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BCOV THEORY VIA GIVENTAL GROUP ACTION ON COHOMOLOGICAL FIELDS THEORIES

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Abstract. In a previous paper, Losev, the author, and Shneiberg constructed a full descendant potential associated to an arbitrary cyclic Hodge dGBV algebra. This contruction extended the construction of Barannikov and Kontsevich of solution of the WDVV equation, based on the earlier paper of Bershadsky, Cecotti, Ooguri, and Vafa.

In the present paper, we give an interpretation of this full descendant potential in terms of Givental group action on cohomological field theories. In particular, the fact that it satisfies all tautological equations becomes a trivial observation.

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1. Introduction

In this paper, which is partly a survey and introductory paper, we revise our previous work [31] on Hodge field theory in terms of the Givental group action on cohomological fields theories. Let us first remind the history and motivation of the whole subject.

1.1. BCOV theory. In [5], Bershadsky, Cecotti, Ooguri, and Vafa introduced an approach to construct a mirror partner for the Gromov–Witten potential of a mirror dual Calabi–Yau manifold. They inroduced what is called now BCOV action on the space of homorphic polyvector fields with coefficients in antiholomorphic forms on a Calabi–Yau manifold.

In [3], Barannikov and Kontsevich proved that indeed the critical value of BCOV action is a solution of the WDVV equation. In fact, their argument works in the settings of an abstract Hodge dGBV algebra (the algebraic structure that formalizes the properties of the space of polyvector fields on Calabi–Yau manifolds), see [32], [33].

In [30], Losev and the author represented the Barannikov–Kontsevich solution of the WDVV equation as a sum over trivalent trees and gave a new proof that it satis-

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fies the WDVV equation. Also we considered its genus expansion (the perturbative expansion of the BCOV action). It has appeared that the additional 1/12-axiom allows to prove some other PDEs coming from the geometry of the moduli spaces of curves (in addition to WDVV) for the higher genus amplitudes.

In order to explain this additional axiom we introduced [30] a special TCFT on the real blow-up of the moduli space of curves (Kimura–Stasheff–Voronov space [24]). Following ideas of Getlzer, we showed [30] that indeed, the relations between the Dehn twists in the mapping class group imply the properties of the BV operator in dGBV algebra and 1/12 axiom in the same fashion [12]; and using the induction procedure closed to Zwiebach's idea of splitting the moduli space into the interior part and the boundary [42], we got a geometric construction that could degenerate to the expansion of BCOV action in the simplest possible case.

This gave a hint of how one can introduce descendants in order to extend BCOV theory to the full descendant potential. First, one could do this by hand in genus 0 using the topological recursion relation, second one could introduce descendants in this special TCFT (called 'Zwiebach invariants' in [30], [31]) and look what happens at degeneration. Both approaches give the same answer (in terms of graphs), and after a sequence of extrimely complicated calculations with graphs [37], [38], [39] Losev, the author, and Shneiberg proved in [31] that the full descendant potential that we defined indeed satisfies all expected relations coming from geometry of the moduli space of curves.

The complete construction is called in [31] Hodge field theory.

1.2. Givental group action. Givental group action on cohomological field theories has first appeared in the papers of Givental [17], [18], [19]. He observed that the localization formulas for the Gromov–Witten potentials of projective spaces and some Fano complete intersections have the structure that can be formulated in terms of the quantization of some group action on genus 0 parts of potentials.

He explained the group action in genus 0 in terms of symplectic geometry. In particular, all semi-simple theories form one orbit of this action (or a small family of orbits [21]). Then, using the quantization procedure for the group action, Givental suggested a universal formula for the genus expansion of an arbitrary Frobenius manifold.

In particular, Givental conjectured that his universal formula, when applied to the genus 0 Gromov–Witten potential of a target manifold with the semi-simple quantum cohomology ring, reconstructs the full Gromov–Witten potential. This conjecture was proved in [40] via a complete classification of semi-simple cohomological field theories.

Meanwhile, Faber, the author, and Zvonkine proved in [11] that after quantization Givental's group action preserves all universal constrains coming from geometry of the moduli space of curves (tautological relations). In particular, if one applies a (quantized) element of the Givental group to the potential of a cohomological field theory, one gets the potential of another cohomological field theory.

Our argument was generalized by Kazarian [21] and Teleman [40], who explained a part of the Givental group action as an action on the cohomology classes in the moduli space of curves.

1.3. Plan of the paper. In Section 2, we briefly introduce all necessary basic notions such as moduli space of curves, cohomological field theory, tautological relations and so on. In Section 3, we discuss different ways to introduce a part of Givental's action (this part is called 'upper triangular group' in the literature). In Section 4, we remind the construction of the full descendant potential of Hodge field theory. Finally, in Section 5, we give an alternative construction of this potential using the Givental group action.

1.4. Remarks. There are several remarks. First, there is a particular mathematical trouble — usually, the cyclic Hodge dGBV algebras are infinite-dimentional, and there is a problem of convegence of all the tensor expressions used in Sections 4 and 5. We don't worry about it, since there exist different approaches to renormalization of BCOV theory [6], [7], and they seem to be very well compatible with our technique.

Second, we don't know, whether 1/12-axiom is valid in known examples of Hodge dGBV algebras. It has never appeared before since people used to study only genus 0. In the only example [7], where all computations can be done explicitely (elliptic curve), it is valid, but in appears to be a statement of the type $0 = 0$, so the coefficients are not essential.

Third, there is another family of examples of Hodge dGBV algebras. It appears to be the structure on the space of differential forms on symplectic manifolds satisfying the hard Lefschetz condition [34]. However, these examples seems not be studied yet elsewhere.

Forth, the axioms of Hodge dGBV algebra also have another motivation. Losev's description [29] of Frobenius structures on the base space of the universal unfolding of a simple singularity leads to some algebraic formalism, whose immediate nonlinear generalization appears to be exactly the axioms of Hodge dGBV algebra.

Fifth, the full descendant potential that we study here (a genus expansion of Barannikov–Kontsevich construction) is not exactly the mirror partner of the Gromov–Witten potential of a mirror dual Calabi–Yau manifold. Barannikov observed [1], [2] that there is a family of possible changes of variables, parametrized by semi-infinite variations of Hodge structure, and one of them is required to complete the mirror construction. In fact, Barannikov construction seems to be nothing but again the upper-triangular part of the Givental group action with the additional restriction that is preserves the weights of homogeneity of the formal variables in the potential. We are going to discuss it elsewhere.

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2. Basic Notions

2.1. The moduli space of curves. The moduli space of curves $\mathcal{M}_{g,n}$, $g \geqslant 0$, $n \geqslant 0$, $2g - 2 + n > 0$, parametrizes smooth complex curves of genus g with n ordered marked points. It is a smooth complex orbifold of dimension $3g - 3 + n$.

