Unknitting the black hole: black holes as effective geometries
Messamah, I.

Citation for published version (APA):

General rights
It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

Disclaimer/Complaints regulations
If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: http://uba.uva.nl/en/contact, or a letter to: Library of the University of Amsterdam, Secretariat, Singel 425, 1012 WP Amsterdam, The Netherlands. You will be contacted as soon as possible.
The central subject of this thesis is the investigation of the idea that black holes are effective descriptions of some underlying microscopic system. We will study how concretely is this realized in the fuzzball scenario. In such scenario, this microscopic system will be some subset of all possible smooth supergravity solutions with the same asymptotic quantum numbers as the black hole. From now on, we will call these smooth geometries “black hole states”. Furthermore, these states are assigned different weights depending on the nature of the black hole under study. The black hole states with their weights define an ensemble that we call, henceforth, “the black hole ensemble”.

In this chapter we will try to explain some basic concepts of the fuzzball scenario that we will be using later on in this thesis. We will be using the strong analogy between black hole ensembles and thermodynamical ensembles, a result of AdS/CFT duality [17]. We will first start by describing and motivating the kind of black holes we are interested in. Then, we will elaborate on the black hole ensemble setting up the stage for subsequent concepts. After that, we will specify the common characteristics of black hole states, leaving details to specific examples discussed later on in this thesis. A first test that a black hole ensemble should pass is to reproduce the entropy of the associated black hole. This leads us to our next task: to figure out a way to count black hole states. Our approach will be to quantize the phase space of the associated smooth solutions. Another test of the black hole ensemble is that after coarse graining, one should get an effective geometry that resembles, to a great precision, the one of the black hole outside the horizon. Where by coarse graining we mean a process where insignificant “microscopic” details are washed away. Explaining how to do coarse graining will be the subject of the last section of
In this chapter the reader is assumed to be familiar with statistical physics, group theory, quantum field theory and spinor representation.

\section*{2.1 Black Holes of Interest to Us}

So far, we were not that clear about which black holes are we going to study. It is time to lift the curtains on our main protagonists. We will do that step by step. First, we are going to motivate why are we interested in BPS black holes. Then, we will spell out our preferred class of black holes and explain their special status, for the fuzzball program, among their cousins.

\subsection*{2.1.1 BPS-ness, Linearity and AdS/CFT}

As alluded to earlier, our winning horse in dealing with the complicated nature of gravity dynamics is symmetries. Although spherical and axial symmetries were powerful enough to allow for exact solutions to the Einstein and Einstein-Maxwell theories, we will be needing an even stronger symmetry for our considerations: supersymmetry. This is a symmetry that links fermions with bosons. The goal of this subsection is to give a taste of the most important qualities of supersymmetry that we will be needing in the remaining of this thesis.

Reviewing supersymmetry and its beautiful structure is not the intended goal of this subsection. The interested reader should consult standard books on the subject e.g \cite{48, 49, 50, 51}. We will be practical in our exposition here, restricting ourselves to describing the needed supersymmetry properties that will be used in subsequent sections and chapters of this thesis. We will start by introducing the supersymmetry algebra. After that, we are going to introduce one of the most important notions in this thesis: BPS states and some of their most useful properties. At the end, we will mention some quick words about our “black holes” that we will study later on, leaving more details to the second and third parts of the thesis.

\textbf{Extended Supersymmetry and BPS States}

Conservation laws play an important role in the search for the fundamental laws of particle interactions. Thanks to Noether theorem, these conservation laws are ultimately connected to global symmetries of the Lagrangian. During the mid of the last century two classes of symmetries were in the heart of the development
of high energy physics: external and internal symmetries. On one hand, we have Poincaré symmetries acting on spacetime, hence the name external. These symmetries manifest themselves through the conservation of energy, momenta and angular-momenta. On the other hand, internal symmetries were put forward to explain the absence of some interaction channels that cannot be accounted for by spacetime considerations. When supplemented with locality, internal symmetries give rise to gauge theories.

In the 1960’s, physicists attempted to combine both classes of symmetries by trying to find a non-trivial symmetry that can fuse both of them. Such efforts accumulated in the Coleman-Mandula no go theorem [52], which asserts that such a non-trivial symmetry does not exist given generic and simple assumptions. Of importance to us, the assumption that physical symmetries are described by Lie groups i.e. restricting to commutation relations between the different generators of the symmetries. The story took a different turn when the last assumption was dropped [53] i.e. include anti-commutation relations in the algebra of symmetries. In such cases, it was possible to construct a non-trivial combination of internal and external symmetries. Since anti-commuting variables are associated with fermions, the anti-commuting generators $Q$ are fermionic and hence map bosons to fermions and vice versa. Such a symmetry is called “supersymmetry”. The Coleman-Mandula no go theorem then restricts the spin of $Q$ to be 1/2.

