Unknitting the black hole: black holes as effective geometries
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Driven by our search for black holes that are close enough to realistic ones, but still under enough control, we are led to study $\mathcal{N} = 2$ four-dimensional supergravity \cite{119}. The latter turns out to be describing type-II string theories compactified on Calabi-Yau threefolds. Although, smooth solutions can exist only when we uplift to $\mathcal{N} = 1$ five-dimensional supergravity \cite{120}, these solutions can be traced back to four dimensional multi-center solutions \cite{121, 122, 123} through the 4d-5d connection \cite{62, 63}. This connection is a result of the equivalence of type-IIA on a Calabi-Yau and M-theory on the same Calabi-Yau $\times S^1$. As a result, when uplifting four-dimensional solutions to five dimensions, the resulting solutions have a $U(1)$ isometry. Since all the known five-dimensional smooth solutions have a $U(1)$ isometry, moving up or down in dimensions does not have any effect on the number of states. This allows us to restrict our attention to the four-dimensional solutions \cite{124}. The basic example of a four-dimensional solution that becomes smooth when it is uplifted to five dimensions is the solution describing a D6-brane. The four-dimensional solution is singular, whereas the five dimensional uplift is a smooth solution known as the “Taub-Nut geometry”.

This chapter is a short summary of what is known about multi-center black hole solutions to the $\mathcal{N} = 2$ four-dimensional supergravity theory. As usual, we will be very brief inviting the interested reader to check the literature. See for example \cite{119, 125, 96, 22, 27, 30, 126} and references therein. See also the references mentioned throughout this chapter.
We start by reviewing the construction of the four dimensional action. Then, we discuss the simplest BPS black hole solution i.e. the static spherically symmetric one. In the process, we uncover a famous behavior of the scalar fields which is known as the “attractor mechanism” [127]. It turns out that there are other BPS solutions that are essentially a bound state of many black holes [121, 122, 123]. Such solutions do not exist always and may disappear when we cross co-dimension one hypersurfaces in the moduli space of scalar fields. These hypersurfaces are called “walls of marginal stability”. The final section will deal with characterizing the possibility of the disappearance of multi-center black holes by a generalization of the attractor mechanism to these new solutions [122]. Along the way, we will be able to count the number of states that have disappeared using the wall crossing formula [30].

Familiarity with string theory, differential geometry and compactification is assumed. Some concepts about these subjects are summarized in appendices B and D.

5.1 FROM TEN TO FOUR DIMENSIONS

To get the four dimensional action, one starts with the ten dimensional one (B.1), then reduces over a Calabi-Yau threefold. The derivation of the massless field content of our four dimensional theory is carried out in appendix D, section D.3.2. A further simplification that we are going to take advantage of, is that by restricting ourselves to the two derivative effective action, supersymmetry restricts the allowed interactions between hypermultiplets and vectormultiplets to gravitational ones [119]. Since the hypermultiplets enter in the action through their derivative, we can put them to constants and decouple them. As a consequence, the black hole solutions we will derive are characterized by the vectormultiplets. For example, the dilaton will be a constant throughout this part of the thesis as it belongs to the universal hypermultiplet. For a discussion of the inclusion of hypermultiplets in the study of black holes see [128]. A quick look at the four-dimensional massless fields that one gets after reducing over a Calabi-Yau threefold (section D.3.2) reveals that, all we need to know about the Calabi-Yau is its even-cohomology and its complexified Kähler form.

5.1.1 WALKING THE PATH OF REDUCTION

In this section we will go through the main steps in deriving the bosonic part of the four-dimensional action. As explained in appendix D section D.3.2, the vectormultiplets are in one to one correspondence with the basis elements of $H^{(0,0)}$ and
The complexified Kähler moduli, and where $F$ is the scalar curvature of the ten-dimensional metric and the Ricci-flatness of Calabi-Yau manifolds. Let $\alpha_A: A = 1, \ldots, h^{1,1}$ be a harmonic basis for the $H^{(2)}(X, \mathbb{Z})$ cohomology, and let us call $1 = \alpha_0$ the generator of the $H^{(0)}(X, \mathbb{Z})$ cohomology. We will collectively denote by $\alpha_A: A = 0, 1, \ldots, h^{1,1}$ the harmonic basis of $H^{(0)}(X, \mathbb{Z}) \oplus H^{(2)}(X, \mathbb{Z})$ cohomology.

Following the discussion in section D.3.2, we parametrize the ten-dimensional fields as follows:

$$C^{(1)} = A^0(x) \alpha_0, \quad C^{(3)} = A^3(x) \alpha_A, \quad B + i J = (h^A(x) + i j^A(x)) \alpha_A,$$

where $C^{(1)}$ and $C^{(3)}$ are the ten-dimensional RR-forms, $J$ is the Kähler form, and $j^A$ parametrize the Kähler deformations of the metric (section D.3.2). To carry out the reduction of (B.1) to four dimensions, we need to know the action of $*_{10}$ and the form of $\sqrt{-G} R_{10}$. First, we go to the Einstein frame by rescaling the metric $G_s$ as $G_E = e^{-\phi/2} G_s$. Next, using that the ten-dimensional metric $G$ takes the following bloc diagonal form $G = g_M \oplus g_X$, where $g_X$ is the Kähler metric of the Calabi-Yau (section D.3.1), one concludes that $*_{10} = *_M *_X$. The evaluation of the scalar curvature $R_{10}$ simplifies drastically as a result of the bloc diagonal form of the ten-dimensional metric and the Ricci-flatness of Calabi-Yau manifolds i.e. $R_{ij} = 0$.

