\]

\[
\sum_{i \geq 2} (T_i)^{\alpha\beta\gamma\beta_1\ldots\beta_{16} x_i} (T_i)^{\alpha'\beta'\gamma'\beta_1'\ldots\beta_{16}' x_i} \lambda^{\alpha'} \lambda^{\beta'} \lambda^{\gamma'} \theta^{\beta_1'} \ldots \theta^{\beta_{16}'}
\]

where \( x_i \) are the indices representing the representation. To obtain the above expansion one first needs to compute the tensor product Gam\(^3\)16 Asym\(^5\)16. As discussed in section 3.4.1 this contains one scalar. One also finds there is one 45 in the tensor product, hence the second line. The sum in the last line runs over all the other irreducible representations in the tensor product, each one has an invariant tensor \( (T_i) \) associated to it. Furthermore all the \( (T_i) \)’s satisfy

\[
T^{\alpha\beta\gamma\beta_1\ldots\beta_{16}}(T_i)_{\alpha\beta\gamma\beta_1\ldots\beta_{16} x_i} = 0. \tag{5.119}
\]

This can be proved by contracting both sides of (5.118) with \( T^{\alpha\beta\gamma\beta_1\ldots\beta_{16}} \). The integrations in (5.117) can be evaluated by Lorentz invariance:

\[
\left( \int d^{16}\theta \theta^{\beta_1} \ldots \theta^{\beta_{16}} \right) \left( \int [d\lambda][dC] \lambda^\alpha \lambda^\beta \lambda^\gamma C^1_{\beta_1} \ldots C^1_{\beta_{11}} \delta(C^1\lambda) \ldots \delta(C^{11}\lambda) \right) =
\]

\[
e^{\beta_1\ldots\beta_{16}} (\epsilon T)^{\alpha\beta\gamma}_{\beta_1\ldots\beta_{11}} = T^{\alpha\beta\gamma\beta_1\ldots\beta_{16}} \tag{5.120}
\]

After using (5.119) one sees all the non-scalar terms in (5.118) are annihilated by the integration. It is therefore a projection on the scalar as claimed. The final expression for the amplitude becomes

\[
A = T^{\alpha\beta\gamma\beta_1\ldots\beta_{16}} f^{(5)}_{\alpha\beta\gamma\beta_1\ldots\beta_{16}}(a,k). \tag{5.121}
\]
Decoupling of $Q_S$ exact states at tree level

After integrating out the nonzero modes, the amplitude containing a $Q_S$ exact state becomes,

$$
\int [d\lambda] d^{16} \theta (Q_S \Omega(\lambda, \theta, a, k)) \theta^{\beta_1} \cdots \theta^{\beta_{11}} C_{\beta_1}^{1} \cdots C_{\beta_{11}} \delta(C^{1} \lambda) \cdots \delta(C^{11} \lambda),
$$  \hspace{1cm} (5.122)

for some $\Omega$, where all fields are zero modes. The above integral will be shown to vanish for any $\Omega$.

Since only the terms with five $\theta$’s and three $\lambda$’s in $Q_S \Omega$ contribute, let us focus on terms in $\Omega$ with two $\lambda$’s and six $\theta$’s. The upshot of the proof is that no Lorentz scalar can be constructed from two $\lambda$’s and six $\theta$’s. Therefore there will be no scalar in $Q_S(\lambda)^2(\theta)^6$ and since the integration projects on the scalar the amplitude vanishes.

In order to make this argument precise let us write:

$$
\Omega|_{(\lambda)^2(\theta)^6} = \lambda^\alpha \lambda^\beta \theta^{\beta_1} \cdots \theta^{\beta_6} \tilde{f}_{\alpha \beta \beta_1 \cdots \beta_6}(a, k)
$$  \hspace{1cm} (5.123)

for some $\tilde{f}$. The next step is writing the tensor product $(\lambda)^2(\theta)^6$ in terms of its irreducible representations:

$$
\Omega|_{(\lambda)^2(\theta)^6} = \tilde{f}_{\alpha \beta \beta_1 \cdots \beta_6}(a, k) \left( \sum_i (\tilde{T}_i)^{\alpha \beta \beta_1 \cdots \beta_6 y_i} (\tilde{T}_i)^{\alpha' \beta' \beta'_1 \cdots \beta'_6 y_i} \lambda^{\alpha'} \lambda^{\beta'} \theta^{\beta'_1} \cdots \theta^{\beta'_6} \right).
$$  \hspace{1cm} (5.124)

