Unknitting the black hole: black holes as effective geometries

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CHAPTER 1

FOUR DIMENSIONAL BLACK HOLES

This chapter is an attempt to summarize what is known about black holes in general relativity and semi-classical quantum gravity. We start by reviewing standard four-dimensional black holes: Schwarzschild and Kerr-Newman black holes. Then, a summary of the laws of black hole mechanics will be given suggesting the possible thermodynamical nature of black holes which closes the classical side of the story. Although a full quantum theory of gravity is still out of reach, one can still see some glimpses of its effects by resorting to semi-classical quantum gravity. Following this line of thoughts, we are going to quantize a free scalar field in a black hole background. This leads to the Hawking radiation of black holes making the thermodynamical nature of black holes physical. We close this chapter by discussing some implications of the Hawking radiation.

In this chapter familiarity with general relativity and black holes is assumed. The reader is also required to be familiar with canonical quantization of field theories in flat spacetime.

1.1 BLACK HOLES IN GENERAL RELATIVITY

As opposed to the other fundamental interactions, gravity distinguishes itself by treating spacetime as a dynamical entity. As a result, generalizations of well established notions in Mink spacetime are not that trivial as we will see later. In the
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The first exact solution to the vacuum Einstein equation $R_{\mu\nu} = 0$ was found by Schwarzschild. This solution is spherically symmetric and it turns out to be describing a black hole. The metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where $M$ is a free parameter that equals the mass of the black hole. A strange thing seems to happen at $r = r_h = 2M$ where naively the metric degenerates. Finiteness of the Ricci scalar at $r_h$ reveals that such degeneracy is nothing more than an artifact of the chosen coordinates. The surface defined by $r = r_h$ is called a “horizon” (or “event horizon”) and marks a “point of no return”; if one crosses it, he is forced to crush into the singularity at $r = 0$ where the curvature $R$ blows up. This can be seen easily by drawing the associated Penrose diagram, see picture 1.1.

![Penrose Diagram](image)

**Figure 1.1:** The Penrose diagram of the fully extended Schwarzschild black hole. $I^+$ (black) and $I^-$ (Grey) are future and past lightlike infinity respectively. The horizon is depicted in a dotted line. The future and past singularities are depicted in dashed lines. The small wedges represent future directed light cones.
A generalization of the Schwarzschild black hole is the Kerr-Newman one which is charged and rotating. This is a solution of the vacuum Einstein-Maxwell theory

\[ S = \frac{1}{16 \pi} \int d^4x \sqrt{-g} \ (R - F_{\mu\nu} F^{\mu\nu}) \ . \]  

The Kerr-Newman black hole metric and Maxwell one-form read

\[ ds^2 = -\frac{\Delta}{\Sigma} (dt - a \sin^2 \theta \, d\varphi)^2 + \frac{\sin^2 \theta}{\Sigma} \left[ adt - (r^2 + a^2) \, d\varphi \right]^2 + \frac{\Sigma}{\Delta} dv^2 + \Sigma d\theta^2 \ , \] 

\[ A = \frac{1}{\Sigma} \left( Q \ r \ (dt - a \sin^2 \theta \, d\varphi) - P \ \cos \theta \ [a \, dt - (r^2 + a^2) \, d\varphi] \right) \ , \]

where

\[ \Sigma = r^2 + a^2 \cos^2 \theta \ , \] (1.5)

\[ \Delta = r^2 - 2M r + a^2 + (P^2 + Q^2) \ , \] (1.6)

and \( a = J/M \), \( J \) is the angular momentum, \( M \) is the mass, \( Q \) is the electric charge, and \( P \) is the magnetic charge. One of the striking results in general relativity concerning black holes is

**The No-Hair Theorem** If \((M, g)\) is an asymptotically flat stationary vacuum Einstein-Maxwell spacetime that is non-singular on and outside an event horizon, then \((M, g)\) is a member of the three-parameter Kerr-Newman family of black holes described above.