The fermionic generators, $Q$, called also “supercharges”, have the structure of multiplets $Q^a_\alpha$, where $\alpha$ is a spinor index and $a$ is a multiplet label, as they are 1/2-spin representation of the Lorentz group. If $\mathcal{N}$ the number of multiplets is bigger than one we have an extended supersymmetry. The latter plays an important role in what follows as it allows for a spacial class of massive representations of the supersymmetry algebra, called “BPS-states”, which we will turn to now.

For the sake of the arguments, we will restrict ourselves to $\mathcal{N} = 2$ supersymmetry in four dimensions. In this case we have eight supercharges $Q^a_\alpha$ and $(Q^b_\alpha)^\dagger$, which are taken to be in the two-dimensional Weyl (called also “chiral”) representation, where $\alpha$ is a spinor index and takes the values 1 and 2. The part of the supersymmetry algebra that we will be needing is:

$$\{Q^a_\alpha, (Q^b_\beta)^\dagger\} = \sigma^{\mu}_{\alpha\beta} P_\mu \delta^\beta_b, \quad \{Q^a_\alpha, Q^b_\beta\} = \epsilon_{\alpha\beta} \epsilon^{ab} Z, \quad \{(Q^a_\alpha)^\dagger, (Q^b_\beta)^\dagger\} = \epsilon_{\alpha\beta} \epsilon^{ab} \overline{Z},$$

where $\sigma^0 = -1_{2\times2}$, $\sigma^i; i = 1, 2, 3$ are the Pauli matrices, $\epsilon_{\alpha\beta}$ is the totally antisymmetric tensor where $\epsilon_{12} = -1$, and $Z$ is a complex number called a “central charge”. In general, we will have instead of $Z$ an operator which commutes with all operators in the supersymmetry algebra.

For massive representations and by going to the center of mass frame $P = (-M, \vec{0})$,
positivity of the operators $O^\pm_\alpha = \{a^\pm_\alpha, (a^\pm_\alpha)\dagger\}$ defined as

$$a^\pm_\alpha = Q^1_\alpha \pm \epsilon_{\alpha \beta} (Q^2_\beta)\dagger,$$  \hspace{1cm} (2.2)

implies that $M \geq |Z|$. As a result, the mass of field representations of the supersymmetry algebra are bounded from below by $|Z|$. Such a bound is called the “BPS bound” and it is saturated by a special class of states called “BPS states”. It is clear from the way we derived the bound that these states are annihilated by two out of the four supercharges. In such a situation the BPS state is called $1/2$-BPS. In general, we will have other kinds of BPS states according to the number of independent linear combinations of supercharges that annihilate them.

### Estimating the Number of BPS States

In a supersymmetric theory, fields come in multiplets as they are representations of the supersymmetry algebra, in accordance with the general lore that fields in a quantum field theory are representations of the global symmetries of the Lagrangian. One of the powerful results of supersymmetry is that the number of fermions and bosons in a multiplet is the same. This is a straightforward result of the first commutator in (2.1), and the fact that $P_\mu$ is a one-to-one operator. Taking advantage of such pairing, Witten showed in a fascinating paper [54] that one can define a parameter-dependant weighted sum over states that is invariant under a continuous change of this parameter, provided that supersymmetry is not broken.

For the sake of the argument, we will restrict ourselves in the following to one supercharge and its Hermitian conjugate. One has the following commutators

$$\{Q, Q^\dagger\} = E, \quad [Q, \mathcal{E}] = [Q^\dagger, \mathcal{E}] = 0,$$  \hspace{1cm} (2.3)

where $E$ is the energy. The first equation is a special case of (2.1), and the other two were not mentioned earlier as they did not play a role in the previous argument. It is easy now to show that the following weighted sum, usually called an “index”,

$$I(\beta) = \text{Tr}_{\text{states}} (-1)^F e^{-\beta E},$$  \hspace{1cm} (2.4)

where $F$ is the fermion number, and $\beta$ is some continuous parameter, does not receive contributions from states that are not annihilated by the supercharges. Suppose the inverse is true, there exists a states $|\psi\rangle$ that contributes to $I(\beta)$ such that $Q|\psi\rangle \neq 0$ and $Q^\dagger |\psi\rangle \neq 0$. By acting with a supercharge on this state, one can generate another state $|\chi\rangle$ with the wrong fermion number and the same energy, a consequence of (2.3). In doing so, we managed to construct a state $|\chi\rangle$ whose contribution exactly cancels the contribution of $|\psi\rangle$. According to this argument,
the only contributing states to the index \( I(\beta) \) are the ones that are annihilated by supercharges, which, according to the first equation in (2.3), have zero energy. We will call such states "ground states".