In the above formula it is important to note that there are no scalars in the tensor product of two pure spinors and six fermionic spinors. This is reflected by the fact that $y_i$ represents (a positive number of) indices for every $i$. Now one can perform the $Q_S$ transformation:

$$
Q_S \Omega|_{(\lambda)^2(\theta)^6} = \tilde{f}_{\alpha \beta \beta_1 \cdots \beta_6}(a, k) \left( \sum_i (\tilde{T}_i)^{\alpha \beta \beta_1 \cdots \beta_6 y_i} (\tilde{T}_i)^{\alpha' \beta' \beta_1 \cdots \beta_6 y_i} \lambda^{\alpha'} \lambda^{\beta'} \lambda^{\gamma'} \theta^{\beta'_1} \cdots \theta^{\beta'_6} \right).
$$  \hspace{1cm} (5.125)

After invoking (5.120) one finds

$$
\int [d\lambda] d^{16} \theta (Q_S \Omega|_{(\lambda)^2(\theta)^6}) \theta^{\beta_1} \cdots \theta^{\beta_{11}} C_{\beta_1}^{1} \cdots C_{\beta_{11}} \delta(C^{1} \lambda) \cdots \delta(C^{11} \lambda) = 
\tilde{f}_{\alpha \beta \beta_1 \cdots \beta_6}(a, k) \sum_i (\tilde{T}_i)^{\alpha \beta \beta_1 \cdots \beta_6 y_i} (\tilde{T}_i)^{\alpha' \beta' \beta_1 \cdots \beta_6 y_i} T^{\alpha' \beta' \beta'_1 \cdots \beta'_6} = 0
$$

This vanishes because

$$
T^{\alpha' \beta' \beta'_1 \cdots \beta'_6} = 0,
$$  \hspace{1cm} (5.127)

which follows from the statement that there are no scalars in $(\lambda)^2(\theta)^6$. This concludes the proof that (5.122) vanishes.
5.4.2 Higher-loop amplitudes

In order to prove decoupling of unphysical states at higher-loop amplitudes one can take similar steps to the tree-level case. This means that one first reduces the amplitude to a zero mode integral, which is effectively a projection onto a scalar and then one shows there is no scalar when one started with a $Q_S$ exact state. In the higher-loop case an additional ingredient is needed for the second step which is a symmetry possessed by the integrand of the functional integral. This symmetry is closely related to the transformations in (4.50).

Additional symmetry

The amplitude prescription contains products of PCOs $Z_B$ and $Z_J$. The main observation is that

$$Z_BZ_J = B_{mn} \lambda \gamma^{mn} d \delta(B_{mn}N^{mn})(\lambda d)\delta(J)$$

is invariant under

$$\delta B_{mn} = (\lambda \gamma_{[m})\alpha f^\alpha_{n]}.$$  \hspace{1cm} (5.129)

where $f^{n\alpha}$ are constants. This transformation acts on the $B_{mn}N^{mn}$ and $B_{mn}\lambda \gamma^{mn}d$ as,

$$\delta B_{mn}N^{mn} = (\lambda \gamma_{m})\alpha f^\alpha_{n}(\lambda \gamma^{mn}w) = (\lambda \gamma^{n} f_{n})(\lambda w),$$  \hspace{1cm} (5.130)

$$\delta B_{mn}(\lambda \gamma^{mn}d) = (\lambda \gamma_{m})\alpha f^\alpha_{n}(\lambda \gamma^{mn}d) = (\lambda \gamma^{n} f_{n})(\lambda d).$$  \hspace{1cm} (5.131)

Since all these transformations contain either $(\lambda w)$ or $(\lambda d)$ and $Z_J$ contains both $\delta(\lambda w)$ and $\lambda d$:

$$\delta(Z_BZ_J) = 0.$$ \hspace{1cm} (5.132)

Now recall that at genus $g$, $3g - 3$ $B$’s (one at genus one) enter via the $b$ ghost. These $B$’s are taken to be inert. The remaining $7g + 3$ $B$’s (9 at genus one) are taken to transform as in (5.129). Note that at one loop, the factor of $(Z_B)^9Z_J$ is placed at a single point on the worldsheet. At two-loop order, the additional factor of $(Z_B)^7Z_J$ is placed at a second point on the worldsheet. And at each additional loop order, one places the new factor of $(Z_B)^gZ_J$ at a $g^{th}$ point on the worldsheet. With this choice, (5.129) is an invariance of the theory for $7g + 3$ $B$’s and the amplitudes must respect this symmetry.