The space $\overline{\mathcal{M}}_{g,n}$ is a compactification of $\mathcal{M}_{g,n}$. It parametrizes stable curves of genus g with n ordered marked points. A stable curve is a possibly reducible curve with possible nodes, such that the order of its automorphism group is finite. Genus of a stable curve is the arithmetic genus, namely, the genus of the smooth curve that we get if the replace each node (given locally by the equation $xy = 0$) with a cylinder (given locally by the equation $xy = \epsilon$). The space $\overline{\mathcal{M}}_{g,n}$ is a smooth compact complex orbifold.

The space $\overline{\mathcal{M}}_{g,n}$ has a natural stratification by the topological type of stable curves.

2.2. Natural mappings. There is a number of natural mappings between the moduli spaces of curves.

First, there are projections $\pi\colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ that forget the last marked point. Note that there is a subtlety related to the fact that when we forget a marked point a stable curve can become unstable.

Second, there is a 2-to-1 mapping $\sigma\colon \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$ whose image it the boundary divisor of irreducible curves with one node. Third, there are mappings $\rho\colon \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g,n}$, $g_1 + g_2 = g$, $n_1 + n_2 = n$, whose images are the other irreducible boundary divisors of the compactification of $\overline{\mathcal{M}}_{g,n}$.

That gives a complete description of strata in codimension 1. Also we see that the maps ρ_* and σ_* induce the structure of modular operad [16] on the (co)homologies of the moduli spaces of curves.

2.3. Strata in $\overline{\mathcal{M}}_{g,n}$. Any irreducible boundary stratum of codimension k in $\overline{\mathcal{M}}_{g,n}$ is represented as an image $p(S)$ of the product $S = \overline{\mathcal{M}}_{g_1,n_1} \times \cdots \times \overline{\mathcal{M}}_{g_a,n_a}$. Here p is a composition of the k mappings σ and/or ρ described above. We can associate a dual graph G to the mapping p. The vertices v_i, $i = 1, \ldots, a$, correspond to the spaces $\overline{\mathcal{M}}_{g_i,n_i}$; v_i is marked by g_i (a non-negative integer that keeps track of genus) and its index is n_i. The marked points on curves in $\overline{\mathcal{M}}_{g_i,n_i}$ that remain marked point in the image of p correspond to the leaves (free half-edges) of G. The pairs of marked points that turn into nodes in the image of p form edges of the graph G.

We see that there are exactly k edges in G and G is connected. Moreover, the requirements on the arithmetic genus and the number of marked points of curves in the image of p mean that $\sum_{i=1}^{a} g_i + b_1(G) = g$ and $\sum_{i=1}^{a} n_i = n + 2k$.

2.4. Tautological classes. The cohomology of the moduli space of curves is a complicated object that is not yet studied well enough. Instead of it, people usually consider a special system of subalgebras of the cohomology algebras of the moduli spaces called tautological rings.

The system of tautological rings $RH^*(\overline{\mathcal{M}}_{g,n}) \subset H^*(\overline{\mathcal{M}}_{g,n})$ is defined as a minimal system of subalgebras of the algebras of cohomology that is closed under the push-forwards and pull-backs via the natural mappings between the moduli spaces.

The cohomology classes in $RH^*(\overline{\mathcal{M}}_{g,n})$ are called tautological classes. It follows immediately from the definition that $RH^*(\overline{\mathcal{M}}_{g,n})$ also form a modular operad.

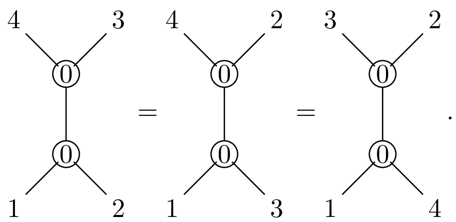
2.5. Additive generators of the tautological ring. We describe a system of additive generators of the tautological ring of the space $\overline{\mathcal{M}}_{g,n}$.

Let L_i the line bundle over $\overline{\mathcal{M}}_{g,n}$, whose fiber over a point $x \in \overline{\mathcal{M}}_{g,n}$ represented by a curve C_g with marked points $x_1, \ldots, x_n$ is equal to $T^*_{x_i}C_g$. Denote by $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n})$ the first Chern class of L_i. It is easy to show that ψ_i is a tautological class. Denote by κ_j, $j = 1, 2, \ldots$, the class $\pi_*(\psi_{n+1}^{j+1})$, $\pi\colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$. Since ψ-classes are tautological, $\kappa_j \in RH^{2j}(\overline{\mathcal{M}}_{g,n})$.

Now the system of additive generators of $RH^*(\overline{\mathcal{M}}_{g,n})$ can be described in terms of ψ-κ-strata. Let $p(S)$ be an irreducible stratum of codimension k in $\overline{\mathcal{M}}_{g,n}$, $S = \overline{\mathcal{M}}_{g_1,n_1} \times \cdots \times \overline{\mathcal{M}}_{g_a,n_a}$, as in Section 2.3. Equip each $\overline{\mathcal{M}}_{g_i,n_i}$, $i = 1, \ldots, a$, with a monomial of ψ- and κ-classes. Consider push-forward p_* of the class that we get in the cohomology of S. What we obtain is called ψ-κ-stratum, and all ψ-κ-strata form a system of additive generators of $RH^*(\overline{\mathcal{M}}_{g,n})$.

2.6. Tautological relations. Of course, ψ-κ-strata are not free additive generators. There are plenty of relations that are called tautological relations. Let us give a few examples. In examples, we use dual graphs defined in Section 2.3, decorated by ψ- and κ-classes.

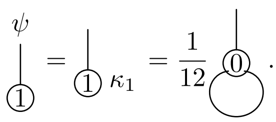
In $\overline{\mathcal{M}}_{0,4}$ we have: $\psi_1 = \psi_2 = \psi_3 = \psi_4 = \kappa_1$. Moreover all these classes are also equal to



(1)

Leaves of graphs are labeled by the numbers of the corresponding marked points.

In $\overline{\mathcal{M}}_{1,1}$ we have:



(2)

There is a very small number of basic tautological relations known explicitely [41], [22], [35], [13], [14], [4], [23]. Also, there are general theorems proving the existence of some special families of tautological relations [14], [20], [10].