Suppose now that, for some reason, by changing the parameter \( \beta \) some ground states cease to be so. These states should leave the ground state in fermion-boson pairs to preserve supersymmetry. Hence using the argument before, the change in the index \( I(\beta) \) vanishes. One concludes then, that \( I(\beta) \) is invariant under a continuous change in \( \beta \) as promised.

There are generalizations of the simple index (2.4) where, instead of one parameter \( \beta \), we have a parameter space \( \mathcal{M}_\beta \), and the weights are chosen such that the only contributing states belong to a specific class of BPS states. These generalized indices are sometimes called "elliptic genera". Unfortunately, it turns out that BPS states annihilated by at most four supercharges do suffer a discrete jumps in the their index [55]. This happens at specific values of the continuous parameters \( \beta_i \) that enter in the definition of the index. These special values of \( \beta_i \) define a co-dimension one hypersurface in \( \mathcal{M}_\beta \) which is called a "wall of marginal stability". In some cases, like the 1/2-BPS sector of \( \mathcal{N} = 2 \) four-dimensional supergravity of interest to us, there are ways to quantify such a jump [30] (see also section 5.4).

Making the supersymmetry local, forces us to study gravity at the same time. Such theories are called "supergravity" theories. In this case one can construct BPS solutions that have only bosonic excitations, which are the class of solutions that we will be dealing with. But, how is that possible? Naively one would think that supersymmetry transformations will turn on fermionic degrees of freedom. The trick is that these BPS solutions exist whenever there is a supersymmetry transformation parameter \( \epsilon \), that does not generate fermionic degrees of freedom when acting on the bosonic fields of the BPS solution. In other words, this spinor \( \epsilon \), called a "Killing spinor", is such that the variations of fermions \( \psi \) under supersymmetry transformations vanish \( \delta_\epsilon \psi = 0 \), for the given values of the bosonic fields of the BPS solution. In the following, we will be calling the equation \( \delta_\epsilon \psi = 0 \) the "Killing spinor constraints".

Our interest in these BPS-solutions is threefold:

i- Generically, and for enough preserved supersymmetries, the index gives the actual number of BPS states. In such a situation, the number of BPS states is invariant under continuous variations of parameters baring walls of marginal stability. This will play an important role in the arguments of section 2.3 below.

ii- On the technical level, and at least for the class of theories we will be dealing with, these solutions are much simpler than the non-BPS ones for the following reason. We have already seen that the gravity field equations are second
Chapter 2 - The Fuzzball Machinery

order non-linear differential equations, which makes them hard to solve. The simplification that BPS solutions bring is that, the Killing spinor constraint is first order. In some cases, a further simplification does occur: solving the set of field equations and Killing spinor constraints in the right order linearizes the equations to be solved at each step. The non-linearity manifests itself in a source term that depends only on solutions to the equations of the previous steps. This happens in the examples we will be studying in the next two parts of the thesis. Using this strategy, a large number of smooth solutions with the same asymptotic quantum numbers as a black hole was built, opening the door for a fuzzball study of such solutions.

iii- The Last motivation comes from totally different considerations. A powerful tool to study properties of black holes in string theory is AdS/CFT duality [9]. This duality asserts the equivalence of string theory (gravity) on AdS$_{d+1}$ backgrounds and a CFT living on the boundary of AdS$_{d+1}$. A way to get such backgrounds is to take a decoupling limit of a certain class of black holes that includes BPS ones. This decoupling limit amounts to decoupling the physics in the near horizon region of the black hole from that of the asymptotically flat region by scaling the appropriate Planck length, $l_p$, to decouple the asymptotic gravitons from the bulk. At the same time one needs to scale appropriate spatial coordinates with powers of $l_p$ to keep the energies of some excitations finite. Two lessons learned in the AdS/CFT duality will be of importance to us in what follows. On one hand, it has been shown that “BPS black holes behave like thermodynamical ensembles [17], even though they do not Hawking radiate”. The chemical potentials of such ensembles, including the temperature, are just parameters to describe the black hole ensemble without any physical meaning. On the other hand, it is only in the realm of AdS/CFT duality that one can really be confident that the fuzzball scenario might be after all a correct way to think about black holes. We already mentioned that black holes are described by thermodynamical ensembles of states on the CFT side. The AdS/CFT duality then tells us that these states, if they are accessible in gravity, should be given in terms of smooth geometries.

