Fundamentals of the pure spinor formalism
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Appendix A

Detailed computations of $I_k$

This appendix contains the details of the $\lambda$ integrals that appear at one loop. In particular those that play a role in computations involving a $Q_S$ exact state. A typical integral one encounters in such an amplitude is given by

\[
(I_k)_{a_1 \cdots a_{2k}, \beta_2 \cdots \beta_{11}} = \int \left[ \frac{1}{\lambda^{k-2}} \right] \lambda^{\beta_1} \lambda_{a_1 a_2} \cdots \lambda_{a_{2k-1} a_{2k}} \Lambda_{\delta_1 \delta_2 \delta_3} (\epsilon T)^{\delta_1, \delta_2, \delta_3} \beta_1 \cdots \beta_{11}. \quad (A.1)
\]

By charge conservation one can conclude that at most two choices for $\beta_2 \cdots \beta_{11}$ lead to a non-vanishing $I_k'$ for any $k$. This follows from

\[
0 = N(I_k)_{a_1 \cdots a_{2k}, \beta_2 \cdots \beta_{11}} = [(k-3)\frac{5}{4} + k(-\frac{1}{4}) + N(\beta_2 \cdots \beta_{11})](I_k)_{a_1 \cdots a_{2k}, \beta_2 \cdots \beta_{11}}. \quad (A.2)
\]

This fixes the total charge of the $\beta$ indices, which implies there are only two choices. For example for $k = 3$ equation (A.2) implies only the only non-vanishing components satisfy $N(\beta_2 \cdots \beta_{11}) = -\frac{1}{4}$. Thus $\beta_2 \cdots \beta_{11}$ must consist of either seven 10's and three 5's or a +, five 10's and four 5's.

In section A.1 we first compute all integrals of the form

\[
(I_k')^{\beta_1}_{a_1 \cdots a_{2k}, \delta_1 \delta_2 \delta_3} = \int \left[ \frac{1}{\lambda^{k-2}} \right] \lambda^{\beta_1} \lambda_{a_1 a_2} \cdots \lambda_{a_{2k-1} a_{2k}} \Lambda_{\delta_1 \delta_2 \delta_3}. \quad (A.3)
\]

Since $I_k$ vanishes for $k < 3$ (cf. (5.51)), we are only interested in $I_k'$ for $k \geq 3$. By a similar argument the $I_k'$'s are also only non-vanishing for at most two choices of $\delta_1 \delta_2 \delta_3$. In the last subsection half of the non-vanishing components of $I_3$ and all components of $I_5$ are computed.

A.1 Coefficients in $\lambda$ integrals

For a given $k$ at most two components of $\Lambda_{\alpha \beta \gamma}$ give non-vanishing results. One can make three choices for $\beta_1$ in $I_k'$, all three choices lead to an integral of the form (not
necessarily for the same $k$):
\[
(I''_k)_{a_1\ldots a_2k\delta_1\delta_2\delta_3} = \int [d\lambda] \frac{1}{(\lambda^+)^{k-3}} \lambda_{a_1a_2} \cdots \lambda_{a_{2k-1}a_{2k}} \Lambda_{\delta_1\delta_2\delta_3}. \tag{A.4}
\]

After some algebra one finds the only non-vanishing components of the $I''_k$'s are:
\[
(I''_1)_{a_1\ldots a_8+d_1d_2} = \frac{1}{20} \epsilon_{a_1a_2a_3a_4(g_1\epsilon_d_2_1)a_5a_6a_7a_8} + 2 \text{ perms}, \tag{A.5}
\]
\[
(I''_4)_{a_1\ldots a_8} d_1d_2d_3d_4 d_5 = \frac{1}{5} \epsilon_{a_1a_2a_3a_4d_5} \delta_{a_5}^{[d_1} \delta_{a_6}^{d_2} \delta_{a_7}^{d_3} \delta_{a_8]^{d_4]} + 11 \text{ perms}, \tag{A.6}
\]
\[- \frac{1}{20} \epsilon_{a_1a_2a_3a_4d_5} \delta_{a_5}^{d_1} \delta_{a_6}^{d_2} \delta_{a_7}^{d_3} \delta_{a_8]^{d_4]} + 5 \text{ perms}
\]
\[
(I''_5)_{a_1\ldots a_{10}d_1d_2} d_3d_4 = \frac{1}{20} \epsilon_{(d_1) a_1a_2a_3a_4 \epsilon_2_1_2_3_4 a_5a_6a_7a_8} \delta_{a_5}^{d_3} \delta_{a_6}^{d_4} + 14 \text{ perms}, \tag{A.7}
\]
\[
(I''_6)_{a_1\ldots a_{12}d_1d_2d_3} = \frac{1}{60} \epsilon_{(d_1) a_1a_2a_3a_4 \epsilon_2_1_2_3_4 a_5a_6a_7a_8} \delta_{a_5}^{d_4} + 14 \text{ perms}. \tag{A.8}
\]