2.7. Gromov–Witten theory. Gromov–Witten theory associates to a compact Kähler manifold X some algebraic structure on its cohomology $H^*(X)$ that generalizes the standard structure of a graded Frobenius algebra (that is, a structure of graded commutative associative algebra with a unit compatible with a non-degenerate even scalar product). From the algebraic point of view, this structure on a vector space V $(= H^*(X))$ equipped with a scalar product is given by a unital representation of a modular operad of the cohomologies $H^*(\overline{\mathcal{M}}_{g,n})$ of the moduli spaces of curves $\overline{\mathcal{M}}_{g,n}$.

Let us describe the operations on $H^*(X)$. We consider

$$\alpha_{g,n}^{u,\beta}(v_1,\, \dots,\, v_n) := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} f^*(u) ev_1^*(v_1) \dots ev_1^*(v_1). \tag{3}$$

as a linear map $H^*(X)^{\otimes n} \to \mathbb{C}$; it depends on $g \geqslant 0$x, $\beta \in H_2(X)$, and $u \in H^*(\overline{\mathcal{M}}_{g,n})$.

Here $\overline{\mathcal{M}}_{g,n}(X,\, \beta)$ is the moduli space of stable maps of curves of genus g with n marked points to X, such that the image of the fundamental class is equal to $\beta \in H_2(X)$. Such spaces consist of several irreducible components of different dimension, so the integral is taken over a particular homology class $\overline{\mathcal{M}}_{g,n}(X,\, \beta)$ that is calles the virtual fundamental class and replace the usual fundamental class.

There is a forgetful map $f\colon \overline{\mathcal{M}}_{g,n}(X,\, \beta) \to \overline{\mathcal{M}}_{g,n}$ that associate to a stable map $(C_g,\, x_1,\, \dots,\, x_n,\, \phi\colon C_g \to X)$ the stabilization of the source curve with marked points. The map $ev_i\colon \overline{\mathcal{M}}_{g,n}(X,\, \beta) \to X$, $i = 1,\, \dots,\, n$ is defined as $\phi(x_i)$.

2.8. CohFT. The main properties of Gromov–Witten theory are captured by the notion of cohomological field theory (CohFT). Roughly speaking, a CohFT is a system of cohomology classes on the moduli spaces of curves with the values in the tensor powers of V, compatible with all natural mappings between the moduli spaces.

The formal definition is the following. We fix a vector space $V = \langle e_1,\, \dots,\, e_s \rangle$ (e_1 will play a special role) with a non-degenerate scalar product η. A cohomological field theory is a system of cohomology classes $\alpha_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n},\, V^{\otimes n})$ satisfying the properties:

1. (1) $\alpha_{g,n}$ is equivariant with respect to the action of S_n on the labels of marked point and component of $V^{\otimes n}$.
2. (2) $\sigma^*\alpha_{g,n} = (\alpha_{g-1,n+2},\, \eta^{-1})$; $\rho^*\alpha_{g,n} = (\alpha_{g_1,n_1+1} \cdot \alpha_{g_2,n_2+1},\, \eta^{-1})$ (in both cases we contract with the scalar product the two components of V corresponding to the two points in the preimage of the node under normalization).
3. (3) $(\alpha_{0,3},\, e_1 \otimes e_i \otimes e_j) = \eta_{ij}$, $\pi^*\alpha_{g,n} = (\alpha_{g,n+1},\, e_1)$ (again, we contract the component of V corresponding to the last marked point with e_1).

2.8.1. *TFT.* Topological field theory is a special case of CohFT, when all cohomology classes $\alpha_{g,n}$ are in degree 0. One can easily show that in this case the whole system $\{\alpha_{g,n}\}$ is determined by $\alpha_{0,3} \in V^{\otimes 3}$. Of course, there are some restrictions for the choice of the 3-tensor $\alpha_{0,3}$. One can show that a particular 3-tensor can be exteded to a TFT if an only if this 3-tensor determines a structure of commutative Frobenius algebra with the unit on the vector space V with the scalar product η [8].

2.9. Correlators. We want to assign to a CohFT a numerical information that would allow to reconstruct completely its restriction to the tautological ring. Actually, it appears that it is enough to know all the integrals of the type

$$\langle \tau_{d_1}(e_{i_1}) \dots \tau_{d_n}(e_{i_n}) \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \prod_{j=1}^{n} \psi_j^{d_j} \cdot \Big(\alpha_{g,n},\, \bigotimes_{j=1}^{n} e_{i_j}\Big) \tag{4}$$

(correlators) in order to be able to reconstruct $\int_{\overline{\mathcal{M}}_{g,n}} \beta \cdot (\alpha_{g,n},\, \bigotimes_{j=1}^{n} e_{i_j})$ for an arbitrary tautological class β.

Indeed, using the property (3) of CohFT, we can express the integrals with the monomials of ψ- and κ-classes via the integrals with the monomials of ψ-classes only. Then, using the property (2), we can combine the integrals with monomials of ψ- and κ-classes in order to get the integrals with the class β represented with an arbitrary dual graph decorated by ψ- and κ-classes.

Usually, it is convenient to gather the correlators into a generating series (potential of CohFT):

$$\begin{aligned} Z &= \exp\Bigg(\sum_{g\geqslant 0} \hbar^{g-1} F_g\Bigg) \qquad (5)\\ &= \exp\Bigg(\sum_{g,n} \hbar^{g-1} \sum \frac{1}{n!} \sum_{\boldsymbol{d},\boldsymbol{i}} \Big\langle \prod_{j=1}^{n} \tau_{d_j}(e_{i_j}) \Big\rangle_g \prod_{j=1}^{n} t_{d_j,i_j}\Bigg). \end{aligned}$$

Here $t_{d,i}$ are some formal variables. In the second line, the first sum is taken over g, $n \geqslant 0$, $2g-2+n>0$, and the second sum is taked over all possible tuples of indices $\boldsymbol{d} = (d_1, \ldots, d_n)$, $\boldsymbol{i} = (i_1, \ldots, i_n)$.

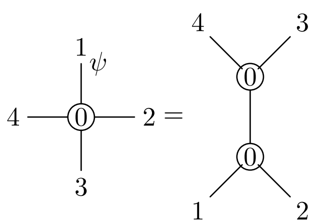
2.10. Tautological equations. First, there is a freedom in the choice of expressions of the intergrals with an arbitrary tautological class in terms of correlators. Second, ψ-κ-strata are not free additive generators of the tautological ring, as discussed in Section 2.6. This implies that correlators should satisfy a huge number of universal relations. 'Universal' means that the realtions depend only on the choice of the vector space with the scalar product and the unit.