2.1.2 BEYOND AdS/CFT?

Although considerations in AdS/CFT were the origin of the fuzzball program [32, 33, 34, 35], we would like to weaken such dependence if not get rid of it. The hope at the end is to apply coarse graining ideas to realistic black holes (without supersymmetry) for which AdS/CFT as we know it now is not applicable. Although we cannot, for the moment, disregard the AdS/CFT because some of the building
blocks we are going to use are well established in this duality (such as the thermodynamical nature of BPS black holes and general characterization of black hole states, see section 2.2), we are not going to use AdS/CFT explicitly in the following. We should always keep in the back of our mind that, in the cases we are interested in, we can embed the whole discussion in an AdS/CFT duality frame. Such a duality then supplements us with the missing building blocks that we will be needing.

Another point of view would be that the fuzzball paradigm, as we are going to use it, is complementary to AdS/CFT. The former can explore questions beyond the reach of the latter. As an example, the fuzzball conjecture predicts that quantum effects in gravity gets enhanced to large distances (horizon size) [56], completely unseen by the AdS/CFT. Such enhancement opens the door to re-question the applicability of local effective field theories whenever gravity is around even if the curvature is small everywhere.

2.1.3 The Guinea Pigs

We will be dealing in the bulk of this thesis with two kinds of black holes: the D1-D5 “small” black holes in five dimensions (part II) and the $\mathcal{N} = 2$ four dimensional $1/2$-BPS black holes (part III). The word “small” in small black hole means that the horizon area of the black hole vanishes if one is using the two derivative effective action. It was shown that, including higher order terms will generate a proper albeit small horizon [57, 58, 59, 60, 61, 26].

We are interested in these systems mainly because they are among few that have known large space of smooth solutions with the same asymptotic charges as a black hole. Actually, smooth solutions exist provided we go one dimension higher: six for the D1-D5 and five of the $\mathcal{N} = 2$ black holes. The uplift of the D1-D5 system is still described using type-IIB string theory, however, the uplift of the $\mathcal{N} = 2$ black holes is now described using M-theory. This is a consequence of the fact that type-IIA string theory is the compactification of the eleven-dimensional M-theory over an $S^1$ whose radius is related to type-IIA coupling constant: the M-theory is the large coupling limit of type-IIA where the $S^1$ circle decompactifies. As a result, if one compactifies type-IIA on a Calabi-Yau, the resulting physics is equivalent to compactifying M-theory on the same Calabi-Yau $\times S^1$. This relation allows us to map four-dimensional solutions to a special class of five dimensional solutions that have a $U(1)$ isometry. This map goes under the name of “4d-5d uplift” [62, 63]. Since we are keeping the $U(1)$ isometry, going back and forth between five and four dimensions will not affect the number of states. Based on this observation, we will be studying the four dimensional solutions in this thesis.
Another reason behind our interest in these class of solutions is that we can em-
bed them in an asymptotic AdS$_{3} \times S^{d}$ spacetime, with, $d = 3$ for the uplifted six-
dimensional D1-D5 solutions e.g. [24] and $d = 2$ for the uplifted five-dimensional
1/2-BPS black holes [64]. Such embedding makes us confident about the possible
applicability of the fuzzball ideas to these class of solutions.

From a different perspective, the study of the D1-D5 system is physically interesting
as one can hope to describe the so called “small black rings” [65] using an appropri-
ate thermodynamical ensemble of the D1-D5 states [66]. We have already explained
the use of the word small but what about the word black ring? and why are they
interesting? Black rings are the first gravity solutions with a horizon whose topology
is not a sphere [5, 6]. They are five dimensional solutions with a horizon that is
topologically $S^{2} \times S^{1}$ instead of $S^{3}$. These black rings are the first known gravity so-
lutions that violate the no-hair theorem [67]. The hair of these black rings is a local
non-conserved charge, called the “dipole charge”, that is not visible at the asymptotic
flat region. Even though we cannot measure these dipole charges at infinity, we can
still find their imprint as they enter in the first law of thermodynamics of black rings
[67, 68]. Such a non expected role of dipoles makes them special as, based on gen-
eral grounds, they should enter somehow in the characterization of the black ring
ensemble.

