Unknitting the black hole: black holes as effective geometries

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CHAPTER 4

SIMPLE ENSEMBLES AND THEIR COARSE GRAINING

In this chapter, we are going to explore some simple ensembles of the D1-D5 states and their effective geometries. In the examples we are going to discuss, the weights of the D1-D5 states in the ensemble under study can be parametrized by giving a density matrix. Since we are looking for an effective geometry description of these ensembles, we will be led to discuss the phase space counterpart of the defining density matrix. This is the so-called “phase space density” [92] (section 4.1). Having that at our disposal allows us to propose a map from quantum states to geometries (section 4.2), opening the possibility to look for the effective geometry description of D1-D5 ensembles.

Since our Hilbert space is constrained (3.21), we should take into account such a constraint in the definition of our density matrix. This is very much like dealing with a microcanonical ensemble. In some cases, we will switch to the canonical ensemble description as it is easier to deal with. For large quantum numbers, we expect that the two descriptions will give the same physics at leading order. For a thorough discussion on this point see section 4.2.2. We start by the ensemble that associates the same weight to all the states. Dealing with its canonical ensemble version allows us to derive its effective geometry description. This turns out to be approximately the D1-D5 naive geometry except in a region around the origin (section 4.3). After that, we will discuss a simple yet an interesting class of thermodynamical ensembles (section 4.4). In this class of ensembles, each oscillator $c_k$ is occupied thermally with a temperature $\beta_k$. It turns out that most of the details of the ensemble disappear leaving behind three quantum numbers with well understood physics (section
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4.4). However, to arrive at this conclusion we overlooked an important physical phenomenon: the Bose-Einstein condensate (section 4.5). As we will see, when the temperatures defining a toy model ensemble are tuned appropriately, a condensation of specific modes can be induced (section 4.5). Finding out the geometric description of such condensates will be the subject of the next-to-last section (section 4.6). In the last section (section 4.7), we will set foot on a path to construct a metric that describes a conical defect with an arbitrary deficit angle whose existence was ruled out based on regularity requirements [97, appendix: C]. We will try to relax such a requirement by looking for an effective description of a specific ensemble of (possibly non-smooth) metrics.

Throughout this chapter familiarity with quantum mechanics and statistical physics is assumed.

4.1 INTERLUDE: PHASE SPACE DENSITIES

As is well known, the uncertainty principle destroys the classical notion of phase space as coordinates and their conjugate momenta cannot be defined at the same time. Since ultimately we want to deal with geometries which are—in our set up—points in a phase space, we would like to have an approximate quantum description of a phase space. It turns out that such a description can be achieved with the help of the “phase space density” (see [92] and references there in). Explaining this concept and its use is the subject of this section.

4.1.1 WHY PHASE SPACE DENSITIES?

Quantum mechanics, as we know it, is a set of abstract rules; “axioms” which gives a recipe for results to be expected from carrying out an experiment. Due to its probabilistic nature, the average of measured quantities are nicely expressed as expectation values of the associated operators. Usually, in such a process, these operators are integrated against probabilities that are expressed in terms of either position coordinates or dual momenta but not both. Such a traditional approach is clearly not applicable when we want to treat the phase space as a whole. This is precisely the case we are dealing with, because the geometries (3.8) are points in a phase space. Another physical situation where such a treatment is very welcome is statistical mechanics. This was the reason behind Wigner’s suggestion to reformulate Schrödinger’s quantum mechanics in a way such that coordinates and momenta are treated on the same footing [105]. This proposal goes under the name of “phase space density” approach.
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Statistical and quantum mechanics share the same probabilistic nature though these probabilities have different origins: the probability in statistical mechanics originates in our ignorance about the full details of our system on the contrary to quantum mechanics whose probabilistic nature is of a fundamental origin. This clearcut difference does not forbid us from migrating some technical tools from statistical to quantum mechanics. Let us follow this line of thoughts and see what we can learn.

A particle (or a statistical system) in the quantum theory is described by giving its density matrix \( \rho \). The result of any measurement can be seen as an expectation value of an appropriate operator which is given explicitly by

\[
\langle O \rangle_\rho = \text{Tr}(\rho \, O).
\]

(4.1)

This is reminiscent of classical statistical mechanics where the measurements are averages of appropriate quantities using some statistical distribution \( w(p,q) \)

\[
\langle O \rangle_w = \int dp \, dq \, w(p,q) \, O(p,q),
\]

(4.2)

where the integration is over the full phase space. One can wonder at this point if it is possible to construct a density \( w(p,q) \) so that one can rewrite equation (4.1) as equation (4.2). The answer is affirmative: for every density matrix \( \rho \) there is an associated phase space distribution \( w_\rho \) such that the following equality holds for all operators \( A \)

\[
\int dp \, dq \, w_\rho(p,q) \, A(p,q) = \text{Tr}(\rho \, A(\hat{p}, \hat{q})).
\]

(4.3)

In defining the phase space density \( w_\rho(p,q) \), there is a subtlety that needs to be addressed. This will be the subject of the next subsection.

4.1.2 Wigner vs Husimi Distribution

Recall that in the quantum theory we have to face the question of operator ordering. This comes about because the operators \( \hat{q} \) and their dual momenta \( \hat{p} \) do not commute with each other. This means that the distribution \( w_\rho \) should somehow include information about the chosen order of \( \hat{p} \) and \( \hat{q} \). As a result, there does not exist a unique phase space distribution. However, for semi-classical states, which by definition are states for which the classical limit is unambiguous, \( w_\rho(p,q) \) should be independent of the choice of ordering prescription in the classical limit as well, so this is not actually much of a problem for these class of states.

Wigner was the first one to introduce a phase space density [105]. His motivation was the study of quantum corrections to classical statistical mechanics. This distribution is commonly known right now as the "Wigner distribution" and is given
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by

\[ w(p, q) \sim \int dy \langle q - y | \rho | q + y \rangle e^{2ipy}, \]  

(4.4)

where \( \rho \) is the quantum density matrix. It turns out that such a distribution corresponds to Weyl ordering [92], given by the prescription:

\[ \hat{O}(\hat{q}, \hat{p}) = \int d\sigma d\tau \alpha(\sigma, \tau) e^{i (\sigma \hat{q} + \tau \hat{p})}, \]  

(4.5)

where the hats in \( \hat{q}, \hat{p} \) stand for operators and \( \alpha(\sigma, \tau) \) is the Fourier transform of the classical quantity \( O(q, p) \).

Wigner distribution suffers from the fact that it is not positive-definite in general. It is also quite sensitive to the physics at a quantum scale [92, 73], as it usually has fluctuations of order \( \hbar \). Another drawback of this distribution is that it is difficult to work with from a computational standpoint.

These issues stimulate us to look for another distribution. We need to have a good requirement so that we can narrow down the possible candidates. Going back to our starting point, “phase space quantization”, supplements us with a possible starting point. Remember that we want to be able to map states to geometries which are points in phase space. This suggests to look for a smooth distribution that can get us close enough to the notion of a point in phase space. A possible candidate is a Gaussian with width \( \hbar \), which is the best we can do according to the uncertainty principle. However, these distributions should not be associated to any quantum state. If this was the case, then, quantum mechanics will be a small deformation of classical mechanics which we know is not the case: Quantum mechanics is fundamentally different from classical mechanics. There is however a special class of states, the so called “semi-classical states”, for which quantum mechanics measurements reduce to the classical ones in the limit \( \hbar \rightarrow 0 \) limit. These are the class of states we wish to associate a Gaussian distribution of width \( \hbar \) to them.

Finding semi-classical states is usually far from trivial, but fortunately our Hilbert space is similar to the one of a Harmonic oscillator. And luckily enough, we know precisely what we mean by a semi-classical state in this case. These are the so called “coherent states” \( |z \rangle \) defined by:

\[ |z\rangle = e^{-|z|^2/2} e^{za^\dagger} |0\rangle; \quad a|z\rangle = z|z\rangle. \]  

(4.6)

Using the properties of such states, a natural guess for the sought after distribution is the projection on the associated coherent state. This is not a \( \delta \)-function as we all know that the coherent states are not orthonormal. Indeed their innerproduct is given by

\[ \langle z | w \rangle = \exp \left[ -\frac{1}{2} |z - w|^2 \right], \]  

(4.7)
which clearly shows that our distribution reduces to a Gaussian centered around a classical solution when evaluated on a semi-classical state. It can be checked that, after re-introducing \( \hbar \) and normalizing the distribution, in the \( \hbar \to 0 \) limit this distribution reduces to a \( \delta \)-function.

