Unknitting the black hole: black holes as effective geometries
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CHAPTER 3

THE LUNIN-MATHUR (LM) GEOMETRIES

The D1-D5 system was and still is an active laboratory for different ideas and aspects of quantum gravity in string theory. In order to better understand it, using AdS/CFT duality, Mathur and collaborators ended up suggesting an unorthodox idea that was baptized the “fuzzball” program [32, 33, 34, 35]. In this chapter we will try to describe this fertile system leaving the study of the implications of the fuzzball ideas when applied to this system to the next chapter.

We will start by describing the stringy/brane construction of the D1-D5 system. Then, by using T and S duality we will be able to construct the most general D1-D5 supergravity solution through its connection with the FP system. The latter describes the closed fundamental string (F) which carries a momentum charge (P) as a traveling wave. Finally, we will discuss the quantization of this solution space.

Familiarity with string theory is assumed throughout this chapter. See appendix B for a summary of our conventions.

3.1 The Five Dimensional “Small” Black Hole

The simplest stringy system that bears a lot of resemblance to black holes is the so called “two charges system”. A prototype is the D1-D5 system which, after back reaction, describes a five dimensional black hole with zero horizon area. Because of the last property it is dubbed a “small” black hole. We are going to describe the
brane set-up, and its ten dimensional backreacted geometry in the following. We will be very brief in our survey, for more details see e.g [93] and references there in.

### 3.1.1 The Set-Up

We start with type-IIB string theory on $M^{(1,4)} \times S^1 \times X_4$, where $M^{(1,4)}$ is a non-compact five dimensional spacetime and $X_4$ is either a $T^4$ or $K_3$. In the following, we will have $T^4$ in our mind as it allows us to be explicit. Things should carry on to $K_3$ with slight changes. In the above geometry, the size of the $S^1$ is much bigger than the size of $X_4$, which is itself much bigger that the typical string scale so that we can trust supergravity to leading order. Such scale hierarchy allows us to compactify on $X_4$ reducing every thing to a six dimensional supergravity theory. The latter will be the frame work in the next chapter.

Among the fields of type-IIB supergravity, we are going to turn on the dilaton $\phi$ and, of all possible R-R forms, the two-form $C^{(2)}$ corresponding to the D1 and the D5 D-branes. Furthermore, we are going to wrap $N_5$ D5 branes on $S^1 \times T^4$ and $N_1$ D1 branes on $S^1$. Since we do not want to keep track of the position of the D1 in the internal space $T^4$, we are going to uniformly distribute the D1-branes on this space. This is called “smearing”.

### 3.1.2 The Geometry

To construct the supergravity solution that corresponds to the D1-D5 system, we need to solve the equations of motion coming from the action (section B.1.2)

$$ S = \frac{1}{2\kappa_0^2} \int d^{10}x \, (-G)^{1/2} \left( e^{-2\phi} \left[ R + 4(\nabla \phi)^2 \right] - \frac{1}{12} dC^{(2)} \wedge \ast dC^{(2)} \right), $$  

(3.1)

where we should be careful when dealing with $C^{(2)}$ because it describes both D1 and D5 branes. We should require that

$$ \int_{T^4 \times S^3} \ast dC^{(2)} \sim Q_1, \quad \int_{S^3} dC^{(2)} \sim Q_5, $$  

(3.2)

where $Q_1$ ($Q_5$) is the D1 (respectively, D5) charge, and $S^3$ is a three-sphere that sits at large radius on the spatial part of $M^{(1,4)}$. We should also require the existence of a Killing spinor predicted by the probe brane analysis. In such analysis, the string coupling constant that sets the strength of gravity is very small. This means that the geometry will not feel the presence of the D-branes. As a result, the non-compact part of the space, $M^{(1,4)}$, will be the flat five-dimensional Minkowski spacetime. In
such a situation, one can use the physics of open strings ending on the D1 and/or D5 branes to study the D1-D5 system.

In the following, we are not going to use the strategy outlined before. Instead, we are going to take advantage of the explicit form of the solutions describing the response of spacetime to the existence of D-branes (section B.2). In some cases, one can apply a simple algorithm to figure out the metric of orthogonally intersecting branes called the “harmonic function rule” [94, 95]. The D1-D5 system does satisfy the requirements needed to apply such an algorithm that we are going to describe in some detail. Essentially, each D-brane solution is characterized by a harmonic function $H$ that depends on the non-smeared transverse directions. In the string frame, it enters in the metric as $H^{-1/2}$ in front of the parallel directions and as $H^{1/2}$ in front of the transverse directions (see section B.2). The full solution then contains all the associated harmonics as if each D-brane exists on its own. In the case of the D1-D5 system the solution reads [93]