The D1-D5 small black ring carries a non-trivial angular momentum in contrast to
the naive D1-D5 small black hole. In the presence of this angular momentum, the
D1-D5 system starts to look like a ring rather than a point-like object in five dimen-
sions [69, 33]. Therefore, it is expected that such a system, D1-D5 with angular
momentum, will describe a black ring with a small horizon [65, 70, 71, 72] when
higher order terms are included in the gravity action, in the same spirit as the small
D1-D5 black hole develops a small horizon.

On the negative side of the story, both our classes of solutions (D1-D5 and $\mathcal{N} = 2$
four dimensional 1/2-BPS black holes) suffer from some drawbacks. The D1-D5,
as was mentioned before, corresponds to a five dimensional solution with vanishing
horizon area which raises doubts about its possible usefulness to realistic black holes
i.e. black holes with large classical horizon. On the other hand, the $\mathcal{N} = 2$ four-
dimensional 1/2-BPS solutions include such large black holes, but due to the small
amount of preserved supersymmetries (only four supercharges), they are technically
more demanding. Certainly for these class of solutions there is much work that still
needs to be done.

Historically, the D1-D5 system was the main inspiration of the whole fuzzball pro-
gram and by far the most fruitful testing ground of the fuzzball ideas (see [36, 37,
11, 38, 39, 40] and references therein). This is mainly due to the high supersym-
metry that the system preserves, eight preserved supercharges. However, the first
Chapter 2 - The Fuzzball Machinery

class of solutions that was subject to coarse graining considerations [73] describes the $1/2$-BPS sector of type-IIB supergravity on $\text{AdS}_5 \times S^5$. These solutions go under the name of “Lin-Lunin-Maldacena geometries” (LLM in short) [74]. These are the simplest solutions to deal with as they preserve sixteen supercharges. Despite that, there is no known large black hole that preserves the same symmetries. We will not mention this system beyond this point. On the other hand, the $\mathcal{N} = 2$ black holes are still in the beginning of the journey (see [38, 39, 40] and references therein).

2.2 Black Hole Ensemble

Black holes are solutions of gravity theories with the metric playing the role of the fundamental field. This suggests that the black hole states might be characterized by their geometry. This is not entirely obvious and seems to fail for $1/4$-BPS black holes. For the moment, let us forget about this failure leaving the discussion it deserves to the third part of this thesis and proceed with the general idea.

2.2.1 The No-Hair Theorem, Entropy and Geometry

As we have seen in the first chapter, black holes have an entropy that is in clear clash with the no hair theorem. The latter states that the metric is completely fixed given stationarity and a small number of fixed charges at infinity. A familiar situation that bears a lot of resemblance to the black hole is thermodynamics. There, the system is also characterized by few parameters (temperature, energy, pressure, ...). Still the statement that such a system has a non-zero entropy does not raise any objections. This is because we know that Newtonian mechanics is the “fundamental” theory, and thermodynamics is just an effective description. In this realm, the entropy is a measure of the degrees of freedom that the thermodynamical system can access. Given such success, one is tempted to repeat the same philosophy for the black hole. However, things are not that trivial.

Let us restrict ourselves to the Schwarzschild black hole for the sake of the argument. Such a black hole emerges in the context of general relativity where the metric is the sole field. The no-hair theorem states that, given the mass, the geometry is unique. In other words, if general relativity is the fundamental theory this seems to be the end of the story.

An ambitious idea that the fuzzball proposal seems to suggest is that the classical picture of the region behind the horizon is not the right one. We do not have a clear notion of geometry there, the spacetime is fuzzy which is the origin of the name
coined by Mathur to such proposal, “the fuzzball proposal” [32, 33, 34, 35]. Such a drastic change in our way of perceiving black holes goes against our intuition from classical gravity, and even worse, from the point of view of effective field theory. The gravity objection to such picture is that the horizon is a global concept, locally there is nothing special about it. While the effective field theory objection is due to the intuition that for a large horizon with a small curvature everywhere, there will not be any drastic change of the classical picture due to quantum effects. However, quantum mechanics outruns such objections by appealing to the discrete nature of the quantum phase space. Remember that, due to the uncertainty principle, the classical phase space is discretized to $\hbar$-sized cells. All the states that belong to the same cell are indistinguishable classically. If these classically indistinguishable solutions differ from each other over large distances in the real spacetime, they will manifest large macroscopic quantum effects. We will see an example of such effects in the following two parts of the thesis.