Our starting point is then a distribution that, in the case of a Harmonic oscillator, projects a state \( |\psi\rangle \) on coherent states:

\[
Hu_\psi(z) = |\langle \psi | z \rangle |^2.
\] (4.8)

This is the so called “Husimi distribution” [92, 73]. To prove that such a distribution is appropriately normalized, we use the over-completeness relation of coherent states

\[
\int d^2z |z\rangle \langle z| = \mathbb{I},
\] (4.9)

where \( d^2z = -i/(2\pi) dz \wedge d\bar{z} \), which is the convention we are going to use in the remaining of this chapter. It can be shown that the Husimi distribution is always smooth and positive definite [92]. The price to be payed for these nice properties is that the average of classical quantities, using Husimi distribution, corresponds to anti-normal ordered operators [92, 73]. This is a result of the following general identity

\[
\int d^2z Hu_\psi(q,p) z^m \bar{z}^n = \langle \psi | a^m (a^\dagger)^n |\psi\rangle,
\] (4.10)

where we used (4.8), (4.6) and its Hermitian conjugate, and finally (4.9). In the following, we are going to use the Husimi distribution because of its nice properties mentioned above. It turns out also that dealing with such distribution is a lot easier technically than the Wigner distribution.

### 4.2 Mapping States to Geometries

We succeeded above in constructing a dictionary between states in the Hilbert space and distributions in the classical phase space. This will allow us to carry on an averaging process for our LM geometries (3.8, 3.9) starting from our Hilbert space (3.22, 3.19). Describing such a process will be the subject of this section.

#### 4.2.1 From States to Geometries

The first step towards the implementation of coarse graining will be to construct a map from states to solutions through an adaptation of the Husimi distribution to the
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phase space (3.19, 3.10). Given a state, or more generically a density matrix

$$\rho = \sum_i |\psi_i\rangle\langle\psi_i|,$$  \hspace{1cm} (4.11)

we wish to associate to it a density on phase space. The phase space is given by the space of classical curves (section 3.2.2), which we will parametrize as (note that $d$ and $\bar{d}$ are now complex numbers, not operators as in (3.19))

$$F(s) = \mu \sum_{k=1}^{\infty} \frac{1}{\sqrt{2k}} \left( d_k e^{i 2\pi k s} + \bar{d}_k e^{-i 2\pi k s} \right),$$  \hspace{1cm} (4.12)

and which obey the classical constraint (3.10).

We now propose to associate to a density matrix of the form (4.11) a phase space density of the form [106]

$$f(d, \bar{d}) = \sum_i \prod_k e^{-|d_k|^2} \langle 0| e^{d_k c_k} |\psi_i\rangle \langle\psi_i| e^{\bar{d}_k c_k^\dagger} |0\rangle.$$  \hspace{1cm} (4.13)

This is just the Husimi distribution described in the previous section (equation 4.8). Notice that this phase space density, as written, is a function on a somewhat larger phase space as $d, \bar{d}$ do not have to obey (3.10). Let us ignore this issue for the moment and return to it later. As an example, the distribution corresponding to a generic state

$$|\psi\rangle = \prod_{k=1}^{\infty} \frac{1}{\sqrt{N_{k_i}}} \left( c_k^{\dagger} \right)^{N_{k_i}} |0\rangle,$$

can be easily computed to be

$$f(d, \bar{d}) = \prod_{k, \bar{d}} e^{-d_k \bar{d}_{\bar{k}}} \frac{(d_k \bar{d}_{\bar{k}})^{N_{k_i}}}{N_{k_i}!}.$$  \hspace{1cm} (4.14)

The density (4.13) has the property that for any function $g(d, \bar{d})$

$$\int d, d \ f(d, \bar{d}) g(d, \bar{d}) = \sum_i \langle \psi_i | :g(c, c^\dagger):_A |\psi_i\rangle,$$  \hspace{1cm} (4.15)

where $:O:A$ is the anti-normal ordered operator associated to $O$, and $\int d, d$ is an integral over all variables $d_k$. This is just a straightforward generalization of (4.10).

As expected, (4.13) associates to a coherent state a density which is a Gaussian centered around a classical curve (see section 4.1.2), which totally agrees with the usual philosophy that coherent states are the most classical states. It is then clear that given a classical curve (4.12), we wish to associate to it the density matrix

$$\rho = P_N \prod_k e^{-|d_k|^2} \left( e^{d_k c_k} |0\rangle \langle 0| e^{\bar{d}_k c_k^\dagger} \right) P_N,$$  \hspace{1cm} (4.16)
where $P_N$ is the projector onto the actual Hilbert space of states of energy $N$ as defined in (3.22). Because of this projector, the phase space density associated to a classical curve is not exactly a Gaussian centered around the classical curve, but, there are some corrections due to the finite $N$ projections. Obviously, these corrections will vanish in the $N \to \infty$ limit.

Since the harmonic functions appearing in (3.9) can be arbitrarily superposed, we finally propose to associate to (4.11) the geometry defined using the functions

$$
\begin{align*}
    f_5 &= 1 + \frac{Q_5}{L} \int_0^L \int_{d,\bar{d}} \frac{f(d, \bar{d}) ds}{|x - \mathbf{F}(s)|^2}, \\
    f_1 &= 1 + \frac{Q_5}{L} \int_0^L \int_{d,\bar{d}} \frac{f(d, \bar{d}) |\mathbf{F}'(s)|^2 ds}{|x - \mathbf{F}(s)|^2}, \\
    A_i &= \frac{Q_5}{L} \int_0^L \int_{d,\bar{d}} \frac{f(d, \bar{d}) \mathbf{F}'_i(s) ds}{|x - \mathbf{F}(s)|^2}.
\end{align*}
$$

See [107] for a discussion about the validity of such proposal for a generic state.

As we have already mentioned at the end of section 3.2.2, the geometries corresponding to a classical curve are regular provided $|\mathbf{F}'(s)|$ is not vanishing and the curve is not self-intersecting [97]. In our setup, we sum over continuous families of curves which generically smooths the singularities. The price that one pays for this is that the solutions will no longer solve the vacuum type-IIB equations of motion, instead a small source will appear on the right hand side of the equations. We defer a discussion on this point to section 4.2.2 below.

### 4.2.2 Avoiding Red Traffic Lights

We will first take care of some subtleties and inconveniences, that otherwise will make our lives complicated, before going further and discuss the effective geometric description of some simple ensembles.

**Canonical vs Microcanonical Ensemble**

The first subtlety that we need to address is that we wish to study the phase space of curves of fixed length. The phase space of curves of arbitrary length is very easy, it simply equals to that of an infinite set of harmonic oscillators. The length of the curve is measured by some operator $\hat{N}$. The constraint $\hat{N} = N$ is however first class in the language of Dirac, because $[\hat{N}, \hat{N}] = 0$ (or in classical language, the length Poisson commutes with itself). First class constraints generate a gauge invariance. In
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the present case, the operator $\hat{N}$ also generates a gauge invariance, which is simply the shift of the parametrization of the curve,

$$F(s) \rightarrow F(s + \delta s).$$

This follows immediately from the commutation relations of $\hat{N}$ with the oscillators. Therefore, we have two possibilities:

- We can either forget about the length constraint, and include an extra factor $\exp(-\beta \hat{N})$ in the calculations, where we choose $\beta$ such that the expectation value of $\hat{N}$ is precisely $N$. This would be like doing a canonical ensemble, and for many purposes this is probably a very good approximation.

- Or, we can insist on fixing the length taking into account the gauge invariance. Therefore, once we include the length constraint, it is impossible to distinguish curves whose parametrization is shifted by a constant. In particular, the only meaningful quantities to compute are those of gauge invariant operators such as $f_1, f_5$ and $A$.

If we insist on fixing the length of the curve $F$, we also need to improve the map we discussed above a little bit: we need to project the measure (4.13) on loop space onto the submanifold of phase space of curves of fixed length. It is not completely trivial to determine the right measure. To get an idea we will do the simple example of two oscillators.