The first step to obtain these results is finding the number of invariant tensors with the appropriate symmetries, this is one in all cases but the second. Finding the coefficients requires more work, this is done in subsection A.1.1. All these coefficients are fixed by (5.24), including the overall factor. Two corollaries are
\[
(I''_3)_{a_1\ldots a_6d_1d_2} d_3d_4 = (5 \delta_{(d_1) a_1a_2a_3a_4} \delta_{a_5}^{d_3} \delta_{a_6}^{d_4] + 2 \text{ perms},} \tag{A.9}
\]
\[
(I''_4)_{a_1\ldots a_8d_1d_2d_3} = \frac{1}{12} \delta_{(d_1) a_1a_2a_3a_4} \delta_{a_5}^{d_3} \delta_{a_6}^{d_4} + 2 \text{ perms.} \tag{A.10}
\]

### A.1.1 Proof of equations (A.5) and (A.6)

By Lorentz invariance one can write
\[
\int [d\lambda] \frac{1}{\lambda^+} \lambda_{a_1a_2} \cdots \lambda_{a_7a_8} \Lambda_{+1d_1d_2} = c_3 \epsilon_{a_1a_2a_3a_4(d_1\epsilon_2_1_2_3_4 a_5a_6a_7a_8} + 2 \text{ perms} \tag{A.11}
\]
and
\[
\int [d\lambda] \frac{1}{\lambda^+} \lambda_{a_1a_2} \cdots \lambda_{a_7a_8} \Lambda_{d_1d_2d_3d_4} d_5 = c_4 (\epsilon_{a_1a_2a_3a_4d_5} \delta_{a_5}^{d_1} \delta_{a_6}^{d_2} \delta_{a_7}^{d_3} \delta_{a_8]^{d_4]} + 11 \text{ perms}) + 
\]
\[
c_5 (\epsilon_{a_1a_2a_3a_4d_5} \delta_{a_5}^{d_1} \delta_{a_6}^{d_2} \delta_{a_7}^{d_3} \delta_{a_8]^{d_4]} + 5 \text{ perms}). \tag{A.12}
\]

for some coefficients $c_3, c_4, c_5$. They can be determined from the defining equation of $\Lambda_{\alpha\beta\gamma}$, (5.24). After evaluating the r.h.s. of that equation for the relevant components one finds
\[
\int [d\lambda] \lambda^a \lambda^b \lambda^{+d_1d_2} = \delta_{d_1}^{(a} \delta_{d_2}^{b)} - \frac{2}{5} \delta_{d_1}^{(a} \delta_{d_2}^{b)} = \frac{3}{5} \delta_{d_1}^{(a} \delta_{d_2}^{b)}, \tag{A.13}
\]

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\[ \int [d\lambda] \lambda^a \lambda^b \lambda^c \Lambda_{d_1 d_2 d_3 d_4}^{d_5} = \frac{1}{5} \epsilon_{d_1 d_2 d_3 d_4}^{a b} \delta_{d_5}^c, \quad (A.14) \]

\[ \int [d\lambda] \lambda_{a_1 a_2} \lambda_{a_3 a_4} \lambda^a \Lambda_{d_1 d_2 d_3 d_4}^{d_5} = (\delta_{[a_1}^{d_1} \delta_{a_2}^{d_2} \delta_{a_3}^{d_3} \delta_{a_4]}^{d_4} + 1 \text{ perm}) \]

\[ -\frac{1}{5} \delta_{[a_1}^{d_1} \delta_{a_2}^{d_2} \delta_{a_3}^{d_3} \delta_{a_4]}^{d_4} \delta_{d_5}^c + (\frac{1}{5} \delta_{[a_1}^{d_1} \delta_{a_2}^{d_2} \delta_{a_3}^{d_3} \delta_{a_4]}^{d_4} + 3 \text{ perms}). \]