In the fuzzball scenario, our starting point is an ensemble of geometries with the following characteristics:

- The pure state geometries are smooth everywhere as the existence of a singularity is expected to lead to an entropy as follows. Naked singularities are unpleasant objects as they destroy the predictability of physics even long distances far away. So, one expects that they will be shielded by a horizon once higher order corrections are included. This suggests, following general considerations for black holes, that such solutions will have an entropy proportional to the area of this horizon. As such, they should be seen as a sub-ensemble rather than a pure state. We can sometimes relax the smoothness requirement to include solutions with zero entropy.

- The smooth geometries should carry the same asymptotic charges as a black hole. In some cases refinements of this condition will be needed. We will see some examples in the next two parts of the thesis. Although we do not know a precise general enough formulation of these refinements, a natural proposal would be to include all the quantities that appear in the first law of black holes.

- A weight that depends on the black hole under consideration is associated, in principle, to each smooth geometry.

### 2.2.2 At Which Level Can we Trust Our Geometries

The fuzzball scenario with its unorthodox ideas is subject to some criticism that we will not discuss all of it here. For more details see [36, 11, 75, 40] and the conclusions at the end of the thesis. We have already discussed the large quantum
effects objection in the previous subsection. The other objection that we are going
to address here concerns the possible deference in the results of measurements done
with respect to the black hole ensemble effective geometry, or with respect to the
naive black hole geometry. If these were so, then the usual black hole could not be
a good effective description, and there would be a massive violation of our usual
expectations from effective field theory. Fortunately, one can show that in any sce-
nario where the entropy of a black hole has a statistical interpretation in terms of
states in a microscopic Hilbert space, the variance of finitely local observables over
the Hilbert space will be suppressed by a power of $e^{-S}$ [76]. Thus, even if the mi-
crostates of a black hole are realized in spacetime as some sort of horizon free bound
states, finitely local observables with finite precision, of the kind that are accessible
to semiclassical observers, would fail to distinguish between these states. Indeed,
the semiclassical observer, having finite precision, might as well take an ensemble
average of the observables over the microstates, as this would give the same answer.
The ensemble of microstates gives a density matrix with entropy $S$, and will be de-
scribed in spacetime as a black hole geometry. In this sense, the black hole geometry
will give the effective description of measurements made by semiclassical observers.

Even with this positive situation, one can still question the validity of describing the
black hole ensemble using an effective geometry. In general, one is not sure that
such geometry will be free of regions with a large curvature, and hence, will not be
trustworthy. Unfortunately, such a possibility is non-vanishing as typical states tend
to have such problems. In such a situation, one should be careful about the kind
of questions he is asking. Presumably, a corrected version of this effective geometry
will be the honest effective description of the black hole ensemble.

2.3 PHASE SPACE QUANTIZATION

After we (partially) specified our black hole ensemble, we need a way to count the
number of states among other things. We are going to do so by quantizing the space
of solutions. Among the reasons to follow such approach is that the counting of
states will be more transparent. We will also be able to evaluate quantum effects in
view of identifying the scales at which they become important. By doing so, we will
be able to check when does classical geometry break down. A third reason is that by
having an associated quantum Hilbert space, one can hope to coarse grain.

It turns out that, the space of solutions in the cases of interest to us, comes equipped
with a symplectic form. This allows us to use the so called “covariant phase space
quantization” to quantize our systems. In such an approach, we will quantize the
actual space of solutions rather than the fluctuations around these solutions. This
is equivalent to the study of Landau levels of an electron in a large magnetic field, which amounts to neglecting quadratic terms while keeping the linear term in momentum.

2.3.1 Solution Space, Phase Space and the Symplectic Form

Shifting the quantization from phase space to solution space relies on two fundamental observations:

- On general grounds, one can identify the space of solutions of a general field theory with its canonical phase space. Heuristically, this is because a given point in the phase space, comprised of a configuration and associated momenta, can be translated into an entire history by integrating the equations of motion against this initial data. Likewise, by fixing a spatial foliation, any solution can be translated into a unique point in the phase space by extracting configuration and momentum from the solution evaluated on this spatial slice.

- The existence of a symplectic form on the space of solutions that can be derived starting form the Lagrangian. This is an old idea [77], see also [78] for an extensive list of references and [79, 80, 81, 82] for more recent work. In the case of Lagrangians that only depend on the fields and their derivatives $L = L(\phi_a, \partial \phi_a)$, which is the class of Lagrangians we will be working with, this symplectic form is given by

$$\omega = \int d\Sigma J^l ; \quad J^l = \delta \left( \frac{\partial L}{\partial \partial_t \phi_a} \right) \wedge \delta \phi_a , \quad (2.5)$$

where the integration is over an initial Cauchy surface and $\delta$ is the exterior derivative in field space.