These universal relations can be gathered in an infinite number of very complicated PDEs on the series F_g, $g \geqslant 0$. These PDEs are the following:

$$\exp\Big(-\sum t_{0,i}\eta^{ij}t_{0,j}\Big)\frac{\partial Z}{\partial t_{0,1}} = \sum t_{d+1,i}\frac{\partial Z}{\partial t_{d,i}} \quad \text{(string);} \qquad (6)$$

$$\frac{\partial F_g}{\partial t_{1,1}} - (2g-2)F_g = \sum t_{d,i}\frac{\partial F_g}{\partial t_{d,i}}, \quad g \geqslant 0 \quad \text{(dilaton);} \qquad (7)$$

and a system of PDEs associated to each tautological relation in the sense of Section 2.6. Each system of PDEs collects the relations for correlators that are implied by all pull-backs of a given relation multiplied by an arbitrary monomial of ψ-classes. For example, the relation



(8)

implies the following system of PDEs for F_0:

$$\frac{\partial^3 F_0}{\partial t_{a+1,i}\partial t_{b,j}\partial t_{c,k}} = \sum \frac{\partial^2 F_0}{\partial t_{a,i}\partial t_{0,\alpha}}\eta_{\alpha,\beta}\frac{\partial^3 F_0}{\partial t_{0,\beta}\partial t_{b,j}\partial t_{c,k}} \qquad (9)$$

(topological recursion relation in genus 0, or just TRR-0 for short).

3. Givental's Group Action on CohFT

In this section we describe a part of the Givental theory of the group action on CohFT [18], [19].

3.1. Action on cohomology classes. Let $R(z) = \mathrm{Id} + zR_1 + z^2R_2 + \cdots \in \mathrm{Hom}(V, V) \otimes \mathbb{C}[[z]]$ be a series of endomorphisms of the space V satisfying the identity $R^*(-z)R(z) = \mathrm{Id}$. In other words, $R(z) = \exp(r_1 z + r_2 z^2 + \cdots)$, where $r_l \in \mathrm{Hom}(V, V)$ is (graded) symmetric matrix for odd l and skew-symmetric for even l, with respect to the scalar product η.

Following [21] and [40], we associate to $\sum_{l=1}^{\infty} r_l z^l$ an infinitesimal deformation of CohFT. Denote by $(r_l z^l)\hat{}\,\alpha_{g,n}$ the following class on $\overline{\mathcal{M}}_{g,n}$:

$$
\begin{aligned}
-\pi_*\big(\alpha_{g,n+1}\psi_{n+1}^{l+1},\, r_l(e_1)\big) &+ \sum_{k=1}^{n} \psi_k^l r_l^{(k)} \alpha_{g,n} \\
&+ \frac{1}{2}\sum_{i=0}^{l-1}(-1)^{i+1}\sigma_*\big(\alpha_{g-1,n+2}\psi_{n+1}^{i}\psi_{n+2}^{l-1-i},\, \eta^{-1}r_l\big) \\
&+ \frac{1}{2}\sum_{\mathrm{div}}\sum_{i=0}^{l-1}(-1)^{i+1}\rho_*\big(\alpha_{g_1,n_1+1}\psi_{n_1+1}^{i}\cdot\alpha_{g_2,n_2+1}\psi_{n_2+1}^{l-1-i},\, \eta^{-1}r_l\big). \quad (10)
\end{aligned}
$$

The last sum here ($\sum_{\mathrm{div}}$) is taken over all irreducible boundary divisors, whose generic points are represented by two-component curves.

We define $\alpha'_{g,n} := \exp(\sum_{l=1}^{\infty}(r_l z^l)\hat{}\,)\alpha_{g,n}$ applying successively formula (10) simultaneously to the whole system of classes, for all g and n. Kazarian [21] and Teleman [40] proved that (1) $\alpha'_{g,n}$ are well-defined cohomology classes with the values in the tensor powers of V and (2) $\alpha'_{g,n}$ form a CohFT.

3.2. Action on correlators. We start with the same operator $R(z) = \exp(r_1 z + r_2 z^2 + \cdots)$ as before, but now we denote by $(r_l z^l)\hat{}$ the following differential operator:

$$
-(r_l)_\mu^1 \frac{\partial}{\partial t_{l+1,\mu}} + \sum_{d=0}^{\infty} t_{d,\nu}(r_l)_\mu^\nu \frac{\partial}{\partial t_{d+l,\mu}} + \frac{\hbar}{2}\sum_{i=0}^{l-1}(-1)^{i+1}(r_l)_{\mu,\nu}\frac{\partial^2}{\partial t_{i,\mu}\partial t_{l-1-i,\nu}}. \quad (11)
$$

Let $Z = \exp(\sum_{g=0}^{\infty}\hbar^{g-1}F_g)$ be the potential associated to a CohFT (see Section 2.9). Givental proved [18] that $Z' := \exp(\sum_{l=1}^{\infty}(r_l z^l)\hat{}\,)Z$ is also a well-defined formal power series of the same type as Z, namely Z' can be represented in the form $\exp(\sum_{g=0}^{\infty}\hbar^{g-1}F'_g)$. Moreover, Faber, the author, and Zvonkine proved in [11] that Z' is also the potential of a CohFT.

In fact, the theorem of Kazarian mentioned above is dual to our theorem, and it is even more general, since it doesn't require to restrict CohFT to the tautological ring.

3.3. Weyl quantization. We want to explain, following Givental, why the operator given by (11) (or, the dual operator on cohomology classes given by (10)) is denoted by $(r_l z^l)\hat{}$.

3.3.1. *Action in genus* 0. Consider the genus 0 part F_0 of the potential of a CohFT. It should satisfy only string, dilaton, and TRR-0 (see Section 2.10). It is possible to encode these properties in geometric terms. Let $\mathcal{V}$ be the space $V \otimes \mathbb{C}((z^{-1}))$ with the natural symplectic structure

$$\omega(f,\, g) = \frac{1}{2\pi\sqrt{-1}} \oint (f(-z),\, g(z))\, dz, \qquad f(z),\, g(z) \in \mathcal{V}. \tag{12}$$

This allows to identify $\mathcal{V}$ with the space $T^*\mathcal{V}_+$, $\mathcal{V}_+ = V \otimes \mathbb{C}[z]$. The natural coordinates $q_{n,i}$ in the space $z^n V$ (dual to a chosen basis $\langle e_1, \ldots, e_s\rangle$ in V) are identified with $t_{n,i}$; the only exception is $q_{1,1} = t_{1,1} - 1$.

Givental proved [18], [19] that the graph $\mathcal{L}$ of dF_0 in $\mathcal{V}$ is the germ of a Lagrangian cone with the vertex at 0 such that its tangent spaces L are tangent to $\mathcal{L}$ exactly along zL. This immediately implies that the group of linear symplectic operators compatible with the multiplication by z, that is, of the type $M(z) \in \mathrm{End}(V) \otimes \mathbb{C}((z^{-1}))$, $M^*(-z)M(z) = \mathrm{Id}$, acts on on the formal power series satisfying string, dilaton, and TRR-0. The group of such operators is called the twisted loop group.