where stationary means that there exists a timelike Killing vector. Before moving on, let us pause for a moment to point out an important difference between the Schwarzschild black hole and the more general Kerr-Newman black one. We have seen that the Schwarzschild black hole has two special spacetime regions: the singularity which for the Kerr-Newman black hole sits at \( r = 0 \) and \( \theta = \pi/2 \), and, the horizon whose counterpart for Kerr-Newman is located at the zeros of \( \Delta \). The latter being a second order polynomial in \( r \) (1.6) leads to three possibilities: two horizons, one horizon or no horizon at all. The second possibility is very special and the corresponding black hole is called “extremal”. A class of them can be embedded in a very special class of solutions to supergravity theories called BPS solutions (see section 2.1.1). These are solutions that preserve a fraction of the supersymmetries of the full theory. They (and their five dimensional cousins) will play an important role in the bulk of the thesis. The third possibility above (no horizon) suggests that there is a naked singularity. The existence of such a naked singularity will destroy the predictability of the future given initial data on a spacelike hypersurface. Based on the behavior of physically reasonable matter and in order to avoid such disaster, Penrose suggested the
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**Cosmic Censorship Principle** Under suitable physical conditions, naked singularities cannot form from a gravitational collapse in an asymptotically flat spacetime that is not singular on an initial spacelike hypersurface.

Proving or disproving this conjecture is one of the important open questions in general relativity. Actually, there are some gravitational collapse scenarios where the end result is a naked singularity. For a thorough discussion on the status of the cosmic censorship principle see e.g. [13, 14] and references therein.

1.1.2 **Killing Horizons and Surface Gravity**

Null hypersurfaces enjoy a lot of peculiar properties and play an important role in the study of the causal structure of spacetime. An important class of these surfaces are Killing horizons. These are null hypersurfaces $\mathcal{H}$ that admit a normal Killing vector $\xi$. It can be shown that on $\mathcal{H}$, $\xi$ satisfies

$$\xi^\nu \nabla_\nu \xi^\mu = \kappa \xi^\mu,$$

(1.7)

where $\xi$ is normalized appropriately in the asymptotic flat region. $\kappa$ is called the surface gravity. Our interest in Killing horizons is twofold

$\rightarrow$ Usually black hole horizons are Killing horizons.

$\rightarrow$ $\kappa$ is constant on the horizon and it will play the role of a temperature.

1.2 **The Laws of Black Hole Mechanics**

The laws of black hole mechanics are obtained by studying the reaction of the solution to perturbations of its parameters (Mass, angular momentum, charge ...). Our first task then will be to define what we mean by these quantities. This is not as simple as it sounds because gravity does carry energy which makes it hard to distinguish black hole contributions from gravity ones. Luckily, there is a well defined prescription for asymptotic flat solutions of direct interest to us.

1.2.1 **Kumar Integrals**

Using the analogy with the ambiguity in defining the potential energy in classical mechanics, we are going to choose a reference geometry for which the mass, charge, and angular momentum are set to zero. For asymptotically flat black holes, the flat
Minkowski spacetime does the job. There are different methods to put this convention into work. A covariant formulation goes under the name of Kumar integrals which is as follows. Given a Killing vector $\xi$, for a spacelike hypersurface $\Sigma$ one defines the conserved quantity

$$Q_\xi(\Sigma) = \frac{c}{16\pi} \int_\Sigma dS_\mu g^{\mu \rho} \nabla_\nu \nabla_\rho \xi^\nu,$$  \hspace{1cm} (1.8)

provided that $J^\mu(\xi) = g^{\mu \rho} R_{\rho \nu} \xi^\nu$ vanishes on the boundary of $\Sigma$. In the equation above $S_\mu$ is the normal vector to $\Sigma$ and $c$ is a constant. For example, the energy (angular momentum) is evaluated using the Killing vector that generates time translations (space rotations) and choosing $c = -2$ ($c = 1$) respectively. These values of the constant $c$ are chosen so that one ends up with the right normalization of mass and angular momentum.

1.2.2 BLACK HOLE LAWS

The following laws are specific to asymptotically flat black holes. We will be a little bit loose here, for exact formulation and proofs see [2].

**Zeroth Law**  $\kappa$, the surface gravity is a constant on the horizon.

**First Law**  The perturbation of a stationary black hole with mass $M$, charge $Q$ and angular momentum $J$ satisfies

$$\delta M = \frac{\kappa}{8\pi} \delta A + \Omega_h \delta J + \Phi_h \delta Q$$  \hspace{1cm} (1.9)

where $\kappa$ is the surface gravity, $A$ is the area of the horizon, $\Omega_h$ is the angular velocity, and $\Phi_h$ is the electric surface potential.