An important subtlety in our application of such quantization approach is that we will not quantize the entire solution space, but rather, a subspace of the solutions with a certain amount of preserved supersymmetry (see part II and III of the thesis).

In general, quantizing a subspace of the phase space will not yield the correct physics, as it is not clear that the resultant states do not couple to states coming from other sectors. It is not even clear that a given subspace will be a symplectic manifold with a non-degenerate symplectic pairing. As discussed in [83], we expect the latter to be the case only if the solutions belonging to the subspace have a non-trivial momentum. For gravitational solutions we thus expect stationary but not static solutions to possibly yield a non-degenerate phase space. This is because the canonical momenta of these class of solutions are non-vanishing. To see this,
we will discuss the metric part of the solution. We start by putting the metric in the canonical form \[ ds^2 = -N^2 dt^2 + f_{ij} (dx^i + N^i dt) (dx^j + N^j dt), \] (2.6) where \( x^i; i = 1, 2, 3 \) span the spatial part of the metric. This separation of time \( t \) and space allows us to define the canonical momenta dual to the components of the spatial part of the metric \( f_{ij} \). These turn out to be given by \[ \pi^{ij} = \sqrt{-g} f^{ik} f^{jl} \left( \Gamma_{kl}^t - f^{mn} \Gamma_{mn}^t f_{kl} \right), \] (2.7) where \( f^{ij} \) is the inverse of \( f_{ij} \) i.e \( f^{ij} f_{jk} = \delta^i_k \) which is different from \( g^{ij} \) the spatial part of the inverse of the metric (2.6), and \( \Gamma_{\mu\nu}^{\rho} \) is the usual Christoffel connection. In the case of a stationary metric, the coordinates \( t \) and \( \{x^i\} \) can be chosen such that the functions \( N, f_{ij} \) and \( N^i \) entering in the expression of the metric (2.6) are independent of time \( t \). In this case we have non-trivial momenta \( \pi^{ij} \) if \( \Gamma_{ij}^t \) is non-vanishing, which in turn requires \( N^i \) to be different from zero. The last condition fails in the case of a static metric. Therefore, in the case of stationary but non-static metric we do have non-trivial dual momenta.

This still does not address the issue of consistency as states in the Hilbert space derived by quantizing solutions along a constrained submanifold of the phase space might mix with modes transverse to the submanifold. When the submanifold corresponds to the space of BPS solutions one can argue, however, that this should not matter. The number of BPS states is invariant under continuous deformations that do not cross a wall of marginal stability or induce a phase transition. Thus, if we can quantize the solutions in a regime where the interaction with transverse fluctuations is very weak, then, the energy eigenstates will be given by perturbations around the states on the BPS phase space. Although, these will change character as parameters are varied, the resultant space should be isomorphic to the Hilbert space obtained by quantizing the BPS sector alone. In some cases, nailing precisely the regime where the interaction of BPS states with transverse fluctuations is weak, harbors a lot of subtleties that we are not going to discuss here. However, there is a much quicker argument in favor of the safety of restricting the quantization to the BPS submanifold. Our approach consists of enforcing the BPS-constraints at the classical level then quantize the resultant constrained system. The other, more correct, way of quantization is to quantize the full space of solutions then enforce the BPS-constraints on the resultant quantum states. Most of the time, these two approaches are the same. For an example of the relation between states obtained by considering the BPS sector of the fully quantized Hilbert space and the states obtained by quantizing just the BPS sector phase space see [85].

Let us emphasize that the validity of this decoupling argument depends on what questions one is asking. If we were interested in studying dynamics, then, we would
have to worry about how modes on the BPS phase space interact with transverse modes. For the purpose of enumerating or determining general properties of states, however, as we have argued, it should be safe to ignore these modes.

### 2.3.2 Quantization

So far we spoke about the symplectic form in the classical theory, how do we proceed to the quantum theory? The symplectic form encodes the quantum information through its connection with the Poisson bracket as follows. In the phase space, the canonical Poisson bracket is given by

$$\{q^a, p^b\} = \delta^{ab}.$$  

This can be encoded in the symplectic form $\Omega = dp^a \land dq^a$. Now, given a coordinization $x^\alpha$ of the phase space, the symplectic form becomes

$$\Omega = \omega_{\alpha\beta} dx^\alpha \land dx^\beta; \quad \omega_{\alpha\beta} = \frac{1}{2} \left( \frac{\partial p^\beta}{\partial x^\alpha} \frac{\partial q^\alpha}{\partial x^\beta} - \frac{\partial q^\beta}{\partial x^\alpha} \frac{\partial p^\alpha}{\partial x^\beta} \right). \tag{2.8}$$

It is easy to see from the equation above that, the Poisson bracket of $x^\alpha$ and $x^\beta$ is just the inverse of the $\omega_{\alpha\beta}$. Explicitly

$$\{x^\alpha, x^\beta\} = \frac{1}{2} \omega^{\alpha\beta}. \tag{2.9}$$

The only missing connection to the quantum theory is then, to replace the Poisson bracket by the (anti-)commutator following the standard canonical quantization procedure.

