CHAPTER 6

SETTING THE STAGE FOR FUZZBALLS

So far, we have described a rich family of four dimensional BPS states that asymptotically look like a single center black hole (5.3). It will be very interesting if one could choose a subset of all these possible solutions and declare them to be black hole states. As was already mentioned at several places, although for smoothness considerations one needs to go one dimension higher we are not going to do so here. Our space of solutions that we will be working with, will be the set of all possible center positions $\vec{x}_a$, subject to the constraint (5.41). Unfortunately, due to the complicated nature of our space of solutions, things are not as concrete as was the case in the D1-D5 system. Actually, what we have been able to do is to quantize a special class of all possible $\mathcal{N} = 2$ BPS solutions. This is a first step toward an implementation of the fuzzball ideas to a $1/2$-BPS macroscopically large black hole of the $\mathcal{N} = 2$ four-dimensional supergravity.

This chapter will deal mainly with quantizing the $\mathcal{N} = 2$ four-dimensional solution space. Although, the latter is very complicated and might have a rich topology, we managed to carry out the quantization in some simple cases. These are the three-center (section 6.3) and the so called “dipole halo solutions” (section 6.4). Our results agree with what one expects from wall-crossing considerations when the latter is applicable (section 5.4.2). In the cases where the wall-crossing considerations fail, like in the case of scaling solutions (section 6.3.2), our quantization gives a prediction of the number of BPS states.

Even though the class of solutions we managed to quantize is very restricted, a
special case of the dipole halo systems turns out to be very interesting (section 6.5). These are dipole halos that develop a scaling behavior. It was argued that such solutions can be seen as a geometric manifestation of D4-D0 black hole states [116, 117, 118]. Unfortunately, after counting the number of BPS states of these class of solutions, we seem to get far less entropy than the corresponding entropy of the D4-D0 with the same total charges. As a result, we are facing two possibilities: either there are other supergravity solutions that we did not include in our counting and these will account for the missing states, or we need stringy degrees of freedom to reach the needed number of states. We are inclined to believe the second possibility. We will present an estimate of an upper limit on the possible supergravity BPS states in section 6.5.3 in support of our claim.

Before diving into the details of these exciting results, we start by constructing the symplectic form which gives us a clear criterion to when a solution space is a phase space. Armed with this, we go ahead and describe the quantization method we will be using. This is the so called “geometric quantization”. Some details will be left to appendices E and F.

In this chapter, the reader is assumed to have some knowledge of differential geometry and two-dimensional conformal field theory.

6.1 From the Symplectic Form to Quantization

In this section, we will study the general features of the quantization approach that we will use, later on, to quantize a special class of our solution spaces (5.40), where the centers positions $\vec{x}_a$ are subject to the bubble constraint (5.41). Following the general approach discussed in section 2.3, we need to derive the restriction of the symplectic form (2.5) to our solution space. Due to the complicated nature of both the supergravity action (5.3) and our solution space, we will take another approach to get the symplectic form relying on open/closed string duality. We will be using the dual open string picture of our multi-center solutions to derive our symplectic form.

Let us try to describe in simple words what kind of field theory one gets in the open string picture. We will be sketchy in the following, for more details see [85]. The story is a little bit involved but its spirit is simple as we will describe now. Remember that the multi-center solutions describe the geometry response to a set of D-branes that wrap different cycles inside the compact Calabi-Yau. These D-branes are characterized by the the charge vector $\Gamma_a$, where $a$ labels the different centers, as described below (5.24). In the open string picture, we start with the same D-brane configuration as the one of our gravity solution, however, we will not backreact these
D-branes. In other words, we go to a regime where the gravitational interaction is so weak that the geometry will not feel the presence of these D-branes. This can be achieved by decreasing the string coupling constant.

Summarizing, in the open string picture we have the same brane configuration as in the gravity side, but now the background geometry is a Calabi-Yau times a four-dimensional Minkowski spacetime. Each stack of these D-branes $\Gamma_a$, that generates the charge of the center $(\alpha)$, contributes a $U(N_a)$ gauge field where $N_a$ is the greatest common factor of the component of $\Gamma_a$ in the harmonic basis $\alpha^\Lambda$, $\alpha^\Lambda$ of $H^*$. On top of these gauge fields, we have fields that describe open strings stretching between different stacks of D-branes i.e. open strings stretching between the stacks described by $\Gamma_a$ and $\Gamma_b$ where $a \neq b$. After reducing this theory over the Calabi-Yau, one ends up with a one-dimensional theory with a couple of $U(N_a)$ gauge fields and fields that transform in the (fundamental, anti-fundamental) representation of $U(N_a) \times U(N_b)$ where $a \neq b$. Such a theory is called “quiver quantum mechanics” (QQM in short) [85].

After deriving the symplectic form from the dual quiver quantum mechanics, one has to study to modification of such symplectic form once the string coupling constant is increased. Usually, this is quiet non-trivial, but luckily in our case, the terms in the quiver quantum mechanics action that contribute to the symplectic form are protected. It turns out that they do not receive neither perturbative nor non-perturbative corrections beyond one-loop [85, 56]. Motivated by this non-renormalization theorem [85], we propose that the same symplectic from should be derivable from the supergravity action following the logic in [151] (see also [81] and references therein). This will be further confirmed by an exact agreement of our state counting with Denef and Moore’s wall-crossing formula [56]. Actually we can recognize a term in the supergravity action that might lead to the same symplectic form as in the open string picture. However, there are other terms in the supergravity action besides this term. So we can rephrase our conjecture in the following way: the other putative terms contributing to the symplectic form from supergravity cancel, or only change the normalization as has been seen in [151].

To proceed further, we need to choose a polarization to split our phase space in coordinates and momenta in order to be able to quantize it. Unfortunately, there is no universal strategy to accomplish this. However, the examples we are going to discuss later on (sections 6.3, 6.4 and 6.5) come with a common beautiful mathematical structure that allows us to quantize them in a particularly nice way. Their solution spaces turn out to be “toric Kähler” (see section 6.2.1 for some general facts about such geometries). In these cases we can use geometric quantisation approach, see e.g. [152, 89, 90, 91], to carry out our quantization.