We consider $\mathbb{C}^2$ with the usual measure. We wish to restrict to the submanifold $N = a_1|z_1|^2 + a_2|z_2|^2$, and we wish to gauge fix the $U(1)$ symmetry that maps $z_k \rightarrow e^{i\alpha_k} z_k$. What is the measure that we should use? In general, if we have a three-manifold with a $U(1)$ action, and we gauge fix this $U(1)$ the measure on the gauge-fixed two-manifold is simply the induced measure as long as the $U(1)$ orbits are normal to the gauge fixed two-manifold. As a result, integrating a gauge-invariant operator over the gauge fixed two-manifold is the same as integrating it over the entire three-manifold, but dividing by the length of the $U(1)$ orbit through each point. Call the length of this orbit at the point $P$, $\ell(P)$. On the three-manifold (given by $N = a_1|z_1|^2 + a_2|z_2|^2$) we have the induced measure $d^4x \delta(f)|df|$, with $|df|$ the norm of the differential $df$, and $f = 0$ is the defining equation of the three-manifold. So, all in all we can write the integral of a gauge invariant quantity $A$ on the two-dimensional submanifold as

$$\int d^4x A(x) \frac{\delta(f)|df|}{\ell(P)}. \quad (4.19)$$

The length of the $U(1)$ orbit is rather tricky, for general $a_1, a_2$ the orbits do not even close. So, we will assume that these numbers are integers. Then, up to an overall
constant that depends only on $a_i$, the length of the orbit is almost everywhere

$$\ell(P) = \sqrt{\sum a_i^2 |z_i|^2}, \quad (4.20)$$

with some pathologies if some of the $z_i$ vanish.

Interestingly enough, we now see that $|df|$ and $\ell(P)$ cancel each other. Thus, the only modification in the measure will be to include an extra delta function of the form

$$\delta(N - \sum_k kd_k \bar{d}_k), \quad (4.21)$$

in phase space density. As long as we integrate gauge invariant quantities, this will yield the right answer. Thus, in (4.13) and in (4.14) we should include the appropriate delta function.

Inserting the delta function is just like passing from a canonical to a microcanonical ensemble. For many purposes, the difference between the two is very small, and not relevant as long as we consider the classical gravitational equations of motion only. We will therefore, in the remainder of this chapter, work predominantly in the canonical picture, commenting on the difference with the (more precise) microcanonical picture when necessary.

**The Ghost of Operator Ordering**

We have already pointed out that our phase space density corresponds to anti-normal ordered operators. As our theory behaves like a 1+1 dimensional field theory, such a prescription will lead to infinities. To get rid of such unwanted behavior, we resort to calculating normal ordered quantities. To further motivate such decision, we note that everything we do is limited by the fact that our analysis is in classical gravity, and therefore, can at best be valid up to quantum corrections.

But, how do we proceed to implement such a modification? One can for example use another distribution that guarantees normal ordering. However, such a distribution suffers from a lot of problems, and is not appropriate for all density matrices [92]. Instead, we are going to adopt the following procedure. We just rewrite our normal ordered operator in terms of anti-normal ordered ones, and use the classical expression of the latter in evaluating the left hand side of (4.15). As an example, let us verify (3.21). The implementation of our prescription for $|F'(s)|^2$ simply amounts to the replacement:

$$d_k^\dagger d_k^\dagger \rightarrow d_k^\dagger d_k^\dagger - 1. \quad (4.22)$$

We will continue to write expressions like $|F'(s)|^2$ in order to not clutter the notation, but always keep in mind that a modification according to our prescription to get rid
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of infinities is implemented. Using (4.22), it is then easy to show that (3.21) is satisfied. Indeed, (4.22) is equivalent to the following condition

\[ Q_1 = \frac{Q_5}{L} \int_0^L \int_{d,d} f(d,d) |F'(s)|^2 ds, \]  

(4.23)

and this is satisfied as a consequence of \( \sum kN_k = N_1 N_5 \). More explicitly

\[ \frac{Q_5}{L} \int_0^L \int_{d,d} f(d,d) |F'(s)|^2 ds = \frac{Q_5}{L} \int_{d,d} f(d,d) \left( \mu^2 \frac{4\pi^2}{L^2} \sum_{k=1}^{\infty} k(d_k d_k^\dagger) - 1 \right) \]

\[ = \mu^2 \frac{4\pi^2}{L^2} Q_5 \left( \sum_k kN_k \right) = Q_1. \]   

(4.24)

To go from the first line to the second we have used the following identity

\[ \int_{d,d} (dd^\dagger)^k e^{-dd^\dagger} = 2 \int_0^\infty dr r^{2k+1} e^{-r^2} = k!. \]   

(4.25)

**SEMI-CLASSICAL VALIDITY**

The last red light we need to take care of is that the average will no longer solve the vacuum type-IIB equations of motion, instead, a small source will appear on the right hand side of the equations. Since these sources are subleading in the \( 1/N \) expansion and vanish in the classical limit, they are in a regime where classical gravity is not valid and they may well be cancelled by higher order contributions to the equations of motion. To have an idea about these sources, let us study the circular profile.

**Classical Treatment** We consider the following profile

\[ F_1(s) = a \cos \frac{2\pi q}{L} s, \quad F_2(s) = a \sin \frac{2\pi q}{L} s, \quad F_3(s) = F_4(s) = 0, \]   

(4.26)

which describes a circular curve winding \( q \) times around the origin in the 12-plane. In order to simplify our discussion, we focus on the simplest harmonic function \( f_5 \). Plugging (4.26) into (3.8), it is straightforward to compute

\[ f_5 = 1 + \frac{Q_5}{\sqrt{(x_1^2 + x_2^2 + x_3^2 + x_4^2 + a^2)^2 - 4a^2(x_1^2 + x_2^2)}}, \]   

(4.27)

where the value of \( a \) is fixed by the condition (3.10) to be

\[ Q_1 = Q_5 \left( \frac{2\pi q}{L} a \right)^2. \]
In order to evaluate the various integrals, it will be convenient to Fourier transform the $x$-dependence. Using

$$\frac{1}{|x|^2} = \frac{1}{4\pi^2} \int d^4 u \frac{e^{iu \cdot x}}{|u|^2},$$

we can write $f_5$ in the following equivalent way

$$f_5^{clas} = 1 + \frac{Q_5}{4\pi^2} \int d^4 u \frac{e^{iu \cdot x}}{|u|^2} J_0(u \sqrt{a^2 + a^2})$$

$$= 1 + J_0(a \sqrt{-\partial_1^2 - \partial_2^2}) \frac{Q_5}{|x|^2}.$$  \hspace{1cm} (4.29)

Writing $f_5$ in this somewhat formal way has the advantage of being more easily compared to the quantum expression obtained below. As explained in appendix C, the other harmonic functions can be obtained from the “generating harmonic function”

$$f_v = Q_5 J_0 \left( a \sqrt{\left( \frac{2\pi q}{L} v_2 + i\partial_1 \right)^2 + \left( \frac{2\pi q}{L} v_1 - i\partial_2 \right)^2} \right) \frac{1}{|x|^2}.$$  \hspace{1cm} (4.30)

For example, putting $v_1 = v_2 = 0$ immediately reproduces (4.29). The geometry can be written in a more familiar form by performing the following change of coordinates

$$x_1 = (r^2 + a^2)^{1/2} \sin \theta \cos \varphi, \quad x_2 = (r^2 + a^2)^{1/2} \sin \theta \sin \varphi, \quad x_3 = r \cos \theta \cos \psi, \quad x_4 = r \cos \theta \sin \psi.$$  \hspace{1cm} (4.31)

In terms of these coordinates, the harmonic functions $f_{1,5}$ become

$$f_5 = 1 + f_v|_{v=0} = 1 + \frac{Q_5}{r^2 + a^2 \cos^2 \theta}, \quad f_1 = 1 - \partial_\psi f_v|_{v=0} = 1 + \frac{Q_1}{r^2 + a^2 \cos^2 \theta}.$$  \hspace{1cm} (4.32)