**Second Law**  The area of the horizon of an asymptotically flat spacetime is a non-decreasing function of time.

These laws are analogous to thermodynamics laws if one identifies the temperature (the entropy) with the surface gravity $\kappa$ (respectively, the horizon area $A$). At this level, the resemblance is just in terms of formulas and seems not to have any sort of physical explanation whatsoever. After all, black holes are black objects, they do not radiate and they do not have hair. The status changes drastically when field theory is quantized in a black hole background, which we will turn to in the following section.
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1.3 Black Holes as Thermodynamical Objects

Diffeomorphism invariance of gravity theories leads to an ambiguity in defining time. Such an ambiguity triggers a chain of contamination all the way to the definition of creation/annihilation operators as will be explained below (subsection 1.3.1). Generically, there will be a mixing between creation/annihilation operators that are constructed starting from different choices of time. This is precisely the origin of Hawking radiation of black holes. The transformation that governs the mixing between these operators is called the "Bogoliubov transformations".

In the following, we will start by discussing a free scalar field in a stationary curved background. This will shed light on the origin of Bogoliubov transformations and their physical consequences. As an application, we will deal with the Schwarzschild black hole background. Even if the full analytic solution is out of reach, by studying the asymptotic behavior of the solution (the asymptotic flat region and the horizon) we can derive the Hawking radiation. The material in this section follows very closely [41].

1.3.1 A Free Scalar Field and Bogoliubov Transformations

Our aim at the end is the derivation of the Hawking radiation; i.e., particle creation by a black hole. To understand this phenomenon, we first need to specify the definition of particles. Armed with our knowledge from flat Minkowski spacetime, we will quantize a free scalar field in a curved spacetime. It turns out that the generalization of the notion of particles to curved spacetimes is in general ambiguous, which leads to Bogoliubov transformations.

Curved Background

Going from a flat to a curved background involves the task of covariantizing formulas. In the following, we are going to restrict ourselves to stationary spacetimes which admit a timelike Killing vector $\partial_t$. The associated coordinate will be our time coordinate $t$. By choosing the other spatial coordinates judiciously, one can always bring the metric to the form

$$ ds^2 = -h (dt + \omega)^2 + h_{ij} dx^i dx^j, \quad (1.10) $$

where $h$, $h_{ij}$ and $\omega = \omega_i dx^i$ depend only on the spatial coordinates $x^i$. Such a rewriting of the metric supplements us with a nice separation between temporal $t$
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and spatial coordinates $x^i$, which allows us to perform a canonical quantization of our free scalar field. The action of the scalar field reads

$$S = \int dt \ L, \quad L = \int d^3x \sqrt{-g} \frac{1}{2} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \mu^2 \phi^2) \ , \quad (1.11)$$

where $g = \det g_{\mu\nu}$ is the determinant of $g_{\mu\nu}$, and $g^{\mu\nu}$ is the inverse of the metric $g_{\mu\nu}$. As usual in the canonical quantization approach, one defines the conjugate momentum as

$$\pi = \frac{\delta L}{\delta (\partial_t \phi)} = \sqrt{-g} g^0_\mu \partial_\mu \phi \ . \quad (1.12)$$

Then, one requires that the field $\phi$ and its conjugate momentum $\pi$, promoted now to Hermitian operators, to satisfy the following equal-time commutation relations

$$[\phi(x, t), \phi(y, t)] = [\pi(x, t), \pi(y, t)] = 0 \ , \quad [\phi(x, t), \pi(y, t)] = i \delta^3(x - y) \ , \quad (1.13)$$

where $\delta^3(x - y)$ satisfies

$$\int d^3x \delta^3(x - y) f(x) = f(y) \ ,$$

for any scalar function $f(x)$. The transition from quantum fields to particles goes through creation/annihilation operators, which are defined as follows. One starts by constructing a basis for the solution space of the scalar wave equation, given below (equation 1.23). A convenient way to find a basis is to require it to be orthonormal with respect to a well defined inner product. Generalizing the one used in the flat spacetime case, the Klein-Gordon inner product, to general backgrounds gives

$$\langle f, h \rangle = i \int \Sigma d\Sigma_{\mu} \sqrt{-\tilde{g}} \ g^{\mu\nu} (f^* \partial_{\nu} h - h \partial_{\nu} f^*) \ , \quad (1.14)$$