If one now uses equations (A.11) and (A.12) to evaluate the l.h.s. of the above integrals the values of \( c_3, c_4, c_5 \) are completely determined. In fact the integrals (A.13)-(A.15) lead to more than three equations, but they include only three independent conditions as they should. To obtain \( c_3 \) one has to write out \( \lambda^a \) and \( \lambda^b \) in (A.13) and then perform all the contractions of the two \( \epsilon \)'s with the r.h.s. of (A.11):

\[ \frac{3}{5} \delta_{d_1}^{(a} \delta_{d_2}^{b)} = \int [d\lambda] \lambda^a \lambda^b \lambda^c \Lambda_{d_1 d_2 d_3 d_4}^{d_5} = 12 c_3 \delta_{d_5}^{(a} \delta_{d_2}^{b)} \Rightarrow c_3 = \frac{1}{20}. \quad (A.16) \]

Finding \( c_4 \) and \( c_5 \) is more involved. The l.h.s. of (A.14) can be evaluated as

\[ \frac{1}{5} \epsilon_{d_1 d_2 d_3 d_4}^{a b} \delta_{d_5}^c = \int [d\lambda] \lambda^a \lambda^b \lambda^c \Lambda_{d_1 d_2 d_3 d_4}^{d_5} = (4c_4 + 12c_5) \delta_{d_5}^{(a} \delta_{d_2}^{b)} \quad (A.17) \]

This gives the first equation for \( c_4, c_5 \). In order to completely determine them, one has to work out the l.h.s. of (A.15):

\[ \frac{1}{8} e^{a a_5 a_6 a_7 a_8} \int [d\lambda] \frac{1}{\lambda^c} \lambda_{a_1 a_2} \lambda_{a_3 a_4} \lambda_{a_5 a_6} \lambda_{a_7 a_8} \Lambda_{d_1 d_2 d_3 d_4}^{d_5} = \quad (A.18) \]

\[ \frac{c_4}{8} \left( 24 \delta_{d_5}^{a} \delta_{[a_1}^{d_1} \delta_{a_2}^{d_2} \delta_{a_3}^{d_3} \delta_{a_4]}^{d_4} + 1 \text{ perm} \right) + 8 \epsilon_{d_1 d_2 d_3 d_4}^{a b} \delta_{a_1 a_2 a_3 a_4 d_5} + 16 \delta_{d_5}^{a} \delta_{[a_1}^{d_1} \delta_{a_2}^{d_2} \delta_{a_3}^{d_3} \delta_{a_4]}^{d_4} + 1 \text{ perm} \right) + \left( 8 \delta_{d_5}^{a} \delta_{[a_1}^{d_1} \delta_{a_2}^{d_2} \delta_{a_3}^{d_3} \delta_{a_4]}^{d_4} + 3 \text{ perm} \right) \]

\[ \frac{c_5}{8} \left( 24 \delta_{d_5}^{a} \delta_{[a_1}^{d_1} \delta_{a_2}^{d_2} \delta_{a_3}^{d_3} \delta_{a_4]}^{d_4} + 24 \epsilon_{d_1 d_2 d_3 d_4}^{a b} \delta_{a_1 a_2 a_3 a_4 d_5} + 16 \delta_{d_5}^{a} \delta_{[a_1}^{d_1} \delta_{a_2}^{d_2} \delta_{a_3}^{d_3} \delta_{a_4]}^{d_4} \right) \]

To be able to read off equations for the \( \epsilon \)'s one has to rewrite the invariant tensors in terms of the ones appearing in (A.15). It turns out the space of invariant tensors with the indices and symmetries of (A.15) is four dimensional. Hence the invariant tensors in (A.18) can be written out on a basis that contains the three invariant tensors that are present in (A.15) plus a fourth one, that does not lie in the span of the first three. After using

\[ \epsilon_{d_1 d_2 d_3 d_4}^{a b} \delta_{a_1 a_2 a_3 a_4 d_5} = \delta_{d_5}^{a} \delta_{[a_1}^{d_1} \delta_{a_2}^{d_2} \delta_{a_3}^{d_3} \delta_{a_4]}^{d_4} + (\delta_{[a_1}^{d_1} \delta_{a_2}^{d_2} \delta_{a_3}^{d_3} \delta_{a_4]}^{d_4} + 1 \text{ perm}), \quad (A.19) \]