One can understand the origin of this symmetry by going back to the first principles derivation of the amplitude prescription in chapter 4. As shown there, PCO insertions arise from gauge fixing the invariance due to pure spinor zero modes. The auxiliary fields in the gauge fixing terms have gauge invariances (cf. (4.50)).
symmetry (5.129) is a remnant of these invariances. This suggests that the amplitudes are also invariant under transformations of the \((3g - 3)\) (one when \(g = 1\)) factors of \(B\) involved in the \(b\) insertions\(^7\), but this will not be proved or used here.

**One-loop amplitudes**

After integrating out all nonzero modes, as well as the \(d_\alpha\) zero modes, every one-loop amplitude can be written as

\[
\int [d\lambda][dN][dC][dB]d^{16}\theta\lambda^{\alpha_1}\ldots\lambda^{\alpha_{11}}B_{m_1n_1}^1\ldots B_{m_{10}n_{10}}^{10}f_{\alpha_1\ldots\alpha_{11}}^{m_1\ldots m_{10}n_{10}}(\theta, a, k) \tag{5.133}
\]

where all fields are zero modes and the integrand is invariant under the \(B\) transformation (5.129). As in the tree amplitude, the integration over the nonzero modes does not affect the \((Y_C)^{11}\) factor since this factor can be written in terms of only zero modes. In this expression, eleven factors of \(\lambda\) originate as follows: one from the unintegrated vertex operator, one from \(Z_J\) and nine from the nine factors of \(Z_B\). In general the zero mode integral can contain additional factors of the Lorentz currents \(N\), higher powers of \(B\) and higher derivatives of \(\delta(BN)\). These additional factors can be put into the form of (5.133) by integrating by parts using that

\[
N^{pq}B^{mn}\partial\delta(BN) = -\delta^{[p}\delta^{q]}\delta(BN).
\]

One can show that the integral in (5.133) is also a projection on a scalar. To see this first note that there is one scalar in \(\Gamma_{\alpha_1\alpha_2\ldots\alpha_{11}}^{16}\otimes\text{Asym}_5^{16}\otimes\text{Asym}_4^{45}\). This implies one can write

\[
\lambda^{\alpha_1}\ldots\lambda^{\alpha_{11}}\theta^{\beta_{12}}\ldots\theta^{\beta_{16}}B_{m_1n_1}^1\ldots B_{m_{10}n_{10}}^{10} = \tag{5.134}

(\text{TR})^{\alpha_1\ldots\alpha_{11}\beta_{12}\ldots\beta_{16}}_{m_1n_1\ldots m_{10}n_{10}}((\text{TR})(\lambda)^{11}(\theta)^5(B)^{10}) + \\
\sum_i (S_i)^{\alpha_1\ldots\alpha_{11}\beta_{12}\ldots\beta_{16}}_{m_1n_1\ldots m_{10}n_{10}x_i} (S_i(\lambda)^{11}(\theta)^5(B)^{10})^{x_i},
\]

where the notation \(((\text{TR})(\lambda)^{11}(\theta)^5(B)^{10})\) means that all indices of \((\text{TR})\) have been contracted with those of \(\lambda, \theta\) and \(B\) and \((S_i(\lambda)^{11}(\theta)^5(B)^{10})^{x_i}\) denotes an object that has \(x_i\) as its only free index and which transforms in some non-scalar representation. Similar to the tree-level case the invariant tensors \(S_i\) satisfy

\[
((\text{RT})(S_i))^{x_i} = 0. \tag{5.135}
\]

Note that since \(B\) is not a covariant tensor this is not the decomposition of a Lorentz invariant object into a lot of Lorentz invariant terms like (5.118). However this does

---

\(^7\)Recall that \((3g - 3)\) (one when \(g = 1\)) of the \(Z_B\) factors are absorbed into the \(b\)-insertions.
not matter, the point of performing this expansion is that all the non scalar terms vanish due to the integration. The last point follows from (5.135) and

$$\int [d\lambda][dC][dB][dN]\lambda^{\alpha_1} \ldots \lambda^{\alpha_{11}} B^1_{m_1 n_1} \ldots B^{10}_{m_{10} n_{10}} C_1^\beta \ldots C^{11}_{\beta_{11}}$$

(5.166)

$$\delta(C^1\lambda) \cdot \ldots \delta(C^{11}\lambda)\delta(B^1 N) \cdot \ldots \delta(B^{10} N)\delta(J) = (\epsilon T R)^{\alpha_1 \cdot \alpha_{11}}_{\beta_1 \cdot \beta_{11} m_1 n_1 \ldots m_{10} n_{10}},$$

which is also a consequence of the fact there is only one Lorentz scalar in $\text{Gam}^{11} \mathbf{16} \otimes \text{Asym}^5 \mathbf{16} \otimes \text{Asym}^{10} \mathbf{45}$.