This section is subdivided into two parts. In the first one, we are going to summarize
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the derivation of the symplectic form [56]. While we are going to discuss some key points of geometric quantization in the second part.

6.1.1 OPEN STRINGS AND SYMPLECTIC FORM

The symplectic form can, in principle, be derived from the supergravity action as was done, for instance, in [151]. In our case, however, it is far more tractable to take a different approach [56]. As discussed in [85], the four dimensional multi-centered solutions can also be analyzed in the probe approximation by studying the quiver quantum mechanics of D-branes in a multi-centered supergravity background. Moreover, a non-renormalization theorem [85] implies that the terms in the quiver quantum mechanics Lagrangian linear in the velocities do not receive corrections, either perturbatively or non-perturbatively beyond one-loop. We can use this fact to calculate the symplectic form in the probe regime and extend it to the fully back-reacted solution; this is because, for time-independent solutions, the symplectic form depends only on the terms in the action linear in the velocity.

For this approach to be consistent it is necessary that the BPS solution space, which we interpret as a phase space, of the four-dimensional supergravity theory, as well as that of the probe theory, all match. This follows from the fact that they are all governed by the same equation, (5.41) [85]. For instance, one can see that a probe brane of charge $\Gamma_a$ in the background generated by a charge $\Gamma_b$ is forced off to infinity as a wall of marginal stability is approached [85], analogous to what was described around equation (5.44) for the corresponding supergravity solution.

In [56], the symplectic form on the solution space is determined. We will not review the derivation in detail but simply note that it arises from the term coupling the probe brane to the background gauge field, $\dot{x}^i A_i$, giving

$$\omega = \frac{1}{2} \sum_p \delta x_p^i \wedge (\Gamma_p, \delta A^i(x_p)) \ . \ (6.1)$$

where $A_i$ is the “spatial” part of the gauge field given by (5.40)

$$dA = \ast dH \ , \ (6.2)$$

with the “$\ast$” above is the flat three-dimensional Hodge star. This descends naturally to the spatial part of the 4-d gauge field. Using the definition of $A$, we can further manipulate this expression [56] and put it in the form

$$\omega = \frac{1}{4} \sum_{p \neq q} \langle \Gamma_p, \Gamma_q \rangle \epsilon_{ijk} (\delta(x_p - x_q)^i \wedge \delta(x_p - x_q)^j \cdot (x_p - x_q)^k \cdot [x_p - x_q]^i) \ . \ (6.3)$$
This is a two form on the \((2N-2)\)-dimensional solution space which is a submanifold of \(\mathbb{R}^{3N-3}\) defined by (5.41). Moreover, one can show that, on this submanifold, this form is closed and, in the cases we will investigate below, non-degenerate. Thus, it endows the solution space with the structure of a phase space. Note that, as anticipated, the center of mass degrees of freedom do not appear in the symplectic form above and hence decouple in the quantization of the system.

Although the constraint equations (5.41) are invariant under global SO(3) rotations, these are nonetheless (generically) degrees of freedom of the system, and this is reflected in the symplectic form. If we contract (6.3) with the vector field that generates rotations around the 3-vector \(n^i\) (i.e. we take \(\delta x^i_{pq} = \epsilon^{ijk}n^j x^k_{pq}\)), then the symplectic form reduces to

\[
\omega \rightarrow n^i \delta J^i, \tag{6.4}
\]

where \(J^i\) are the components of the angular momentum vector defined in (5.42).

This is nothing more than the statement that the components \(J^i\) are the conjugate momenta associated to global SO(3) rotations. In general, the symplectic form on any of our phase spaces will have terms like the above coming from the global SO(3) rotations, in addition to terms depending on other degrees of freedom. This does not hold for solution spaces with unbroken rotational symmetries, such as solution spaces containing only collinear centers or a single center. In these cases some SO(3) rotations act trivially, they do not correspond to genuine degrees of freedom nor do they appear in the symplectic form. We close this subsection by noting that (6.4) implies that solution spaces with \(\vec{J} = \vec{0}\) everywhere will have a degenerate symplectic form, and therefore, will not constitute a proper phase space. This happens when all the intersection products between charges vanish \((\langle \Gamma_a, \Gamma_b \rangle = 0 ; \forall a, b\). In such situations, the centers are free to move anywhere and hence they are not bound. These systems are not amenable to quantization using the methods that will be developed in this chapter.

In situations like these, one could try to include small velocities for the centers in order to arrive at a well-defined phase space. It is clearly an interesting question whether this modified system will give rise to BPS states upon quantization. Superficially, the momenta increase the energy while leaving the charges invariant, and they therefore violate the BPS condition. However, if the Hilbert space has a continuous spectrum of momenta, it is possible that there is a BPS bound state at zero momentum in the spectrum. This is difficult to analyze in general, but in our case, we do not expect this to happen, at least not in asymptotically AdS spaces, since AdS effectively provides a box and will therefore put an IR cutoff on the admissible momenta. Thus our proposal is that solution spaces with a degenerate symplectic form should not be thought of as describing proper BPS bound states.
This immediately leads to another issue; namely, we know that, for example, $N$ D0-branes can form a marginal bound state [153, 154], but the symplectic form for such a configuration (for example in the presence of a D4-brane) would vanish identically. This clearly conflicts with the statements of the preceding paragraph.

We would like to argue that the resolution of this inconsistency lies in the fact that the marginal bound state of D0-branes cannot be understood purely from a low-velocity expansion. Rather, the presence of the non-Abelian degrees of freedom is essential for the bound state to exist. This is supported both by the analysis of [154], as well as by the size of the bound state (see e.g. [155]). Again, it would be interesting to explore this further. In this chapter, we will take the point of view that a solution containing a marginal bound state of e.g. $N$ individual D0-branes should be counted separately from a similar solution where the marginal bound state has been replaced by $N$ D0-branes. This will be crucial for identifying the number of states of the non-scaling dipole-halo solution, that we will get using our quantization method, with the number of states predicted by wall-crossing formula (section 6.4).