As a consistency check, we notice that $\Box f_5$ is a delta function with a source at the location of the classical curve, to be precise

$$\Box |x - F(s)|^{-2} = -4\pi^2 \delta^{(4)}(x - F(s)).$$

One indeed finds

$$\Box f_5 = -\frac{Q_5}{4\pi^2 L} \int_0^L ds \int d^4 u e^{iu \cdot (x - F(s))}$$

$$= -\frac{Q_5 4\pi^2}{L} \int_0^L ds \delta(x_1 - a \cos \frac{2\pi q}{L} s) \delta(x_2 - a \sin \frac{2\pi q}{L} s) \delta(x_3) \delta(x_4)$$

$$= -4\pi^2 Q_5 \delta(x_1^2 + x_2^2 - a^2) \delta(x_3) \delta(x_4).$$  \hspace{1cm} (4.33)
Quantum Treatment In a quantum theory, it is impossible to arbitrarily localize wave packets in phase space. Therefore, in the quantum theory we expect to obtain a profile that is something like a minimal uncertainty Gaussian distribution spread around the classical curve. If we take the classical circular curve (4.26), then associate to it the density matrix (4.16), and subsequently the phase space density (4.13), we find out that

$$f(d, \bar{d}) = \left(\frac{d^+ \bar{d}^-}{(N/q)!}\right)^{N/q} e^{-\sum d^+_k \bar{d}^-_k} ,$$  (4.34)

where we used the notation: $$d^\pm_k = \frac{(d^+_1 \pm d^+_2)}{\sqrt{2}}$$. We have ignored the delta function (4.21) here and expect (4.34) to be valid for large values of $N/q$. It is therefore better thought of as a semi-classical profile rather than the full quantum profile.

The distribution above, (4.34), is similar to (C.15) in appendix C. This observation allows us to borrow the results derived there. For example (C.11) reads in this case

$$f_5 = 1 + \frac{Q_5}{4\pi^2} \int d^4u e^{iu \cdot x} \frac{\rho^{2N/q + 1}}{(N/q)!} e^{-\rho^2} J_0 \left( \mu \frac{1}{\sqrt{q}} \sqrt{u^2 + \rho^2} \right) ,$$  (4.35)

where $\rho = |d^+_k|$. The $\rho$ integration is easily done using the identity

$$L_n(x) = \frac{e^x}{n!} \int_0^\infty e^{-t} t^n J_0(2\sqrt{tx}) dt ,$$  (4.36)

with $L_n$ the Laguerre polynomial of order $n$. At the end, we are left with the following expression for $f_5$ (see also (C.18))

$$f^\text{quantum}_5 = 1 + L_{N/q} \left( \frac{a^2 \rho^2}{4N/q} \frac{\partial^2}{\partial^2} \right) \frac{Q_5}{|x|^2} .$$  (4.37)

Notice that, beside the approximation of ignoring the $\delta$ function (4.21) in the distribution, this result is exact in $N/q$. In order to relate both results recall that

$$L_n(x) = \sum_{m=0}^n \frac{(-1)^m n!}{(n-m)! (m!)^2} x^m ,$$

which allows to find the following expansion for large values of $N/q$

$$L_{N/q} \left( \frac{a^2 \rho^2}{4N/q} \right) = J_0(a\rho) - \frac{1}{N/q} a^2 \rho^2 J_2(a\rho) + \ldots .$$  (4.38)

From this, we see explicitly that in the limit $N/q \gg 1$ the quantum geometry coincides with the classical one. More precisely, around asymptotic infinity the harmonic functions can be written as a series expansion in $a^2/r^2$. If we focus on a given term
for some fixed (but arbitrarily large) \( p \), then, the coefficient of such term tends to the classical coefficient as \( \frac{N}{q} \) tends to infinity. Note, however, that for finite \( \frac{N}{q} \) the quantum harmonic function is a finite order polynomial in \( \frac{a^2}{r^2} \) (of degree \( \frac{N}{q} \)) which contains a large number of terms that are singular at the origin (and that will re-sum only in the strict \( \frac{N}{q} \) infinite limit). These divergences at \( r = 0 \) may sound like a disaster, but they are actually unphysical and due to the fact that we ignored the delta function (4.21) in the distribution (4.34). Including the delta function will impose a cutoff on the \( \rho \) integral in (4.35), and since all singular terms are due to the large \( \rho \) behavior of the integrand in (4.35), the cutoff will remove the singularities in \( f_5 \).

From this discussion, it is clear that we can trust our semi-classical computation provided \( \frac{N}{q} \) is large and we do not look at the deep interior of the solution.

As for the case of the classical curve, it is instructive to compute \( \Box f_5 \) for this case

\[
\Box f_5 = -4\pi^2 Q_5 \delta(x_3) \delta(x_4) A(x_1, x_2), \tag{4.39}
\]

\[
A(x_1, x_2) = \int_0^\infty d\rho \rho J_0(\sqrt{x_1^2 + x_2^2} \rho) L_{\frac{N}{q}} \left( \frac{a^2 \rho^2}{4\frac{N}{q}} \right), \tag{4.40}
\]

Until here we have not used any approximation. Using identity (4.36), and approximating \( \exp(\frac{a^2 \rho^2}{4\frac{N}{q}}) \approx 1 \), one obtains

\[
A(x_1, x_2) = e^{-(N/q)(r^2/a^2)} \left( \frac{N/q}{N/q - 1} \right)^{N/q}, \tag{4.41}
\]

with \( r^2 = x_1^2 + x_2^2 \). In the limit \( \frac{N}{q} \to \infty \), \( A(x_1, x_2) \) approaches \( \frac{\delta(r^2/a^2 - 1)}{a^2} \), and the classical and quantum results agree. For large \( \frac{N}{q} \), \( A(x_1, x_2) \) is approximately a Gaussian around \( r^2 \approx a^2 \), and width \( 1/\sqrt{N/q} \). Indeed, using Stirling’s formula

\[
A(x_1, x_2) \approx \frac{\sqrt{N/q}}{\sqrt{2\pi}} e^{-(N/q)(r^2/a^2 - 1)} \frac{(r^2/a^2)^{N/q}}{(N/q - 1)^{N/q}}. \tag{4.42}
\]

So, the quantum geometry corresponds to a solution of the equations of motion in presence of smeared sources. The width of the smeared source goes to zero in the limit \( \frac{N}{q} \to \infty \), as expected.

### 4.3 A First Look at Thermodynamical Ensembles

We managed above to lay out the tools to tackle the question of thermodynamical ensembles effective description. The simplest ensemble would be our full Hilbert space (3.22) equally weighted. This is believed to be describing a massless BTZ black
hole \([106, 40]\) after taking a decoupling limit. We propose that such an ensemble is describing, in the full geometry, the small five dimensional black hole after reducing the six dimensional metric over the \(U(1)\) parametrized by \(y\) in \((3.8)\).

In principle, one should consider a microcanonical ensemble with states of fixed level
\[
\hat{N}|\psi\rangle \equiv \sum_k k c_k^\dagger c_k |\psi\rangle = N|\psi\rangle .
\]
We will, instead, consider a canonical ensemble, since in the large \(N\) limit the difference between the two should vanish. In the following, we are going to ignore the \(i\)-index in some equations where it does not play any role to avoid cluttered equations. The corresponding thermal ensemble is characterized by the following density matrix
\[
\rho = \sum_{N_k, \hat{N}_k} \frac{|N_k\rangle\langle N_k| e^{-\beta \hat{N}} e^{-\beta \hat{N}_k}}{\text{Tr} e^{-\beta \hat{N}}} ,
\]
where \(|N_k\rangle\) is a generic state labelled by collective indices \(N_k\)
\[
|N_k\rangle = \prod_k \frac{1}{\sqrt{N_k!}} (c_k^\dagger)^{N_k} |0\rangle ,
\]
and we have chosen a normalization so that \(\langle N_k|\hat{N}_k\rangle = \delta_{N_k, \hat{N}_k}\). The value of the potential \(\beta\) has to be adjusted such that \(\langle \hat{N} \rangle = N\). It is clear that
\[
\rho = \prod_n \rho_k , \quad \rho_k = (1 - e^{-\beta k}) \sum_{n=0}^\infty e^{-nk \beta} |k, n\rangle \langle k, n| ,
\]
with \(|k, n\rangle = \frac{1}{\sqrt{n!}} (c_k^\dagger)^n |0\rangle\). Then, the full distribution will simply be the product
\[
f(d, d\bar{d}) = \prod_k f^{(k)}_{d_k, d_k\bar{d}_k} ,
\]
with
\[
f^{(k)}_{d_k, d_k\bar{d}_k} = (1 - e^{-k \beta}) e^{-d_k \bar{d}_k} \sum_{n=0}^\infty \frac{e^{-nk \beta}}{n!} (d_k \bar{d}_k)^n = (1 - e^{-k \beta}) \exp \left( - (1 - e^{-k \beta}) d_k \bar{d}_k \right) .
\]
This is a special case of \((C.19)\), where, in this case \(\beta_k^\pm = k/\beta\). In the present case, the generating function \(f_v\) reads \((C.24)\)
\[
f_v = Q_5 e^{-\frac{\pi^2}{4} N^2 |w|^2} \left( 1 - e^{-\frac{\pi^2}{4} \beta^2} \right) .
\]
where,
\[
N = 4 \sum_k k \frac{e^{-\beta k}}{1 - e^{-\beta k}} ; \quad D = 4 \sum_k \frac{1}{k} \frac{e^{-\beta k}}{1 - e^{-\beta k}} .
\]
Using the expression (4.46), it is easy to evaluate the different harmonic functions and one-forms (3.9) that enter in the characterization of our solution (3.8), to be