**Decoupling of $Q_S$ exact states**

Decoupling of unphysical states will be shown by proving that if

$$\lambda^{\alpha_1} \ldots \lambda^{\alpha_{11}} B^1_{m_1 n_1} \ldots B^{10}_{m_{10} n_{10}} f^{m_1 n_1 \ldots m_{10} n_{10}}_{\alpha_1 \ldots \alpha_{11}} (\theta, a, k)$$

(5.137)

can be written as $Q_S \Omega$ where $\Omega$ is invariant under the $B$ transformation then (5.133) vanishes.

Note $\Omega$ must contain ten $\lambda$’s, six $\theta$’s and ten $B$’s. There are two scalars in $\text{Gam}^{10} \mathbf{16} \otimes \text{Asym}^6 \mathbf{16} \otimes \text{Asym}^{10} \mathbf{45}$. Since $\text{Gam}^{11} \mathbf{16} \otimes \text{Asym}^5 \mathbf{16} \otimes \text{Asym}^{10} \mathbf{45}$ contains only a single scalar and $Q_S$ maps scalars to scalars, there is a basis of invariant tensors such that one of the scalars is annihilated by $Q_S$ and the other one, call it $\Omega_1$, has a nonzero variation, $Q_S \Omega_1 \neq 0$. This scalar is\footnote{Another possible candidate, $(T(\lambda)^2(\theta)^6) (R(B)^{10}(\lambda)^8)$, vanishes identically because of (5.127).} $\Omega_1 = (T(\lambda)^3(\theta)^5) (R(B)^{10}(\lambda)^7(\theta)^1)$.

Here $(R(B)^{10}(\lambda)^7(\theta)^1)$ denotes the unique scalar obtained by contracting all indices of the objects involved. The state $Q_S \Omega_1$ is a candidate exact state that may not decouple. The scalar $\Omega_1$ however is not invariant under the transformation (5.129) for nine of the ten $B$’s. In fact, one can show that $\Omega_1$ is invariant under the transformation (5.129) for only six of the ten $B$’s. To see this, note that $(R(B)^{10}(\lambda)^7(\theta)^1)$ can be expressed as

$$(\lambda^m \gamma_{m_1 \ldots m_5} \lambda)(\lambda^m \gamma_{m_6 \ldots m_{10}} \lambda)(\lambda^m \gamma_{m_{11} \ldots m_{15}} \lambda)(\lambda^m \gamma_{m_{16} \ldots m_{20}} \lambda)$$

(5.139)

cropped with the 20 vector indices of $(B)^{10}$. If both indices of $B_{pq}$ are contracted with $m_1 \ldots m_{15}$, then $\Omega_1$ is invariant under the transformation (5.129) for that $B$ since $(\lambda^m \gamma_{m_1 \ldots m_4} \lambda)(\lambda^m \gamma_m)_{\alpha} = 0$. However, if at least one index of $B_{pq}$ is contracted with $m_1 \ldots m_{20}$, then $\Omega_1$ is not invariant under the transformation (5.129) for that $B$. Using the definition of $R_{m_1 \ldots m_{20}}^{\alpha_1 \ldots \alpha_{11}}$, one finds there are four $B$’s whose indices are contracted with $m_1 \ldots m_{20}$, so $\Omega_1$ is invariant under the transformation (5.129) for six of the ten $B$’s.
But since the gauge parameter must be invariant under (5.129) for nine of the ten $B$’s, there is no way to generate $Ω_1$ as a possible gauge parameter. Thus one can conclude that if it is $Q_S$ exact and invariant under the $B$ transformation,

$$ f_{α_1}^{m_1 n_1} \cdots m_{10} n_{10} (θ, a, k) λ^{α_1} \cdots λ^{α_{11}} B_{m_1 n_1}^1 \cdots B_{m_{10} n_{10}}^{10} $$

(5.140)

does not contain any scalars constructed from eleven $λ$’s, five $θ$’s and ten $B$’s. Since the integration projects on the (single) scalar the total zero mode integral vanishes. The precise argument is analogous to the steps in section 5.4.1.