$$ds^2 = \frac{1}{\sqrt{f_1 f_5}}[-dt^2 + dy^2] + \sqrt{f_1 f_5}(dr^2 + r^2[d\theta^2 + \cos^2 \theta d\phi^2 + \sin^2 \theta d\psi^2]) + e^\Phi dz^2,$$

$$e^{2\Phi} = \frac{f_1}{f_5}, \quad f_i = 1 + \frac{Q_i}{r^2}, \quad (3.3)$$

$$G^{(3)} = dC^{(2)} = Q_5 \sin 2\theta \, d\theta \wedge d\phi \wedge d\psi - \frac{2 Q_1}{r^3 f_1} dr \wedge dt \wedge dy,$$

where $y$ is the coordinate along $S^1$, and $\{z^i\}$ are the $T^4$ coordinates. We use throughout the remaining of this part of the thesis boldface notations to indicate four-dimensional vectors. Innerproducts are also understood in expressions like $|F|^2$.

It can be checked that this geometry when reduced to five dimensions describes a black hole with vanishing horizon area [96]

$$A \sim \lim_{r \to 0} \left( e^3 \sqrt{\frac{Q_1 Q_5}{r^4}} \right) = 0.$$

As for the supersymmetries, one can either start from the zero string coupling limit (probe D-branes in flat spacetime), or solve the Killing spinor equations [93, Appendix: D]. In both cases, the conclusion is that the Killing spinors of the D1-D5 system satisfy the following two projections [93]

$$\Gamma_\hat{t} \Gamma_\hat{y} \epsilon = -i\epsilon^*, \quad \Gamma_\hat{t} \Gamma_\hat{y} \prod_{i=1}^{4} \Gamma_{\hat{z}_i} \epsilon = -i\epsilon^*, \quad (3.4)$$

where the hat stands for flat local frame directions.
3.2 The Lunin-Mathur (LM) Geometries

The reduction of the solution described above to six dimensions describes the geometry of an effective string wrapped around the $S^1$. This suggests that we can construct other solutions by including the possibility of a non-trivial profile. The story is not as simple as it sounds because we should make sure that we are not adding an extra charge by doing so. Using the large duality group that the D1-D5 system has, [69, 35, 97] managed to construct a generalization of the solution (3.3) by mapping it to a fundamental string carrying a right (or left) momentum, called FP system [98, 99, 100]. Such a construction is the subject of this section.

3.2.1 Switching to the FP System

There are different duality chains that map a D1-D5 system to an FP one. A possible chain is [97]

$$
\begin{align*}
\left( \begin{array}{c}
D1(y) \\
D5(y, \{z_i\})
\end{array} \right) & \xrightarrow{S} \left( \begin{array}{c}
F1(y) \\
\text{NS5}(y, \{z_i\})
\end{array} \right) \\
& \xrightarrow{T_y} \left( \begin{array}{c}
P(y) \\
\text{NS5}(y, \{z_i\})
\end{array} \right) \\
& \xrightarrow{S \prod T_{z_i}} \left( \begin{array}{c}
P(y) \\
\text{D1}(y)
\end{array} \right) \\
& \xrightarrow{S} \left( \begin{array}{c}
P(y) \\
\text{F1}(y)
\end{array} \right).
\end{align*}
$$

(3.5)

where $T^*$ stands for T-duality in the "*" direction and $S$ stands for S-duality (section B.3). At the end, we get an FP system whose general supergravity solution is well known [98, 99, 100]

$$
ds^2 = H(x, v) \left( -du dv + K(x, v) dv^2 + 2 A_i(x, v) dy_i dv \right) + dx^2 + dz^2,
$$

(3.6)

$$
b_{uv} = -g_{uv} = \frac{1}{2} H(x, v), \quad b_{vi} = -g_{vi} = -H(x, v) A_i(x, v), \quad e^{-2\phi} = H^{-1}(x, v),
$$

where $b_{\mu\nu}$ is the NS-NS B-field, $g_{\mu\nu}$ is the metric, $v = t - y$ and $u = t + y$ are null coordinates. The functions $H(x, v), K(x, v)$ and $A_i(x, v)$ depend on $F(v)$, the profile of the string in the following way

$$
H^{-1}(x, v) = 1 + \frac{Q}{|x - F(v)|^2}, \quad K(x, v) = \frac{Q |\dot{F}(v)|^2}{|x - F(v)|^2}, \quad A_i(x, v) = -\frac{Q \dot{F_i}(v)}{|x - F(v)|^2},
$$

(3.7)

where the dot in $\dot{F}$ stands for derivative with respect to $v$.