(A.18) becomes

\[ (5c_4 \delta_{d_5}^{a} \delta_{[a_1}^{d_1} \delta_{a_2}^{d_2} \delta_{a_3}^{d_3} \delta_{a_4]}^{d_4} + 1 \text{ perm}) + (c_4 \delta_{d_5}^{a} \delta_{[a_1}^{d_1} \delta_{a_2}^{d_2} \delta_{a_3}^{d_3} \delta_{a_4]}^{d_4} + 3 \text{ perm}) + \]

\[ (A.20) \]

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The coefficients in equations (A.7) and (A.8) follow in the same way. The second relevant component is given by:

\[\lambda\]

The idea of this section is simple, use the explicit form of the gamma matrices. Now one can read off four equations for \(c_4, c_5\) by comparing to (A.15). Combined with the equation from (A.17) this gives:

\[5c_4 = 1, \quad c_4 + 8c_5 = -\frac{1}{5}, \quad c_4 = \frac{1}{5}, \quad c_4 + 4c_5 = 0, \quad 4c_4 + 12c_5 = \frac{1}{5}. \quad (A.21)\]

These equations are solved by

\[c_4 = \frac{1}{5}, \quad c_5 = -\frac{1}{20}. \quad (A.22)\]

The coefficients in equations (A.7) and (A.8) follow in the same way.

### A.2 Computing the \(I_k\)’s

The idea of this section is simple, use the explicit form of the gamma matrices and the \(\lambda\) integrals (A.5)-(A.10) to evaluate \(I_k\). In practice this involves a lot of computation. The integrals \(I_0, I_1, I_2\) and \(I_6\) have already been shown to vanish in chapter 5. By the charge conservation property there is only one choice of \(\beta_2 \cdots \beta_{11}\) for which \(I_5\) does not vanish. For \(I_3\) and \(I_4\) there are two possibilities. Let us explicitly compute \(I_3\) for

\[\beta_2, \ldots, \beta_{11} = +, c_1, c_2, c_3, c_4, b_1b_2, \ldots, b_9b_{10}. \quad (A.23)\]

The integral \(I_3\) consists of three terms, two for \(\beta_1 = b_1b_2\) and one for \(\beta_1 = b_1\). The first of three relevant components of \(\epsilon T\) is given by\(^1\)

\[\frac{1}{16}8(\epsilon_{10})b_1 \cdots b_20c_{1}c_{2}c_{3}c_{4} = \quad (A.24)\]

\[\frac{1}{2}8(\epsilon_{10})b_1 \cdots b_20c_{1}c_{2}c_{3}c_{4}b_{11}b_{12}b_{13}b_{14}b_{15}b_{16}b_{17}b_{18}b_{19}b_{20}. \]

The second relevant component is given by:

\[\frac{1}{16}8(\epsilon_{10})b_1 \cdots b_20c_{1}c_{2}c_{3}c_{4} = \quad (A.25)\]

\[\frac{1}{2}8(\epsilon_{10})b_1 \cdots b_20c_{1}c_{2}c_{3}c_{4}b_{11}b_{12}b_{13}b_{14}b_{15}b_{16}b_{17}b_{18}b_{19}b_{20}. \]

\[+8\frac{1}{16}(\epsilon_{10})b_1 \cdots b_20c_{1}c_{2}c_{3}c_{4}b_{11}b_{12}b_{13}b_{14}b_{15}b_{16}b_{17}b_{18}b_{19}b_{20}. \]