Higher-loop amplitudes

The argument for $g > 1$ is exactly analogous. After integrating out all nonzero modes, as well as the zero modes of $d_α$, every $g > 1$ loop amplitude can be written as

$$ \int d^{16}θ |dC| λ^{α_1} |dC| λ^{α_2} λ^{α_3} θ^{β_1} \cdots θ^{β_{11}} C_{β_1}^1 \cdots C_{β_{11}}^{11} δ(C^1 λ) \cdots δ(C^{11} λ) $$

$$ \prod_{I=1}^{g} \left( [dB^I][dB^I] λ^{α_I}^{α_I} \cdots λ^{α_{11}} B_{m_{11} n_{11}}^{1} \cdots B_{m_{10} n_{10}}^{10} δ(B^{1I} N) \cdots δ(B^{10I} N) δ(J^I) \right) $$

$$ f_{α_1}^{m_1 n_1} \cdots m_{10} n_{10}^g (θ, a, k) $$

(5.141)

where all fields are zero modes and the integrand is invariant under the $B$ transformation (5.129). Now the factors $λ$ originate from the $(7g + 3)$ factors of $Z_B$ and the $g$ factors of $Z_J$. Additional factors of $N$, $B$ and derivatives of $δ(BN)$ can be removed as in the one-loop case.

In this case the analogue of (5.136) is

$$ \int |dC| λ^{α_1} |dC| λ^{α_2} λ^{α_3} C_{β_1}^1 \cdots C_{β_{11}}^{11} δ(C^1 λ) \cdots δ(C^{11} λ) $$

$$ \prod_{I=1}^{g} \left( [dB^I][dB^I] λ^{α_I}^{α_I} \cdots λ^{α_{11}} B_{m_{11} n_{11}}^{1} \cdots B_{m_{10} n_{10}}^{10} δ(B^{1I} N) \cdots δ(B^{10I} N) δ(J^I) \right) $$

$$ = (εTR^g)^{α_1 α_2 α_3 \cdots α_{11}}_{β_1 \cdots β_{11} m_{11} n_{11} \cdots m_{10} n_{10}} $$

where $(εTR^g)$ is the generalisation of (5.71) involving $g$ factors of $R$.

There are $g$ candidate $Q_S$ exact states that may not decouple, which are the analogs of (5.138) and are given by

$$ Ω_J = (T(λ)^3(θ)^5) \prod_{I=1}^{J-1} (R(B^I)^{10}(λ)^8) \ (R(B^I)^{10}(λ)^7(θ)^1) \ \prod_{I=J+1}^{g} (R(B^I)^{10}(λ)^8) $$

(5.143)
where $B^I$ denotes the $B$'s associated with the $I^{th}$ zero mode. As in the one-loop case, the term $(R(B^I))^{10}(\lambda)^7(\theta)^4$ is at most invariant under six of the ten $B^J$ transformations. But invariance under (5.129) requires invariance under seven of the ten $B^J$ transformations.

This concludes the proof that unphysical states decouple to all orders in $g$.

5.5 Origin of the problems

Based on the BRST methods of chapter 4 one would expect that the PCOs are $Q_S$ closed, since they originate from the gauge fixing term which is $Q_S$ exact. However it has been proved in this chapter that the PCOs are not closed inside correlators. In order to explain this paradox let us go back to the first principles derivation of the amplitude prescription in chapter 4. Both the minimal and the non-minimal amplitude prescriptions were obtained by first coupling the pure spinor sigma model to topological gravity and then proceeding to BRST quantise this system. The BRST quantisation was applied to all gauge invariances, including the zero mode shifts of the worldsheet fields. As shown in this section the gauge fixing condition for these zero modes implicit in $L_3$ (cf. (4.46)) sets all the zero modes to $\lambda^\alpha = 0$. However including this point in the target space of the curved $\beta\gamma$ system that the pure spinor sector is, leads to anomalies. More precisely Nekrasov showed that the target space of curved $\beta\gamma$ systems is subject to certain conditions, which are necessary for conformal invariance of the worldsheet theory [21]. These conditions dictate that the point $\lambda^\alpha = 0$ cannot be part of the target space of the pure spinor sigma model.