which can be proved to be independent of the Cauchy surface “$\Sigma$” when $f$ and $h$ are solutions to the wave equation (1.23) given below. This is the equivalent statement of time independence of the Klein-Gordon inner product in the flat spacetime case. For stationary metrics (1.10) “$\Sigma$” can be chosen to be the spatial part $\{x^i\}$ of spacetime. In this case, the inner product simplifies to

$$\langle f, g \rangle = i \int d^3x \sqrt{\tilde{h}} (f^* \partial_t g - g \partial_t f^*) \ , \quad (1.15)$$

where $\tilde{h}$ is the determinant of the induced spatial metric ($\tilde{h}_{ij} = h_{ij} - h_{\omega_i \omega_j}$). Furthermore, in view of characterizing particles by their energy (mass) we want to isolate the time dependence of our wave functions. A nice guess stemmed from the flat spacetime experience is to use frequencies; solutions of the form $f(x, t) = f(x) e^{-i\omega t}$. There is a well defined generalization of frequencies to a stationary spacetime called...
a “Killing frequency”. The latter is defined with respect to the timelike Killing vector which is in our case $\partial_t$.

Before going on with our discussion, we need to address a small subtlety here. There is a redundancy in the solution space of (1.23) due to the reality of the wave equation. As is custom in quantum field theory, we will use the positive frequency $\omega > 0$ solutions to define the needed operators. To complete the picture, one uses the Hermitian conjugates of the already constructed operators. For example, our scalar field will have the following expansion

$$\phi = \sum_f (a(f) f + a^\dagger(f) f^*) ,$$

where $f$ is the positive frequency part of a complete basis of solutions to (1.23) that satisfy $\langle f, f' \rangle = \delta_{ff'}$, $\sum_f$ is a sum/integral over all possible $f$’s, and $a(f)$ ($a^\dagger(f)$) are the annihilation (respectively, creation) operator as they satisfy

$$[a(f), a(h)] = -(f, h^*) = 0 ,$$
$$[a^\dagger(f), a^\dagger(h)] = -(f^*, h) = 0 ,$$
$$[a(f), a^\dagger(h)] = \langle f, h \rangle = \delta_{fh} .$$

To get these commutators, we used (1.13), (1.12) and the inverse of (1.16) which is given, using the orthonormality of $f$’s, by

$$a(f) = \langle f, \phi \rangle , \quad a^\dagger(f) = -\langle f^*, \phi \rangle .$$

We are now ready to define the notion of particles. First, we define the vacuum state $|0\rangle$ as the unique state annihilated by all the annihilation operators

$$\forall f ; \quad a(f) |0\rangle = 0 .$$

Particles are than constructed by the action of an appropriate combination of creation operators on the vacuum $|0\rangle$. For example, a particle with “characteristic” $f$ is given by

$$|f\rangle = a^\dagger(f) |0\rangle .$$

**Frequency Ambiguity and Bogoliubov Transformation**

The definition of particles above starts to be problematic when our timelike Killing vector is not globally well defined. This happens in the case of spacetimes with a horizon such as black holes. Before discussing the special case of the Schwarzschild black hole, let us look for general lessons to be learned.
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Creation and annihilation operators are defined through equation (1.20) given a basis of solutions to the wave equation

$$\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \right) \phi - \mu^2 \phi = 0.$$  \hspace{1cm} (1.23)

Suppose that our timelike Killing vector is not globally well defined. This means that, there are at least two patches where the expression of the Killing vector differs. On the overlap we have two notions of time leading to two different basis of positive frequency solutions $f$ and $h$

$$\phi = \sum_f \langle f, h \rangle a(f) f + a^\dagger(f) f^*,$$ \hspace{1cm} (1.24)

$$\phi = \sum_h \langle a(h) h + a^\dagger(h) h^* \rangle.$$ \hspace{1cm} (1.25)

Using the definitions of creation and annihilation operators (1.20), one finds

$$a(f) = \sum_h \langle f, h \rangle a(h) + \sum_h \langle f, h^* \rangle a^\dagger(h),$$ \hspace{1cm} (1.26)

$$a^\dagger(f) = \sum_h \langle h^*, f \rangle a(h) + \sum_h \langle h, f \rangle a^\dagger(h),$$ \hspace{1cm} (1.27)

which are called Bogoliubov transformation. An important consequence of the mixing between creation and annihilation operators in this kind of transformation is the ambiguity in defining the vacuum, which leads to particle production. To see this, let us calculate the average number of particles of type $f$ “$N(f) = a^\dagger(f) a(f)$” in the $h$-vacuum “$|0(h)\rangle$” defined as $a(h) |0(h)\rangle = 0$. One easily finds using (1.26, 1.27)

$$\langle 0(h) | N(f) | 0(h) \rangle = \sum_h |\langle f, h^* \rangle|^2.$$ \hspace{1cm} (1.28)

This relation clarifies what we mean by particle creation, the $h$-vacuum is unstable against emission of particles of type $f$.