\(^1\)To evaluate \(\epsilon T\) the following convention for \(\epsilon_{\beta_{11}}\) is used, \((\epsilon_{16})_{+a_1 \cdots a_5}b_1b_2 \cdots b_{19}b_{20} = (\epsilon_{5})_{a_1 \cdots a_5}(\epsilon_{10})b_1 \cdots b_{20}.\]
Appendix A - Detailed computations of $I_k$

\[
8 \frac{1}{2}(\epsilon_{10})^{b_1 \cdots b_{20}} \epsilon_{c_1 c_2 c_3 c_4 b_17} \epsilon_{b_19 d_1 d_2 b_{13} b_{14}} \epsilon_{b_{20} d_3 d_4 b_{15} b_{16} d_{b_{18}}} + \\
8 \frac{1}{4}(\epsilon_{10})^{b_1 \cdots b_{20}} \epsilon_{c_1 c_2 c_3 c_4 b_19} \epsilon_{d_1 d_2 b_{13} b_{14}} \epsilon_{b_{20} d_3 d_4 b_{15} b_{16} d_{b_{18}}} + \\
8 \frac{1}{4} \frac{1}{2}(\epsilon_{10})^{b_1 \cdots b_{20}} \epsilon_{c_1 c_2 c_3 c_4 d_2 d_1} \epsilon_{b_{15} b_{19} b_{20} b_{18}} \epsilon_{b_{17} d_3 d_4 b_{13} b_{14} d_{b_{16}}} + (d_1 d_2 \leftrightarrow d_3 d_4). 
\]

The third relevant component is given by:

\[
(\epsilon T)^{d_1 d_2} d_{3 d_4} b_{1 \cdots b_{10}} c_1 c_2 c_3 c_4 = \quad (A.26)
\]

\[
-8 \frac{1}{32}(\epsilon_{10})^{b_1 \cdots b_{20}} \epsilon_{c_1 c_2 c_3 c_4 b_{17}} \gamma_{b_{11} b_{12}}^{k_1} d_{12} d_{2} b_{13} b_{14} d_{b_{16}}^{b_{17} b_{18} b_{19} b_{20}} = \\
8 \frac{1}{4} \epsilon_{c_1 c_2 c_3 c_4 b_{12} d_{3 d_4} b_{15} b_{16} b_{17} d_{b_{18}} d_{b_{19}} b_{20} \delta_{b_{12}}^{d_{1 d_{2}}} \delta_{b_{14}}^{d_{2 d_{3}}}.
\]

In the above components a factor of eight is extracted coming from the $SU(5)$ decomposition (cf. (3.22)). The powers of $\frac{1}{2}$ compensate for double counting in expressions like $x_{ab} y_{ab}$ in each line. Using the explicit form of the components of $(\epsilon T)$ and the $\lambda$ integrals, $I_3$ can be written out as

\[
I_3 = \frac{1}{23} \int [d\lambda] \frac{1}{\lambda} \lambda_{b_{11} b_{12}} \lambda_{a_{1} a_{2}} \lambda_{a_{3} a_{4}} \lambda_{a_{5} a_{6}} \Lambda_{d_{1 d_{2}}}^{d_{1 d_{2}}} d_{b_{12}}^{b_{12}} b_{13} b_{14} b_{15} b_{16} b_{17} b_{18} b_{19} b_{20} + \\
\frac{1}{8} \frac{1}{3} \int [d\lambda] \frac{1}{\lambda} \lambda_{b_{11} b_{12}} \lambda_{a_{1} a_{2}} \lambda_{a_{3} a_{4}} \lambda_{a_{5} a_{6}} \Lambda_{d_{1 d_{2}}}^{d_{1 d_{2}}} d_{b_{12}}^{b_{12}} b_{13} b_{14} b_{15} b_{16} b_{17} b_{18} b_{19} b_{20} + \\
\frac{3}{2} \frac{1}{2} \int [d\lambda] \frac{1}{\lambda} \lambda_{b_{11} b_{12}} \lambda_{a_{1} a_{2}} \lambda_{a_{3} a_{4}} \lambda_{a_{5} a_{6}} \Lambda_{d_{1 d_{2}}}^{d_{1 d_{2}}} d_{b_{12}}^{b_{12}} b_{13} b_{14} b_{15} b_{16} b_{17} b_{18} b_{19} b_{20} +
\]