Focussing on the tree level case for a moment the gauge fixing Lagrangian for the zero mode invariances is given by (after the BRST ghosts have been integrated out):

$$L'_3 = \pi_\alpha \lambda^\alpha + \tilde{\pi}_\alpha \theta^\alpha.$$  \hspace{1cm} (5.144)

5.5.1 Minimal formalism

To express the fact that $\pi_\alpha$ and $\tilde{\pi}_\alpha$ have eleven independent components they were parametrised as

$$\pi_\alpha = p_I C^I_\alpha, \quad \tilde{\pi}_\alpha = \tilde{p}_I C^I_\alpha, \quad I = 1, \ldots, 11,$$  \hspace{1cm} (5.145)

where $C^I_\alpha$ is a matrix that must have maximal rank. Thus the gauge fixing condition is given by

$$C^I_\alpha \lambda^a = 0.$$  \hspace{1cm} (5.146)
The eleven constant spinors $C^I_\alpha$ are the ones that enter in the minimal pure spinor prescription. Indeed, using (5.145) one finds that the path integral contains
\[
\int [dp_I] [d\bar{p}_I] \exp \left( p_I C^I_\alpha \lambda^\alpha + \bar{p}_I C^I_\alpha \theta^\alpha \right) = \prod_{I=1}^{11} (C^I_\alpha \theta^\alpha) \delta(C^I_\alpha \lambda^\alpha)
\] (5.147)
which are the eleven picture changing operators $Y_C$.

Implicit in (5.147) there is an analytic continuation in the field variables. A Weyl spinor in ten Euclidian dimensions cannot be real, hence $\lambda$ is complex and in the minimal formulation only the holomorphic part appears. In equation (5.147) one analytically continues $\lambda$ to be real and considers $\pi_I$ to be purely imaginary. This can be done if the explicit expressions appearing in the amplitude computations are not singular. Typical integrals in the minimal formalism at tree level are of the form
\[
\int_{-i\infty}^{i\infty} [dp] \int_{-\infty}^{\infty} [d\lambda] f(\lambda) e^{p_I C^I_\alpha \lambda^\alpha} = \int_{-\infty}^{\infty} [d\lambda] f(\lambda) \delta(C^1_\lambda) \cdots \delta(C^{11}_\lambda).
\] (5.148)
where $f(\lambda)$ contains $\lambda$ but not its complex conjugate. For this expression to be well-defined $f(\lambda)$ should not contain any $(C^I)$ poles and moreover there should not be any poles that obstruct the analytic continuation of $\lambda$ to real values.

At higher loops the conjugate momentum has zero modes as well and gauge fixing this invariance leads exactly to the insertion of PCOs $Z_B, Z_J$, where the tensors $B_{mn}$ enter through the gauge fixing condition, as discussed in chapter 4. In addition, one needs a composite $b$ field satisfying (2.34). In the minimal formulation, a solution of (2.34) is given by [55]
\[
b = \lambda^\alpha \frac{G^\alpha}{C^\alpha \lambda^\alpha}
\] (5.149)
where $G^\alpha$ is given in (3.139). This is however too singular to be acceptable. One can obtain a non-singular $\tilde{b}$ field by combining the $b$ field with the PCO and solving instead (2.36). Note that this $\tilde{b}$ field now depends on the $B_{mn}$ constant tensors but not on $C_\alpha$.

## 5.5.2 Non-minimal formalism

The same expression (5.144) leads to the so-called regularisation factor in (5.73). This time one has to choose $\pi_\alpha$ to be a pure spinor of opposite chirality to $\lambda^\alpha$, usually called $\bar{\lambda}_\alpha$. This indeed has eleven independent components, as required. The field $\bar{\pi}_\alpha$, usually called $r_\alpha$, automatically follows because it is the $Q_S$ variation of $\pi_\alpha$,
\[
r_\alpha = Q_S \bar{\lambda}_\alpha.
\] (5.150)
This leads to the non-minimal formalism. To see this explicitly note that the factor $e^{-L_3}$, which is given by
\[
e^{-\bar{\lambda}_\alpha \lambda^\alpha - r_\alpha \theta^\alpha},
\] (5.151)
is precisely \( \mathcal{N} \). The additional factors \( N_{mn} \bar{N}^{mn} + \frac{1}{4} S_{mn} \lambda \gamma^{mn} d + J \bar{J} + \frac{1}{4} S \lambda d \) originate from gauge fixing the zero modes of \( w_\alpha \).