Let us also introduce the following quantities for later convenience

$$D_{ABC} = \int_X \alpha_A \wedge \alpha_B \wedge \alpha_C, \quad j_{AB} = \int_X \alpha_A \wedge \alpha_B \wedge J = D_{ABC} j^C, \quad (5.2)$$

$$j_A^2 = \int_X \alpha_A \wedge J \wedge J = D_{ABC} j^B j^C, \quad j^3 = \int_X J \wedge J \wedge J = D_{ABC} j^A j^B j^C. \quad (5.3)$$

Notice that $D_{ABC}$ is the intersection number of three four-cycles $y^4A$ that are Poincaré dual to the forms $\alpha_A$. We are now almost in the position of getting the reduced four-dimensional effective action. First, we plug the expressions (5.1) in the action (B.1) (after going to the Einstein frame). Then, we perform the integration over the Calabi-Yau to get [129, 130, 131]

$$2 S = \int R*1 - \int (\mathrm{Im} N_{A\Sigma}(t) F^A \wedge * F^\Sigma + \mathrm{Re} N_{A\Sigma}(t) F^A \wedge F^\Sigma + G_{AB} dt^A \wedge * dt^B), \quad (5.3)$$

where $F^A = dA^A$ is the field strength of the Abelian gauge field $A^A$, $t^A = b^A + i j^A$ is the complexified Kähler moduli, and [131]

$$G_{AB} = \frac{3}{2j^3} \int_X \alpha_A \wedge * \alpha_B = - \frac{3}{2} \left( \frac{j_{AB}}{j^3} - \frac{3}{2} \frac{j_A^2 j_B^2}{(j^3)^2} \right) = - \frac{\partial}{\partial t^A} \frac{\partial}{\partial t^B} \ln \left( \frac{4}{3} j^3 \right), \quad (5.4)$$

Chapter 5 - Black Constellations in Four Dimensions

$H^{(1,1)}$ cohomology of the Calabi-Yau threefold $X$. String theory requires an integer version of these cohomology groups which is defined only of real cohomology. Since the Calabi-Yau we will be working with has no $H^{(2,0)}$, $H^{(1,0)}$ and $H^{(0,1)}$ cohomologies (see equation D.11), we can identify $H^{(1,1)}(X, \mathbb{C})$ with $H^{(2)}(C, \mathbb{R})$. Let $\alpha_A: A = 1, \ldots, h^{1,1}$ be a harmonic basis for the $H^{(2)}(X, \mathbb{Z})$ cohomology, and let us call $1 = \alpha_0$ the generator of the $H^{(0)}(X, \mathbb{Z})$ cohomology. We will collectively denote by $\alpha_A: A = 0, 1, \ldots, h^{1,1}$ the harmonic basis of $H^{(0)}(X, \mathbb{Z}) \oplus H^{(2)}(X, \mathbb{Z})$ cohomology.

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where $C^{(1)}$ and $C^{(3)}$ are the ten-dimensional RR-forms, $J$ is the Kähler form, and $j^A$ parametrize the Kähler deformations of the metric (section D.3.2). To carry out the reduction of (B.1) to four dimensions, we need to know the action of $*_{10}$ and the form of $\sqrt{-G} R_{10}$. First, we go to the Einstein frame by rescaling the metric $G_s$ as $G_E = e^{-\phi/2} G_s$. Next, using that the ten-dimensional metric $G$ takes the following bloc diagonal form $G = g_M \oplus g_X$, where $g_X$ is the Kähler metric of the Calabi-Yau (section D.3.1), one concludes that $*_{10} = *_M *_X$. The evaluation of the scalar curvature $R_{10}$ simplifies drastically as a result of the bloc diagonal form of the ten-dimensional metric and the Ricci-flatness of Calabi-Yau manifolds i.e. $R_{ij} = 0$.

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where $F^A = dA^A$ is the field strength of the Abelian gauge field $A^A$, $t^A = b^A + i j^A$ is the complexified Kähler moduli, and [131]

$$G_{AB} = \frac{3}{2j^3} \int_X \alpha_A \wedge * \alpha_B = - \frac{3}{2} \left( \frac{j_{AB}}{j^3} - \frac{3}{2} \frac{j_A^2 j_B^2}{(j^3)^2} \right) = - \frac{\partial}{\partial t^A} \frac{\partial}{\partial t^B} \ln \left( \frac{4}{3} j^3 \right), \quad (5.4)$$
whereas the expression of $N_{\Lambda \Sigma}$ is given below in (5.14). To get the expression of $G_{AB}$ above we used that [132, 129]:

$$
* X_{\alpha A} = -J \wedge \alpha_A + \frac{3}{2} \frac{j^2}{j^3} J \wedge J ,
$$

which can be derived using that the volume-form of the Calabi-Yau is given by

$$
\text{dv}_X = \frac{1}{6!} J \wedge J \wedge J ,
$$

and that $\alpha_A$ is harmonic i.e. closed and co-closed. It is clear from (5.4) that the scalar moduli space is Kähler. However, this is not the end of the story, as super-symmetry requirements tell us that this manifold should be special Kähler, see e.g. [129, 133, 134]. Discussing the whole geometric structure of special Kähler manifolds is beyond the scope of this thesis, we will content ourselves by introducing some practical formulas that we will be needing later on.

### 5.1.2 The Special Kähler Geometry

It turns out that all the information about the action (5.3) can be nicely expressed in terms of a single function $F$ called the “prepotential”. Before spelling out formulas, let us first go back to the integer even-cohomology of the Calabi-Yau threefold. First, we are going to enlarge the cohomology we worked with so far ($H^{(0,0)} \oplus H^{(1,1)}$) to include $H^{(2,2)} \oplus H^{(3,3)}$. The absence of the $H^{(3,1)}$ and the $H^{(1,3)}$ cohomologies allows us to identify $H^{(2,2)}(X, \mathbb{C})$ with $H^{(4)}(X, \mathbb{R})$. Furthermore, we will choose the harmonic basis of $H^{(4)}(X, \mathbb{Z}) (H^{(6)}(X, \mathbb{Z}))$ denoted by $\alpha^A; A = 1, \ldots, h^{1,1}$ (respectively, $\alpha^{(i)}$) such that

$$
\int_X \alpha_\Lambda \wedge \alpha_\Sigma = \delta^\Sigma_\Lambda ,
$$

where $\Lambda = 0, 1, \ldots, h^{1,1}$. In the following, we are going to abbreviate $H^{(2n)}(X, \mathbb{Z})$ by $H^{(2n)}$. We will also denote by $H^*$ the total even-cohomology of the Calabi-Yau

$$
H^* = H^{(0)} \oplus H^{(2)} \oplus H^{(4)} \oplus H^{(6)} .
$$

It turns out that the even-cohomology $H^*$ comes equipped with a skew-symmetric pairing $\langle \cdot, \cdot \rangle$ between its elements, that appears naturally in supergravity. It is defined as follows; first, we expand each element $\Gamma$ of $H^*$ in the harmonic basis $\alpha_A, \alpha^A$ as

$$
\Gamma = \Gamma_0 \alpha_0 + \Gamma^A \alpha_A + \Gamma_A \alpha^A + \Gamma_0 \alpha^0 .
$$