\[
\left[ -\frac{1}{4}(\epsilon_{10})^{b_1 \cdots b_{20}} \epsilon_{c_1 c_2 c_3 c_4 b_{17}} \delta_{b_{12}}^{d_{1 d_{2}}} \delta_{b_{14}}^{d_{1 d_{2}}} \epsilon_{b_{13} b_{15} b_{17} b_{19} b_{20}} + 
\frac{3}{40} \left( \epsilon_{a_{1} a_{2} a_{3} a_{4} (d_{1 d_{2}} a_{5} a_{6} b_{11} b_{12}) + 2 \text{ perms} \right) \right] + \\
\left[ 4(\epsilon_{10})^{b_1 \cdots b_{20}} \epsilon_{c_1 c_2 c_3 c_4 b_{17}} \epsilon_{b_{19} d_{1} d_{2} b_{13} b_{14}} \epsilon_{b_{20} d_{3} d_{4} b_{15} b_{16} \delta_{b_{18}}^{d_{1 d_{2}}} + 
2(\epsilon_{10})^{b_1 \cdots b_{20}} \epsilon_{c_1 c_2 c_3 c_4 b_{17}} \epsilon_{b_{19} d_{1} d_{2} b_{13} b_{14}} \epsilon_{b_{20} d_{3} d_{4} b_{15} b_{16} \delta_{b_{18}}^{d_{1 d_{2}}} + 
(\epsilon_{10})^{b_1 \cdots b_{20}} \epsilon_{c_1 c_2 c_3 c_4 \epsilon_{d_{1} d_{2}} b_{15} b_{19} b_{20} b_{18} \epsilon_{b_{17} d_{3} d_{4} b_{13} b_{14} \delta_{b_{18}}^{d_{1 d_{2}}} + (d_1 d_2 \leftrightarrow d_3 d_4) + 
\frac{3}{2} \left( 5 \delta_{b_{12}}^{d_{1 d_{2}}} a_{1} a_{2} a_{3} a_{4} \delta_{a_{5} a_{6}}^{d_{3 d_{4}}} + \delta_{b_{12}}^{d_{1 d_{2}}} \delta_{b_{14}}^{d_{1 d_{2}}} \epsilon_{b_{11} b_{13} b_{18} b_{20} \delta_{b_{18}}^{d_{1 d_{2}}} \delta_{b_{14}}^{d_{2 d_{3}}} + 2 \text{ perms} \right)
\left[ 2 \epsilon_{c_1 c_2 c_3 c_4 \epsilon_{d_{1} d_{2}} b_{15} b_{17} b_{19} b_{20} \delta_{b_{18}}^{d_{1 d_{2}}} \delta_{b_{14}}^{d_{2 d_{3}}} =
\]

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The first relevant component of $\epsilon T$ are obtained by computing components. This component of $b_{1}^{b_{1}}b_{2}^{b_{2}}\epsilon_{b_{1}b_{2}b_{3}b_{4}b_{5}b_{6}b_{7}b_{8}b_{9}b_{10}} + 2$ perms +
\frac{12}{5}(\epsilon_{10})^{b_{1}^{b_{1}}b_{2}^{b_{2}}}\epsilon_{b_{1}b_{2}a_{3}a_{4}a_{5}a_{6}a_{7}a_{8}a_{9}a_{10}} + 2$ perms +
\frac{3}{5}(\epsilon_{10})^{b_{1}^{b_{1}}b_{2}^{b_{2}}}\epsilon_{b_{1}b_{2}a_{3}a_{4}a_{5}a_{6}a_{7}a_{8}a_{9}a_{10}} + 2$ perms +
\frac{6}{5}\epsilon_{a_{1}a_{2}a_{3}a_{6}b_{16}}(\epsilon_{10})^{b_{1}^{b_{1}}b_{2}^{b_{2}}}\epsilon_{c_{1}c_{2}c_{3}c_{4}b_{12}} + 2$ perms +
\frac{6}{5}\epsilon_{b_{11}b_{12}a_{1}a_{2}b_{16}}(\epsilon_{10})^{b_{1}^{b_{1}}b_{2}^{b_{2}}}\epsilon_{c_{1}c_{2}c_{3}c_{4}a_{4}a_{5}a_{6}a_{7}a_{8}a_{9}a_{10}} + 2$ perms +
60(\epsilon_{10})^{b_{1}^{b_{1}}b_{2}^{b_{2}}}\epsilon_{b_{14}a_{1}a_{2}a_{3}a_{4}a_{5}a_{6}a_{7}a_{8}a_{9}a_{10}} + 2$ perms +
12\epsilon_{b_{14}a_{1}^{a_{1}}a_{2}a_{3}a_{4}a_{5}a_{6}a_{7}a_{8}a_{9}a_{10}} + 2$ perms =
\left(-\frac{3}{5}(1) + \frac{12}{5}(\frac{1}{2}) + \frac{3}{5}(\frac{1}{2}) - \frac{6}{5}(\frac{1}{2}) + \frac{6}{5}(\frac{27}{2}) + 60(1) + 12(0)\right)
\epsilon_{a_{1}a_{2}a_{3}a_{4}b_{14}b_{16}a_{5}a_{6}a_{7}a_{8}a_{9}a_{10}} + 2$ perms +
\frac{129}{2}\epsilon_{a_{1}^{a_{1}}a_{2}a_{3}a_{4}b_{14}b_{16}a_{5}a_{6}a_{7}a_{8}a_{9}a_{10}} + 2$ perms +
Since Asym$^{5}10 \otimes$ Sym$^{3}10 \otimes$ Asym$^{4}5$ contains one scalar all seven tensors in the penultimate step are proportional to each other. The constants of proportionality are obtained by computing components.