1.3.2 Hawking Radiation

As a quantum physicist, one would like to know the answer to the following question “during the collapse of matter field in a state $|\psi\rangle$ to become a black hole, what will be the response of an observer at asymptotic infinity long after the black hole forms?” During the collapse there will be different modes that get excited. The modes near the would be horizon, just before the black hole forms, are special as they can escape to infinity at the cost of spending a long time to overcome the strong gravitational field. By the time these modes reach infinity, there will be a black hole that has
been formed ages ago. This suggests that we will measure radiation, even after the formation and stabilization of the black hole. On the contrary, if the original state is the vacuum, one would have expected that no radiation should be measured. This conclusion turns out to be erroneous as was first shown by Hawking [1]. The reason resides in the difference between the definition of positive frequencies for the horizon and the observer. This leads to particle creation as explained above.

In the following, we are going to discuss the case of a free scalar field in the background of the Schwarzschild black hole. We are going to follow closely [41]. For another approach following the original work of Hawking, see [42]. Our starting point will be the Schwarzschild metric (1.1) written in different coordinate systems

\[ ds^2 = \left(1 - \frac{r_h}{r}\right) dt^2 + \left(1 - \frac{r_h}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right) \]

(1.29)

\[ = \left(1 - \frac{r_h}{r}\right) (dt^2 - dr^2) - r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right) \]

(1.30)

\[ = \left(1 - \frac{r_h}{r}\right) du dv + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right) \]

(1.31)

where \( r_h = 2M \) is the horizon radius, \( r_s = r + r_h \ln(r/r_s - 1) \) is the Regge-Wheeler radial coordinate, and \( u = t - r_s \) (\( v = t + r_s \)) is the outgoing (respectively, ingoing) null coordinate. Taking advantage of the spherical symmetry of the problem, one decomposes \( \phi \) in spherical harmonics as

\[ \phi(t, r, \theta, \phi) = \sum_{l,m} \frac{\psi_{l,m}(t, r)}{r} Y_{lm}(\theta, \phi) , \]

(1.32)

which reduces the wave equation (1.23) to [41]

\[ (\partial_t^2 - \partial_r^2 + V_{l,m}(r)) \psi_{l,m} = 0 , \]

(1.33)

where

\[ V_{l,m}(r) = \left(1 - \frac{r_h}{r}\right) \left(\frac{r_s}{r^3} + \frac{l(l+1)}{r^2} + \mu^2\right) . \]

(1.34)

We are mainly interested in the observer region (asymptotic infinity), and the horizon region. In the observer region \((r \to \infty \text{ or } r_s \to \infty)\), the potential goes to \( \mu^2 \). In this case, solutions with positive frequency will be of the form \( \xi(r_s) \exp(-i\omega t) \). On the other hand, near the horizon \((r_s \to -\infty \text{ or } r - r_h \to r_h e^{\gamma_i/r_h})\) the potential vanishes exponentially. In this case, the solutions are of the form \( f(u) + g(v) \).

Jumping ahead of ourselves, suppose that our observer measures an outgoing wave packet narrowly peaked around a frequency \( \omega \). Its form will be \( P \sim \exp(-i\omega t) \).