Note that \( \lambda \) is now holomorphic and \( \pi_\alpha \equiv \bar{\lambda}_\alpha \) is considered as its complex conjugate variable. Typical integrals one encounters at tree level in the non-minimal formalism are therefore

\[
\int [d\lambda][d\bar{\lambda}] f(\lambda) e^{-\bar{\lambda}\lambda}.
\]

At higher loop order the \( b \) field enters the amplitudes. In the non-minimal formalism, equation (2.34) has a solution that depends on both \( \lambda \) and \( \bar{\lambda} \). It is however singular as \( \lambda \lambda \to 0 \) and this causes problems to certain amplitudes as explained in section 2.2. Note that the \( b \) field does not depend on how the gauge invariances due to the zero modes of \( w_\alpha \) are treated. This is similar to the \( b \) field in (5.149) but different from \( \tilde{b} \) which depends on the gauge fixing of the invariance due to zero modes of the conjugate momentum through \( B_{mn} \).

To summarise, the minimal and non-minimal are related by field redefinitions and an analytic continuation in field space. In particular, starting from the non-minimal formalism one obtains the minimal formalism by taking \( \bar{\lambda}_\alpha = C^I_\alpha \pi^I \) and analytically continuing \( \pi^I \) to be imaginary while at the same time analytically continuing \( \lambda \) to be real. There are similar redefinitions and analytic continuations in the sector related with the conjugate momentum. Furthermore, the non-minimal \( b \) field combined with part of \( \mathcal{N} \) is related to \( \tilde{b} \). Clearly, the two formalisms would be equivalent if the analytic continuations had not been obstructed by singularities in the amplitudes. Finally, note that the underlying gauge choice for the invariance due to pure spinor zero modes is the same: the gauge fixed action is the same, only the reality condition of the fields is different.

### 5.5.3 Toy example

Given the formal equivalence between the minimal and non-minimal formalisms one may wonder why the PCOs are not \( Q_S \) closed in the minimal formalism, but the corresponding object in the non-minimal formalism is \( Q_S \) closed. This issue is discussed here by analyzing a toy example that has almost all features of the actual case. Consider the following integral

\[
I = \int dxdpe^{-xp}.
\]

To compare with the expressions in the previous subsection \( p \) corresponds to the BRST auxiliary field and \( x \) to the pure spinor.

If one wants to evaluate the above integral, contours have to be chosen for \( x \) and \( p \). Choosing \( p = ip_1 \) and \( x = x_1 \) with \( p_1, x_1 \) to be real, gives

\[
I = i \int dx_1 dp_1 e^{ix_1p_1} = i \int_{-\infty}^{\infty} dx_1 2\pi \delta(x_1) = 2\pi i.
\]
Another choice is to consider $x$ complex and take $p = x^*$. In this case $I$ becomes

$$I = \int dx dx^* e^{-xx^*} = 2i \int_0^\infty r dr \int_0^{2\pi} d\theta e^{-r^2} = 2\pi i. \quad (5.155)$$

This agrees nicely with the general property of contour integrals, that one is free to deform them as long as no poles are encountered. Note that (5.154) resembles a zero mode integral in the minimal formalism and (5.155) a non-minimal one.

The difference between the two prescriptions is exposed by considering the integral

$$I_{\text{min}}[f] = i \int_{-\infty}^\infty dx_1 \int_{-\infty}^\infty dp_2 e^{ix_1p_1} f(x_1) = i \int_{-\infty}^\infty dx_1 2\pi \delta(x_1) f(x_1) = 2\pi i f(0). \quad (5.156)$$

Now rotate the contour, $p = x^*$, so that the integral becomes

$$I_{\text{non-min}}[f] = \int dx dx^* e^{-|x|^2} f(x) = 2i \int_0^\infty r dr e^{-r^2} \int_0^{2\pi} d\theta f(re^{i\theta}), \quad (5.157)$$

$I_{\text{min}}$ is the analogue of (5.148) and $I_{\text{non-min}}$ the analogue of (5.152). $I_{\text{min}}$ and $I_{\text{non-min}}$ give exactly the same answer if $f(x)$ is non-singular but (5.156) is ill defined for any choice of singular $f(x)$ whereas (5.157) may be well defined. For example, for the function

$$f(x) = \frac{1}{x}, \quad (5.158)$$

(5.156) yields $\infty$ but (5.157) gives $0$. More precisely, (5.157) is well defined for all functions $f(z) = \sum_n c_n z^n$, with $c_n = 0$ for $n < -1$. For the $n < -1$ terms the $\theta$ integral vanishes and the $r$ integral diverges, which makes $I_{\text{non-min}}$ ambiguous for these kind of functions.