### 6.1.2 Kähler Geometry and Geometric Quantization

Classical physics being our daily life experience is well understood. However, going to the micro-world requires a new theory with its own “rules of the game”, this is “quantum mechanics”. Unfortunately, the only part that we really understand in quantum mechanics is its limit which is classical physics. Since, in our search for a quantum theory we are trying to build a theory starting from its limit not the other way around, it makes it a challenging task and possibly with a non-unique prescription. After all, the only criterion that we have to check if we got the right quantum mechanics theory, or not, is by confronting its predictions to the results of experiments. In the following, we will be giving a taste of one of the possible approaches to quantization, the so called “geometric quantization”. We will be following closely [89].

Geometric quantization is a perfect example of “the beauty and the beast” i.e. its fundamental ideas are elegant and simple, however, things become quickly mathematically more demanding. Fortunately, as far as counting degrees of freedom is concerned, which is what we will be doing, the elegant part is more than enough. Geometric quantization builds on the symplectic structure of classical physics which will be the subject of the next subsection. A discussion of the first step towards quantization, the so called “prequantization” will follow. In this step, an attempt to construct the Hilbert space will be carried out. This space turns out to be too large and needs to be “halved”. In such a procedure, a polarization will be chosen that
distinguishes coordinates from momenta. In our cases of interest, a natural polarization will be favorable. This is the “holomorphic” (called also “Kähler”) polarization which is suitable for Kähler manifolds.

**Linking Classics to Quantum: Symplectic Geometry**

We have already mentioned very briefly the connection between classical physics and symplectic geometry in section 2.3.2. In the Hamiltonian formulation of classical physics, the dynamics is governed by a function \( H \) called the “Hamiltonian”, while the degrees of freedom of the system parametrize a space called the “phase space”. A central element in such formulation is the Poisson bracket \( \{ \cdot, \cdot \} \), which can be nicely encoded in a symplectic form \( \omega \) (section 2.3.2).

All in all, we have a \((2n)\)-dimensional manifold \( M \) called phase space, equipped with a non degenerate closed symplectic form \( \omega \). Using the latter, we can associate to functions \( f \) on the phase space a vector field \( X_f \), known as the “Hamiltonian vector field” of \( f \), as follows

\[
i_{X_f} \omega \equiv \omega(X_f, \cdot) = -df,
\]

where \( i_{X_f} \) stands for a contraction of \( \omega \) by \( X_f \). Due to closure of \( \omega \), \( X_f \) generates a flow on \( M \) that preserves \( \omega \) i.e. \( \mathcal{L}_{X_f} \omega = 0 \), where \( \mathcal{L}_X \) is the Lie derivative along the vector \( X \). The symplectic form \( \omega \) provides a skew-symmetric pairing between functions in \( M \) through (6.5) given by

\[
\{ f, g \} = \omega(X_f, X_g).
\]

This is the acclaimed Poisson bracket. The laws of classical physics read

\[
\frac{df}{dt} = \{ H, f \} = X_H f,
\]

where \( H \) is the Hamiltonian. One last important identity that will play a key role in the following is that it can be proven easily that

\[
[X_f, X_g] = X_{\{ f, g \}},
\]

where \([ \cdot, \cdot \] is the Lie bracket.

Let us close this section by specifying what we mean by quantization, see e.g [89, 86]. In such a process, we are looking for a map \( Q \) from real functions \( f \) in \( C^\infty(M) \) to self-adjoint operators \( Q(f) \) that satisfy:

**C1. \( \mathbb{R} \)-linearity:** \( \forall r \in \mathbb{R}, \forall f, g \in C^\infty(M); \quad Q(r f + g) = r Q(f) + Q(g). \)
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C2. The constant function is mapped to the identity operator.

C3. The operator $Q(f)$ is self adjoint if $f$ is real.

C4. The quantum condition: 
\[ [Q(f), Q(g)] = -i \hbar Q\{f, g\} \]

C5. \( \{Q(f_1), \ldots, Q(f_n)\} \) is a complete set of operators if \( \{f_1, \ldots, f_n\} \) is a complete set of observables.

It turns out that we cannot satisfy the last two conditions (C4.) and (C5.) for all functions $f_i$ at the same time, see e.g. [89, 86]. To get our from this unfortunate situation, we should look for a weak version of one of these two conditions (or both).

Two widely known approaches to remedy such a conflict are deformation and geometric quantization. The first one tries to modify (C4.) above by higher order terms in $\hbar$ keeping the last requirement, while the second approach – of interest to us – weakens the last condition (C5.) by requiring (C4.) to hold for a restricted class of functions. In the following, we are going to skim over the main steps of geometric quantization leaving details to the literature, see for example [89, 86] and references there in. Our aim is to reach a point where we can count, or even give explicit expressions of quantum states.

Prequantization

An important observation that will ignite the whole geometric quantization machinery is the similarity between (6.8) and the quantum condition (C4). This suggests to associate to the observable $f$ the differential operator, $-i \hbar X_f$, in the quantum theory. However, this turns out to fail the test of (C2.), as any constant function is mapped to the zero vector field (not identity operator). Some logical simple modifications of our first proposal, keeping in mind conditions (C1–C4), leads to the following “prequantum” assignment to a classical observable $f$

\[ f \longrightarrow \mathcal{P}Q(f) = -i \hbar \mathcal{D}(X_f) + f, \quad (6.9) \]

where $\mathcal{D}$ can be seen as some sort of a covariant derivative which reads for a local trivialization of $\omega = d\theta$

\[ \mathcal{D} = d - \frac{i}{\hbar} \theta. \quad (6.10) \]

We should stress here that the prequantum assignment (6.9) does not work for all functions $f$. For a discussion on this point see e.g. [89, 86] and references therein. We will turn to this issue after introducing the notion of “polarization” below. For now, we proceed with our exposition keeping in mind this issue. Putting the assignment (6.9, 6.10) in the appropriate mathematical language brings us to the following definition of the prequantization [89].
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A **prequantization** of a symplectic manifold \((M, \omega)\) is a pair \((\mathcal{L}, \mathcal{D})\), where \(\mathcal{L}\) is a complex Hermitian line bundle over \(M\), and \(\mathcal{D}\) is a compatible connection with curvature \(\omega\). The “itprequantum Hilbert space” is the completion of the space of square integrable smooth sections of \(\mathcal{L}\) with the natural integration measure \(\omega^n/(n!)\).