Ultimately we want to evaluate

\[ \langle N(P) \rangle = \langle \psi | a^\dagger(P) a(P) | \psi \rangle , \]

(1.35)
where, for a normalized wave packet $P$, the operator $a(P)$ is given by (1.20)

$$a(P) = (P, \phi),$$

(1.36)
evaluated on a Cauchy surface $\Sigma_f$ corresponding to $t = t_f$. Due to our ignorance about $|\psi\rangle$ except at earlier times, we need to propagate $P$ backward in time. Ideally, one needs to rewind the evolution until before the formation of the black hole. However, as has been shown first by Unruh [43], it is enough to go back in time until $t = t_i$, long after the black hole formed and long before the measurements. The backward time evolved wave packet $P$ will split, at certain stage, to two components the reflected one $R$ with support asymptotic infinity, and a transmitted one $T$ whose support is a narrow region just above the horizon, see picture 1.2. In the language of creation/annihilation operators, one has

$$a(P) = a(T) + a(R),$$

(1.37)

Figure 1.2: The observer measures a wave packet $P$ narrowly peaked around the frequency $\omega$. When this wave packet is evolved backward in time, it splits into two wave packets: $T$ with support just above the horizon of the black hole, and $R$ with support asymptotic infinity.

where now the evaluation is done with respect to the Cauchy surface $\Sigma_i$ corresponding to $t = t_i$, using that the Klein-Gordon innerproduct is independent of the chosen Cauchy surface. Based on the conservation of the Killing frequency in stationary spacetimes, and that we assumed the absence of incoming radiation at $t = t_i$, one
concludes that \( a(R)\psi = 0 \). So our number of particles \( N(P) \) (1.35) reduces to

\[
\langle N(P) \rangle = \langle \psi | a^\dagger(T) a(T) | \psi \rangle ,
\]

where \( T \) has the form \( T \sim \exp(-i\omega u) \). Suppose that \( |\psi\rangle \) is the vacuum state of the collapsing matter. This singles out the proper time of an in-falling observer \( \tau \) as a preferred time coordinate. We need then to decompose \( T \) into two parts: the negative frequencies and the positive ones with respect to \( \tau \). Assuming that the in-falling observer crosses the horizon at \( \tau = 0 \), one finds that

\[
T \sim \exp \left( \frac{i\omega}{\kappa} \log[-\tau] \right) ; \quad \tau < 0 ,
\]

and zero otherwise, where \( \kappa = 1/(2r_h) \) is the surface gravity of the black hole. \( T \) has both positive and negative \( \tau \)-frequencies because it vanishes on the horizon, otherwise, it will be identically zero everywhere. Next, we use the result that a bounded analytic function in the lower half plane has purely positive frequencies following Unruh [44]. First, we construct the positive/negative frequency extension of \( T \) from \( \tau < 0 \) to \( \tau > 0 \) giving

\[
T_\pm = T(-\tau) e^{\mp\pi \omega/\kappa} . \tag{1.40}
\]

Then, we define a new wave packet that has support only inside the horizon by \( \tilde{T}(\tau) = T(-\tau) \). The argument above shows that the positive/negative frequency components of \( T \) are

\[
T^+ = \frac{T + e^{-\pi \omega/\kappa} \tilde{T}}{1 - e^{-2\pi \omega/\kappa}} , \tag{1.41}
T^- = \frac{T + e^{\pi \omega/\kappa} \tilde{T}}{1 - e^{2\pi \omega/\kappa}} . \tag{1.42}
\]

We are almost there. The only missing link in the chain is the identity

\[
a(T) = a(T^+) + a(T^-) = a(T^+) - a^\dagger \left( |T^-\rangle^* \right) . \tag{1.43}
\]

The last part of this identity comes about because in our definitions of creation/annihilation operators, we used always the positive frequency part of the basis. One can think about it as having a complex scalar field where \( T^+ \) and \( T^- \) are related to the particle and its anti-particle. The vacuum \( |\psi\rangle \) is defined by

\[
a(T^+) |\psi\rangle = a(\tilde{T}^-)|\psi\rangle = 0 . \tag{1.44}
\]

Now it is a matter of plugging the equations derived until now in the equation (1.38) to get

\[
\langle N(P) \rangle = \frac{\langle T, \tilde{T} \rangle}{e^{2\pi \omega/\kappa} - 1} , \tag{1.45}
\]

where we used that \( \langle T, \tilde{T} \rangle = 0 \) (they have different supports), \( \langle \tilde{T}, \tilde{T} \rangle = -\langle T, T \rangle \) and some properties of the inner product (1.15). This is just a thermal state at the Hawking temperature \( T_H = (\kappa/2\pi) \) multiplied by the grey-body factor \( \Gamma = \langle T, T \rangle \).
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The Horizon Vacuum vs the Observer Vacuum