\[ f_5 = 1 + Q_5 \frac{1 - e^{-\frac{\sqrt{2}\mu}{\pi} x^2}}{x^2}, \quad f_1 = 1 + Q_1 \frac{1 - e^{-\frac{\sqrt{2}\mu}{\pi} x^2}}{x^2}, \quad A_i = 0, \tag{4.48} \]

The only remaining step is to express \( D \) in terms of \( N \), which is itself fixed in this ensemble to be \( N = N_1 N_5 \). In the thermodynamical limit \( \beta \ll 1 \), the expressions (4.47) above become

\[ N \approx \frac{2\pi^2}{3} \frac{1}{\beta^2}, \quad D \approx \frac{2\pi^2}{3} \frac{1}{\beta}, \tag{4.49} \]

which gives \( D \approx \pi \sqrt{2N/3} \).

A final comment is in order. The geometry obtained differs from the naïve D1-D5 geometry (3.3) by an exponentially suppressed correction that renders it smooth at \( x = 0 \). Following [33, 36], we could put a stretched horizon at the point where this exponential factor becomes of order one, so that the metric deviates significantly from the classical D1-D5 one. We could also interpret this radius as the scale where quantum effects start to become important. Thus, using this criterion we find for the radius of the stretched horizon

\[ r_0 \approx \frac{\mu}{\beta^{1/2}}, \tag{4.50} \]

with a corresponding entropy that is different from the one of the mixed state from which the geometry was obtained (\( S = N^{1/2} \)). This does not contradict any known laws of physics, and in addition, we should remember that the notion of stretched horizon depends on the choice of the observer. It is quite likely that for a suitable choice of the observer, the entropy of the stretched horizon agrees with the entropy obtained from the dual CFT. For a further discussion of this point see [106, 108].

### 4.4 The Survival of the No-Hair Theorem

As was already mentioned (see section 2.1.3), the no-hair theorem fails in five dimensions [67]. This is due to a new class of solutions called black rings [5, 6] whose horizon has an \( S^2 \times S^1 \) topology. The violation of the no-hair theorem is due to a local charge called a “dipole charge”, which is not visible at the asymptotic infinity. Such a dipole charge is mysterious as it enters in the generalization of the first law of black hole mechanics to black rings [67, 68]. This suggests that such a black hole hair i.e. dipole charge, should be visible in the definition of any possible ensemble that might describe black rings.

The D1-D5 system gives us a good opportunity to address the possibility of describing a black ring as an effective geometry of an appropriate ensemble. This is because,
as we have already discussed (section 2.1.3), the D1-D5 system allows for a small version of black rings, the so called “small black ring” \cite{65, 70, 71, 72}. We will leave the characterization of the small black ring ensemble to the next section (section 4.5), and the derivation of its effective geometry to section 4.6. In this section, we will deal with a much pressing question, could one put hair on black holes by putting more information in the weights defining the D1-D5 ensemble?

To address this question, let us be completely generic and assume that each oscillator $c_k$ is occupied thermally with a temperature $\beta_k$. The latter is a function of $k$ in general. In case we want such ensemble to be a small deformation of the thermal ensemble, discussed in the previous section, at large occupation number, the temperatures $\beta_k$ will take the form

\begin{equation}
\beta_k = \sum_{n=-1}^{\infty} \beta_n k^{-n} .
\end{equation}

One of the quantum numbers that we want to turn on is the angular momentum, as it can be measured at asymptotic infinity. Since we have restricted ourselves to a non-trivial profile in the four non-compact directions $x^i$, the best we can do is to have rotations in the (12) and (34)-planes. It is easy to see, using the form of the angular momentum (3.11), that we will have such rotations once we have different temperatures $(\beta_a)_{\pm k}$ for the oscillator $(c^a)_{\pm k}$; $a = 1, 2$ defined as:

\begin{equation}
(c^1)_{\pm k} = \frac{1}{\sqrt{2}} (c^1_k \pm ic^2_k), \quad (c^2)_{\pm k} = \frac{1}{\sqrt{2}} (c^3_k \pm ic^4_k).
\end{equation}

A further simplification that we are going to adopt in the following is to set the two temperatures $(\beta_a)_{\pm k}$; $a = 1, 2$ equal. This will allow us to get analytical expressions for the averages (4.17). In the following, we are going to suppress the superscript $a$ as it does not play any role.

We are led then to consider a distribution that is the product of

\begin{equation}
f_k(d, \bar{d}) = (1 - e^{-\beta_k^+}) (1 - e^{-\beta_k^-}) \exp \left(-((1 - e^{-\beta_k^+})|d_k^+|^2 - (1 - e^{-\beta_k^-})|d_k^-|^2\right). \end{equation}

Using (C.20, C.4), such a distribution leads to

\begin{align}
f_5 &= Q_5 \frac{1 - e^{-2|\bar{d}|^2/\mu^2D}}{|x|^2}, \\
f_1 &= Q_1 \left(1 - e^{-2|\bar{d}|^2/\mu^2D} - \frac{J^2}{4N\mu^4D} e^{-2|\bar{d}|^2/\mu^2D}\right), \\
A &= \frac{\mu^2JR}{2} \left(2 \frac{e^{-2|\bar{d}|^2/\mu^2D}}{\mu^2D} - \frac{1 - e^{-2|\bar{d}|^2/\mu^2D}}{|x|^2}\right) \left(\cos^2 \theta d\phi + \sin^2 \theta d\psi\right),
\end{align}
where \((|x|, \theta, \phi, \psi)\) are spherical coordinates for \(\mathbb{R}^4\), in terms of which, the Euclidean metric of \(\mathbb{R}^4\) reads
\[
ds^2 = dr^2 + r^2(d\theta^2 + \cos^2 \theta d\phi^2 + \sin^2 \theta d\psi^2).
\]
We see that, rather surprisingly, the geometry depends only on few quantum numbers \(J, D\) explicitly and \(N\) through its relation with \(Q\), i.e., \(N = N_1 N_5\). In terms of the temperatures, these quantum numbers are given by the relations (C.21, C.22, C.23) which we re-quote here:

\[
N = 2 \sum_k \left( \frac{e^{-\beta_k^+}}{1 - e^{-\beta_k^+}} + \frac{e^{-\beta_k^-}}{1 - e^{-\beta_k^-}} \right), \tag{4.56}
\]

\[
J = 2 \sum_k \left( \frac{e^{-\beta_k^+}}{1 - e^{-\beta_k^+}} - \frac{e^{-\beta_k^-}}{1 - e^{-\beta_k^-}} \right), \tag{4.57}
\]

\[
D = 2 \sum_k \frac{1}{\beta_k} \left( \frac{e^{-\beta_k^+}}{1 - e^{-\beta_k^+}} + \frac{e^{-\beta_k^-}}{1 - e^{-\beta_k^-}} \right). \tag{4.58}
\]

As a result, the information carried by the geometry is much less than that carried by the ensemble of microstates. In fact, only \(N\) and \(J\) are visible at infinity while \(D\) sets the size of the “core” of the geometry. We interpret this as a manifestation of the no-hair theorem for black holes.