Since line bundles \(\mathcal{L}\) are classified by their first Chern class \(c_1 \in H^2(M, 2\pi \mathbb{Z})\) which can be represented by the curvature form of any connection on \(\mathcal{L}\), a necessary and sufficient condition for the possibility of prequantization is that \(\omega/2\pi\) represents an integral cohomology class.

**Polarization**

Although our prequantization was natural and cute, our prequantum Hilbert space is too large. This is because our declared states are square integrable functions on the whole phase space which is in clear contradiction with the uncertainty principle. This comes about because such freedom in constructing functions allows us to cook up very localized ones. As a result, we need a way to half our phase space. Such a procedure is called choosing a **polarization**. A way of doing this is to choose a \(n\)-dimensional sub-bundle \(\mathcal{P}\) of the complexified tangent bundle \(T^cM\) of \(M\) and pick states \(\psi\) that are covariantly constant:

\[
\forall X \in \mathcal{P} : \quad \mathcal{D}(X)\psi = 0 . \tag{6.11}
\]

This cannot be done in general. Actually, such a condition requires, using that \([\mathcal{D}(X), \mathcal{D}(Y)]\psi = 0\), that \(\mathcal{P}\) is integrable and Lagrangian i.e.

\[
\forall X, Y \in \mathcal{P} ; \quad [X, Y] \in \mathcal{P} \quad \text{and} \quad \omega(X, Y) = 0 . \tag{6.12}
\]

In our case of interest where the phase space is a Kähler manifolds (see section D.1), there is a natural polarization called the “Kähler polarization”. One starts by choosing complex coordinates such that \(\omega\) is the Kähler form \((D.5)\), locally

\[
\omega = i\partial\bar{\partial}K , \tag{6.13}
\]

where

\[
\partial = dz^k \wedge \frac{\partial}{\partial z^k} , \quad \bar{\partial} = d\bar{z}^k \wedge \frac{\partial}{\partial \bar{z}^k} ,
\]

and \(z^k\) are complex coordinates. Equation \((6.13)\) allows us to choose \(\theta = i\partial K\) or \(\theta = -i\bar{\partial} K\). We are ready to define our polarization:

The “**holomorphic**” (Kähler) polarization \(\mathcal{P}\) is spanned by the vectors \(\partial/\partial \bar{z}^k\). Using the choice \(\theta = -i\partial K\), which vanishes on \(\mathcal{P}\), reduces the condition \((6.11)\) to just holomorphicity i.e. \(\mathcal{D}|_\mathcal{P} = \bar{\partial}\). As a result, in the case of a Kähler manifold it is
natural to require that our states are holomorphic sections. This is the result we are going to use in the following.

Unfortunately, geometric quantization comes with some drawbacks. We will only very briefly discuss one problem which is of relevance to us. For a thorough discussion on other shortcomings of geometric quantization see [86] and references therein. As was mentioned in the second chapter of this thesis, section 2.3, one of the important reasons we decided to perform a quantization of our space of solutions was to check the scales at which quantum effects become important. One way to proceed to find such scales is to evaluate the variance of different semi-classical observables in the resulting Hilbert space, and see when they become large. In such a strategy, we will need, on top of the quantum states, an adequate definition of quantum operators that are associated to these observables. We have already alluded below equation (6.10), and when we defined what we mean by quantization, that geometric quantization procedure, specifically the rule (6.9, 6.10), is not applicable for all classical functions \( f \). It turns out that, in general, it works only for functions that depend linearly on the wrong polarization [89, 86]. For example, in our case of Kähler polarization, the prescription we gave in this section for quantization works for functions of the form \( f(z, \bar{z}) = g(z) + \bar{z} h(z) \). For some special classes of phase spaces, like being Kähler which is the case of interest to us, there is an involved prescription to extend the rule (6.9, 6.10) to quadratic functions in \( \bar{z} \) i.e. \( f(z, \bar{z}) = g(z) + \bar{z} h(z) + \bar{z}^2 l(z) \). For more details, the reader should consult [89, 86] and references therein.

What we should take from this section about quantization is that, in the case of phase spaces with symplectic form \( \omega \) that are Kähler, the geometric quantization approach leads to

- The Hilbert space is the completion of the space of integrable holomorphic section of a line bundle whose first Chern class is \( \omega \).
- For linear functions in \( \bar{z} \), the associated quantum operator is given by (6.9, 6.10).

### 6.2 Quantization at Work

We are ready to start the quantization of our space of solutions defined by (5.40) subject to (5.41). Strictly speaking, our quantization works only for centers that do not carry intrinsic degrees of freedom as it does not see them. However, we can still apply the same procedure of quantization in general, where one can see our approach as quantizing the external degrees of freedom only.
The spaces of solutions we managed to quantize turn out to share the same underlying mathematical features [56, 156], that make them toric Kähler manifolds. These solution spaces, being Kähler, allow us to use geometric quantization techniques developed in the previous section to quantize them. Furthermore, our life is made simpler as mathematician have devised a simple way to construct the needed complex coordinates, and the Kähler potential, in the case of toric Kähler manifolds [157, 158, 159], [160, 161].

A complication that we need to take care of is the inclusion of fermions. These arise because the open string picture [85] requires the addition of fermionic degrees of freedom in order to account for all the BPS states. This is because, in the open string description, the centers are described by \( \mathcal{N} = 4, d=1 \) supersymmetric quiver quantum mechanics (QQM), with the position of each center encoded in the scalars of a vector multiplet, and the latter also includes fermionic components which must be accounted for in any quantization procedure. It turns out that one can summarize the fermionic contribution into a modification of the line bundle whose holomorphic sections are our states. This modification of the line bundle can in turn be encoded in a modification of the integration measure.

In the following, we start by introducing toric Kähler manifolds building up to reach the construction of complex coordinates and the Kähler potential. Then, we will deal with the question of fermionic degrees of freedom. At the end, we combine all the knowledge developed to derive constraints on a set of “integers” \( n_i \) that encode the sought after number of states. Some details are left to the appendix E.

