Decomposing generalized quantifiers

Westerståhl, D.

DOI
10.1017/S1755020308080234

Publication date
2008

Published in
Review of Symbolic Logic

Citation for published version (APA):
DECOMPOSING GENERALIZED QUANTIFIERS

DAG WESTERSTÅHL
Department of Philosophy, University of Gothenburg

Abstract. This note explains the circumstances under which a type \langle 1 \rangle quantifier can be decomposed into a type \langle 1, 1 \rangle quantifier and a set, by fixing the first argument of the former to the latter. The motivation comes from the semantics of Noun Phrases (also called Determiner Phrases) in natural languages, but in this article, I focus on the logical facts. However, my examples are taken among quantifiers appearing in natural languages, and at the end, I sketch two more principled linguistic applications.

1. Introduction. The motivation for the results in this note comes from linguistics: Determiners and other phrases with a similar function in natural languages denote type \langle 1, 1 \rangle (generalized) quantifiers, and noun phrases (NPs or DPs) denote type \langle 1 \rangle quantifiers, often but not always obtained by restricting or freezing the noun argument of a \langle 1, 1 \rangle quantifier. I study the ‘inverse’ of freezing: decomposition of a type \langle 1 \rangle quantifier (when possible) by means of a set and a type \langle 1, 1 \rangle quantifier.

In this article, I focus on the logical facts: characterizations of classes of decomposable quantifiers are provided and questions of uniqueness of the decomposition addressed. The results are not difficult, but they yield a fairly clear picture of the situation and can be directly applied to certain issues in semantics. Two such applications—to reciprocals and to possessives—are sketched in the final section; the latter are dealt with extensively in Peters & Westerstål (in preparation).

Section 2 gives background on (generalized) quantifiers and their properties, section 3 provides the relevant notions of freezing and decomposition, section 4 presents and proves the results, and section 5 presents linguistic applications. Some observations on decomposition and freezing were made in Peters & Westerstål (2006, chap. 4.5.5); here, they are generalized and put in a coherent picture.

2. Background on quantifiers. To make the presentation more self-contained, central notions and definitions are provided in this section. Notation is as in Peters & Westerstål (2006), where (much) more background information on quantifiers in language and logic can be found.

2.1. Quantifiers of types \langle 1, 1 \rangle and \langle 1 \rangle. Except for one example in section 5, we only deal with quantifiers of these two types.

Definition 2.1. A (generalized) quantifier of type \langle 1, 1 \rangle \( Q \) associates with each universe \( M \) a binary relation \( Q_M \) between subsets of \( M \). In the type \langle 1 \rangle case, \( Q_M \) is a unary relation.
Logicians are more used to see $Q$ a class of structures. This is essentially just a notational difference; in the type $(1, 1)$ case:

$$(M, A, B) \in Q \iff Q_M(A, B).$$

Type $(1, 1)$ quantifiers interpreting English determiner phrases are conveniently named by those phrases, as in the examples below. For all $M$ and all $A, B \subseteq M$,

- \(all_{\text{ei}}(A, B) \iff A \subseteq B\)
- \((\text{all}_{\text{ei}})(A, B) \iff \emptyset \neq A \subseteq B\)
- \(\text{no}(A, B) \iff A \cap B = \emptyset\)
- \(\text{at least two}(A, B) \iff |A \cap B| \geq 2\)
- \(\text{exactly five}(A, B) \iff |A \cap B| = 5\)
- \(\text{all but three}(A, B) \iff |A - B| = 3\)
- \(\text{more than two-thirds of the}(A, B) \iff |A \cap B| > 2/3 \cdot |A|\)
- \(\text{most}(A, B) \iff |A \cap B| > |A - B|\)
- \(\text{the ten}(A, B) \iff |A| = 10\) and $A \subseteq B$
- \(\text{John's}(A, B) \iff \emptyset \neq A \cap \{a: \text{John 'possesses' } a\} \subseteq B\)
- \(\text{no except John}(A, B) \iff A \cap B = \{j\}\)
- \(\text{infinitely many}(A, B) \iff A \cap B \text{ is infinite}\)
- \(\text{an even number of}(A, B) \iff |A \cap B| \text{ is even.}\)

\(\text{all}_{\text{ei}}\) is ‘all with existential import’: It is the quantifier that Aristotle (as well as many modern linguists, but few modern logicians) took ‘every’ to mean. Of course, lots of type $(1, 1)$ quantifiers are much more remotely related to natural language:

- \(\text{sqrt}(A, B) \iff |A \cap B| > \sqrt{|A|}\)
- \(I_M(A, B) \iff |A| = |B|\) (the Härtig quantifier).

As to type $(1)$ quantifiers, all but the last two of the following are familiar from logic:

- \(\forall_M(B) \iff B = M\)
- \(\exists_M(B) \iff B \neq \emptyset\)
- \((\exists_{\text{all}})_M(B) \iff |B| = n\)
- \((I_a)_M(B) \iff a \in B\) (a Montagovian individual)
- \(Q^R_M(B) \iff |B| > |M - B|\) (the Rescher quantifier)
- \((Q_0)_M(B) \iff B \text{ is infinite}\)
- \(C^{\text{pl}, a}_M(B) \iff \emptyset \neq C \subseteq B\)
- \((C^{\text{pl}, e})_M(B) \iff C \cap B \neq \emptyset\).

For each $M$, $(I_a)_M$ is the principal filter on $P(M)$ generated by $a$; it is also what Montague used to interpret proper names, so that the sentences:

1. John smokes.
2. Some students smoke.

are interpreted on the same principle, as:

$$[[\text{NP}]]_M([[\text{smoke}]]),$$

where $M$ is the discourse universe, and $[[\text{NP}]] = \text{some}[[\text{student}]]$ (defined in section 3) in (2) and $= I_j$ in (1).
The quantifiers $C^{pl,u}$ and $C^{pl,e}$ can be used to obtain the universal and existential interpretation of *bare plurals* along the same lines, as in, respectively,

(3) Firemen are brave.
(4) Firemen are available.

It should be emphasized that quantifiers are *global*: With each universe $M$, they associate a second-order relation (a local quantifier) over $M$. This is faithful to the meaning of the corresponding phrases in natural languages. To know what ‘some’ means is not to know different things for different universes of discourse; it is to know just one thing, namely, the quantifier *some*.\(^1\) In the case of Montagovian individuals, this means that if $j \notin M$, $(I_j)_M(B)$ is always false, which seems perfectly reasonable.