The quantum number \(D\) defined above has similar properties as the dipole charge. Surprisingly enough, it is the same one proposed in [66] to describe the CFT dual of the “small” black ring dipole charge. This suggests that any general enough density matrix will describe a small black ring. Unfortunately, a quick look at the geometry reveals the failure of such proposal. At this point, we should be very careful before completely dismissing such a proposal because we have neglected—in the discussion above—a very interesting physical phenomenon: the Bose-Einstein condensation. As we will show in the next section, such a phenomenon does occur in our class of thermodynamical ensembles discussed in this section. We will leave the geometric interpretation of such condensate to section 4.6.

### 4.5 Thermal Ensembles and Condensation

Quantum mechanics being the ruler of the microscopic world does not mean that its footprints cannot be seen at the macroscopic level. A famous example of such imprint is superfluidity and superconductivity. In the heart of these phenomena lies the Bose-Einstein condensate. This happens because bosons tend to occupy the same state with increasing probability as their number increases. In the extreme case, it could happen that macroscopically many bosons will occupy the same ground state. This is exactly the notion of Bose-Einstein condensate. The signal of such a
condensate is the blowing up of the occupation number of the associated mode in our ensemble.

In this section, we will try to study the possibility of the occurrence of a Bose-Einstein condensate in the class of ensembles discussed previously. Ultimately, we are looking for an ensemble whose effective description is a small black ring. Since the latter is characterized by three quantum numbers: mass, angular momentum and dipole charge, we will study the simplest toy model that can accommodate for three independent quantum numbers. We already know the part that will reproduce the mass and angular momentum. For the dipole charge, one can use the quantum number $D$ (4.58), found in the previous section, as a first guess. After all, this is the only extra quantum number that appears in the general case, and it does share the same properties as the dipole charge. In the following, we are going to forget for a moment about our Hilbert space and possible geometric interpretation of the results to be derived here. Such an interpretation will be the subject of the next section.

The partition function we want to study is given by

$$Z = \text{Tr}_{\mathcal{H}}(e^{-\beta H + \mu J + \nu D}) ,$$  \hfill (4.59)

where $\mu$, $\nu \sim \beta$. The Hilbert space $\mathcal{H}$ consists of a Fock space built out of $(d+2)$ free oscillators $\alpha_{-n}^\pm$ and $\alpha_{-n}^i$, $i = 1, \ldots, d$, carrying the following charges [66]

$$[H, \alpha_n^\pm] = n \alpha_n^\pm , \quad [H, \alpha_n^i] = n \alpha_n^i , \quad [J, \alpha_n^\pm] = \pm \alpha_n^\pm , \quad [J, \alpha_n^i] = 0 , \quad [D, \alpha_n^+] = \frac{1}{n} \alpha_n^- .$$ \hfill (4.60)

The charge of the other oscillators with respect to $D$ will not be relevant for the discussion below, but will be relevant for the subleading behavior of the entropy. The definition of the operator $D$ is chosen to mimic the expression of the quantum number $D$ in (4.58).

Let us focus on the $\alpha^+$ oscillator, its contribution to the partition function is

$$\log Z = -\sum_{n=1}^{\infty} \log (1 - e^{-n\beta + \mu + \nu/n}) = \sum_{n=1}^{\infty} C_n ,$$ \hfill (4.62)

where $C_n$ can be rewritten as

$$C_n = \sum_{l=1}^{\infty} \frac{e^{-n l \beta}}{l} \left( \sum_{j,k=0}^{\infty} \frac{(\mu l)^j (\nu/n)^k}{j! k!} \right) = \sum_{j,k=0}^{\infty} \frac{\mu^j \nu^k}{j! k! n^k} L_{n-1-j-k}(e^{-\beta n}) .$$ \hfill (4.63)

After changing variables $k + j = s$, and summing over $0 \leq j \leq s$, we get

$$C_n = \sum_{s=0}^{\infty} L_{n-s}(e^{-\beta n}) \frac{(\nu + n \mu)^s}{n^s s!} .$$ \hfill (4.64)
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Up to this point, the above computation is exact. In order to proceed, we approximate, in the limit $\beta \ll 1$, the polylogarithm $Li_{1-s}$ for $s \geq 1$ by

$$Li_{1-s}(e^{-\beta n}) \approx \frac{(s-1)!}{\beta^s n^s}.$$  \hspace{1cm} (4.65)

Then,

$$\bar{c}_n = \sum_{s=1}^{\infty} Li_{1-s}(e^{-\beta n}) \frac{(\nu + n \mu)^s}{n^s s!} \approx \log \left(1 - \frac{\mu}{n \beta} - \frac{\nu}{n^2 \beta}\right).$$ \hspace{1cm} (4.66)

The contribution from $s = 0$ can be taken care of separately, and the sum over $n$ can easily be performed and it gives the usual term depending only on $\beta$. Taking into account all the oscillators, we get

$$\log Z \approx \frac{(d+2)\pi^2}{6 \beta} - \sum_{n=1}^{\infty} \log \left(1 - \frac{\mu}{n \beta} - \frac{\nu}{n^2 \beta}\right).$$ \hspace{1cm} (4.67)

The first term here is obtained by summing over all “d+2” oscillators, but the second term is due only to $\alpha^+$. There are similar $\mu, \nu$-dependent terms for the other oscillators as well, but the reason for not including their contribution will become clear momentarily. Computing the level $N$, the average angular momenta $J$, and the average dipole charge $D$ from (4.67), we get

$$N = -\frac{\partial \log Z}{\partial \beta} = \frac{1}{\beta} \left(\frac{(d+2)\pi^2}{6 \beta} + \mu J + \nu D\right),$$ \hspace{1cm} (4.68)

$$J = \frac{\partial \log Z}{\partial \mu} = \sum_{n=1}^{\infty} \frac{n}{n^2 \beta - n \mu - \nu},$$ \hspace{1cm} (4.69)

$$D = \frac{\partial \log Z}{\partial \nu} = \sum_{n=1}^{\infty} \frac{1}{n^2 \beta - n \mu - \nu}.$$ \hspace{1cm} (4.70)

The expression for $J$ appears to diverge, but that is due to the approximation that we made. If we include the contribution from $\alpha^- n$, which is similar to that of $\alpha^+$ in (4.67) except that $\mu$ is replaced by $-\mu$, the expression for $J$ will be convergent. This $\alpha^-$ contribution will not be relevant for most of what follows though. The expressions for $J, D$ are at first sight of order $\sqrt{N} \sim \beta^{-1}$. To see this we need to include the contribution from $\alpha^-$ in $J$. In order for $J, D$ to be of order $N$, one term in the sum must be very large; if this happens for the term with $n = q$ then in order to have $J, D \sim N$ we need that

$$q^2 - q \mu' - \nu' \sim \beta \ll 1,$$ \hspace{1cm} (4.71)

where $\mu' = \mu/\beta$ and $\nu' = \nu/\beta$. Notice that, this will imply condensation of modes with $n = q_3$. Indeed

$$<0|\alpha^+_n \alpha^-_{n'}|0> = \frac{e^{\beta(-n+\mu'+\nu'/n)}}{1 - e^{\beta(-n+\mu'+\nu'/n)}},$$ \hspace{1cm} (4.72)
which has a pole at \( n = q \) for \( q^2 - q \mu' - \nu' = 0 \). Obviously, the combination 
\[ n^2 - n \mu' - \nu' \]
has to be greater than 0 for all \( n \), otherwise the thermodynamical system is ill-defined. If we also require that this quantity has a minimum obeying (4.71) at \( n = q \), we find 
\[ \mu' \approx 2q, \quad \nu' \approx -q^2. \]  
(4.73)

With these values of \( \mu', \nu' \), the term with \( n = q \) will dominate the sum that appears in the partition function in (4.67). If we keep only this term together with the other contribution \( \pi^2/\beta \), we can compute the entropy and find 
\[ S = \beta(N - \mu' J - \nu'D) + \log Z \sim \frac{1}{\beta} \sim \sqrt{N - \mu' J - \nu'D} = \sqrt{N - q J}. \]  
(4.74)

This scales exactly like the small black ring entropy for a general dipole charge \( q \) [66]!

What we have here is similar to the Bose-Einstein condensate with a slight difference. Instead of a macroscopically large number of bosons occupying the ground state, as is the case in the Bose-Einstein condensate, the condensate state in our case can be chosen to be any excited state, provided we tune the temperatures \( \beta, \mu \) and \( \nu \) appropriately as explained around equation (4.71).