To this element, $\Gamma \in H^*$, we will associate a new element, $\tilde{\Gamma} \in H^*$, defined as

$$
\tilde{\Gamma} = \Gamma_0 \alpha_0 - \Gamma^A \alpha_A + \Gamma_A \alpha^A - \Gamma_0 \alpha^0 .
$$
The skew-symmetric pairing is then given by

\[
\langle \Gamma, \Delta \rangle = \int_X \Gamma \wedge \widetilde{\Delta} = -\Gamma^0 \Delta_0 + \Gamma^A \Delta_A - \Gamma_A \Delta^A + \Gamma_0 \Delta^0 .
\] (5.10)

The last bit of information we need concerns the scalars \( t^A \). Since we want to treat the whole \( H^* \), we need to have \( 2(1 + h^{1,1}) \) complex scalars. Following the same strategy as above, let us first extend the scalar content so it becomes in one to one correspondence with basis elements of \( H^{(0)} \oplus H^{(2)} \), and call these scalars \( X^\Lambda \); \( \Lambda = 0, 1, \ldots, h^{1,1} \). The latter turn out to be describing the same physics if they are multiplied by the same complex number, in agreement with the actual number of physical degrees of freedom \( t^A \). \( X^\Lambda \) are called projective coordinates and we choose them such that, if \( X^0 \neq 0 \) then \( t^A = X^A/X^0 \). The needed “dual” scalars \( Y_\Lambda \) to cover the whole \( H^* \) turn out to be given in terms of \( X^\Lambda \) through a single function \( \mathcal{F} \) called the prepotential. The new scalars \( Y_\Lambda \) are given by

\[
Y^0 = -\frac{\partial \mathcal{F}}{\partial X^0} = -\mathcal{F}_0 \quad \text{and} \quad Y^A = \frac{\partial \mathcal{F}}{\partial X^A} = \mathcal{F}_A .
\] (5.11)

For the modification of this expression to include instanton corrections, see e.g. [135, 136].

We are now ready to discuss the special Kähler geometry underlying the \( \mathcal{N} = 2 \) four-dimensional supergravity. A special kähler geometry is characterized by the existence of a holomorphic section \( \Omega_{\text{hol}} \) of \( H^* \) such that the Kähler potential \( K = -\ln(4j^3/3) \) of the Kähler metric (5.4) is given by

\[
e^{-K} = i\langle \Omega_{\text{hol}}, \overline{\Omega}_{\text{hol}} \rangle = i \left[ X^\Lambda \mathcal{F}_\Lambda - X^A \mathcal{F}_A \right] .
\] (5.12)

The last expression corresponds to the rewriting of \( K \) in terms of the scalars \( X^\Lambda \). It is easy to see that \( \Omega_{\text{hol}} \) can be written as:

\[
\Omega_{\text{hol}} = X^\Lambda \alpha^A - \mathcal{F}_A \alpha^A + \mathcal{F}_0 \alpha^0 = -e^{t^A \alpha^A} .
\] (5.13)

The last expression in the equation above is a formal one and is valid for \( X^0 = -1 \). Since supersymmetry relates \( h^{1,1} \) combinations of the gauge fields \( A^\Lambda \) to the scalars \( t^A \) as they belong to the same multiplet (see the end of section D.3.2), one expects that the metric \( \mathcal{N}_{\Lambda \Sigma} \) will be expressed in terms of \( \mathcal{F} \) and \( X^\Lambda \) only. This turns out to be true and the expression reads [119]

\[
\mathcal{N}_{\Lambda \Sigma} = \mathcal{F}_{\Lambda \Sigma} + 2i \frac{\Im(\mathcal{F}_{\Lambda \Lambda'}) X^\Lambda' \Im(\mathcal{F}_{\Sigma \Sigma'}) X^\Sigma'}{\Im(\mathcal{F}_{\Lambda \Sigma'}) X^\Lambda' X^\Sigma'} ,
\] (5.14)

where \( \mathcal{F}_\Lambda = \partial_\Lambda \mathcal{F} \) and \( \mathcal{F}_{\Lambda \Sigma} = \partial_\Lambda \partial_\Sigma \mathcal{F} \).
In the following, it will be more useful to work with a normalized version of $\Omega_{\text{hol}}$ defined as

$$\Omega = e^{K/2} \Omega_{\text{hol}} = -\frac{1}{\sqrt{4j^3/3}} e^{(b_A + ij^A)} \alpha_A .$$  (5.15)

Notice that, under Kähler transformations $K \rightarrow K + f$, $\Omega_{\text{hol}}$ transforms like $\Omega_{\text{hol}} \rightarrow e^{-f} \Omega_{\text{hol}}$. This transformation motivates us to introduce a covariant derivative acting on $\Omega$ as:

$$\mathcal{D}_A \Omega = \left( \partial_A + \frac{1}{2} [\partial_A K] \right) \Omega ,$$  (5.16)

such that $\Omega$ and $\mathcal{D}_A \Omega$ transform in the same way under Kähler transformations. It turns out that $\{ \Omega, \mathcal{D}_A \Omega, \mathcal{D}_B \Omega, \Omega \}$ constitute an “orthonormal” basis for $H^*$ with respect to the skew-symmetric pairing (5.10) as they satisfy

$$\langle \Omega, \Omega \rangle = -i, \quad \langle \mathcal{D}_A \Omega, \mathcal{D}_B \Omega \rangle = G_{AB}, \quad \langle \Omega, \mathcal{D}_A \Omega \rangle = 0 .$$  (5.17)