### 2.2. Basic properties of quantifiers.

A crucial feature of quantifiers involved in natural language semantics, and in logic too for that matter, is that they *mean the same thing on every universe*. The following property comes very close to capturing this intuitive idea.

**Definition 2.2.** $Q$ satisfies extension (EXT) iff whenever $A, B \subseteq M \subseteq M'$, we have $Q_M(A, B) \iff Q_{M'}(A, B)$. Similarly for the type $\langle 1 \rangle$ case; EXT makes sense for quantifiers of any type.

Thus, extending the universe (but not the arguments of the quantifier) has no effect. It is easy to define quantifiers that mean different things on different universes (e.g., one meaning *some* on universes with fewer than 10 elements, and *all* on larger universes), but these do not appear in natural language semantics. All type $\langle 1, 1 \rangle$ quantifiers mentioned above and all type $\langle 1 \rangle$ quantifiers, except $\forall$ and $Q^R$, are EXT. In general, it seems that all type $\langle 1 \rangle$ natural language quantifiers are EXT, except some of those that explicitly refer to the universe, for example, by means of the predicate *thing*: *John, firemen, three cats, all but five students, some things, at most three things* are all EXT, whereas *everything, most things* are not.

For EXT quantifiers, mention of the universe $M$ can be omitted; this is done whenever possible below.

**Definition 2.3.** $Q$ is closed under isomorphism (ISOM) iff the corresponding class of structures is closed under isomorphism. In the type $\langle 1, 1 \rangle$ case, this is equivalent to the requirement that $Q_M(A, B)$ only depends on the cardinals of $A - B$, $A \cap B$, $B - A$, and $M - (A \cup B)$.

Many but not all natural language quantifiers are ISOM; the exceptions mentioned above are those explicitly mentioning some individual or property, like *Mary, all except John, every student’s*, and so forth. For logicians, it is natural to treat these individuals or properties as arguments too. For example, one could think of the quantifier at work in *every student’s* as having type $\langle 1, 1, 1 \rangle$: *every C’s A is B*. Similarly, one could think of *Mary’s* as taking two sets and one individual as arguments. These quantifiers—we could call them *fully abstracted*—would be ISOM, but the problem is that they have the wrong category. *Every student’s* and *Mary’s* are in important ways similar to *some* or *most*: They take a noun argument to form an NP, just like other determiners, so from a linguistic point of

\(^1\) In the present context, ‘meaning’ stands for extension; intensions, possible worlds, and so forth, will not be relevant.
view, they should have type $\langle 1, 1 \rangle$. Similarly, NPs formed by a determiner and a noun should have type $\langle 1 \rangle$, not type $\langle 1, 1 \rangle$.

The natural way to deal with this issue is to have a background model $M_0$ fix the interpretation of the relevant proper names or nouns. Provided the fully abstracted quantifiers are EXT, there is a unique way to treat expressions like every student’s, Mary’s, John, three cats, and so forth, as quantifiers of the desired types ($\langle 1, 1 \rangle$ or $\langle 1 \rangle$), relative to $M_0$ (Peters & Westerståhl, 2006, chap. 3.5). Naturally, however, ISOM is lost.

ISOM and EXT apply to arbitrary quantifiers. The next property, on the other hand, concerns only quantifiers having a restriction argument (usually a set argument), in particular, those of type $\langle 1, 1 \rangle$.

**Definition 2.4.** A type $\langle 1, 1 \rangle$ quantifier $Q$ is conservative (CONSERV) iff for all $M$ and all $A, B \subseteq M$, $Q_M(A, B) \iff Q_M(A, A \cap B)$.

All determiner interpretations are CONSERV and EXT. This fact is responsible for much of the special behavior of natural language quantification, and in particular for all the results in the present note. For such quantifiers, $Q_M(A, B)$ depends only on the sets $A - B$ and $A \cap B$. If $Q$ in addition is ISOM, only the cardinality of those two sets matter.

Boolean operations on quantifiers (of the same type) are defined in the obvious way. In addition to ordinary negation, sometimes called outer negation, there is also an inner negation for quantifiers of these types; define, in the type $\langle 1 \rangle$ case, the quantifier $Q\neg$ by:

$$(Q\neg)_M(B) \iff Q_M(M - B).$$

In the type $\langle 1, 1 \rangle$ case, we replace the second argument by its complement.

Finally, we shall need suitable notions of triviality.

**Definition 2.5.** A local quantifier $Q_M$ is trivial iff it holds either of all subsets of $M$ or of no such subsets. $Q$ is trivial iff each $Q_M$ is trivial.

Below, we sometimes restrict attention to finite universes; this is indicated by writing FIN.

3. Freezing and decomposition.

3.1. Freezing. The first argument of a type $\langle 1, 1 \rangle$ quantifier denoted by a determiner is called the noun or restriction argument, and the idea of freezing is simply to fix that argument to some given set $C$. However, the result should be a global type $\langle 1 \rangle$ quantifier, defined on every universe $M$, whether $M$ contains $C$ or not. This is done as follows.

**Definition 3.1.** If $Q$ is any type $\langle 1, 1 \rangle$ quantifier and $C$ any set, define the type $\langle 1 \rangle$ quantifier $Q^C$, the freezing of $Q$ to $C$, by:

$$(5) \ (Q^C)_M(B) \iff Q_M(M \cup C, C, B),$$

for every $M$ and every $B \subseteq M$.

Thus, the universe is expanded, if necessary, so that it includes $C$. Though the idea of freezing is familiar, the definitions in the literature are often vague as to how to read $Q^C$ on an arbitrary universe. One could, instead, keep $M$ and cut down $C$ to $C \cap M$, or one could simply stipulate that $(Q^C)_M(B)$ is false when $C \not\subseteq M$. Note that all three alternatives are equivalent when $C \subseteq M$. It is argued in Peters & Westerståhl (2006) that (5) is the correct definition, mainly because it preserves EXT:

**Fact 3.2.** If $Q$ is EXT, so is $Q^C$. 

Proof. If \( B \subseteq M \subseteq M' \): \((Q^C)_M(B) \leftrightarrow Q_{C \cup M}(C, B) \leftrightarrow Q_{C \cup M'}(C, B) (\text{EXT}) \leftrightarrow (Q^C)_{M'}(B).