To get a better understanding of the Hawking radiation process, let us compare the horizon vacuum $|\psi\rangle$ with the observer vacuum $|0\rangle$. It is easy to show, using (1.44) and the relations (1.41, 1.42), that

$$a(T) - e^{-\pi \omega / \kappa} a^\dagger(\tilde{T}^*) |\psi\rangle = 0,$$  \hspace{1cm} (1.46)

$$a(\tilde{T}^*) - e^{-\pi \omega / \kappa} a^\dagger(T) |\psi\rangle = 0,$$  \hspace{1cm} (1.47)

whereas the vacuum $|0\rangle$ satisfies

$$a(T) |0\rangle = a(\tilde{T}^*) |0\rangle = 0.$$  \hspace{1cm} (1.48)

It is not hard to check that (1.46, 1.47) are satisfied given (1.48) and

$$|\psi\rangle = \left(1 - e^{-2\pi \omega / \kappa}\right)^{1/2} \exp\left(e^{-\pi \omega / \kappa} a^\dagger(T) a^\dagger(\tilde{T}^*)\right) |0\rangle,$$  \hspace{1cm} (1.49)

which can be interpreted as follows. A pair of particle and anti-particle is spontaneously created near the horizon. The particle escapes to infinity where it can be measured, and the anti-particle disappears behind the horizon. Despite being a pure albeit entangled state, the information that the observer gathers corresponds only to the incoming particles. So, as far as the black hole is there, the best the observer can do is to describe the $|\psi\rangle$ state as a density matrix by tracing over the anti-particles. This leads to the following matrix

$$\rho = \left(1 - e^{-2\pi \omega / \kappa}\right) \sum_n e^{-2\pi \omega / \kappa} |n\rangle \langle n|,$$  \hspace{1cm} (1.50)

where $|n\rangle$ stands for a state with $n$ particles. The above density matrix describes a thermal canonical ensemble at the Hawking temperature. We end this section with a small remark of relevance later. The vacuum $|\psi\rangle$ can also be seen as a vacuum squeezed state (Appendix A). The reason is that the equations (1.47, 1.48) look precisely like equation (A.6) in appendix A.

1.4 Consequences of the Black Hole Radiation

After the discovery of the black hole radiation, the physics of black holes entered a new era marked by unresolved or partially resolved puzzles. The information loss paradox occupies the top of the list. On the other hand, the thermodynamical nature of black holes is taken as an indication that the black hole geometry could be just an effective description of an underlying microscopic system. Following this line of thoughts, Ted Jacobson [45] derived the Einstein equations assuming the thermodynamical properties of black holes generalized to local horizons.
1.4.1 Black Hole Evaporation and Information Loss

There are two different sources of information loss in the background of a black hole. The first one is the information about the collapsed matter after they cross the horizon. After all, the horizon delimits a region that is causally disconnected from the outside. The other source involves the entangled nature of the vacuum near the horizon (1.49). This state remains pure as far as the black hole is there: the missing information are, in principle, stored behind the horizon even if we cannot access them. However, the black hole does evaporate leading to the disappearance of all the stored information. But, why does a black hole evaporate in the first place? Roughly, this happens because the black hole loses its mass through Hawking radiation due to the conservation of total energy. This leads to a decrease in the horizon area through backreaction. Because the radiation does not depend on the black hole, this process continues until all the mass is converted into energy. This conclusion seems counter intuitive as the black hole does eat particles in the process of radiation and so its mass should increase. This last observation is misleading because the energy density of our field near the horizon is negative [46, 47]. This is not so strange because, as we have noted at the end of section 1.3.2, the local vacuum state of the horizon is a squeezed state, and we know that these kind of states have always a region in space where their energy density is negative. See appendix A for an example in the flat spacetime.

1.4.2 Gravity as an Effective Description

One of the main lessons that we learned from statistical mechanics is that thermodynamical systems are an effective description of an underlying complex microscopic system where most of the details are ignored. In this paradigm the entropy is a measure of the number of degrees of freedom accessible to the system. As black holes behave like thermodynamical systems, one cannot help himself but wonder: “could black holes be an effective description of some sort?” Answering this question is the main interest of this thesis following ideas stemmed from the fuzzball proposal. The short answer seems to be yes, however, the access to the details of the underlying ensemble changes drastically depending on the black hole under study. We leave the details to the bulk of this thesis (part II and III).