## 4.6 The "Small" Black Ring

We argued successfully in the previous section that, one should be careful when dealing with general thermodynamical ensembles of the kind discussed in section 4.4, as condensates of certain modes may appear. The aim of this section is to shed some light on the possible geometric manifestation of such condensates. First, we are going to describe the kind of density matrix that can describe a thermodynamical ensemble with a condensate. Then, we are going to turn on our machinery, developed so far, to translate this density matrix to a geometry.

### 4.6.1 Describing the "Condensate" Ensemble

When a condensate occurs, a part of all possible degrees of freedom freezes in the condensate state leaving a reduced thermal ensemble. To be concrete let us treat the situation described in the previous section where \( J \) oscillators \( a_{-q}^+ \) have condensed leaving a thermal ensemble of effective level \( N - q J \). It is easy to see that, the density matrix associated with such a system will be the thermal one (4.43) in the
excited state $|q, J\rangle$. Explicitly:

$$\rho = \sum_{N_k, \bar{N}_k} |N_k\rangle\langle N_k| e^{-\beta \hat{N}} |\tilde{N}_k\rangle\langle \tilde{N}_k| \otimes |q^+, J\rangle\langle q^+, J| , \quad (4.75)$$

where the prime in $\sum'$ means that the sum does not include states coming from the oscillator $a^+_{\pm q}$, and we use the notation $|N, k\rangle$ to denote the state $|N, k\rangle = \frac{1}{\sqrt{N!}} (a^+ - k) N|0\rangle$. In the following, we use the index $q$ to denote quantities related to the special oscillator $a^+_{\pm q}$ to avoid messy formulas. A careful look at the density matrix above reveals that it is just a tensor product between the thermal one (section 4.3) and the one associated with the circular profile (section 4.2.2). This inevitably leads to a phase space density that is the product of the ones associated to each component. The only task that we are left with here is to combine the two calculations. For example, the phase space density is a combination of (4.34) and (4.45). It reads

$$\rho = e^{-|d_q|^2} \frac{|d_q|^{2J}}{J!} \prod_{k} (1 - e^{-k\beta}) \exp \left[-(1 - e^{-k\beta}) |d_k|^2 \right] . \quad (4.76)$$

4.6.2 THE SMALL BLACK RING EFFECTIVE GEOMETRY

It is time now to discuss the effective geometry description of the density matrix (4.75), given above, using our general rule (4.17). It is enough to evaluate the generating function $f_v$ given by (C.1), which turns out to be:

$$f_v = Q_3 L J \left( \frac{\mu^2}{4q} \left[ \left( \frac{2\pi q}{L} \sigma_2 + i \partial_1 \right)^2 + \left( \frac{2\pi q}{L} \sigma_1 - i \partial_2 \right)^2 \right] \right) e^{-\frac{2|x|^2}{\pi^2 D^2}} (N - qJ) 1 - e^{-\frac{2|x|^2}{\pi^2 D^2}} , \quad (4.77)$$

where $D = \pi \sqrt{2/3(N - qJ)^{1/2}}$, which indicates that the geometry is purely expressed in terms of the macroscopic quantities $N, J$ and $q$. The form (4.77) that $f_v$ takes above is easily understood as follows. The condensate behaves as the circular profile (section 4.2.2) so the integral over $d_q, d_{\bar{q}}$ can be evaluated in a completely analogous way giving rise to the Laguerre polynomial above. The extra terms look the same as the thermal contribution (4.46). The reason we have $N - qJ$ above instead of $N$ is due to the restricted level of our thermal part as a result of condensation.

We would like to make contact between this geometry and the geometry corresponding to small black rings studied in [66]. As we will see, in the limit of large quantum numbers both geometries reproduce the same asymptotics.

In order to see this, first note that the exponential factor $e^{-\frac{2|x|^2}{\pi^2 D^2}}$ will not contribute (as it vanishes faster than any power at asymptotic infinity). Secondly, one has the
formal expansion

\[ L_J \left( \frac{\mu^2}{4q} O \right) = J_0(\mu \sqrt{\frac{J}{q}} O^{1/2}) + ... \] (4.78)

In order to estimate the validity of this approximation, we can think of \( O \) as being proportional to \( 1/|x|^2 \). On the other hand, \( \mu \sqrt{J/q} \) can be roughly interpreted as the radius of the black ring (see [6, 66], where this parameter is called \( R \)). Hence, this approximation is valid for large values of \( J \) at a fixed distance compared to the radius of the ring.

Using the above approximations, and the change of coordinates (4.31), it is straightforward to compute the harmonic functions

\[ f_5 = 1 + \frac{Q_5}{r^2 + \mu^2 \frac{J}{q} \cos \theta}, \quad f_1 = 1 + \frac{Q_1}{r^2 + \mu^2 \frac{J}{q} \cos \theta}. \] (4.79)

This result could have been guessed based on the observation that, up to the exponentially suppressed terms, which we got rid of because we are mainly interested in the asymptotics, the generating function (4.77) is similar to the one of the circular profile (C.18). In this situation, and for large quantum numbers, the “quantum” geometry reduces to the classical one as argued in section (4.2.2). This means that under our assumptions, the effective geometry of the condensate ensemble should be similar to (4.32) which is the case. Hence, in this approximation the geometry reduces exactly to that of the small black ring studied in [66].

Let us summarize the key points that the ensemble characterized by the density matrix (4.75) share with an ensemble that could describe a small black ring. First of all, the statistical entropy of the ensemble (4.74) is the same as the entropy of the small black ring with dipole charge \( q \) and angular momentum \( J \) [66]. The second important property is that its effective geometry description is the same as the naive geometry of a small black ring at large distances. Based on such key points, one is confident to declare that the density matrix (4.75) is the right thermodynamical description of a small black ring with dipole charge \( q \) and angular momentum \( J \).

### 4.6.3 Avoiding the No-Hair Theorem

We have seen that the existence of a condensate changes drastically the thermodynamical ensemble, and hence, its corresponding effective geometry. One would like to know what will happen to generic ensembles and the associated no-hair theorem discussed in section 4.4. One would expect that by tuning the temperatures, it will be possible to condense one (like in the small black ring case) or more oscillators. If this happens, we should perform a more elaborate analysis than what have
been done previously in section 4.4. Naively, one would guess that the generating function now will take the form of multiple Laguerre polynomials with differential operator arguments acting on the generating function of the naive D1-D5 thermal ensemble (4.46).

From the geometry point of view, we expect that the effective geometrical description to correspond to concentric small black rings. In this case the configuration will depend on more quantum numbers than just $N, J, D$, in particular we will find solutions where the small black rings carry arbitrary dipole charges. Thus, once we try to put hair on the small black hole by tuning chemical potentials appropriately, we instead find a phase transition to a configuration of concentric small black rings, each of which still is characterized by just few quantum numbers.

### 4.7 The Conical Defect Metric

We have already seen that coarse graining over simple thermodynamical ensembles gave rise to effective geometries that look like known geometries far away from the origin. The aim of this section is to shed some light on the claim of [97] appendix C, where it is shown that there is no conical defect metric with arbitrary opening angles. Our aim here is to answer the following question: “is there a phase space density of D1-D5 geometries that gives as an effective description a conical defect metric with arbitrary opening angles after coarse graining?”