The other definitions suggested above do not preserve EXT. In addition, it can be seen that the facts about freezing and decomposition to be established below, as well as a number of other similar results, depend on defining \( Q^C \) in the correct way, that is, as in (5).

### 3.2. Decomposition.

**Definition 3.3.**

(i) A type \( \langle 1 \rangle \) quantifier \( Q \) is decomposable iff there is a set \( C \) and a CONSERV and EXT type \( \langle 1, 1 \rangle \) quantifier \( Q_1 \) such that \( Q = Q_1^C \).

(ii) If \( Q_1 \) is also ISOM, \( Q \) is said to be ISOM decomposable.

Since \( Q_1 \) is required to be CONSERV and EXT, decomposition is an inverse to freezing in a precise sense explained in Peters & Westerståhl (2006, chap. 4, fact 4). Note that without any requirements on \( Q_1 \), the notion of (ISOM) decomposability would be empty.

**Fact 3.4.** For every type \( \langle 1 \rangle \) quantifier \( Q \), there is a set \( C \) and a type \( \langle 1, 1 \rangle \) quantifier \( Q_1 \) such that \( Q = Q_1^C \).

**Proof.** Define:

\[(Q_1)_M(A, B) \iff A = \emptyset \land Q_M(B).
\]

Then, \((Q_1)^\emptyset(B) \iff (Q_1)_M(\emptyset, B) \iff Q_M(B)\), so \( Q = (Q_1)^\emptyset \). Note that \( Q_1 \) is not CONSERV, but it is EXT (respectively, ISOM) if \( Q \) is EXT (ISOM). □

Here are some familiar examples of ISOM decomposable quantifiers:

1. \( I_a = \text{all } [a] = (\text{all}_{\text{sg}})[a] = (\text{the}_{\text{sg}})[a] = \text{some } [a] \).
2. \( I_a \land I_b = \text{all } [a, b] = (\text{all}_{\text{pl}})[a, b] = (\text{the}_{\text{pl}})[a, b] (a \neq b) \).
3. \( I_a \lor I_b = \text{some } [a, b] \).
4. \( C^{pl,u} = (\text{all}_{\text{pl}})^C \).
5. \( C^{pl,e} = \text{some }^C \).

Examples of decomposable but not ISOM decomposable quantifiers and of EXT but not decomposable type \( \langle 1 \rangle \) quantifiers are given below.

A decomposable quantifier is only ‘active’ on the set to which the underlying type \( \langle 1, 1 \rangle \) quantifier is frozen. The following definition and result make this claim precise.

**Definition 3.5.** Let \( Q_1, Q_2 \) be CONSERV and EXT type \( \langle 1, 1 \rangle \) quantifiers.

(i) \( Q_1 \) and \( Q_2 \) agree on \( C \) iff for all sets \( B, Q_1(C, B) \iff Q_2(C, B) \).

(ii) \( Q_1 \) is ISOM on \( k \) (where \( k \) is any cardinal), iff for all \( M \) and \( A, B \subseteq M \), and all \( M' \) and \( A', B' \subseteq M' \), if \( |A| = |A'| = k, |A \cap B| = |A' \cap B'| \), and \( |A - B| = |A' - B'| \), then \((Q_1)_M(A, B) \iff (Q_1)'_{M'}(A', B')\).

**Fact 3.6.** For CONSERV and EXT \( Q_1 \) and \( Q_2 \):

(a) If \( Q = Q_1^C \) and \( Q_1 \) and \( Q_2 \) agree on \( C \), then \( Q = Q_2^C \).

(b) If \( Q_1 \) is ISOM, then \( Q_1 \) is ISOM on \( k \) for every \( k \).

(c) If \( Q = Q_1^C \) where \( Q_1 \) is ISOM on \( |C| \), then \( Q \) is ISOM decomposable.
Proof. Parts (a) and (b) are immediate. As to (c), if $Q$, $Q_1$, and $C$ are as assumed, define $Q_3$ by:

$$(Q_3)_M(A, B) \iff \begin{cases} (Q_1)_M(A, B) & \text{if } |A| = |C| \\ A \cap B \neq \emptyset \text{ (say) otherwise.} \end{cases}$$

Clearly, $Q_3$ is CONSERV and EXT. Distinguishing the cases $|A| = |C|$ and $|A| \neq |C|$, we see that $Q_3$ is also ISOM (in the former case, since $Q_1$ is ISOM on $|C|$, and in the latter case, since $Q_3$ then amounts to some). In addition, $Q_1$ and $Q_3$ agree on $C$, so it follows from part (a) that $Q = Q_3^C$. □

Fact 3.6 has the following immediate consequence.

**Corollary 3.7.** $Q$ is ISOM decomposable if and only if $Q = Q_1^C$ for some $C$ and some CONSERV and EXT $Q_1$, which is ISOM on $|C|$.

4. Results.

4.1. Global quantifiers living on sets. We are looking for some property of type $\langle 1 \rangle$ quantifiers, which gives a necessary and sufficient condition for decomposability. By Fact 3.2, one necessary condition is EXT; $\forall$ and the Rescher quantifier (everything and most things) are not decomposable already because they are not EXT. It turns out that the right place to start is the live-on property introduced in Barwise & Cooper (1981). Conservativity is crucial to most facts about global quantifiers in natural language, but Barwise and Cooper, who were among the first to realize its importance, thought of it as a property of local quantifiers.

**Definition 4.1.** Let $C$ be any set. A local type $\langle 1 \rangle$ quantifier $Q_M$ lives on $C$ iff for all $B \subseteq M$, $Q_M(B) \iff Q_M(C \cap B)$.

The connection with conservativity (which is immediate if the right notion of freezing has been chosen) is the following.

**Fact 4.2.** A type $\langle 1, 1 \rangle$ quantifier $Q$ is CONSERV iff for each set $C$ and each universe $M$, $(Q^C)_M$ lives on $C$.

Here are some easily verified facts about the live-on property (cf. Peters & Westerståhl, 2006, chap. 3, lemma 1):

**Fact 4.3.**

(a) $Q_M$ always lives on $M$, but need not live on any proper subset of $M$.

(b) $Q_M$ lives on $\emptyset$ iff it is trivial.

(c) If $Q_M$ lives on $C$ and $D$, it lives on $C \cap D$.

(d) If $Q_M$ lives on $C$ and $C \subseteq D$, $Q_M$ lives on $D$.