Using this new basis, it is easy to work out the decomposition of an arbitrary element $\Gamma$ of $H^*$ to be

$$\Gamma = 2 \text{Im} \left( \langle \Omega \rangle \Omega - G^{AB} \mathcal{D}_A \Omega \mathcal{D}_B \Omega \right) ,$$  (5.18)

where $G^{AB}$ is the inverse of $G_{AB}$ and $\langle \Omega \rangle$ is the central charge of $\Gamma$ given by

$$\langle \Gamma, \Omega \rangle = \frac{1}{\sqrt{4j^3/3}} \left( t^3 - \frac{t_A^3}{2} \Gamma_A + t^A \Gamma_A - \Gamma_0 \right) .$$  (5.19)

The easiest way to understand the reason behind calling such a combination a central charge is to study the action of a supersymmetric probe brane $\Gamma$ in a $\mathcal{N} = 2$ four-dimensional background, which we will write down in a moment. Before doing so, we need to introduce a new element in $H^* \otimes \Omega^{(2)}(M^{(1,3)})$, where $\Omega^{(2)}(M^{(1,3)})$ stands for the space of two-forms in $M^{(1,3)}$ the four-dimensional non-compact part of our ten-dimensional geometry. Taking advantage of the form of the gauge field part in (5.3), we introduce the following even-form

$$F = F^A \alpha_A - G_A \alpha^A + G_0 \alpha^0 ,$$  (5.20)

where $G_A$ is defined by

$$G_A = \text{Re} \langle N_{A\Sigma} F^\Sigma + \text{Im} N_{A\Sigma} \ast F^\Sigma \rangle ,$$  (5.21)

The signs in (5.20) are chosen such that the gauge field part of the action (5.3) is written as

$$S_F = \frac{1}{2} \int \langle F_{(0,2)} , F_{(4,6)} \rangle ,$$  (5.22)

where a four dimensional wedge product is understood in the expression above, and the even forms $F_{(0,2)}$ and $F_{(4,6)}$ are given by:

$$F_{(0,2)} = F^A \alpha_A, \quad F_{(4,6)} = -G_A \alpha^A + G_0 \alpha^0 .$$  (5.23)
Using the expression (5.20) and that $\alpha^A, \alpha_A$ are harmonic, the Bianchi identity and the gauge field equations can be combined in the equation $dF = 0$. The latter implies the conservation of the following charge

$$\Gamma = \frac{1}{4\pi} \int F = p^\Lambda \alpha_A + q_A \alpha^A,$$

(5.24)

where $p^0$ is due to a D6-brane wrapping the whole Calabi-Yau $X$, $\{p^A\}$ are due to a D4-brane wrapping the four-cycle dual to the two-form $\beta^{(2)} = p^A \alpha_A$, $\{q_A\}$ are due to a D2-brane wrapping the two-cycle dual to the four-form $\beta^{(4)} = q_A \alpha^A$ and $q_0$ is the D0-brane charge.

The equation $dF = 0$ can be solved locally to give $F = dA$. The components $A^\Lambda, A_A$ of $A$ in the harmonic basis of $H^*$ are the four dimensional Maxwell fields. Strictly speaking we have only $h^{1,1} + 1$ independent fields as a result of the relation (5.21). The components $A^\Lambda, A_A$ of $A$ are essentially the electric and magnetic parts of the physical Maxwell field. Using the field $A$, the action of a supersymmetric probe brane $\Gamma$ in a $\mathcal{N} = 2$ four-dimensional background is given by [137]

$$S_{\text{probe}} = -\int |Z(\Gamma)| \, ds + \frac{1}{2} \int \langle \Gamma, A \rangle,$$

(5.25)

where $s$ is the line element of the particle. It is clear that the first term is a mass term i.e. the corresponding central charge giving the needed explanation as promised. The second term is like an electron-monopole interaction term, and will play an important role in the multi-center solutions (section 5.3).

### 5.2 Spherical Symmetry and Attractor Flow

In the following, we are going to spell out the simplest 1/2-BPS black hole solution to (5.3) and some of its most important properties. This solution is the static spherically symmetric 1/2-BPS black hole.

#### 5.2.1 Supersymmetry and Attractor Flow

Requiring supersymmetry puts a lot of constraints on our solution [138, 139]. Restricting ourselves further to static spherically symmetric solutions fixes the metric to be of the form

$$ds^2 = -e^{2U(\rho)} \, dt^2 + e^{-2U(\rho)} \, d\vec{x}^2,$$

(5.26)

where $\rho$ is the radial coordinate in the spatial part $\mathbb{R}^3$. Spherical symmetry also reduces the information about the scalar moduli $t^A$ and the gauge fields $A^\Lambda$ to $1 +$
$h^{1,1}$ unknown functions that depend only on $r$ [122]. These functions are $U$ and $t^A$. We will not go through the whole derivation of solution here. The interested reader should consult [122] for details. Rather, we will outline the strategy and describe some important properties of the solution.

The idea of [122] is to take advantage of the staticity and the spherical symmetry of the problem to reduce the action (5.3) to an effective one-dimensional action that depends only on $\tau = 1/r$. Then, one solves the field equations that result from this effective action. Such procedure is in general illegal. Rather, one should first find the equations of motion of the action (5.3) then, reduce them over the sphere using our ansatz. This generally gives rise to more equations than what one gets using the resulting effective action. Luckily for us, it was checked in [122] that solving the effective field equations is enough in the case we are dealing with. A further simplification that [122] used is time independence of our solution. Basically, they rewrote the effective action they got after reducing (5.3) over a sphere $S^2$ as a sum of a square and a boundary term. Then, they used that the Hamiltonian of the system equals the Lagrangian multiplied by $(-1)$ due to time independence to derive the BPS equations. According to supersymmetry, these equations should minimize the energy and hence, are equivalent in this case to the vanishing of the square term in the effective action. These equations turn out to be given by [122]:

\[ \text{Im} \left( \partial_A \mathcal{K} i^{A} \right) + \dot{\xi} = 0, \quad 2\partial_{\tau} \left( e^{-U} \text{Im} \left[ e^{-i\xi} \Omega \right] \right) = -\Gamma, \] (5.27)

where $\mathcal{K}$ is the Kähler potential (5.12), the charge “vector” $\Gamma$ is defined in (5.24), and $\xi$ is the phase of the central charge $Z(\Gamma)$ defined in (5.19).