### 6.2.1 Behind the Scene: Toric Kähler Manifolds

Our starting point in describing our solution spaces was a symplectic point of view. It is however more convenient for geometrical quantization to have a Kähler description, which can always be made in the case of a symplectic toric manifold. The main results in this subsection are the expressions (6.19, 6.20) for the complex coordinates and Kähler potential in terms of the symplectic coordinates on a symplectic toric manifold. Before giving these formulas, we review some of the basics of symplectic toric manifolds and symplectic toric orbifolds.

#### Polytopes

As is customary, we will refer to the convex hull of a finite number of points in \( \mathbb{R}^n \) as a polytope. The boundary of such a polytope is itself the union of various lower dimensional polytopes that are called faces. In particular, a zero-dimensional face
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is called a vertex, a one-dimensional face an edge, and a \((n-1)\)-dimensional face a facet. Note that we can view any polytope as the intersection of a number of affine half spaces in \(\mathbb{R}^n\). A polytope \(P\) can thus be uniquely characterized by a set of inequalities, namely \(\bar{x} \in P\) if and only if \(\forall a = 1, \ldots, m\)

\[
\langle \bar{c}_a, \bar{x} \rangle \geq \lambda_a \iff \sum_j c_{ai} x_j \geq \lambda_a , \tag{6.14}
\]

where \(m\) is the number of facets. It is clear that \(m \geq n + 1\) otherwise we will not have a compact polytope. Given a polytope we will call the set \(\bar{c}_a \in \mathbb{Z}^n\), given by the inward pointing normals to the various facets, the normal fan.

An \(n\)-dimensional polytope is called a “Delzant polytope” if it satisfies the following three conditions

- **simplicity**: In each vertex exactly \(n\) edges meet,
- **rationality**: Each of the \(n\) edges that meet at the vertex \(p\) is of the form \(p + tu_i\), with \(t \in \mathbb{R}^+\), and \(u_i \in \mathbb{Z}^n\),
- **smoothness**: For each vertex the \(u_i\) form a \(\mathbb{Z}\)-basis of \(\mathbb{Z}^n\).

The polytope is called “rational” instead of Delzant if we replace, in the third condition, the requirement of a \(\mathbb{Z}\)-basis by that of a \(\mathbb{Q}\)-basis.

**Symplectic Toric Manifolds**

Before giving the precise technical definition of a symplectic toric manifold, let us first sketch the idea. Roughly speaking, a toric manifold is a \(\mathbb{T}^n\) fibration over a given \(n\)-dimensional polytope, such that at each facet a single \(U(1)\) inside the \(\mathbb{T}^n\) shrinks to zero size. On the intersections of the different facets multiple \(U(1)\)’s collapse, e.g. at the vertices all circles have shrunk. On the interior of the polytope the toric manifold is simply of the form \(P^\circ \times \mathbb{T}^n\), and the full toric manifold is a compactification of this space. On the interior there is, thus, a standard set of coordinates of the form \((x_i, \theta_i)\), with \(x_i \in P^\circ\), and \(\theta_i \in \mathbb{T}\), and the manifold comes with a standard symplectic form

\[
\omega = \sum_i dx_i \wedge d\theta_i . \tag{6.15}
\]

It is of course rather non-trivial that this manifold can be smoothly compactified, but when the polytope is Delzant, it is the case. Let us now state the above ideas more precisely.

A **symplectic toric manifold** is a compact connected \(2n\)-dimensional symplectic manifold \((M, \omega)\), that allows an effective Hamiltonian action of an \(n\)-dimensional torus
Remember that, the action of a Lie group on a symplectic manifold is called Hamiltonian if there exists a **moment map** \( \mu \), from the manifold to the dual Lie algebra, that satisfies
\[
d\langle \mu(p), X \rangle = \omega(\cdot, \tilde{X}),
\]
with \( p \in M \), \( X \) is a generator of the Lie algebra, and \( \tilde{X} \) is the corresponding vectorfield. Furthermore, the moment map should be equivariant with respect to the group action, i.e. \( \mu(g(p)) = \text{Ad}^* g \circ \mu(p) \), with \( \text{Ad}^* \) the coadjoint representation.

By a theorem of Delzant [162], every symplectic toric manifold is uniquely characterized by a Delzant polytope. Given a symplectic toric manifold the corresponding polytope is given by the image of the moment map. To conversely reconstruct the manifold from the polytope is slightly more involved and relies on the technique of symplectic reduction, we refer readers interested in further details to e.g. [157]. Note that the normal fan of the polytope is identical to the fan that is used to characterize toric varieties in algebraic geometry, see e.g. [163] for a nice introduction. This can be useful to identify a symplectic manifold given by a polytope, and furthermore, provides an embedding in projective spaces.

**Toric Kähler Manifolds**

What will be of use to us is that Delzant’s construction also associates a set of canonical complex coordinates to every symplectic toric manifold, effectively implying that every closed symplectic toric manifold is actually a Kähler manifold. As the states that we will count are holomorphic sections, we will describe now the construction of a natural set of complex coordinates, be it without proofs or motivation. Those can be found in references [158, 159].

As mentioned above (6.14), any polytope \( P \) is characterized by a set of inequalities. Given this combinatorial data of the polytope, one can define the associated functions
\[
l_a(x) = \sum_i c_{ai} x_i - \lambda_a, \quad l_\infty = \sum_{i,a} c_{ai} x_i,
\]
which are everywhere positive on \( P \). Using these functions one can define a ‘potential’ as follows
\[
g(x) = \frac{1}{2} \sum_a l_a(x) \log l_a(x). \tag{6.18}
\]
In case the polytope is Delzant, it is shown in [158] that this potential can be used to define good complex coordinates on the toric manifold as follows
\[
z_i = \exp \left( \frac{\partial}{\partial x_i} g(x) + i \theta_i \right). \tag{6.19}
\]
Furthermore, a Kähler potential for the corresponding Kähler metric \( \omega(\cdot, J \cdot) \) is given by

\[
K = \sum_a \lambda_a \log l_a(x) + l_{\infty}.
\] (6.20)

It follows from the construction of [158, 159] that \( K \) is the Legendre transform of \( g \), i.e. \( K(z) = \frac{\partial g}{\partial x} - g(x) \). This can be used to derive that

\[
(\det \partial_i \partial_j K)^{-1} = \exp \left( 2 \sum_i \frac{\partial g}{\partial x_i} \right) \det \frac{\partial^2 g}{\partial x_i \partial x_j},
\] (6.21)

which will be a useful formula later when we discuss the inclusion of fermions.