We did not require that $C \subseteq M$ in the definition of ‘$Q_M$ lives on $C$’, but the aforementioned fact shows that $Q_M$ lives on $C$ if and only if $Q_M$ lives on $M \cap C$. Indeed, it shows that:

$$\{X \subseteq M: Q_M \text{ lives on } X\}$$

is a filter on $P(M)$, which is proper iff $Q_M$ is nontrivial. If $M$ is finite, it follows that $Q_M$ lives on:
\[ W_{Q,M} = \bigcap \{ X \subseteq M : Q_M \text{ lives on } X \}, \]

which then generates the filter.

We now extend the live-on notion to global quantifiers as follows.

**Definition 4.4.** Let \( C \) be a set. A (global) type \( \langle 1 \rangle \) quantifier \( Q \) lives on \( C \) iff for all \( M \), \( Q_M \) lives on \( C \).

Fact 4.3, parts (b)–(d), carry over directly to the extended notion.

**Fact 4.5.**

(a) \( Q \) lives on \( \emptyset \) iff it is trivial.

(b) If \( Q \) lives on \( C \) and \( D \), it lives on \( C \cap D \).

(c) If \( Q \) lives on \( C \) and \( C \subseteq D \), \( Q \) lives on \( D \).

If \( Q \) lives on some set, we let:

\[ W_Q = \bigcap \{ X : Q \text{ lives on } X \} \]

**Fact 4.6.** (FIN) If \( Q \) lives on some set, it lives on \( W_Q \). That is, \( W_Q \) is then the smallest set that \( Q \) lives on.

*Proof.* We are assuming that \( Q \) lives on some finite set \( C \). Then, clearly,

\[ W_Q = \bigcap \{ X \cap C : Q \text{ lives on } X \}. \]

But on the right-hand side, we have an intersection of finitely many sets. Thus, by Fact 4.5, part (b), \( Q \) lives on \( W_Q \). \(\square\)

The next fact relates \( W_Q \) to \( W_{Q,M} \).

**Fact 4.7.** (FIN) If \( Q \) lives on some set then, for all \( M \), \( W_{Q,M} \subseteq W_Q \).

*Proof.* It is enough to show that if \( Q \) lives on a set \( C \), then for all \( M \), \( W_{Q,M} \subseteq C \). But \( Q_M \) lives on \( C \) by definition and hence on \( M \cap C \). And \( W_{Q,M} \) is the smallest subset of \( M \) that \( Q_M \) lives on, so \( W_{Q,M} \subseteq M \cap C \). Thus, \( W_{Q,M} \subseteq C \). \(\square\)

**4.2. Decomposition and the live-on property.** Even though the live-on property for global quantifiers draws on Barwise and Cooper’s local version, it is a very different property. The local version essentially codifies conservativity, and so is to be expected in all natural language contexts. Also, *every* local quantifier lives on some set (its universe). Global quantifiers, on the other hand, ‘normally’ do not live on any sets at all. For example, a nontrivial and ISOM type \( \langle 1 \rangle \) quantifier lives on no set (Fact 4.11). In fact, the ones that do live on some set are essentially the decomposable quantifiers, as we now see.

---

2 This fails without the restriction to finite sets. Let \( Q \) be *infinitely many numbers*, that is, \( Q = \text{infinitely many}^N \), where \( N = \{0, 1, 2, \ldots \} \). \( Q \) lives not only on \( N \) but also on \( N - \{0\}, N - \{0, 1\}, \ldots \). So there is no smallest set on which it lives, and \( W_Q \) as defined above is empty. Note that if \( Q \) lives on some set \( C \), \( W_Q \) is well defined: It is a subset of \( C \) defined by a first-order set-theoretic formula. The notation \( W_Q \) was used in a slightly different sense in Peters & Westerståhl (2006, chap. 4.6), namely, for certain \( Q \) such that \( W_{Q,M} \) is independent of \( M \) (when \( Q_M \) is nontrivial). But when this is the case, that notion coincides with the one defined here.
The next lemma is immediate (either directly from definitions or from Facts 4.2 and 4.5, part (c)).

**Lemma 4.8.** If $Q$ is **CONSER** $Q^C$ lives on $C$ and all its supersets.

We obtain the following characterization.

**Theorem 4.9.** $Q$ is decomposable if and only if it is **EXT** and lives on some set.

**Proof.** Suppose $Q$ is decomposable as $Q_1^C$. Then, $Q$ is **EXT**, and by Lemma 4.8, $Q$ lives on $C$.

In the other direction, suppose $Q$ is **EXT** and lives on $C$. Define $Q_1$ by:

$$(Q_1)_M(A, B) \iff A = C \land Q_M(A \cap B).$$

$Q_1$ is **CONSER** by definition, and **EXT** since $Q$ is **EXT**. We have, for all $M$ and all $B \subseteq M$,

$$(Q^C_1)_M(B) \iff (Q_1)_M(C, B)$$

$$\iff Q_M(C \cap B) \quad \text{(by definition)}$$

$$\iff Q_M(C \cap B) \quad \text{(by **EXT**)}$$

$$\iff Q_M(B) \quad \text{(since $Q_M$ lives on $C$).}$$

That is, $Q = Q^C_1$. □

**Corollary 4.10.**

(a) If $Q$ is decomposable with $C$ as the underlying set, then for every $D \supseteq C$, $Q$ is decomposable with $D$ as the underlying set.

(b) The class of decomposable type $\langle 1 \rangle$ quantifiers is closed under Boolean operations, including inner negation and dual.