The equations (5.27) are equivalent to the Killing spinor equations. They describe a one parameter flow of $t^A$ in the moduli space. They are called the "attractor flow" equations [127]. One can show that during such a flow, the norm of the central charge $|Z(\Gamma)|$ is a decreasing function of $\tau$, and it reaches its minimum with respect to varying all moduli $t^A$ at $\tau \to \infty$. This implies that the value of $|Z(\Gamma)|_{\tau=\infty} = |Z(\Gamma)|_{\text{min}}$ is completely fixed by the value of $\Gamma$ independently from the values of the moduli $t^A$ at $\tau = 0$. This phenomenon is called the “attractor mechanism” [140, 141]. It can happen that there is more than one minimum of $|Z(\Gamma)|$. In such cases, one can end up in any one of them. In this situation, the attractor flow is said to have "multiple-bassins of attraction". Such phenomenon can occur only in singular regions of moduli space [142, 143]. Throughout the remaining of this thesis, we will assume that we are “far” from such singular regions.

It turns out that $\tau = \infty$ ($r = 0$) describes a horizon of the black hole as $e^{-U} \sim |Z(\Gamma)|_{\infty} \tau \to \infty$, [122]. Such a behavior allows us to calculate the associated entropy of the black hole, which turns out to be fixed by $Z(\Gamma)$

\[ S(\Gamma) = \pi |Z(\Gamma)|^2_{\text{min}}. \] (5.28)
This is good news because due to the attractor mechanism, the entropy of a black hole depends only on the charges and does not care about the value of the moduli in the asymptotic flat region. This is in agreement with the no-hair theorem.

5.2.2 The One Centered Black Hole

We are more or less ready to construct our static spherically symmetric solution. We are going to describe the different steps leading to the solution leaving the details to the literature. We will be following [123] and [122] where the whole solution was expressed in terms of a single function \( \Sigma \) called the “entropy function”. Its explicit expression was first derived in [144], in the special case of the large Calabi-Yau volume limit.

The idea is to take advantage of the attractor flow equations (5.27), while using at the same time different properties of \( \Omega, D_A \Omega \) and their complex conjugates (5.17). First, one formally solves the second equation in (5.27) as:

\[
2 e^{-U} \Im (e^{-i\xi} \Omega) = -\Gamma \tau + 2 \Im (e^{-i\xi} \Omega)_{\tau=0} \equiv -H .
\] (5.29)

The trick that allows us to construct the solution is to rewrite all our fields (metric, moduli \( t^A \), and Maxwell one-forms \( A^\Lambda \)) in terms of this new function \( H \). Taking the skew-symmetric pairing of this equation with \( \Omega \) gives

\[
e^{-2U} = |Z(H)|^2 \equiv \Sigma(H) .
\] (5.30)

To get the expressions for \( t^A \), one first plugs (5.30) back into (5.29) to get the imaginary part of \( t^A \). Then, taking the pairing of (5.29) with \( D_A \Omega \) gives their real part [123]. Combining both gives:

\[
t^A = \frac{H^\Lambda - i \partial_{H^\Lambda} \Sigma(H)}{H^0 + i \partial_{H^0} \Sigma(H)} .
\] (5.31)

The only remaining unknown fields to be found are the Maxwell one-forms \( A^\Lambda \). They turn out to be given by [123]:

\[
A^\Lambda = \epsilon \partial_{H^\Lambda} \ln \Sigma(H) \, dt - p^\Lambda \cos \theta \, d\phi ,
\] (5.32)

where \( \epsilon = -1 \) for \( \Lambda = 0 \) and \( +1 \) otherwise. The only remaining thing to do now is to express \( \Sigma(H) \) in terms of \( H^\Lambda \). Using that the former is a homogeneous function of degree two [123], and its asymptotic expression near the horizon \( \tau \to \infty \), one concludes that

\[
\Sigma(H) = \frac{1}{\pi} S(H) ,
\] (5.33)
where $S(H)$ is the same function that gives the black hole entropy when we replace $H$ by the corresponding $\Gamma$. In general, figuring out such a function is a hard task and depends strongly on the form of the prepotential $\mathcal{F}$ (5.13). In the case of a cubic prepotential (5.11), using the expression for $S(\Gamma)$ derived in [144], our static spherically symmetric BPS black hole solution reads

$$\begin{align*}
    ds^2 &= -\frac{1}{\Sigma} \, dt^2 + \Sigma \, d\vec{x}^2, \\
    t^A &= \frac{H^A}{H^0} + \frac{y^A}{Q^{3/2}} \left( i\Sigma - \frac{L}{H^0} \right), \\
    A^0 &= -\frac{L}{\Sigma^2} \, dt + A^0, \\
    A^A &= \frac{H^A \, L - Q^{3/2} \, y^A}{H^0 \, \Sigma^2} \, dt + A^A, \\
    H &= \frac{r}{r_{\infty}} - 2 \, \text{Im} \left( e^{-i\xi} \Omega \right)_{r=\infty}, \\
    dA^A &= *dH^A, \\
    \Sigma &= \sqrt{Q^3 - L^2 \left( \frac{H^0}{H^0} \right)^2}, \\
    Q^3 &= \left( \frac{1}{3} D_{ABC} y^A y^B y^C \right), \\
    L &= (H^0)^2 H_0 + \frac{1}{3} D_{ABC} H^A H^B H^C - H^0 H^A H_A,
\end{align*}$$