**Toric Orbifolds**

As we will also consider quotients of symplectic toric manifolds by a permutation group in this chapter, it will be necessary to introduce the generalization of the above construction of complex coordinates to that of symplectic toric orbifolds. This is because, modding out a manifold by the action of a permutation group leads to a space that belongs to a class of spaces called “orbifolds”. As in the manifold case, a symplectic toric orbifold is a 2n-dimensional symplectic orbifold that allows a Hamiltonian \( \mathbb{T}^n \) action. As was shown in [160], such symplectic toric orbifolds are in one to one correspondence to labeled rational polytopes. Such a labeled rational polytope is nothing but a rational polytope with a natural number attached to each facet. The label \( m_a \) denotes that the \( a \)th facet is a \( \mathbb{Z}_{m_a} \) singularity. Again, the explicit construction of the toric orbifold from the labeled polytope is rather involved and we refer those who are interested to [160]. The labeled polytope corresponding to the quotient of a symplectic toric manifold by a group respecting the torus action, is however easy to find. It is given by the quotient of the original polytope, where we attach a label \( m \) to each facet that is a \( \mathbb{Z}_m \) fixed point under the group action.

Given a labeled rational polytope, one can construct complex coordinates on the toric orbifold in a way similar to the manifold case. The functions \( l_a \) from (6.17) are generalized to [160, 161]

\[
l_a(x) = m_a \left( \sum_i c_{ai} x_i - \lambda_a \right), \quad l_{\infty} = \sum_{i,a} m_a c_{ai} x_i,
\] (6.22)

where \( m_a \) is the label attached to the facet orthogonal to the vector \( \vec{c}_{ai} \). The construction of the complex coordinates and the kähler potential from these functions then carries on analogously to (6.19,6.20). Notice that one can recover the previous case (no orbifolding) by setting \( m_a = 1; \forall a \). In order to keep the discussion as general as possible, we will be using the toric orbifold formulas in the following.
6.2.2 Inviting Fermions to the Party

Naively, our phase space is given by the coordinates, $\vec{x}_p$, subject to the constraint (5.41), which parametrize the space of purely bosonic BPS solutions. But we know from the open string picture that this is not quite true, as we need to include fermions in order to account of all BPS states [85]. Since we expect to see the same number of BPS states in both the open and closed descriptions, and since the bosonic phase spaces in both cases match exactly (and the symplectic forms agree in view the non-renormalization theorem discussed above), we may ask what the closed string analog of the fermions in the QQM is?

Consider our phase space: the coordinates, $\vec{x}_p$, subject to the constraint (5.41), parametrize the space of purely bosonic BPS solutions but, for each such solution, we may still be able to excite fermions if doing so is allowed by the equations of motion. If we consider only infinitesimal fermionic perturbations of the bosonic solutions, then the former will always appear linearly in the equations of motion, acted on by a (twisted) Dirac operator. Thus fermions which are zero modes of this operator may be excited without altering the bosonic parts of the solution (to first order).

Determining the actual structure of these zero modes is quite non-trivial. A natural guess is that the bosonic coordinates of the centers must be augmented by fermionic partners (making the solution space a superspace), as is argued in [164, 165] where there is no potential. The fact that the bosonic coordinates are constrained by a potential complicates the problem in our case, so we will simply posit the simplest and most natural guess and justify it, a posteriori, by reproducing the right degeneracy as expected based on the split attractor conjecture [30].

Thus, we will posit that the full solution space is actually the total space of the spin bundle over the Kähler phase space. The correct phase space densities are now harmonic spinors on the original phase space [166]. Recall (see e.g. [167]) that on a Kähler manifold $\mathcal{M}$ there is a canonical Spin$^c$ structure where the spinors take values in $\Lambda^0,\star(\mathcal{M})$. To define a spin structure, we need to take a square root of the canonical bundle $K = \Lambda^{N,0}(\mathcal{M})$ and twist $\Lambda^{0,\star}(\mathcal{M})$ by that. We also need to remember that the bosonic part of the wave functions were sections of a line bundle, $\mathcal{L}$. Thus altogether, the spinors on the solution space are given by sections of

$$\mathcal{L} \otimes \Lambda^{0,\star}(\mathcal{M}) \otimes K^{1/2}. \quad (6.23)$$

The Dirac operator is given by

$$D = \partial + \partial^\star, \quad (6.24)$$

and we have to look for zero modes of this Dirac operator. These are precisely the harmonic spinors on $\mathcal{M}$, and therefore, the BPS states correspond to $H^{0,\star}(\mathcal{M}, \mathcal{L} \otimes$
By the Kodaira vanishing theorem \cite{168}, $H^{0,n}(\mathcal{M}, \mathcal{L} \otimes K^{1/2})$ vanishes unless $n = 0$. This is true provided $\mathcal{L}$ is “very ample”, which means in our case that we should be working with large quantum numbers $(\Gamma_a, \Gamma_b)$. Thus, finally, the BPS states are given by the global holomorphic sections of $\mathcal{L} \otimes K^{1/2}$.

To find the number of BPS states following the geometric quantization approach, we have to make sure that in the innerproduct we use the norm appropriate for $\mathcal{L} \otimes K^{1/2}$ which is

$$\text{measure} \sim e^{-\mathcal{K}} \left(\det \partial_i \partial_j \mathcal{K}\right)^{-1/2} \omega^n,$$

where $\omega$ is the symplectic form \eqref{6.3} on the solution space, and $n$ is the dimension of the polytope (half the dimension of the solution space). The modification of the natural measure, $\omega^n/n!$, used in the definition of the norm when we discussed the prequantum Hilbert space (subsection 6.1.2) comes about because we are integrating over sections of a non-trivial line bundle. The extra terms incode the transformation rules of the sections. For toric Kähler manifolds we find, after some manipulations using \eqref{6.20}, \eqref{6.18}, \eqref{6.22}, \eqref{6.21} and \eqref{E.1}, that

$$\text{measure} \sim \prod_{a=1}^{m} \left(\sum_{i=1}^{n} \frac{m_a c_{a} - 1}{2} - m_a \lambda_a\right),$$

where $\sim$ stands for equality up to an overall smooth non-vanishing term. This is because, as we will see later, the number of “normalizable” sections is controlled by the singularity structure and the zeros of \eqref{6.25}.