**Proof.** Part (a) follows from the theorem and Lemma 4.8. As to part (b), the claims about negations (and hence duals) follow from the easily verified observations that for **CONSER** and **EXT** $Q_1$,

$$\neg(Q^C_1) = (\neg Q_1)^C.$$

and

$$(Q^C_1)^\neg = (Q_1^\neg)^C.$$

Now suppose $Q = Q^C_1$ and $Q' = Q'^C_2$, for **CONSER** and **EXT** $Q_1$ and $Q_2$. By Lemma 4.8, both $Q$ and $Q'$ live on $C_1 \cup C_2$. But then, it readily follows that $Q \land Q'$ and $Q \lor Q'$ also live on $C_1 \cup C_2$. Since they are also **EXT**, they are decomposable, by the theorem. □

We can use the easy direction of the characterization to show that various type $\langle 1 \rangle$ quantifiers, including many denoted by noun phrases in natural languages, are not decomposable. To begin, *something* and *nothing* ($\exists$ and $\neg \exists$) are **EXT** but not decomposable. More generally, the quantifiers at least $n$ things ($\exists \geq n$), for $n \geq 1$, and all nontrivial Boolean combinations of these, including inner negations and duals, are not decomposable. These include cases like *everything* ($\forall$) and *all but three things* ($\exists \geq 3 \land \exists \leq 3 \neg$), which are not even **EXT**, and so *a fortiori* not decomposable, but also **EXT** noun phrase denotations like *between three and five things* ($\exists \geq 3 \land \exists \leq 5$), not to mention an even number of things, and so forth. All this follows from the simple fact that these quantifiers are (nontrivial and) **ISOM**.
FACT 4.11. If a type (1) quantifier is nontrivial and ISOM, it is not decomposable.

Proof. Suppose $Q$ is nontrivial and ISOM. By Fact 3.2, we may also suppose that $Q$ is EXT.

Case 1: $\neg Q(\emptyset)$. By nontriviality, there is some $M$ and some $B \subseteq M$ such that $Q_M(B)$. By EXT, it follows that $Q_B(B)$. Now suppose that $Q$ lives on some set $C$. Take $B'$ such that $|B'| = |B|$ and $C \cap B' = \emptyset$. It follows, since $Q$ is ISOM, that $Q_{B'}(B')$ and hence, since $Q_{B'}$ lives on $C$, that $Q_{B'}(C \cap B')$. But this contradicts the assumption of the case. Thus, $Q$ does not live on any set and is therefore not decomposable.

Case 2: $Q(\emptyset)$. Then, the previous argument shows that $Q' = \neg Q$ is not decomposable. But then, by (11), neither is $Q$. □

What about EXT but not ISOM quantifiers? Among noun phrase denotations, we find the following. Define, for any set $D$, the quantifier only $D$ by:

\[(13) (\text{only } D)_M(B) \iff \emptyset \neq B \subseteq D.\]

These are at work in the interpretation of sentences like:

(14) Only John left the party. ($D = \{j\}$).
(15) Only firemen wear helmets.

FACT 4.12. Quantifiers of the form only $D$ are EXT, but not decomposable when $D \neq \emptyset$.

Proof. That EXT holds is clear from the definition of only $D$. Suppose only $D$ were decomposable and hence lived on some set $C$. Take any $a \notin C \cup D$. Since (only $D$)($D \neq \emptyset$), we obtain (only $D$)(C∩$D$) and therefore (only $D$)(C∩(D∪{a})), so (only $D$) ($D \cup \{a\}$), which contradicts (13). □

This underscores the familiar observation in linguistics that, first appearances notwithstanding, only is in fact not an English determiner (e.g. Peters & Westerståhl, 2006, p. 139, fn. 15). If it were a determiner, one would expect only firemen to be interpreted as a frozen type (1, 1) quantifier, but Fact 4.12 says it cannot be.

4.3. Characterizing ISOM decomposability. We have seen several examples of ISOM decomposable quantifiers, as well as EXT but nondecomposable quantifiers, but what about quantifiers that are decomposable but not ISOM decomposable? A host of such quantifiers can be identified using the characterization of ISOM decomposability in this subsection.

DEFINITION 4.13. A type (1) quantifier $Q$ does not distinguish subsets of $C$ of the same size iff whenever $B_1, B_2 \subseteq C \cap M$ are such that $|B_1| = |B_2|$ and $|C - B_1| = |C - B_2|$, $Q_M(B_1) \iff Q_M(B_2)$.

THEOREM 4.14. $Q$ is ISOM decomposable if and only if $Q$ is EXT and lives on some set $C$ such that $Q$ does not distinguish subsets of $C$ of the same size.

Proof. If $Q = Q_1^C$ for some CONSERV, EXT, and ISOM $Q_1$, then $Q$ lives on $C$, and it is immediate that $Q$ does not distinguish subsets of $C$ of the same size.

In the other direction, suppose $Q$ is EXT and lives on a set $C$ such that $Q$ does not distinguish subsets of $C$ of the same size. Define $Q_1$ by:

\[(Q_1)_M(A, B) \iff |A| = |C| \& \exists X \subseteq C(Q_M(X) \& |X| = |A \cap X| \& |C - X| = |A - B|).\]
Proof. This is more or less immediate from Fact 3.6: If \( Q = Q_1^C \), just let \( Q_2 \) differ from \( Q_1 \) on some \( D \) with \( |D| \neq |C| \), but be the same as \( Q_1 \) otherwise.

The second part of the question is more interesting. From Corollary 4.10, part (a), we know that it is only in the case of ISOM decomposability that we have any chance to recover the underlying set.

\footnote{But in this case, the conclusion follows directly from the fact that \( Q_1 \) is ISOM: If \( m \in D \), \( Q_1(D, \{j\}) \) implies \( Q_1(D, \{m\}) \) by ISOM.}
LEMMA 4.17. (Fin) Suppose $Q$ is nontrivial and decomposable as $Q_1^C$, where $Q_1$ is ISOM on $|C|$. Suppose further that $Q$ lives on some set $D$. Then, $C \subseteq D$.

Proof. Suppose first that $\neg Q(\emptyset)$ holds. Let $B_0$ be a set of the smallest size such that $Q(B_0)$. (Since $Q$ is EXT by assumption, we can leave out the universe as usual.) Such a set exists by nontriviality. By assumption, $B_0 \neq \emptyset$. Moreover, by the conservativity of $Q_1$ (or the fact that $Q_M$ lives on $C$), we may assume that $B_0 \subseteq C$.

Suppose $a \in C - D$. Take $B \subseteq C$ such that $a \in B$ and $|B| = |B_0|$. Since $Q_1$ is ISOM on $|C|$, $C$ is finite, and $Q_1(C, B_0)$ holds, it follows that $Q_1(C, B)$ and thus $Q(B)$. Since $Q$ lives on $D$, we get $Q(D \cap B)$ and thus $Q(D \cap (B - \{a\}))$ since $a \notin D$. But then, $Q(B - \{a\})$, again because $Q$ lives on $D$. Since $B$ is finite, this contradicts the assumption that it was of the smallest size such that $Q(B)$.