The integral $I_5$ is only non-vanishing if one chooses
\[\beta_2, \ldots, \beta_{11} = b_3 b_4, \ldots, b_{11} b_{12}, 1, 2, 3, 4, 5.\] (A.28)

This component of $I_5$ consists of two terms, one for $\beta_1 = b_1 b_2$ and one for $\beta_1 = +$:

\[\frac{1}{2} \int d\lambda \frac{1}{(\lambda'^{+})^{3}} \lambda_{b_{1}b_{2}} \lambda_{a_{1}a_{2}} \cdots \lambda_{a_{9}a_{10}} \Lambda_{\hat{\delta}_1 \hat{\delta}_2 \hat{\delta}_3} (\epsilon T)^{\hat{\delta}_1 \hat{\delta}_2 \hat{\delta}_3} b_{b_{3}b_{4}} + \frac{1}{2} \int d\lambda \frac{1}{(\lambda'^{+})^{2}} \lambda_{b_{1}b_{2}} \lambda_{a_{1}a_{2}} \cdots \lambda_{a_{9}a_{10}} \Lambda_{\hat{\delta}_1 \hat{\delta}_2 \hat{\delta}_3} (\epsilon T)^{\hat{\delta}_1 \hat{\delta}_2 \hat{\delta}_3} b_{b_{1}b_{2}b_{3}b_{4}}.\] (A.29)

The first relevant component of $\epsilon T$ is given by

\[\epsilon T^{d_{1}d_{2}d_{3}b_{1}b_{2}} + \frac{1}{2} \frac{1}{16} (\epsilon_{10})^{b_{1}^{b_{1}}b_{2}^{b_{2}}} \frac{1}{2} \gamma_{a_{1}a_{2}} a_{1} a_{2} \gamma_{b_{1}b_{2}} b_{1} b_{2} a_{3} a_{4} a_{5} a_{6} a_{7} a_{8} a_{9} a_{10} + \frac{1}{2} \frac{1}{16} (\epsilon_{10})^{b_{1}^{b_{1}}b_{2}^{b_{2}}} \frac{1}{2} \gamma_{a_{1}a_{2}} a_{1} a_{2} \gamma_{b_{1}b_{2}} b_{1} b_{2} b_{3} b_{4} a_{5} a_{6} a_{7} a_{8} a_{9} a_{10} = 0.\] (A.30)

\[\begin{align*}
&= \frac{1}{2} \frac{1}{16} (\epsilon_{10})^{b_{1}^{b_{1}}b_{2}^{b_{2}}} \gamma_{a_{1}a_{2}} \gamma_{b_{1}b_{2}} b_{1} b_{2} a_{3} a_{4} a_{5} a_{6} a_{7} a_{8} a_{9} a_{10} + \frac{1}{2} \frac{1}{16} (\epsilon_{10})^{b_{1}^{b_{1}}b_{2}^{b_{2}}} \gamma_{a_{1}a_{2}} \gamma_{b_{1}b_{2}} b_{1} b_{2} b_{3} b_{4} a_{5} a_{6} a_{7} a_{8} a_{9} a_{10} = 0.
\end{align*}\]
The second relevant component is given by