(5.34)

where $\star$ is the flat three-dimensional $\mathbb{R}^3$ Hodge star, and $y^A$ are solutions to the following equation

$$D_{ABC} y^B y^C = -2 \, H^0 H_A + D_{ABC} H^B H^C. \quad (5.35)$$

From the explicit expression of the metric and $\Sigma$ above (5.34), it is clear that a solution will not exist if $\Sigma^2$ is negative. Since $\Sigma$ is roughly the modulus square of the central charge $Z(\Gamma)$ where the charge $\Gamma$ is replaced now by $H$ (5.30), one expects that the existence of the solution has something to do with $Z(\Gamma)$. This turns out to be true, where [143] showed that there are three possibilities

- $|Z(\Gamma)|_{\text{min}} \neq 0$, a black hole of charge $\Gamma$ exists.
- $|Z(\Gamma)|_{\text{min}} = 0$ at a regular point in moduli space, the solution does not exist.
- $|Z(\Gamma)|_{\text{min}} = 0$ at a singular point in moduli space, more analysis is needed to decide whether the solution exists or not.

The second point led to the following puzzle. Some known microscopic BPS states at weak coupling (D-brane states) do not have a strong coupling counterpart i.e. supergravity solution. Such a situation is confusing as the dilaton lives in the hypermultiplets moduli-space (section D.3.2), which as we argued in the beginning of this chapter has nothing to do with our solutions. The resolution of such puzzle will be the subject of the next section.
5.3 Bubbles and Bound Black Holes

It was realized in [122, 145] (for an earlier attempt see [146]), that there are other 1/2-BPS solutions to the action (5.3) which describe a bound state of black holes. They may not exhaust the list of all possible BPS-solutions to $N = 2$ four dimensional supergravity as they are stationary. These solutions play an important role on different fronts. On top of making the map between microscopic and macroscopic degrees of freedom richer and more subtle e.g. [147, 148, 30], they also describe candidate geometries for black hole states [124, 118, 56]. Although, smooth solutions can only appear when we uplift to five dimensions e.g. [38] (through [62, 63]), the number of these states –as things stand right now– is the same. Furthermore, it was shown in [64] that the uplift can be embedded in an asymptotic AdS$_3 \times$ S$^2$ spacetime, which opens the possibility to apply AdS/CFT duality considerations to these solutions. The latter can be seen as normalizable deformations of AdS$_3$, which, according to the AdS/CFT dictionary, should be mapped to states in the dual CFT theory. Such identification can be seen as another argument for the possible applicability of the fuzzball considerations to these supergravity solutions.

In the following, we are going to discuss these solutions and some of their important properties following [122, 85, 123]. We will be very brief in our exposition as the intermediate steps become a bit technical very quickly inviting the unsatisfied reader to check the literature.

5.3.1 More than one center

Based on our intuition from the probe brane action (5.25), the metric this time will not be static any more. This comes about because of the “electron-monopole” interaction term $(\Gamma, A)$. So our starting point will be the following metric ansatz:

$$ds^2 = -e^{-U} (dt + \omega)^2 + e^{-U} d\vec{x}^2,$$

(5.36)

where $\omega$ is a one-form on the base space $\mathbb{R}^3$. The existence of such term, and the dependence of the solution on the vector $\vec{x}$ and not only on its norm $r$ as in the one center case, makes the BPS analysis more involved. [122] managed to derive the following BPS equations

$$2 e^{-U} \text{Im} \left( e^{-i\xi} \Omega \right) = -H, \quad *d\omega = \langle dH, H \rangle,$$

(5.37)

$$A = 2e^{U} \text{Re} \left( e^{-i\xi} \Omega \right) dt + \mathcal{A}, \quad *dA = dH,$$

(5.38)

which clearly generalize (5.27). In this generic case, $\xi$ is the phase of the total central charge $Z(\Gamma)$ where $\Gamma = \sum_a \Gamma_a$, whereas $H$ is a generic harmonic function.
that, in the case of many center $\vec{x}_a$, takes the form

$$H = \sum_a \frac{\Gamma_a}{r_a} - 2 \text{Im} \left( e^{-i\xi} \Omega \right)$$ \quad (5.39)

where $r_a = |\vec{x} - \vec{x}_a|$. The derivation of the solution describing a multi-center black holes uses the same strategy as before, where most of the intermediate steps remain valid. At the end, and in the case of a cubic prepotential, the following solution is found

$$ds^2 = -\frac{1}{\Sigma} (dt + \omega)^2 + \Sigma d\vec{x}^2, \quad t^A = \frac{H^A}{H^0} + \frac{y^A}{Q^{3/2}} \left( i\Sigma - \frac{L}{H^0} \right),$$

$$A^0 = -\frac{L}{\Sigma^2} dt + A^0, \quad A^A = \frac{H^A L - Q^{3/2} y^A}{H^0 \Sigma^2} dt + A^A,$$

$$H = \frac{\Gamma}{r} - 2 \text{Im} \left( e^{-i\xi} \Omega \right) r=\infty, \quad dA^A = *dH^A, \quad *d\omega = \langle dH, H \rangle,$$ \quad (5.40)

$$\Sigma = \sqrt{\frac{Q^3 - L^2}{(H^0)^2}}, \quad Q^3 = \left( \frac{1}{3} D_{ABC} y^A y^B y^C \right),$$

$$L = (H^0)^2 H_0 + \frac{1}{3} D_{ABC} H^A H^B H^C - H^0 H^A H_A,$$

where $*$ is the flat three-dimensional $\mathbb{R}^3$ Hodge star, and $y^A$ are solutions to (5.35). As before, the solution is valid if $\Sigma^2 > 0$. We will postpone the discussion about this point to the next section, and turn now to the description of two of the most important properties of these solutions.