Thus, $C \subseteq D$. If instead $Q(\emptyset)$ holds, We apply the same reasoning, but this time to $\neg Q$, which also satisfies the assumptions in the lemma. \hfill \square

THEOREM 4.18. (Fin) If $Q$ is nontrivial and ISOM decomposable, its underlying set is uniquely determined. More exactly, there is a unique set $C$ such that for some $Q_1$ which is CONSERV, EXT, and ISOM (on $|C|$), $Q = Q_1^C$.

Proof. Suppose $Q = Q_1^C = Q_2^D$, where $Q_1$, $Q_2$ are CONSERV and EXT, and $Q_1$ is ISOM on $|C|$, whereas $Q_2$ is ISOM on $|D|$. By Lemma 4.8, $Q$ lives on $D$. Thus, by Lemma 4.17, $C \subseteq D$. A symmetric argument shows that $D \subseteq C$, so $C = D$. The corresponding claim of uniqueness when $Q_1$, $Q_2$ are ISOM follows by Fact 3.6, part (b). \hfill \square

Let $Un(Q, Y) \iff Q = Q_1^Y$ for some CONSERV, EXT, and ISOM $Q_1$.

By the theorem, assuming Fin, the relation $Un$ is single-valued for nontrivial $Q$, so we can define a function $U$ from type (1) quantifiers to sets by:

$$U(Q) = \begin{cases} 
Y & \text{if } Q \text{ is nontrivial and } Un(Q, Y) \\
\emptyset & \text{otherwise.}
\end{cases}$$

$U$, restricted to ISOM decomposable quantifiers, can also be defined in terms of $Q$ alone. Recall the definition of $W_Q$ in section 4.1.

COROLLARY 4.19. (Fin) If $Q$ is ISOM decomposable, then $U(Q) = W_Q$.

Proof. If $Q$ is trivial, $U(Q) = W_Q = \emptyset$, by Fact 4.5, part (a). Suppose $Q = Q_1^C$ for some CONSERV, EXT, and ISOM $Q_1$, and $Q$ is nontrivial. Then, $Q$ lives on $C$, and $C = U(Q)$ by the theorem, so, by Fact 4.6, $W_Q \subseteq U(Q)$. Also, since $Q$ lives on $W_Q$, it follows by Lemma 4.17 that $U(Q) \subseteq W_Q$. So in both cases, $U(Q) = W_Q$. \hfill \square

Putting together all the above, we obtain a final characterization of ISOM decomposability.

COROLLARY 4.20. (Fin) $Q$ is ISOM decomposable if and only if $Q$ is EXT, $W_Q$ exists, and $Q$ does not distinguish subsets of $W_Q$ of the same size.

The restriction to finite universes is essential for the results in this subsection. For example, the quantifier mentioned in footnote 2, $Q = \text{infinitely many numbers}$, is nontrivial and ISOM decomposable, $W_Q = \emptyset$, $Q$ does not live on any finite set, and for each $n$, $Q = (\text{infinitely many})^{N-\{0, 1, \ldots, n\}}$ (where $N = \{0, 1, 2, \ldots\}$).
4.5. Some instructive examples. Suppose $Q = Q_1^C$, where $Q_1$ is not ISOM. Can we conclude anything about the ISOM decomposability of $Q$? No: Even if $Q$ is ISOM decomposable, it is equal to $Q_1^C$ for infinitely many $C$ and infinitely many non-ISOM $Q_1$.

We give some examples to illustrate that natural quantifiers of the form $Q_1^C$ with non-ISOM $Q_1$ are sometimes ISOM decomposable and sometimes not.

In section 2.1, we defined simple possessive quantifiers like $John's$ by:

$$John's_M(A, B) \iff \emptyset \neq A \cap R_j \subseteq B,$$

(where $R_j = \{ b : R(j, b) \}$, and $j = John$). This is a special case of:

$$Q_2 \text{ of } John's_M(A, B) \iff \emptyset \neq A \cap R_j \& Q_2(A \cap R_j, B).$$

Compare sentences like:

(17) John’s bikes were (all) stolen.
(18) Most of Mary’s friends are here.
(19) All but two of Henry’s job applications failed.

Here, $Q_2 = all$, $most$, and $all\ but\ two$, respectively. None of the possessive type $(1, 1)$ quantifiers in (17)–(19) is ISOM, but we still have the following.

**FACT 4.21.** If $Q_2$ is CONSERV, EXT, and ISOM, quantifiers of the form $Q = (Q_2 \text{ of } John's)^C$ are ISOM decomposable. Moreover (FIN), $W_Q = C \cap R_j$.

**Proof.** $Q$ is clearly EXT. We have:

(20) $Q(B) \iff \emptyset \neq C \cap R_j \& Q_2(C \cap R_j, B)$.

If $C \cap R_j = \emptyset$, then $Q$ is trivial and hence trivially ISOM decomposable, and $W_Q = \emptyset$ (Lemma 4.5, part (a)). If $C \cap R_j \neq \emptyset$, then, by (20), $Q = (Q_2)^{C \cap R_j}$ and hence is ISOM decomposable by the assumption on $Q_2$. Also, if FIN holds, we see from Corollary 4.19 that $W_Q = C \cap R_j$. $\square$

Thus, the smallest set that, say, $John's\ books$ lives on is not the set of books in the discourse universe, but the set of books owned by John (if $R = owns$) (cf. Corollary 4.20).

What about other frozen possessive quantifiers? I will not try to answer this question in any generality here but only give a few more illustrative examples. Consider the determiner some student’s. This has (at least) two readings, a universal and an existential one, illustrated by:

(21) Some student’s tennis rackets were stolen.
(22) Some student’s books were left in the classroom.

A plausible reading of (21) is that some students had all their tennis rackets stolen, whereas (22) can be taken to mean that some, but not necessarily all, of the students’ books had been left in the room. The two readings are given by, respectively,

(23) $(\text{some } D's)^u(A, B) \iff \exists a \in D (\emptyset \neq A \cap R_a \subseteq B)$.
(24) $(\text{some } D's)^e(A, B) \iff \exists a \in D (\emptyset \neq A \cap R_a \cap B)$.

**FACT 4.22.** $Q = ((\text{some } D's)^e)^C$ is ISOM decomposable, for all $D, C$, and $R$.

**Proof.** We use Theorem 4.14. Let:

(25) $C_0 = C \cap \bigcup_{a \in D} R_a$. 