3.3.2. *Quantization.* Consider an element m of the twisted loop group. It turns the graph of dF_0 into the graph of dF_0', where F_0' is a formal power series that satisfies string, dilaton, and TRR-0. The logarithm $\log m$ of this element is a linear vector field with quadratic Hamiltonian h_m. We can quantize any quadratic Hamiltonian in the following standard way:

$$\begin{aligned} (p_{n_1,i_1} p_{n_2,i_2})\hat{} &= \hbar \frac{\partial^2}{\partial q_{n_1,i_1} \partial q_{n_2,i_2}}, \\ (p_{n_1,i_1} q_{n_2,i_2})\hat{} &= q_{n_2,i_2} \frac{\partial}{\partial q_{n_1,i_1}}, \\ (q_{n_1,i_1} q_{n_2,i_2})\hat{} &= \frac{1}{\hbar} q_{n_1,i_1} q_{n_2,i_2} \end{aligned} \tag{13}$$

(the coordinates $p_{n,i}$ are the Darboux pairs of $q_{n,i}$ with respect to the symplectic structure on $\mathcal{V}$). Then one can consider the operator $\exp(\hat{h_m})$.

Applying this quantization to the operator $R(z) = \mathrm{Id} + zR_1 + z^2R_2 + \cdots \in \mathrm{Hom}(V,\, V) \otimes \mathbb{C}[[z]]$, we obtain the action on full potentials of CohFT. Strictly speaking, it is not obvious that this action is well-defined for the infinite series in z; but actually it is. The key additional property of the full potentials of CohFT is the following: $\langle \tau_{d_1}(e_{i_1}) \ldots \tau_{d_n}(e_{i_n})\rangle_g = 0$ if $\sum_{j=1}^n d_j > 3g-3+n$. It is called $3g-2$ property [9], [15], [18]. It follows from the tautological relations $\psi_1^{d_1} \ldots \psi_n^{d_n} = 0$ in $\overline{\mathcal{M}}_{g,n}$ if $\sum_{j=1}^n d_j > 3g-3+n$, and it can be written down as a system of PDEs. This property guarantees that the action of $\exp(\hat{h_R})$ on the full potentials of CohFT is well-defined.

Explicit formulas for $(r_l z^l)\hat{}$ were first computed and studied in [25], see also [26], [27], [28].

3.4. Remark on gradings. Actually, below we are applying not exactly the Givental group action as it is presented here, but a ($\mathbb{Z}_2$-) graded version of it. Namely, the target space V is a graded vector space. This also requires a graded version of CohFT. In our application these grading will affect some signes in the formulas. We hope that the reader will be able to reconstruct the precise definitions of the graded versions of CohFT and the Givental group action himself, while we are going to ignore this issue.

4. Hodge Field Theory

In this section we present the algebraic part of the construction of the potential associated to cyclic Hodge dGBV-algebra developed by Losev, the author, and Shneiberg.

4.1. Cyclic Hodge algebras. In this section, we give the definition of cyclic Hodge dGBV-algebras, see [30], [31] (cyclic Hodge algebras, for short). A super-commutative associative $\mathbb{C}$-algebra H with the unit is called cyclic Hodge algebra, if there are two odd linear operators Q, $G_-\colon H \to H$ and an even linear function $\int\colon H \to \mathbb{C}$ called integral. They must satisfy the following seven axioms A1–A7:

A1: (H, Q, G_-) is a bicomplex:

$$Q^2 = G_-^2 = QG_- + G_-Q = 0; \tag{14}$$

A2: $H = H_0 \oplus H_4$, where $QH_0 = G_-H_0 = 0$ and H_4 is represented as a direct sum of subspaces of dimension 4 generated by e_α, Qe_α, G_-e_α, QG_-e_α for some vectors $e \in H_4$, i.e.

$$H = H_0 \oplus \bigoplus_\alpha \langle e_\alpha, Qe_\alpha, G_-e_\alpha, QG_-e_\alpha\rangle \tag{15}$$

(Hodge decomposition);

A3: Q is an operator of the first order, it satisfies the Leibniz rule:

$$Q(ab) = Q(a)b + (-1)^{\tilde a} aQ(b) \tag{16}$$

(here and below we denote by $\tilde a$ the parity of $a \in H$);

A4: G_- is an operator of the second order, it satisfies the 7-term relation:

$$\begin{aligned} G_-(abc) = G_-(ab)c + (-1)^{\tilde b(\tilde a+1)} bG_-(ac) + (-1)^{\tilde a} aG_-(bc) \\ - G_-(a)bc - (-1)^{\tilde a} aG_-(b)c - (-1)^{\tilde a+\tilde b} abG_-(c). \end{aligned} \tag{17}$$

A5: G_- satisfies the property called 1/12-axiom:

$$\operatorname{str}(G_- \circ a\cdot) = (1/12)\operatorname{str}(G_-(a)\cdot) \tag{18}$$

(here $a\cdot$ and $G_-(a)\cdot$ are the operators of multiplication by a and $G_-(a)$ respectively, str means supertrace).

Define an operator $G_+\colon H \to H$. We put $G_+H_0 = 0$, and on each subspace $\langle e_\alpha, Qe_\alpha, G_-e_\alpha, QG_-e_\alpha\rangle$ we define G_+ as

$$\begin{aligned} &G_+e_\alpha = G_+G_-e_\alpha = 0, \\ &G_+Qe_\alpha = e_\alpha, \\ &G_+QG_-e_\alpha = G_-e_\alpha. \end{aligned} \tag{19}$$

We see that $[G_-, G_+] = 0$; $\Pi_4 = [Q, G_+]$ is the projection to H_4 along H_0; $\Pi_0 = \mathrm{Id} - \Pi_4$ is the projection to H_0 along H_4.

Consider the integral $\int : H \to \mathbb{C}$.

A6: We require that

$$\begin{aligned} \int Q(a)b &= (-1)^{\tilde{a}+1} \int aQ(b), \\ \int G_-(a)b &= (-1)^{\tilde{a}} \int aG_-(b), \\ \int G_+(a)b &= (-1)^{\tilde{a}} \int aG_+(b). \end{aligned} \tag{20}$$

These properties imply that $\int G_-G_+(a)b = \int aG_-G_+(b)$, $\int \Pi_4(a)b = \int a\Pi_4(b)$, and $\int \Pi_0(a)b = \int a\Pi_0(b)$.

We can define a scalar product on H as $(a, b) = \int ab$.