The next step in the quantization is to construct the Hilbert space. Unfortunately, this is not so trivial and depends on the case under consideration (e.g [86]). The fundamental issue is to distinguish coordinates from momenta by picking a polarization. In the cases we are going to discuss, two approaches are followed

- The D1-D5 system: this case turns out to be easy as one gets a harmonic oscillator algebra. This allows us to introduce creation/annihilation operators leading to the Hilbert space as in the standard way [87, 88].

- The four dimensional multi-center black hole solutions: in this case we resort to geometric quantization techniques (see for example [89, 90, 91]). We defer a small discussion about this approach to the sixth chapter.
2.4 Coarse Graining

In principle, we have succeeded in constructing the Hilbert space of black hole states. However, we do not know yet what is the relation between our classical black hole and these states. To get some insight about this connection, a possible thing to do is to look for a typical state among the semi-classical states. This should be the closest we could get to reproduce the naive black hole physics. However this notion (typical state) usually needs an “average” state to be defined. The latter may also be seen as a close cousin of our black hole. The golden question that begs for an answer is: suppose that we are given a black hole and its ensemble. Suppose also that somehow we have succeeded in finding the average/typical state of the ensemble. Can we treat the three geometries: average state geometry, typical state geometry, and the black hole geometry as being the same when we coarse grain?. If the answer is yes than we can confirm that the black hole is an effective description of its ensemble. But what do we mean by “coarse graining”? In this process we have in mind a measuring device with a limited resolution $\Delta$. Due to such limitation any measurements with deference less than $\Delta$ will be registered as the same. This is called “coarse graining”.

2.4.1 Coarse Graining as an Average

The only way to evaluate averages, that we know of, is when there is a linearity in the system under consideration. Since gravity equations are non-linear, it is not clear how to “average” over geometries. Luckily, in the cases we are going to deal with, our geometries are completely fixed given a set of harmonic functions. In this class of solutions we are offered with a suitable coarse graining procedure, we simply smear these Harmonic functions against appropriate weights. Of course, this is not the end of the story as one has to make sure that the resulting average geometry does solve the original equations of motion at least asymptotically, and is free of pathologies such as closed timelike curves (CTC in short) and regions of high curvature.

To carry on such averaging in practice, we need to specify the weights of the different “geometries” that enter the process of averaging. Such weights have two origins. The first one is classical, and has to do with the nature of the ensemble under study. In the case of black holes these weights will be thermodynamical weights, because of the total body of evidence gathered so far that indicates that black holes behave like thermodynamical ensembles. The second contribution is quantum in nature for the following reason. We are in principle evaluating the average in the quantum theory, but, we want to recast the result in terms of classical quantities (like the geometry) living in the phase space. This transition between quantum Hilbert space to classical phase space can be achieved using the phase space densities, see for
example the review [92] and references there in to the original literature. We will defer the rightful discussion of such a concept to the fourth chapter, section 4.1, as this average procedure will be carried out only for the D1-D5 system. Unfortunately, it is not clear how to extend such an operation to the more interesting $\mathcal{N} = 2$ four-dimensional $1/2$-BPS black holes. This is because, as we will see later, we seem to not have enough states to account for a finite fraction of the entropy.

### 2.4.2 Another Possibility: Typical States

As we pointed out before, another coarse graining candidate is typical states. A typical state in an ensemble is one for which the expectation values of macroscopic observables agree, to within the observable accuracy, with the average of the observable in the entire ensemble. Obviously, this definition depends on the appropriate notions of macroscopic observables and observable accuracy. Given a typical state, we can try to map it directly to a solution of supergravity (this may still be a formidable task), after which, one still needs to verify that the resulting geometry has no pathologies such as closed timelike curves and regions with Planck size curvature. Such typical states can be useful in the case of the $\mathcal{N} = 2$ four-dimensional supergravity to study the physics of $1/2$-BPS black holes, though, we are not going to do so in this thesis.