### 6.2.3 Treasure Hunt: Degeneracy

We collected all the needed mathematical background to address the question of counting “BPS”-states. Remember that, according to geometric quantization, we should be looking for holomorphic sections that are normalizable with respect to the measure \eqref{6.25, 6.26}. A basis of such sections is given by $\left\{\prod_i z_i^{n_i}\right\}$, where $z_i$ are complex coordinates given in our case by \eqref{6.19} and $n_i$, are either integers or half-integers depending on the form of the integrand in \eqref{6.27} below. Essentially, one wants the integrand to be free of branch cuts due to possible square roots that can appear as a result of $\sqrt{\det \partial_i \partial_j \mathcal{K}}$. The number of states will be given by the number of possibilities to choose $\{n_i\}$ so that the integral

$$N_{n_i} \sim \int e^{-\mathcal{K}} \sqrt{\det \partial_i \partial_j \mathcal{K}} \prod_i |z_i|^{2n_i} \omega^n,$$

converges. To proceed further we notice that:

- Since our solution space is toric, one can integrate the $U(1)$ fibers trivially.
Chapter 6 - Setting the Stage for Fuzzballs

- The solution spaces we will quantize will either be compact or can be compactified by adding a harmless boundary. The latter happens in the case of scaling solutions or at walls of threshold stability (section 6.3.2). As a result, divergence of the integral (6.27) can only occur when the integrand becomes singular in the domain of integration.

- It is easy to see that the term \( \omega^n \) does neither vanish nor diverge in the domain of integration using its simple form given by (6.15).

These observations allow us to conclude that the integral (6.27) is convergent as far as its integrand without \( \omega^n \) is free from singularities. Using now the formulas (6.26), (6.19), (6.18) and (6.22), the potential problematic part in the integrand of (6.27) reads

\[
\prod_{a=1}^{m} \prod_{i=1}^{n} (n_i + 1/2) m_a c_{ai} - (m_a \lambda_a + 1/2) .
\]

(6.28)

It is clear from this expression that the absence of singularities translates to the requirement that the set of (half-)integers \( \{n_i\} \) should be chosen such that

\[
\sum_{i=1}^{n} (n_i + 1/2) m_a c_{ai} - (m_a \lambda_a + 1/2) > -1 .
\]

(6.29)

The degeneracy then will be given by the number of all possible ways to choose \( \{n_i\} \) such that (6.29) is satisfied.

So our main task in the following is to calculate the symplectic form \( \omega \) given by (6.3), then construct the polytope associated to the solution space. This can be achieved by choosing appropriate coordinates such that \( \omega \) takes the form \( \omega = \sum dx_i \wedge d\theta_i \), where \( \theta_i \) are the U(1) fiber directions. Finally, we use the formula (6.29) to figure out the number of states.

### 6.3 Simple Bound Black Hole Systems

In this section, we start by “quantizing” the simplest systems possible: the two and three-center solutions. It turns out that these spaces of solutions are toric Kähler manifolds which allow us to use the techniques developed in the previous section. We will be explicit about the different steps of quantization in these two examples to illustrate the general framework discussed in the previous section.
6.3.1 The Two-Center Case

The two-center case is easy to describe. There is only a regular bound state for \( \langle \Gamma_1, \Gamma_2 \rangle \neq 0 \) and \( \langle h, \Gamma \rangle \neq 0 \), and the constraint equations immediately tell us that \( x_{12} \) is fixed and given by

\[
x_{12} = \frac{\langle h, \Gamma_1 \rangle}{\langle \Gamma_1, \Gamma_2 \rangle}.
\]

(6.30)

In other words, \( \vec{x}_1 - \vec{x}_2 \) is a vector of fixed length but its direction is not constrained. The solution space is simply the two-sphere, and the symplectic form is proportional to the standard volume form on the two-sphere. In terms of standard spherical coordinates it is given by

\[
\omega = \frac{1}{2} \langle \Gamma_1, \Gamma_2 \rangle \sin \theta \, d\theta \wedge d\phi \sim j \sin \theta \, d\theta \wedge d\phi = -d(j \cos \theta) \wedge d\phi,
\]

(6.31)

where \( j = |J| \). Comparing this expression with (6.15) suggests to choose \( x = j \cos \theta \), which clearly satisfies \(-j \leq x \leq j\). As a result, our solution space \( S^2 \) is a toric Kähler manifold with the associated polytope given by

\[
j - x \geq 0, \quad j + x \geq 0.
\]

(6.32)

Let us proceed with the quantization of the two-center solution space. We start with the construction of the complex variable \( z \) using the general formula (6.19). We get in this case

\[
z^2 = \frac{1 + \cos \theta}{1 - \cos \theta} e^{2i\phi}.
\]

(6.33)

Next, the Kähler potential corresponding to \( \omega \) can be shown to be

\[
K = -2j \log(\sin \theta) = -j \log \left( \frac{z\bar{z}}{(1 + z\bar{z})^2} \right).
\]

(6.34)

The holomorphic coordinate \( z \) represents a section of the line-bundle \( L \) (over \( S^2 \), the solution space) whose first Chern class equals \( \omega/(2\pi) \). The Hilbert space consists of normalizable holomorphic sections of this line bundle, and a basis of these is given by \( \psi_m(z) = z^m \). Taking fermions into account (section 6.2.2), the norm of \( \psi_m \) is

\[
|\psi_m|^2 \sim \int d\text{vol} \, e^{-K} \sqrt{\det \partial_i \partial_j K} |\psi_m(z)|^2,
\]

(6.35)

where \( d\text{vol} \) is the volume form induced by the symplectic form. In our case, we therefore find

\[
|\psi_m|^2 \sim \int d\cos \theta \, d\phi \left( J + m \right) |j - m + 1|^2,
\]

(6.36)

and clearly \( \psi_m \) only has a finite norm if: \(-|J| \leq m < |J|\). The total number of states equals \( 2|J| \), which is in agreement with the wall-crossing formula \([30]\) (see also section 5.4.2). This result is the same as (6.29) applied to this case of two-center solution.
6.3.2 THE THREE-CENTER CASE

The three-center case is the next non-trivial solution space. Already at this level some new physics emerge. This is the possibility to have “scaling” solutions, which play a distinguished role in trying to construct the geometries of black hole states. These scaling solutions correspond to the case where it is possible for the centers to be arbitrarily close to each other. This section contains three parts. We start by a description of the three-center solution space, some details are left to appendix F. Then, the quantization procedure is described. At the end, a comparison with wall crossing formula [30] will be discussed.