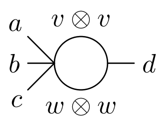
A7: This scalar product is non-degenerate.

Using the scalar product we may turn any graded symmetric or skew-symmetric operator $A\colon H \to H$ into the bivector (that we denote by $[A]$), well-defined up to a sign.

4.2. Tensor expressions in terms of graphs. Here we explain a way to encode some tensor expressions over an arbitrary vector space in terms of graphs.

Consider an arbitrary graph (we allow graphs to have leaves and we require vertices to be at least of degree 3, the definition of graph that we use can be found in [33]). We associate a symmetric n-form to each internal vertex of degree n, a symmetric bivector to each egde, and a vector to each leaf. Then we can substitute the tensor product of all vectors in leaves and bivectors in edges into the product of n-forms in vertices, distributing the components of tensors in the same way as the corresponding edges and leaves are attached to vertices in the graph. This way we get a number (in the case of $\mathbb{Z}_2$-graded vector space there is an additional sign correction, see Section 4.3.1 below).

Let us study an example:



(21)

We assign a 5-form x to the left vertex of this graph and a 3-form y to the right vertex. Then the number that we get from this graph is $x(a, b, c, v, w) \cdot y(v, w, d)$.

Note that vectors, bivectors and n-forms used in this construction can depend on some variables. Then what we get is not a number, but a function.

4.3. Usage of graphs in cyclic Hodge algebras. Consider a cyclic Hodge algebra H. There are some standard tensors over H, which we associate to vertices, edges, and leaves of graphs below. Here we introduce the notations for these tensors.

We always assign the form

$$(a_1, \ldots, a_n) \mapsto \int a_1 \cdots a_n \tag{22}$$

to a vertex of degree n.

There are two bivectors that can be assigned to edges: $[G_-G_+]$ and [Id]. The last one will is used only on loops.

The vectors that we will put at leaves depend on some variables. Let $\{e_1, \dots, e_s\}$ be a homogeneous basis of H_0. In particular, we assume that e_1 is the unit of H. To each vector e_i we associate formal variables $t_{d,i}$, $d \geqslant 0$, of the same parity as e_i. Then we will put at a leaf one of the vectors $E_d = \sum_{i=1}^{s} e_i t_{d,i}$, $d \geqslant 0$.

4.3.1. *Remarks.* There is a subtlety related to the fact that H is a $\mathbb{Z}_2$-graded space. The complete definition of the tensor contraction associated to a graph includes also a sign issue. Suppose we consider a graph of genus g. We can choose g edges in such a way that the graph being cut at these edge turns into a tree. To each of these edges we have already assigned a bivector $[A]$ for some operator $A\colon H \to H$. Now we have to put the bivector $[JA]$ instead of the bivector $[A]$, where J is an operator defined by the formula $J\colon a \mapsto (-1)^{\tilde{a}} a$.

Another subtlety appears when we study infinite-dimensional cyclic Hodge algebras. In this case, we assume that all expressions in graphs that we use do converge.

4.4. Correlators. We are going to define the potential using correlators. Let

$$\langle \tau_{d_1}(v_1) \dots \tau_{d_n}(v_n) \rangle_g \tag{23}$$

be the sum over graphs of genus g with n leaves marked by $\tau_{d_i}(v_i)$, $i = 1, \dots, n$, where $v_1, \dots, v_n$ are some vectors in H. The index of each internal vertex of these graphs is $\geqslant 3$; we associate to it the symmetric form (22). There are two possible types of edges: arbitrary edges marked by $[G_-G_+]$ ('heavy edges') and loops marked by [Id] ('empty loops').

Consider a vertex of such graph. Let us describe all possible half-edges adjusted to this vertex. There are $2g$, $g \geqslant 0$, half-edges coming from g empty loops; m half-edges coming from heavy edges of graph, and l leaves marked $\tau_{d_{a_1}}(v_{a_1}), \dots, \tau_{d_{a_l}}(v_{a_l})$. Then we say that the type of this vertex is $(g,\, m;\, d_{a_1}, \dots, d_{a_l})$. We denote the type of a vertex $\mathfrak{v}$ by $(g(\mathfrak{v}),\, m(\mathfrak{v});\, d_{a_1(\mathfrak{v})}, \dots, d_{a_{l(\mathfrak{v})}(\mathfrak{v})})$.

Consider a graph Γ in the sum determining the correlator

$$\langle \tau_{d_1}(v_1) \dots \tau_{d_n}(v_n) \rangle_g \tag{24}$$

We associate to Γ a number: we contract according to the graph structure all tensors corresponding to its vertices, edges, and leaves (for leaves, we take vectors $v_1, \dots, v_n$). Let us denote this number by $T(\Gamma)$.

Also we weight each graph by a coefficient which is the product of two combinatorial constants. The first factor is equal to

$$A(\Gamma) = \frac{\prod_{\mathfrak{v} \in \mathfrak{V}(\Gamma)} 2^{g(\mathfrak{v})} g(\mathfrak{v})!}{|\mathrm{Aut}(\Gamma)|}. \tag{25}$$

Here $|\mathrm{Aut}(\Gamma)|$ is the order of the automorphism group of the labeled graph Γ, $\mathfrak{V}(\Gamma)$ is the set of internal vertices of Γ. In other words, we can label each vertex $\mathfrak{v}$ by $g(\mathfrak{v})$, delete all empty loops, and then we get a graph with the order of the automorphism group equal to $1/A(\Gamma)$.

The second factor is equal to

$$P(\Gamma) = \prod_{\mathfrak{v}\in\mathfrak{V}(\Gamma)} \int_{\overline{\mathcal{M}}_{g(\mathfrak{v}),m(\mathfrak{v})+l(\mathfrak{v})}} \psi_1^{d_{a_1(\mathfrak{v})}} \dots \psi_{l(\mathfrak{v})}^{d_{a_{l(\mathfrak{v})}(\mathfrak{v})}}. \tag{26}$$

So, the whole contribution of Γ to the correlator is equal to $P(\Gamma)A(\Gamma)T(\Gamma)$. One can check that the non-trivial contribution to the correlator $\langle\tau_{d_1}(v_1)\dots\tau_{d_n}(v_n)\rangle_g$ is given only by graphs that have exactly $3g-3+n-\sum_{i=1}^n d_i$ heavy edges.