### 5.3.2 Useful Properties

The solutions described above distinguish themselves from their one-center cousins by two properties which are:

#### Bubble Constraints

In the derivation of the solution (5.40), we have neglected an important issue. The existence of the one-form $\omega$ is not always trivial. This is because, its defining equation (5.37) combined with the expression of $H$ (5.39) puts constraints on the possible positions of the different centers. Using that $d^2 = 0$, and that $\Delta H = \sum_a \Gamma_a \delta^3 (\vec{x} - \vec{x})$, leads to the following important constraint on the inter-center distances

$$\sum_{b,b\neq a} \frac{\langle \Gamma_a, \Gamma_b \rangle}{r_{ab}} = \langle h, \Gamma_a \rangle; \quad \forall a,$$ \quad (5.41)
where we used the short hand notation \( h = -2 \text{Im} \left( e^{-i \xi} \Omega \right)_\infty \). These equations are not all independent. To check that note, using the expression of \( h \) and the definition of \( \xi \), that the sum of the equations (5.41) is trivial. This can be seen as factoring out the center of mass degrees of freedom. In the case of \( N \) centers, one ends up with a \( 2(N - 1) \)-dimensional space of solutions (called also solution space). The dimension being even will turn out to be important for the considerations of the next chapter.

Having such “bubble” equations complicates our lives. We have to check that the solutions \( r_{ab} \) are physically acceptable; all of them are positive and they should satisfy the triangle inequalities. Such complications, on top of the requirement that \( \Sigma^2 > 0 \), makes a systematic study of such solutions intractable. However, [147, 30] conjectured a simpler way to overcome this murky situation. This will be the subject of the next section.

**Angular-Momentum**

As was already mentioned around (5.36), these solutions are stationary but not static. This is due to the presence of cross terms encoded by \( \omega \). It is clear from its defining equation that its origin resides in the non-trivial angular momentum generated by the electron-monopole interaction. It can be easily shown by studying the asymptotics of the metric around the flat background that, there is a non-trivial angular momentum given by:

\[
\vec{J} = \frac{1}{4} \sum_{a \neq b} \frac{\langle \Gamma_a, \Gamma_b \rangle}{r_{ab}} \vec{x}_{ab},
\]

(5.42)

where \( \vec{x}_{ab} = \vec{x}_a - \vec{x}_b \) and \( r_{ab} = |\vec{x}_{ab}| \). Note that the normalization of \( J \) above is chosen such that after quantization, \( J \) will be quantized in half integer units. The existence of such angular momentum will play an important role in the discussion of the symplectic form in section 6.1.1. Using the constraints (5.41), one can show that the norm of \( \vec{J} \) is given by [56]

\[
|J| = \frac{1}{2} \sqrt{-\sum_{a < b} \langle h, \Gamma_a \rangle \langle h, \Gamma_b \rangle r_{ab}^2}.
\]

(5.43)

This formula will be useful in section 6.3.3 where we will compare the number of states that we will get from quantizing the supergravity solutions to the number that one expects based on the wall crossing formula.
Chapter 5 - Black Constellations in Four Dimensions

5.4 BPS States Counting

So far, we have reviewed a class of four dimensional solutions, but, these solutions are relatively complicated and it is non-trivial to determine if they are well-defined everywhere. In particular, the entropy function $\Sigma$, that appears in the solution involves a square root and may take imaginary values in some regions (when uplifted to five dimensions this can lead to closed timelike curves [63] [149] [150]). In [122] and [30], a simplified criterion was proposed for the well behavedness of such solutions which we will now briefly relate.

5.4.1 The Family Tree

In [122], a conjecture is proposed whereby pathology-free solutions are those with a corresponding attractor flow tree in the moduli space. The latter is a graph in the Calabi-Yau moduli space beginning at the moduli at infinity, $t^A|_{\infty}$, and ending at the attractor points for each center. The edges correspond to single center flows towards the attractor point for the sum of charges further down the tree. Vertices can occur where single center flows (for a charge $\Gamma = \Gamma_1 + \Gamma_2$) cross walls of marginal stability where the central charges are all aligned ($|Z(\Gamma)| = |Z(\Gamma_1)| + |Z(\Gamma_2)|$). The actual flow of the moduli $t^A(\vec{x})$ for a multi-centered solution will then be a thickening of this graph (see [122], [30] for more details). According to the conjecture, a given attractor flow tree will correspond to a single connected set of solutions to the equations (5.41), all of which will be well-behaved. An example of such a flow is given in figure 5.1.

As mentioned before, the main purpose of the attractor flow tree is to allow us to determine if a solution is well defined. For a single centered black hole, the entropy function $\Sigma$ undergoes a monotonic flow from infinity to the horizon. At infinity the value of $\Sigma$ depends on the choice of moduli (boundary conditions), while at the horizon it flows to a fixed value depending only on the charges, as the moduli are fixed by the attractor mechanism. Spherical symmetry dictates that the moduli depend only on a radial variable, so, the flow through moduli space is indeed just a single line from the moduli at infinity to the attractor value. If $\Sigma$ should become imaginary somewhere along this flow, the solution would suffer from pathologies. However, since the flow is monotonic, it need only be checked at its initial (the moduli at infinity) and final points (the attractor point).

For a multi-centered system, the moduli depend on three variables and the flow is no longer monotonic in a straightforward way (it is not even a one dimensional tree but rather a “fat graph”). By assuming that solutions could be built constructively by bringing in centers from infinity, [122] was able to conjecture that even for
Figure 5.1: Three centered attractor flow tree. The system is composed of three center of charge $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ and the moduli at infinity are at the value labelled by the black circle. Each leg of the tree above represents a single center flow towards the attractor value associated with the total charge below that point. Thus the first flow is towards the attractor point for $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$. After the first split the flows are towards the attractor points for charges $\Gamma_3$ and $\Gamma_4 = \Gamma_1 + \Gamma_2$. In each case the split occurs along walls of marginal stability (thick grey lines). The first, horizon, line of MS corresponds to $|Z(\Gamma)| = |Z(\Gamma_3)| + |Z(\Gamma_4)|$ while the second is for $|Z(\Gamma_4)| = |Z(\Gamma_1)| + |Z(\Gamma_2)|$.

multi-centered configurations we can study a flow tree in the moduli space (recall the actual flow will be a “fat” version of this) and study each leg of the flow to check for pathologies. The conjecture is then that if the tree exists (each leg is pathology free) then the full solution is actually well behaved (see [122, 147] for more details). There is considerable evidence for this conjecture [122, 85, 147, 30], and our computation in sections 6.3.3 and 6.4 will provide even further support.