In this section, a decoupling limit is assumed e.g. [93], which amounts in practice to deleting the “1” from the definition of $f_i (3.3, 3.9)$. The end result is a geometry which is asymptotically $AdS_3 \times S^3$. The starting point is the supersymmetric conical metric $[109, 110]$

$$ds^2 = \left( -r^2 + \gamma^2 \right) \frac{dt^2}{R^2} + \gamma^2 \frac{dy^2}{R^2} + \frac{dr^2}{r^2 + \gamma^2} + d\theta^2 + \cos^2 \theta (d\psi + \gamma dy) \frac{2}{R} + \sin^2 \theta (d\phi + \gamma dt) \frac{2}{R},$$

(4.80)

where $N$ is the AdS radius, and $2\pi \gamma$ is the opening angle. It is well known that every supersymmetric conical metric is defined by its angular momentum and $N$. The metric (4.80) is precisely identical to the metric that we would have found in the near-horizon limit in section 4.2.2, if we would also have computed the one-forms $A, B$ and evaluated (3.8), see e.g. [97] for a detailed discussion. The relation between $\gamma$ and $q$ works out to be $\gamma = 1/q$. The construction in section 4.2.2 therefore provides a conical defect metrics with $q$ integer, but for $q$ non-integer the construction in section 4.2.2 fails. The reason is that the classical curve $F(s)$ needs to satisfy $\int_0^1 F(s) ds = 0$, as $F(s)$ does not have a zero-mode, and this is only true if $q$ is an integer and the curve closes.
In order to try to construct a more general conical defect metric, we first notice that according to the $\delta$-function in (4.33), the source for the metric has to be contained in a circle of radius $a$ in the $x_1, x_2$-plane. The most general source term satisfying these requirements is

$$
F_1(s) = a \cos[f(s)], \quad F_2(s) = a \sin[f(s)], \quad F_3(s) = F_4(s) = 0 ,
$$

(4.81)

where $f(s)$ is some arbitrary function which has to satisfy

$$
\int_0^L e^{if(s)} ds = 0 ,
$$

(4.82)

because $F(s)$ does not contain a zero-mode. In addition, the metric (4.80) is invariant under rotations in the $x_1, x_2$-plane. To accomplish this, we need to coarse grain over all $U(1)$ rotations of (4.81). This is most easily done by introducing polar coordinates $x_1 + ix_2 = ue^{i\varphi}, x_3 + ix_4 = ve^{i\psi}$, so that the $U(1)$ average can be expressed as

$$
\begin{align*}
 f_5 &= \frac{Q_5}{2\pi L} \int_0^{2\pi} d\xi \int_0^L \frac{ds}{|ue^{i\varphi} - ae^{if(s)+i\xi}|^2 + v^2} , \\
 f_1 &= a^2 \frac{Q_5}{2\pi L} \int_0^{2\pi} d\xi \int_0^L \frac{f'(s)^2 ds}{|ae^{if(s)}|2 + v^2} , \\
 A &= -a \frac{Q_5}{2\pi L} \int_0^{2\pi} d\xi \int_0^L \frac{i f'(s)e^{if(s)+i\xi} ds}{|ae^{if(s)}+i\xi|^2 + v^2} .
\end{align*}
$$

(4.83)

The constraint (3.10) on the curve now reads

$$
Q_1 = a^2 \frac{Q_5}{2\pi L} \int_0^{2\pi} d\xi \int_0^L f'(s)^2 ds = \frac{a^2 Q_5}{L} < f'^2 > .
$$

(4.84)

Here and in the following by $< g(s) >$ we simply mean

$$
< g(s) > = \int_0^L g(s) ds .
$$

(4.85)

It is straightforward to evaluate the integrals in (4.83) to get

$$
\begin{align*}
 f_5 &= \frac{Q_5}{h} , \\
 f_1 &= \frac{Q_1}{h} , \\
 A &= a Q_5 \left( \frac{< f' >}{L} \right) \left( \frac{u^2 + v^2 + a^2 - h}{2h} \right) d\varphi ,
\end{align*}
$$

(4.86)

with $h^2 = (u^2 + v^2 + a^2)^2 - 4a^2 u^2$. In order to put it in a form which resembles the conical defect one as much as possible, one has to make the following change of coordinates (4.31)

$$
\begin{align*}
 u^2 &= (r^2 + a^2) \sin^2 \theta , \\
 v &= r \cos \theta .
\end{align*}
$$

(4.87)
Using these new coordinates, the various functions appearing in (3.8) become

\[ f_5 = \frac{Q_5}{r^2 + a^2 \cos^2 \theta}, \quad f_1 = \frac{Q_1}{r^2 + a^2 \cos^2 \theta}, \quad C = -\frac{Q_3 r^2 \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} \, d\psi \wedge d\phi, \]

(4.88)

\[ A = \alpha \frac{a\sqrt{Q_1 Q_5}}{r^2 + a^2 \cos^2 \theta} \sin^2 \theta \, d\varphi, \quad B = -\alpha \frac{a\sqrt{Q_1 Q_5}}{r^2 + a^2 \cos^2 \theta} \cos^2 \theta \, d\psi, \]

(4.89)

where

\[ \alpha^2 = a^2 \frac{Q_5}{Q_1} \left( \frac{<f'>}{L} \right)^2 = \frac{1}{L} \left( \frac{<f'>^2}{<f'렇^2>} \right), \]

is a constant introduced for later convenience. Plugging these values into the expression of the metric (3.8) gives

\[ ds_4^2 = (r^2 + a^2 \cos^2 \theta) \left( \frac{dr^2}{r^2 + a^2} + d\theta^2 \right) + r^2 \cos^2 \theta \, d\psi^2 + (r^2 + a^2) \sin^2 \theta \, d\varphi^2. \]

(4.90)

Next, we rescale \( r \) by a factor of \( \frac{\sqrt{Q_1 Q_5}}{R} \), and define \( \gamma = \alpha \frac{2\pi}{L} \), and after some straightforward algebraic manipulations, we end up with

\[ \frac{ds^2}{\sqrt{Q_1 Q_5}} = -(r^2 + \gamma^2) \left( \frac{dt}{R} \right)^2 + r^2 \left( \frac{dy}{R} \right)^2 + \frac{dr^2}{r^2 + \gamma^2} \]

\[ + \left( d\theta^2 + \sin^2 \theta (d\varphi - \alpha \frac{dt}{R})^2 + \cos^2 \theta (d\psi - \alpha \frac{dy}{R})^2 \right) \]

\[ + \frac{(1 - \alpha^2) \gamma^2}{r^2 + \gamma^2 \cos^2 \theta} \left( \sin^2 \theta \, d\Sigma_1^2 + \cos^2 \theta \, d\Sigma_2^2 \right), \]

\[ C \frac{Q_5}{Q_1} = -\alpha \gamma \left( \cos^2 \theta \, \left( \frac{dt}{R} \wedge d\psi + \sin^2 \theta \frac{dy}{R} \wedge d\varphi \right) \right), \]

(4.91)

where we defined

\[ d\Sigma_1^2 = \sin^2 \theta \, d\varphi^2 + (r^2 + \gamma^2 \cos^2 \theta) \left( \frac{dt}{R} \right)^2, \]

\[ d\Sigma_2^2 = -\cos^2 \theta \, d\psi^2 + (r^2 + \gamma^2 \cos^2 \theta) \left( \frac{dy}{R} \right)^2. \]

This metric is a conical defect metric for \( \alpha = 1 \). So, the question is which values of \( \gamma \) are compatible with \( \alpha = 1 \). To analyze this, we recast the constraints on \( f(s) \) for \( \alpha = 1 \) here

\[ \int_0^L e^{f(s)} \, ds = 0, \quad \left( \int_0^L f'(s) \, ds \right)^2 = L \int_0^L (f'(s))^2 = \left( \frac{2\pi}{\gamma} \right)^2. \]

(4.92)
However, according to Schwarz's inequality,

\[
\left( \int_0^L f'(s) ds \right)^2 \leq L \int_0^L (f'(s))^2 ,
\]

(4.93)

for integrable functions \( f'(s) \) with equality if and only if \( f'(s) \) is a constant. Thus, \( \alpha \leq 1 \), and \( \alpha = 1 \) only if \( f'(s) = \text{const} \). Interestingly, the metric (4.91) is in general a perfectly acceptable metric, since \( \alpha \leq 1 \) is precisely the condition for the absence of CTC’s as one can derive using the results in [6]. If \( \alpha = 1 \) then \( f'(s) = \text{const} \) together with (4.92) imply that \( f(s) = 2\pi ks/L \), for some nonzero integer \( k \), and \( \gamma = 1/k \). We can therefore indeed only construct conical defect metrics with \( \gamma = 1/k \), where \( k \) is an integer. For \( k \) noninteger, we find a bound on \( \alpha \)

\[
\alpha^2 \leq \left( \frac{1}{\gamma} \right)^2 \gamma^2 ,
\]

(4.94)

with \( [x] \) the largest integer less than or equal to \( x \). Indeed, we cannot come arbitrarily close to a non-integer conical defect metric in this way.

Such a negative result raises the following puzzle. Even though conical defects with any opening angle are treated on the same footing in gravity, “quantum” gravity seems to restrict the possible opening angles to a specific class, \( \theta = 2\pi/n \), where \( n \) is an integer. It is not clear at all why such a distinction occurs. Is it some non-trivial effect of quantization? or can there be a gravity mechanism that will select such a class of conical defects? An answer to this question will inevitably shed more light on the status of geometries in quantum gravity.