4.5. Potential. We fix a cyclic Hodge algebra and consider the formal power series $Z = Z(t_{d,i})$ defined as

$$\begin{aligned} Z &= \exp\Bigg(\sum_{g=0}^{\infty}\hbar^{g-1}F_g\Bigg) \\ &= \exp\Bigg(\sum_{g=0}^{\infty}\hbar^{g-1}\sum_n \frac{1}{n!}\sum_{d_1,\dots,d_n\in\mathbb{Z}_{\geqslant 0}} \langle\tau_{d_1}(E_{d_1})\dots\tau_{d_n}(E_{d_n})\rangle_g\Bigg). \end{aligned} \tag{27}$$

Losev, the author, and Shneiberg proved in [31] that this formal power series is indeed the potential of a CohFT, that is, its components of fixed genus satisfy string, dilaton, and all tautological relations (particular tautological relations were checked before in [30], [37], [38], [39]).

Our goal in this paper is to revise this theorem completely (both the statement and the prove) using the Givental group action on CohFT.

4.5.1. *Trivial example.* Consider the trivial cyclic Hodge algebra: $H = H_0 = \langle e_1\rangle$, $Q = G_- = 0$, $\int e_1 = 1$. Then $E_a = e_1 {\cdot} t_a$, and the correlator $\langle\tau_{a_1}(E_{a_1})\dots\tau_{a_n}(E_{a_n})\rangle_g$ consists just of one graph with one vertex, g empty loops, and n leaves marked by $a_1, \dots, a_n$. The explicit value of the coefficient of this graph is, by definition,

$$\langle\tau_{d_1}\dots\tau_{d_n}\rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\dots\psi_n^{d_n}. \tag{28}$$

So, in the case of the trivial cyclic Hodge algebra we obtain exactly the Gromov–Witten potential of the point.

5. Hodge Field Theory Revisited

In this section, we reformulate my theorem with Losev and Shneiberg mentioned in Section 4.5, and give a different prove of it.

5.1. Input. We start with a $\mathbb{Z}_2$-graded commutative Frobenius algebra H with a unit $e_1 \in H$. We still require the odd operators Q, G_-, and G_+, where the first two operators determine on H the structure of Hodge bicomplex and the third one is responsible for the particular choice of splitting $H = H_0 \oplus H_4$, where $e_1 \in H_0$. We also require that the operator Q is (graded) skew-symmetric, the operators G_- and G_+ are (graded) symmetric.

The structure of Frobenius algebra on H is equivalent to the structure of TFT on it, see Section 2.8.1. We can choose some basis in the spaces H_0 and H_4, and use it in order to write down the potential associated to this TFT. Denote this potential by Z°.

Proposition 1. *The Leibniz rule for Q (16) is equivalent to the equation $QZ^\circ = 0$. The 7-term relation (17) and 1/12-axiom (18) for G_- are equivalent to the equation $(G_-z)\hat{}\,Z^\circ = 0$.*

5.2. Potential. Consider the formal power series

$$Z := \exp(-(G_-G_+z)\hat{}\,)Z^\circ|_{H_0}. \tag{29}$$

Since it is restricted to H_0, we can consider it as a formal power series in $t_{d,i}$, $d = 0, 1, \dots; i = 1, \dots, s = \dim H_0$ (where the variables a linked to the same basis as was chosen in Section 4.3).

Proposition 2. *The formal power series Z defined here coinsides with the potential defined in Section 4.5.*

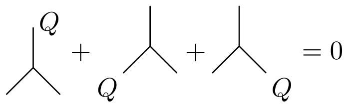
We also want to give an alternative proof of the following statement.

Proposition 3. *The formal power series Z is the potential of a CohFT on H_0.*

5.3. Proofs. In this section, we collect the proofs of Propositions 1–3.

5.3.1. *Proof of the first statement of Proposition 1.* Observe, that in the language of graphs components F_g°, $g \geqslant 0$, of the potential $Z^\circ = \exp(\sum_{g\geqslant 0} \hbar^{g-1} F_g^\circ)$ are represented as sums of one-vertex graphs of genus g. There are g empty loops attached to the vertices of these graphs. The equation $QZ^\circ = 0$ means that if we apply Q to each of the leaves of any particular such graph, we get 0.

Applying Q to each of the leaves of the simplest graph, we get exactly the Leibniz rule for Q:



(30)

It immediately implies that if we apply Q to a genus 0 one-vertex graphs, we also get 0.

Since Q is skew-symmetric, we conclude that if we put Q at two different ends of an empty edge (loop), we get 0. This implies that we also get 0 when we apply Q to the graphs with an arbitrary number of empty loops.

Thus we see that the condition $QZ^\circ = 0$ contains the Leibniz rule for Q as a special case, on the one hand, and the Leibniz rule for Q together with the fact that Q is skew-symmetric imply that $QZ^\circ = 0$.

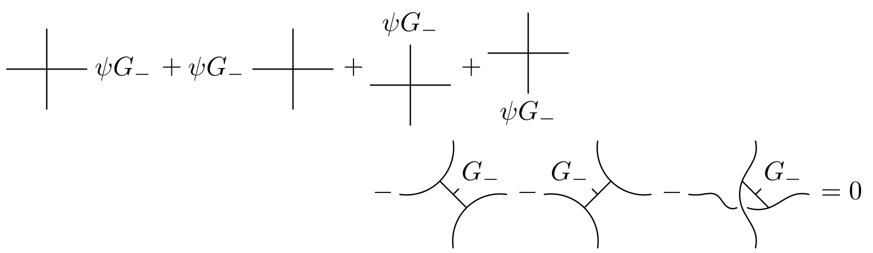
5.3.2. *Proof of the secont statement of Proposition 1.* Now we consider the condition $(G_-z)\hat{}\,Z^\circ = 0$. The operator $(G_-z)\hat{}$ consists of two parts. The linear part, ψG_-, increases the power of ψ-class on a leaf (or, more precisely, it increases the power of ψ-class in the contribution of a leaf in the coefficient $A(\Gamma)$), and applies G_- to the input at this leaf (in particular, since $G_-e_1 = 0$, the first term of (11) disappears). The quadratic part connects two leaves without ψ-classes of the same or of two different graphs with an edge; the bivector on this edge is $-[G_-]$.

So, $(G_- z)\hat{}Z^\circ = 0$ is equivalent to the following. Fix genus g and the number of leaves n. Mark leaves with the powers of ψ-classes $d_1, \ldots, d_n$, such that $\sum_{i=1}^n d_i = 3g - 4 + n$. Then consider the sum of graphs of three possible types:

(1) One-vertex graphs with g empty loops; the degrees of ψ-classes on leaves are $d_1, \ldots, d_{i-1}, d_i + 1, d_{i+1}, \ldots, d_n$, and G_- is applied to the input of i-th leaf.
(2) One-vertex graphs with $g - 1$ empty loops and 1 loop marked by $[G_-]$. The degrees of ψ-classes on leaves are $d_1, \ldots, d_n$.
(3) Two-vertex graphs with g empty loops distributed in an arbitrary way between two vertices; the edge connecting two vertices is marked by $[G_-]$; the degrees of ψ-classes on leaves are $d_1, \ldots, d_n$, and the leaves are distributed between two vertices in an arbitrary way.