The intuition behind this proposal is based on studying the two-center solution for charges $\Gamma_1$ and $\Gamma_2$. The constraint equations (5.41) imply that when the moduli at infinity are moved near a wall of marginal stability (where $Z_1$ and $Z_2$ are parallel), the centers are forced infinitely far apart

$$r_{12} = \frac{\langle \Gamma_1, \Gamma_2 \rangle}{\langle h, \Gamma_1 \rangle} = \frac{\langle \Gamma_1, \Gamma_2 \rangle |Z_1 + Z_2|}{2 \text{Im}(Z_2Z_1)} \bigg|_{\infty}. \quad (5.44)$$

In this regime, the actual flows in moduli space are well approximated by the split attractor trees since the centers are so far apart that the moduli will assume single-center behavior in a large region of spacetime around each center. Thus, in this regime the conjecture is well motivated. Varying the moduli at infinity continuously
should not alter the BPS state count, which corresponds to the quantization of the two center moduli space, so unless the moduli cross a wall of marginal stability we expect solutions smoothly connected to these to also be well defined. Extending this logic to the general $N$ center case requires an assumption that it is always possible to tune the moduli such that the $N$ centers can be forced to decay into two clusters that effectively mimic the two-center case. There is no general argument that this should be the case but one can run the logic in reverse, building certain large classes of solutions by bringing in charges pairwise from infinity, and this can be understood in terms of attractor flow trees. It is clear that not all solutions can be constructed in this way. For example, for some set of charges $\Gamma_a$, it could happen that the constraint equations (5.41) allow for solutions where the centers approach each other arbitrarily closely. These class of solutions cannot be constructed using the strategy explained above. For more discussion on this point, the reader should consult [147].

5.4.2 Missing States: Wall Crossing

For generic charges, the attractor flow conjecture also provides a way to determine the entropy of a given solution space. The idea is that the entropy of a given total charge is the sum of the entropy of each possible attractor flow tree associated with it. Thus, the partition function receives contributions from all possible trees associated with a given total charge and specific moduli at infinity. An immediate corollary of this is that, as emphasized in [30], the partition function depends on the asymptotic moduli. As the latter are varied, certain attractor trees will cease to exist; specifically, a tree ceases to contribute when the moduli at infinity cross a wall of marginal stability (MS) for its first vertex, $\Gamma \rightarrow \Gamma_1 + \Gamma_2$, as is evident from (5.44).

For two-center solutions, one can determine the entropy most easily near marginal stability where the centers are infinitely far apart. In this regime, locality suggests that the Hilbert state contains a product of three factors [30]

$$\mathcal{H}(\Gamma_1 + \Gamma_2; t_{ms}) \supset \mathcal{H}_{int}(\Gamma_1, \Gamma_2; t_{ms}) \otimes \mathcal{H}(\Gamma_1; t_{ms}) \otimes \mathcal{H}(\Gamma_2; t_{ms}).$$

(5.45)

One should be a little bit more careful as attractor flow trees do not have to split at walls of marginal stability. Generally, in such cases there will be other contributions to $\mathcal{H}(\Gamma_1 + \Gamma_2; t_{ms})$ as well. In the following, we will be assuming that such split does happen keeping in mind subtleties aforementioned.

Since the centers move infinitely far apart as $t_{ms}$ is approached, we do not expect them to interact in general. There is, however, a conserved angular momentum carried in the electromagnetic fields sourced by the centers, and this also yields a non-trivial multiplet of quantum states. Thus, the claim is that $\mathcal{H}_{int}$ is the Hilbert
space of a single spin $J$ multiplet where $J = \frac{1}{2}(\langle \Gamma_1, \Gamma_2 \rangle \mid - 1)$. The unusual $(-1)$ in the definition of $J$ comes from quantizing additional fermionic degrees of freedom [85] [56]. $\mathcal{H}(\Gamma_1)$ and $\mathcal{H}(\Gamma_2)$ are the Hilbert spaces associated with BPS brane excitations in the Calabi-Yau, and their dimensions are given in terms of a suitable entropy formula for the charges $\Gamma_1$ and $\Gamma_2$ valid at $t_{\text{ms}}$.

Thus, if the moduli at infinity were to cross a wall of marginal stability for the two-center system above, the associated Hilbert space would cease to contribute to the entropy (or the index). A similar analysis can be applied to a more general multi-centered configuration like that in figure 5.1 by working iteratively down the tree, and treating subtrees as though they correspond to single center with the combined total charge of all their nodes. The idea is, once more, that we can cluster charges into two clusters by tuning the moduli and then treat the clusters effectively like individual charges. We can then iterate these arguments within each cluster. This counting argument mimics the constructive one for building the solutions by bringing in charges from infinity, and is hence, subject to the same caveats, discussed above.

Altogether, the above ideas allow us to determine the entropy associated with a particular attractor tree, which by the split attractor flow conjecture, corresponds to a single connected component of the solutions space. The entropy of a tree is the product of the angular momentum contribution from each vertex (i.e. $\mid \langle \Gamma_1, \Gamma_2 \rangle \mid$, the dimension of $\mathcal{H}_{\text{int}}$) times the entropy associated to each node. When we want to compare against the number of states derived from quantizing the classical phase space, as we are going to do in the next chapter, the latter factor (from the nodes) will not be included as it is not visible in the supergravity solutions.

In the next chapter, we will show that it is also possible—in some cases—to quantize the solution space directly and to match the entropy so derived with the entropy calculated using the split attractor tree. This provides a non-trivial check of both calculations.