3.2.2 The LM Geometries

To get the most general D1-D5 geometry, we need to undo the chain of dualities described above. One slight complication comes from the explicit dependence of the
profile $F$ on $y$ (through $v$) which is used as a T-duality direction. To apply the rules given in section B.3 we need to smear our string over $v$. In practice, this amounts to integrating the functions $H^{-1}, K$ and $A_i$ over $v$. Going through the reverse of the chain of dualities (3.5) leads us to the following D1-D5 solution generalizing (3.3)

$$ds^2 = \frac{1}{\sqrt{f_1 f_5}} \left[ -(dt + A)^2 + (dy + B)^2 \right] + \sqrt{f_1 f_5} dx^2 + \sqrt{f_1 / f_5} dz^2 ,$$

$$e^{2\Phi} = \frac{f_1}{f_5}, \quad C = \frac{1}{f_1} (dt + A) \wedge (dy + B) + C , \quad (3.8)$$

where:

$$dB = *_4 dA, \quad dC = - *_4 df_5, \quad A = \frac{Q_5}{L} \int_0^L \frac{F'(s) ds}{|x - F(s)|^2} ,$$

$$f_5 = 1 + \frac{Q_5}{L} \int_0^L \frac{ds}{|x - F(s)|^2}, \quad f_1 = 1 + \frac{Q_5}{L} \int_0^L \frac{|F'(s)|^2 ds}{|x - F(s)|^2} . \quad (3.9)$$

where the Hodge star $*_4$ is defined with respect to the flat four-dimensional non-compact space spanned by $\{x^i\}$. We have also switched $v$ to $s$ as it is now just a parameter, and the prime in $F'$ stands for derivative with respect to $s$. These solutions (3.8) are asymptotically $R^{1,4} \times S^1 \times T^4$ and they are parametrized in terms of a closed curve $x_i = F_i(s)$ with $0 \leq s \leq L$. In the sequel, we are going to ignore oscillations in the $T^4$ direction as well as fermionic excitations; for a further discussion of these degrees of freedom see [101, 102]. The D1 (DS) charge $Q_1$ (respectively, $Q_5$) satisfy

$$L = \frac{2\pi Q_5}{R}, \quad Q_1 = \frac{Q_5}{L} \int_0^L |F'(s)|^2 ds . \quad (3.10)$$

The first relation is a result of the identification of the D5 with the F1 through duality. In terms of charges, $Q_5$ corresponds to the winding number of the fundamental string. The second relation seems more involved though its origin is very simple. It is due to the identification of the D1 charge with the momentum running around the fundamental string which is given by $F'(s)$. It turns out that these geometries are smooth if the profile $F$ does not self-intersect and has an everywhere non-vanishing derivative $F'$ [97]. The last comment we want to mention here is that the Killing spinors of these geometries satisfy the same projection (3.4) [93]. The only difference resides in the vielbeins, which means that asymptotically they preserve the same supersymmetries.

### 3.3 The Symplectic Form and Quantization

So far, we have succeeded in constructing a large class of smooth solutions (3.8) that look asymptotically like the naive D1-D5 one (3.3), except that they have a non
zero angular momentum given by:

\[ J_{ij} = \frac{Q_5 R}{L} \int_0^L \left( F_i F_j' - F_j F_i' \right) ds, \]  

(3.11)

where \( R \) is the coordinate radius of the \( S^1 \). It can be checked that \( J \leq N_1 N_5 \) in agreement with an upper bound on possible quantum numbers, dubbed the stringy exclusion principle in [17]. The \( N_1 (N_5) \) stand for the number of D1 (respectively, D5) branes. Their relation with the charges \( Q_1 \) and \( Q_5 \) is given by

\[ N_1 = \frac{g_s V_4}{Q_1}, \quad Q_5 = g_s N_5, \]  

(3.12)

where \( V_4 \) is the coordinate volume of \( T^4 \), and \( g_s \) is the string coupling constant.

We want to quantize this space of solutions following the general scheme described in section 2.3. First, we will sketch how to get the symplectic form [87, 88]. Then, we will construct the associated Hilbert space.

### 3.3.1 The Symplectic Form

In evaluating the symplectic form of the solution space described above, we can either use its form (2.5) and evaluate its restriction directly [87], or, we can take advantage of the symmetries of our solutions to reduce the amount of work that has to be done [88]. We are going to follow the latter as it is simpler and does not need any long calculations except to fix an overall prefactor.