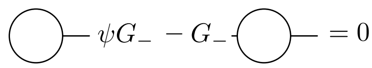
Of course, all graphs that we consider are stable in the sense that the index of each vertex is at least 3. The property $(G_- z)\hat{}Z^\circ$ is equivalent to the property that this sum is equal to 0 for all $g, n, d_1, \ldots, d_n$ as above.

The simplest equation in this system appear when $3g - 4 + n = 0$. This happens when either $g = 0$, $n = 4$, or $g = n = 1$. Let us describe these two equations pictorially:



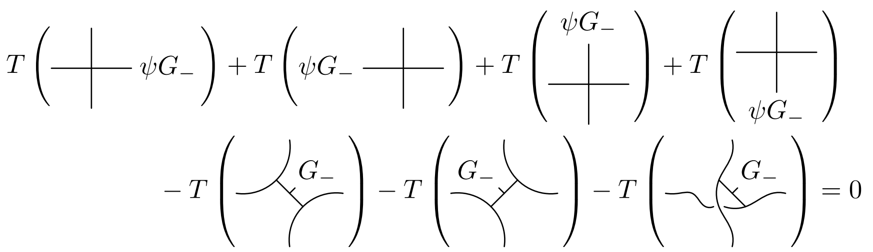
(31)

and



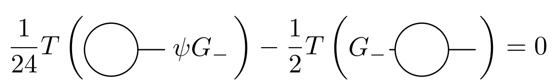
(32)

Recall that each graph Γ denotes a product of three constants related to it, $P(\Gamma)$, $A(\Gamma)$, and $T(\Gamma)$. We compute the coefficients $P(\Gamma)A(\Gamma)$ and rewrite these two formulas as



(33)

and



(34)

This allows us to see immediately, that Equations (31) and (32) are exactly the 7-term relation (17) and 1/12-axiom (18), repectively.

Thus we proved that $(G_-z)\hat{}Z^\circ = 0$ implies the 7-term relation and the 1/12-axiom. The converse statement is proved by essentially the same argument as the Main Lemma (Lemma 4) in [31]. □

5.3.3. *Proof of Proposition* 2. Consider expression $\exp(-(G_-G_+z)\hat{})Z^\circ|_{H_0}$. We have already mentioned in Section 5.3.2 the meaning of both the quadratic and linear part of a differential operator applied to an exponential formal power series. So, applying $-(G_-G_+z)\hat{}$ we either increase the degree of ψ class on a leaf and apply G_-G_+ to the vector on this leaf, or we connect two leaves with no ψ-classes on them with the edge marked by $[G_-G_+]$.

This way we get the following description for $\exp(-(G_-G_+z)\hat{})Z^\circ$. It is a sum of graphs with the vertices described in Section 5.3.1. There are some edges marked by

$$(-\psi)^i \left[\frac{(G_-G_+)^{i+j+1}}{i!\,j!\,(i+j+1)}\right](-\psi)^j,$$

and at each leaf we apply $\exp(-\psi G_-G_+)$ to its input (increase the degree of ψ-class and simulteneously apply the corresponding power of the operator). This, is fact, is a very general description that suites for any operator $\exp(-(Oz)\hat{})$ such that O vanishes the unit.

Since $(G_-G_+)^2 = 0$, we get a substantially simpler picture. First,

$$(-\psi)^i \left[\frac{(G_-G_+)^{i+j+1}}{i!\,j!\,(i+j+1)}\right](-\psi)^j = [G_-G_+],$$

so our edges are just marked by $[G_-G_+]$ (heavy edges from Section 4.4), and there are no ψ-classes on these edges (remember now that in the definition of $P(\Gamma)$ in Section 4.4, the half-edges coming from heavy edges play the role of marked points with no ψ-classes). There are still $\exp(-\psi G_-G_+) = 1-\psi G_-G_+$ on the leaves, but they disappear after the restriction to H_0, since $G_-G_+H_0 = 0$.

The only thing that we have to mention in addition is that the autmorphisms of graphs are always arranged automatically in all expression of this type. This completes the pictorial description of $Z = \exp(-(G_-G_+z)\hat{})Z^\circ|_{H_0}$ that, as we see, coinsides with the descripion of graphs and their contributions from Section 4.4. □

5.3.4. *Proof of Proposition* 3. Since we know that Givental's action preserves string, dilaton, and all tautological equations ([11] and Section 3.2), the problem is only with the restriction of the potential $\exp(-(G_-G_+z)\hat{})Z^\circ$ of a CohFt on H to H_0.

Observe that

$$Q\exp(-(G_-G_+z)\hat{})Z^\circ = \exp(-(G_-G_+z)\hat{})(Q+(G_-z)\hat{})Z^\circ = 0. \tag{35}$$

Now it is a general statement, that the restriction of a Q-closed CohFT-potential to the subspace representing the cohomology of Q gives a CohFT-potential on this spaces.

Indeed, let η denote the scalar product on H and η_0 denote the restriction of the scalar product to H_0. The difference between the bivectors η^{-1} and η_0^{-1} is Q-exact. Consider a tautological equation (as a relation for correlators), that might use η^{-1}.

If we replace η^{-1} with η_0^{-1}, and use the fact that all correlators are Q-closed, we still get a tautological equation, but it is valid now only modulo Q-exact terms. These Q-exact terms vanish when we restrict it to H_0. This completes the proof. $\square$

5.4. Remark. It is obvious that we don't use exactly the axioms described in Sections 4.1 and 5.1. In fact, much weaker system of axioms is to be sufficient. For example, instead of Hodge property, we can just require that there exists an odd symmetric operator A (it is G_-G_+ in the original settings) such that $[Q, \exp(-zA)] = zG_-$ and $A^2 = 0$. This way we get the condition that H splits as $H_0 \oplus H_2 \oplus H_4 = H_0 \oplus \bigoplus_\beta \langle e_\beta, Qe_\beta \rangle \oplus \bigoplus_\alpha \langle e_\alpha, Qe_\alpha, G_-e_\alpha, QG_-e_\alpha \rangle$, where $QH_0 = G_-H_0 = G_-H_2 = 0$. This generalization was considered in [36]. Second, one can observe that we don't use the whole power of Givental's theory, but just a small part of it. This also gives a hint on futher generalizations of BCOV theory, its homotopy version, possible relations to the Zwiebach-type induction procedure considered in [30], [31], and so on. We are going to discuss all these questions in the next paper.

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