It is not hard to verify that:

(26) $Q$ lives on $C_0$.

Also, it follows from the definition of (some D's)$^e$ that:

(27) If $\emptyset \neq B \subseteq C_0$, then $Q(B)$ holds.

This entails that $Q$ does not distinguish subsets of $C_0$ of the same size. Since $Q$ is clearly EXT, it follows from Theorem 4.14 that $Q$ is ISOM decomposable.

Although I will not prove it here, it is a general fact about possessive quantifiers frozen to a set $C$ that they live on the set $C_0$ defined in (25). In particular, $(\text{some } D's)^\mu_C$ does so, and we see that (with $D = \{j\}$) the quantifiers $(Q_2 \text{ of } John's)^C$ do so too. In this latter case, we have $W_Q = C_0 = C \cap R_j$, but in general, $W_Q$ can be a proper subset of $C_0$. To give an example, suppose $D = \{s_1, s_2\}, C = \{b_1, b_2\}$, and $R = \{(s_1, b_1), (s_2, b_1), (s_2, b_2)\}$. Then, $C_0 = C$, and one easily verifies that with $Q = ((\text{some } D's)^\mu)^C$, we have $\neg Q(\emptyset), Q(\{b_1\}), \neg Q(\{b_2\}),$ and $Q(\{b_1, b_2\})$. From this, one sees that:

$$W_Q = \{b_1\}.$$  

For this particular choice of $C, D, R$, we also have that $(\text{some } D's)^\mu_C$ is ISOM decomposable. This is a consequence of Corollary 4.20.

**FACT 4.23. (FIN) If $Q$ is EXT, and $W_Q$ exists and has at most one element, $Q$ is ISOM decomposable.**

**Proof.** Then, $Q$ trivially does not distinguish subsets of $W_Q$ of the same size!  

However, a similar example shows that a slightly different possessive quantifier need not be ISOM decomposable. Consider:

(28) (Exactly) two students’ dorm rooms were burglarized.

One plausible interpretation is:

(29) two D's $(A, B) \iff |D \cap \{a : \emptyset \neq A \cap R_{a} \subseteq B\}| = 2$.

Let $D = \{s_1, s_2, s_3\}, C = \{b_1, b_2\}$, and $R = \{(s_1, b_1), (s_2, b_1), (s_3, b_2)\}$. Again $C_0 = C$, and one sees that with $Q = (\text{two } D's)^C$, we have $\neg Q(\emptyset), Q(\{b_1\}), \neg Q(\{b_2\})$, and $Q(\{b_1, b_2\})$. For example, in the last case, the number of D's such that every $C$ they are related to by $R$ belongs to $\{b_1, b_2\}$ is 3, not 2. But then, $Q$ lives neither on $\{b_1\}$ nor on $\{b_2\}$, although it does live on $C_0 = \{b_1, b_2\}$. So $W_Q = \{b_1, b_2\}$, and it follows that $Q$ does distinguish some subsets of $W_Q$ of the same cardinality (namely, $\{b_1\}$ and $\{b_2\}$). We have shown the following.

**FACT 4.24. There are $D, C,$ and $R$ such that (two $D$'s)$^C$ is not ISOM decomposable.**

We see that the ISOM decomposability (or not) of frozen possessive quantifiers sometimes depends on the choice of the ‘possessor’ relation. For example, we can make the following observation.  

**FACT 4.25. If $R$ is a (partial) function, then for all $D$ and $C$, $Q = ((\text{some } D's)^\mu)^C$ is ISOM decomposable.**

---

4 Thanks to Christian Bennet for the suggestion.
Proof. That $R$ is a partial function means that for all $a$, $R_a$ has at most one element. From this, and the fact that $Q$ is monotone ($Q(B)$ and $B \subseteq B'$ implies $Q(B')$), one verifies that, with $C_0$ as in (25),

\[(30) \text{ for } B \subseteq C_0: Q(B) \iff B \neq \emptyset \& \forall b \in B \ Q(\{b\}).\]

It follows that for all non-empty subsets $B$ of $C_0$ we have $Q(B)$. So in particular, $Q$ does not distinguish subsets of $C_0$ of the same size. Since $Q$ lives on $C_0$ and is EXT, $Q$ is ISOM decomposable, by Theorem 4.14.

However, most ‘possessor’ relations, like real ownership, are not partial functions (one normally owns many different things).

Note finally that although the proof of Fact 4.24 makes use of Corollary 4.20, it identifies a finite set as the smallest set the quantifier lives on. This is independent of the size of the universe; a finite counterexample to ISOM decomposability is given, and this fact, in contrast with positive results of ISOM decomposability using Corollary 4.20, like Fact 4.23, does not rely on FIN. The same holds for our final example.

**FACT 4.26.** There are $D$, $C$, and $R$ such that $Q = ((\text{some } D's)^a)C$ is not ISOM decomposable.

Proof. Let $D = \{s_1, s_2, s_3\}$, $C = \{b_1, b_2, b_3, b_4\}$, and (draw a diagram!) $R = \{(s_1, b_1), (s_2, b_1), (s_2, b_2), (s_3, b_3), (s_3, b_4)\}$. As to subsets of $C_0 = C$, we then have:

(i) $Q$ holds of $\{b_1\}$ and its superset and of $\{b_3, b_4\}$ and its superset.

(ii) $Q$ does not hold of: $\emptyset$, $\{b_2\}$, $\{b_3\}$, or $\{b_4\}$, nor of $\{b_2, b_3\}$ or $\{b_2, b_4\}$.

Using the monotonicity of $Q$, and going through the various cases, one can verify that:

\[(31) \text{ } Q \text{ lives on } \{b_1, b_3, b_4\},\]

and then that

\[(32) \text{ } W_Q = \{b_1, b_3, b_4\}.\]

So $Q$ distinguishes subsets of $W_Q$ of the same size (e.g., $\{b_1\}$ and $\{b_3\}$) and hence is not ISOM decomposable.

Observe also that:

\[((\text{some } D's)^a)^C = \bigvee_{a \in D} (\text{all of } a's)^C,\]

so we have another illustration of the fact that ISOM decomposability is not closed under disjunction (cf. Fact 4.15).