The idea relies on a simple observation regarding the equivalence of the Hamiltonian flows of the total Hamiltonian system \((\mathcal{H}, \Omega)\) on our solution space and the flow of its restriction \((h, \omega)\) on the same solution space. The next input that we are going to use, is that our solutions (3.8) are time independent. We start by writing down the Hamiltonian equations corresponding to the curve \( F \)

\[ \frac{dF_i}{dt} = \{ F_i, h \}, \]  

(3.13)

where \( \{, \} \) is the Poisson bracket that we want to specify using the time independence property. The latter tells us that the only non trivial acceptable equations are of the form

\[ \frac{dF_i}{dt} = \alpha \frac{dF_i}{ds}, \]  

(3.14)

where \( \alpha \) is a constant. This is because \( F \) enters in the solution (3.8) through integration over \( s \) (see equation (3.9)). Next, we need to evaluate the restriction of the
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Hamiltonian to our solution space. This can be done using the expression of energy in the asymptotic flat spacetime. The restricted Hamiltonian turns out to be [88]

\[ h = \frac{R V_4}{g_5^2} \left( Q_5 \int_0^L \left| \mathbf{F}' \right|^2 ds + Q_5 \right) = \frac{R V_4}{g_5^2} (Q_1 + Q_5), \]

(3.15)

where in the last equation we used (3.10). This is in perfect agreement with the fact that our solutions are BPS. Using this explicit expression, it is easy to see that the only possible form of the Poisson bracket which is consistent with equations (3.13) and (3.14) is

\[ \{ F_i(s), F'_j(\tilde{s}) \} = \pi \mu^2 \delta_{ij} \delta(s - \tilde{s}), \]

(3.16)

where \( \mu \) is a constant that cannot be fixed by these general considerations. Looking carefully at some simple examples, [88] showed that \( \mu \) takes the form:

\[ \mu = -\frac{g_s}{R \sqrt{V_4}}. \]

(3.17)

For completeness, we will write down explicitly the symplectic form

\[ \omega = \frac{1}{2\pi \mu^2} \int \delta F'(s) \wedge \delta F(s) \, ds. \]

(3.18)

3.3.2 Quantization

After finding the Poisson bracket (3.16), we can go ahead and construct our Hilbert space. Taking advantage of the periodicity of \( \mathbf{F} \), we can expand it in oscillators as

\[ \mathbf{F}(s) = \mu \sum_{k=1}^{\infty} \frac{1}{\sqrt{2k}} \left( c_k e^{i\frac{2\pi k}{L} s} + c_k^\dagger e^{-i\frac{2\pi k}{L} s} \right), \]

(3.19)

where we have promoted \( \mathbf{F} \) to an operator. One can invert this expression to get \( c_k \) and \( c_k^\dagger \) in terms of \( \mathbf{F} \) using the orthonormality of the different oscillator modes. Now, it is a matter of plugging in these expressions into the commutator that one gets from the Poisson bracket (3.16) through canonical quantization. After things settle down, we get the following non-vanishing commutator between \( c_k \) and \( c_l^\dagger \)

\[ \left[ c_k, \left( c_l^\dagger \right) \right] = \delta^{ij} \delta_{kl}, \]

(3.20)

where \( i, j \) are spacetime indices corresponding to the non-compact spatial part \( x^i \) and \( k, l \) correspond to the level of the oscillator.

Before constructing our Hilbert space, let us pause for a moment and discuss a subtlety that we have ignored until now. The expansion (3.19) is not the most
general one. We have already chosen a gauge where we set the constant term in $F(s)$ to zero. This is because we can absorb such a constant by shifting the origin of the coordinates $x$ which does not change our solution (3.8, 3.9). In a sense, by doing so we decoupled the translation modes of our system. Of course, this is not the only gauge one can choose. The only condition that one should respect is invariance under the shift symmetry $s \to s + \delta s$ (section 4.2.2). As a matter of fact, [103, 104] made another choice based on holographic considerations to treat the decoupled version of our solutions.

Our Hilbert space is not the full Fock space that is comprised of all states that can be built by the action of all possible combinations of the creation operators $c_k^\dagger$ on the vaccum state $|0\rangle$, as the profile $F$ should satisfy the constraint (3.10). The latter, when expressed in terms of the operators $c_k, c_k^\dagger$, becomes

$$ N \equiv N_1 N_5 = \frac{L}{(2\pi)^2} \frac{1}{\mu^2} \int_0^L :|F'(s)|^2 : \, ds = \sum_{k=1}^{\infty} k \langle c_k^\dagger c_k \rangle . \quad (3.21) $$

So our Hilbert space is spanned by

$$ |\psi\rangle = \prod_{i=1}^4 \prod_{k=1}^{\infty} (c_k^\dagger)^{N_{k_i}} |0\rangle, \quad \sum_{k=1}^{\infty} k N_{k_i} = N , \quad (3.22) $$

where $|0\rangle$ is the vacuum state that is annihilated by all the annihilation operators $c_k^\dagger$. 