\[
\begin{align*}
\epsilon T^{d_1d_2d_3b_1...b_{12}}_{d_3d_4} &= -8 \frac{1}{32} \epsilon^{b_3...b_{22}} \gamma_{ad_3d_4b_{13}b_{14}} \gamma_{bd_1} \gamma_{cd_2} \gamma_{bcb_{19}b_{20}b_{21}b_{22}} = \\
&= -\frac{1}{4} \epsilon^{b_3...b_{22}} (-1) \epsilon_{ad_3d_4b_{13}b_{14}} \delta^b_{b_{13}} \delta^d_{b_{15}} \delta^b_{b_{17}} \delta^d_{b_{19}} 12345 \frac{b_3a}{b_{12}}.
\end{align*}
\]

where again a factor of eight and powers of \( \frac{1}{2} \) have been extracted. The above two components of \((\epsilon T)\) can be processed further to give

\[
\begin{align*}
\epsilon T^{d_1d_2d_3b_1...b_{12}}_{12345} &= -8 \frac{1}{2} \epsilon^{b_3...b_{20}} \delta^d_{b_{13}} \delta^d_{b_{15}} \delta^d_{b_{17}} \delta^d_{b_{19}b_{18}b_{19}b_{20}} \\
\epsilon T^{d_1d_2d_3b_1...b_{12}}_{d_3d_4} &= 8 \frac{1}{4} \epsilon^{b_3...b_{20}} \epsilon_{b_{17}d_{3}d_{4}b_{15}b_{16}b_{18}b_{19}b_{20}} \delta^d_{b_{13}} \delta^d_{b_{15}} \delta^d_{b_{17}} \delta^d_{b_{19}b_{20}}.
\end{align*}
\]

The integral \( I_5 \) becomes

\[
\begin{align*}
I_5 &= \int [d\lambda] \frac{1}{(\lambda^+)^3} \lambda^{\beta_1} \lambda_{a_{1}a_{2}} \ldots \lambda_{a_{9}a_{10}} \Lambda_{\delta_{1}b_{1}} \Lambda_{\delta_{2}b_{2}} \Lambda_{\delta_{3}b_{3}} \epsilon T^{d_1d_2d_3b_1...b_{12}}_{12345} = \\
&= \frac{1}{3} \int [d\lambda] \frac{1}{(\lambda^+)^2} \lambda_{a_{1}a_{2}} \ldots \lambda_{a_{9}a_{10}} \Lambda_{d_{1}d_{2}} \Lambda_{d_{3}d_{4}} \epsilon T^{d_1d_2d_3} \epsilon T^{d_1d_2d_3d_4} b_{1}...b_{12} d_{13}45 + \\
&= \frac{1}{2} \int [d\lambda] \frac{1}{(\lambda^+)^3} \lambda_{b_{1}b_{2}} \lambda_{a_{1}a_{2}} \ldots \lambda_{a_{9}a_{10}} \Lambda_{d_{1}d_{2}} \Lambda_{d_{3}d_{4}} \epsilon T^{d_1d_2d_3b_1...b_{12}}_{d_3d_4} d_{13}45 + \\
&= \frac{3}{40} \epsilon(d_{11}a_{1}a_{2}a_{3}d_{4}) \epsilon(d_{2}) \Lambda_{a_{9}a_{7}a_{8}} \Lambda_{a_{9}a_{10}} \Lambda_{d_{1}d_{2}} \Lambda_{d_{3}d_{4}} (A.31)
\end{align*}
\]
Appendix A - Detailed computations of $I_k$

\[
\frac{3}{5} \epsilon b_{2a1a2a3a4} \epsilon b_{14a5a6a7a8} (\epsilon 10)^{b_1 \cdots b_2} \epsilon b_{17a9a10b15b16} \epsilon b_{18b1b13b19b20} + 14 \text{ perms} \\
- \frac{2}{5} \epsilon b_{13a1a2a3a4} \epsilon b_{15a5a6a7a8} \epsilon b_{17a9a10b1b2} (\epsilon 10)^{b_1 \cdots b_2} \epsilon b_{14b10b18b19b20} + 14 \text{ perms} =
\]