5. Applications. Noun phrases abound in most languages, as immediate constituents of other phrases: sentences, verb phrases, prepositional phrases, and so forth. In a compositional semantics, the semantic value of the larger phrase is determined by the values of the immediate constituents (and the ‘mode of composition’). Suppose the value of the NP is of the form $Q^C$, but calculating the value of the larger phrase requires access to $C$ and/or $Q$. Then, we have at least a prima facie problem for compositionality since the phrases whose values are $Q$ and $C$ are not immediate constituents of the larger phrase. Does this situation actually occur? We look at two cases.
5.1. Reciprocals. Consider the sentences:

(33) Most of the pirates stared at each other in surprise.
(34) Most of the boys in my class know each other.

In examples like this, *each other* can be construed as a type \(\langle 1, 2 \rangle\) quantifier \(EO\), where \(EO(A, R)\) says roughly that all individuals in \(A\) ‘\(R\) each other’. But this can mean different things. For example, in (34), we seem to have:

\[
EO_1(A, R) \iff \forall a, b \in A (a \neq b \implies R(a, b)).
\]

(33), on the other hand, rather uses \(EO_2\):

\[
EO_2(A, R) \iff \forall a \in A \exists b \in A (a \neq b \land R(a, b)).
\]

(It is difficult to stare at more than one person at a time.)

The meaning of the whole sentence can then be obtained as a ‘Ramsey lift’ along the following scheme:

\[
Rami(Q)(C, R) \iff \exists X \subseteq C (QC(X) \land EO_1(X, R)).
\]

That is, if \(Q\) is a type \(\langle 1, 1 \rangle\) quantifier like *most* (one usually assumes that the quantifier is increasing in the right argument, like *most*, or John’s), (35) gives a type \(\langle 1, 2 \rangle\) quantifier \(Rami(Q)\) suitable for interpreting certain reciprocal sentences.

The most natural compositional analysis of sentences like (33) and (34) would seem to go like this:

\[
[\text{Det } C] [R \text{ each other}]
\]

\[
QC \quad \lambda X EO(X, R).
\]

But to get the meaning of the sentence, according to (35), at the last step, we in general need access not only to \(QC\) but also to \(C\). Thus, Theorem 4.18 applies, when \(QC\) is \(\text{ISOM}\) decomposable (like *most pirates* or John’s companions) and only finite universes are considered. Since \(C = U(QC) = W_{QC}\), this theorem guarantees the compositionality of the corresponding reciprocal constructions.

5.2. Possessives. Next, consider:

(36) Most planets’ rings are made of ice.

Here too we have an NP, *most planets*, whose interpretation is of the form \(QC\) and which is naturally seen as an immediate constituent of the phrase *most planets’* (which can be taken to be a determiner). But in this case, access to \(C\) does not seem to be enough. This is because *most* in (36) does not really quantify over all the planets, but only over those that have rings. Planets without rings are irrelevant to the truth value of (36). The phenomenon was called *narrowing* in Barker (1995) (from which the example is taken).

---

5 See Dalrymple et al. (1998) for a treatment of the semantics of reciprocals. (33) is a variant of a sentence discussed there.

6 One may note that since \(Q\) is \(\text{CONSERV}\),

\[
Rami_1(Q)(C, R) \iff \exists X (QC(X) \land EO_1(X, R)).
\]

So in this case, access to \(C\) is in fact not needed. This is because \(EO_1\) is defined by a universal condition. But \(EO_2\) is not, and \(C\) cannot be similarly eliminated for \(Rami_2(Q)\).
Thus, we need access to $Q^C$, for some $C' \subseteq C$. But, as we have seen, this is in general not possible. One cannot recover $Q$ from $Q^C$ and therefore not $Q^{C'}$ either, when $C' \neq C$, even if $C'$ is known. In Peters and Westerståhl (in preparation), we analyze the situation and conclude that this appears to constitute a serious problem for the compositionality of the semantics for possessives.

We also note a related problem for possessive semantics. If one construes the semantics so that not $Q^C$ but both $Q$ and $C$ are taken as arguments when calculating the meaning of possessive phrases, one has an additional problem besides compositionality: What to do when the NP is not quantified, as in (37) and (38)?

(37) Mary’s grant applications were successful.
(38) John or Mary’s grant applications were successful.

The obvious answer is: decompose! But one problem is that such decomposition is not unique. For example, $I_m = \text{all } \{m\} = (\text{all } e_i)^{\{m\}}$, and the truth conditions vary slightly with the decomposition chosen.\(^7\) Another issue is that phrases of the form [NP ‘s] are sometimes problematic. Consider:

(39) (?) John, or Mary and Sue’s grant applications were successful.
(40) (?) Some man or every woman’s grant applications were successful.

Many speakers find (39) and (40) meaningless or at least very strange.\(^8\) Our logical results may apply here too. For example, we note that the NPs in (39) and (40) are not ISOM decomposable (Fact 4.15), in contrast with those in (37) and (38). These issues too are discussed further in Peters & Westerståhl (in preparation).

In conclusion, the ubiquity of linguistic constructions involving phrases whose interpretations are type $\langle 1 \rangle$ quantifiers, decomposable or not, makes it likely that the simple logical facts about decomposition established in this note are relevant to various issues in natural language semantics.

6. Acknowledgments. Thanks to Stanley Peters for numerous very useful remarks, suggestions, and corrections. Also, thanks are due to Johan van Benthem and Christian Bennet for helpful comments, and to the audiences of the Amsterdam-Utrecht Logic Seminar (Amsterdam, February 29, 2008), the ASL/APA meeting on Quantifiers in Logic and Language (Pasadena, March 20), and the Gothenburg Logic Seminar (May 23), where

\(^7\) On our account, if one uses $\text{all } \{m\}$, (37) becomes true iff Mary did not make any grant applications or all her applications were successful. With $(\text{all } e_i)^{\{m\}}$, it is—more reasonably—true iff she made at least one grant application and all her applications were successful.

\(^8\) Note, first, that we are of course disregardings readings of (40), which say that either some man was successful or every woman’s grant application was successful, and, second, we realize that there is a tendency to read (40) as the unproblematic:

(i) Some man’s or every woman’s grant applications were successful.

But it could still be maintained that this is just a way to charitably make sense of or ‘save’ the deviant (40).
parts of these results were presented. Work on this article was supported by a grant from the Swedish Research Council.

BIBLIOGRAPHY


DEPARTMENT OF PHILOSOPHY
UNIVERSITY OF GOTHENBURG
GOTHENBURG 405 30, SWEDEN
E-mail: dag.westerstahl@phil.gu.se