A third representation is obtained by noticing that the $\theta$ integral can be rewritten as a contour integral

$$\int_0^{2\pi} d\theta = -i \oint_C \frac{dz}{z} \quad (5.159)$$

where $z = re^{i\theta}$ and the contour $C$ is a circle of radius $r$. Thus for any meromorphic function $f(z)$ the integral over theta is independent of $r$ and

$$I[f] = 2i \left( \int_0^\infty r dr e^{-r^2} \right) \left( -i \oint_C \frac{dz}{z} f(z) \right) = \oint_C \frac{dz}{z} f(z) \quad (5.160)$$

The expression (5.160) are well-defined for all meromorphic functions $f(z)$ whereas (5.156) and (5.157) are not.

Going back to pure spinors and working on the patch with $\lambda^+ \neq 0$ one sees that because of the factor $(\lambda^+)^{-3}$ in the measure (cf. (3.123)) the minimal formalism is expected to have a singularity unless the integrand provides a factor of $(\lambda^+)^3$, but the expressions (5.157) and (5.160) are not necessarily singular.
5.5.4 Singular gauge and possible resolution

As mentioned in the beginning of this section the gauge (5.146) leads to $\lambda^\alpha = 0$ for any choice of the constant spinors $C^I_\alpha$. To see this, recall that the space of pure spinors can be covered with sixteen coordinate patches and on each patch at least one of the components of $\lambda^\alpha$ is nonzero. Let us call this component $\lambda^+ +$ and solve the pure spinor condition as in (3.71). Then,

$$0 = C^I_\alpha \lambda^\alpha = C^I_+ \lambda^+ + C^I_{a\beta} \lambda_{a\beta} + C^I_a \lambda^a = C^I_+ \lambda^+ + C^I_{a\beta} \lambda_{a\beta} + \frac{1}{8} C^I_a \epsilon^{abcde} \lambda_{bc} \lambda_{de} \frac{1}{\lambda^+} \Rightarrow$$

$$C^I_+ (\lambda^+)^2 + C^I_{a\beta} \lambda^+ \lambda_{a\beta} + \frac{1}{8} C^I_a \epsilon^{abcde} \lambda_{bc} \lambda_{de} = 0. \quad (5.161)$$

This system of equations however does not have a solution with $\lambda^+ \neq 0$ and the gauge is singular. To see this, first solve ten of the above equations to obtain $\lambda_{a\beta}$ as a function of $\lambda^+$. A scaling argument implies that these functions are linear in $\lambda^+$. After plugging in the relation $\lambda_{a\beta} = b_{a\beta} \lambda^+$ in the eleventh equation, one finds that $\lambda^+$ vanishes. Thus for any choice $C^I_\alpha$ of maximal rank, the path integral localises at the $\lambda^\alpha = 0$ locus\(^9\), which is the point that should be excised from the pure spinor space for the theory to be non-anomalous [21].

As discussed above, the minimal and non-minimal formalisms are related by analytic continuation in field space. In the toy example in the previous subsection the analytic continuation from the “minimal variables” $x_1, p_1$ to the “non-minimal variables” $x, x^+$ sets to zero certain singular contributions (functions $f(x) \sim x^{-1}$) but the integral still localises at $x = 0$. One would thus expect that the zero mode integrals in the non-minimal formalism localise at the $\lambda^\alpha = 0$ locus, as the minimal ones do, and the problems with the $\bar{\lambda} \lambda$ poles one encounters for certain amplitudes at three loops and higher are a manifestation of this fact.

To avoid these problems one must find a way to gauge fix the zero mode invariances such that the zero mode integrals do not localise at $\lambda^\alpha = 0$. Let us discuss how to achieve this in the minimal formulation. First, in order to avoid the unnecessary analytic continuation to real $\lambda$ one should work with the analogue of the contour representation of the delta function in (5.160) which is appropriate for holomorphic $\lambda$ (and is less singular than (5.156) and (5.157)). In this language the choice of $C$’s translates into a choice of position of poles. Secondly, one must take global issues into account. In particular, as mentioned above, the space of pure spinors can be covered with sixteen coordinates patches. In order to avoid landing in the singular gauge discussed above, one should arrange such that the expression for the path integral insertions valid in any given patch always contains at least one pole that lies in another patch. Relevant related work can be found in [43].

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\(^9\)This also shows that the choice of $C$ in (5.2) that manifestly leads to a factor $\delta(\lambda^+)$ is not special. Any other choice of $C$ will also contain this factor.