DESCRIPTING THE SOLUTION

The three-center solution space is four dimensional. Placing one center at the origin (fixing the translational degrees of freedom) leaves six coordinate degrees of freedom, but, these are constrained by two equations. This leaves four degrees of freedom, of which three correspond to rotations in SO(3) and one of which is related to the separation of the centers.

The constraint equations take the form

\[
\frac{a}{u} - \frac{b}{v} = \frac{\Gamma_{12}}{r_{12}} - \frac{\Gamma_{31}}{r_{31}} = \langle h, \Gamma_1 \rangle =: \alpha ,
\]

\[
\frac{b}{v} - \frac{c}{w} = \frac{\Gamma_{31}}{r_{31}} - \frac{\Gamma_{23}}{r_{23}} = \langle h, \Gamma_3 \rangle =: -\beta ,
\]

in a self-evident notation. The nature of the solution space simplifies considerably if either \(\alpha\) or \(\beta\) vanish, so let us first consider this case. If both vanish, there is an overall scaling degree of freedom and the centers are unbound. This corresponds to a degenerate symplectic form, and is thus, not amenable to quantization using the methods described in this chapter. We have already argued that most likely quantizing these solutions by adding velocities will not probably not lead to BPS states (see section 6.1.1 for more details).

For definiteness, we will take \(\alpha = 0\); in this case \(\sum_p \langle h, \Gamma_p \rangle = 0\) which implies \(\langle h, \Gamma_2 \rangle = \beta\). Thus from (5.42) we find

\[
\vec{J} = \frac{\beta}{2} r_{23} \hat{z} ,
\]

with \(\hat{z}\) defined to be parallel to \(\vec{x}_2 - \vec{x}_3\).

The solution has an angular momentum vector \(\vec{J}\) directed between the centers 2 and 3, and the direction of this vector defines an \(S^2\) in the phase space which we
will coordinatize using $\theta$ and $\phi$. The third center is free to rotate around the axis defined by this vector (since this does not change any of the inter-center distances) providing an additional $U(1)$, which we will coordinatize by an angle $\sigma$, fibred non-trivially over the $S^2$. Finally, the angular momentum has a length, which may be bounded from both below and above, and this provides the final coordinate in the phase space, $j = |\vec{J}|$. This construction is perhaps not the most obvious one from a spacetime perspective but, as we will see, in these coordinates the symplectic form takes a simple and convenient form. When $\alpha = 0$ it is clear from (6.39) that $j$ is a good coordinate on the solution space but, this is not immediately obvious for the more complicated case of $\alpha \neq 0$. This is nonetheless true and, as shown in appendix F, this is always a good coordinatization of the three center solution space (though for $\alpha \neq 0$ the relation between $(j, \sigma, \theta, \phi)$ and the coordinates $\vec{x}_p$ is not as straightforward).

Before quantizing these solution spaces, let us first spend some time describing some physically important special cases of them: the so called “scaling solutions” and “solutions at walls of marginal stability”.

**Scaling Solutions**

As noted in [169] and [30], for certain choices of charges it is possible to have points in the solution space where the centers approach each other arbitrarily closely. Moreover, this occurs for any choice of moduli so it is, in fact, a property of the charges alone. As a consequence, it is not clear how to understand them in the context of attractor flows; the techniques we develop in this chapter provide an alternative method to quantize these solutions that applies even when the attractor tree does not allow us to determine the number of states.

Such solutions occur as follows. We take the leading behavior of inter-center distances to be $r_{ab} \sim \lambda \Gamma_{ab}$ for $\lambda \ll 1$. Clearly, this is only possible if there exists an ordering of the $a, b$ indices such that $\Gamma_{ab} > 0$, and if the positive $\Gamma_{ab}$ satisfy the triangle inequality. In general it is not clear if the solution exists once higher order terms in $\lambda$ are included. However, a detailed analysis of the three-center case in appendix F shows that this is true.

We will in general refer to such solutions as scaling solutions meaning, in particular, supergravity solutions corresponding to $\lambda \sim 0$. The space of supergravity solutions continuously connected (by varying the $\vec{x}_p$ continuously) to such solutions will be referred to as scaling solution spaces. We will, however, occasionally lapse and use the term scaling solution to refer to the entire solution space connected to a scaling solution. We hope the reader will be able to determine, from the context, whether a specific supergravity solution or an entire solution space is intended.
These scaling solutions are interesting because (a) they exist for all values of the moduli; (b) the coordinate distances between the centers go to zero; and (c) an infinite throat forms as the scale factor in the metric blows up as $\lambda^{-2}$. Combining (b) and (c) we see that, although the centers naively collapse on top of each other, the actual metric distance between them remains finite in the $\lambda \to 0$ limit. In this limit, an infinite throat develops looking much like the throat of a single center black hole with the same charge as the total charge of all the centers. Moreover, as this configuration exists at any value of the moduli, it looks a lot more like a single centered black hole (when the latter exists) than generic non-scaling solutions which do not exist for all the values of the moduli at infinity (see section 5.4). On the intuitive level, one can understand such distinction as follows. For non-scaling solutions, there is a minimum inter-center distance which, in principle, allows us to distinguish the different centers as we approach them. On the other hand, for the scaling solutions the centers disappear into the deep throat which makes it harder and harder to distinguish them from a single center black hole.

Unlike the throat of a normal single center black hole the bottom of the scaling throat has non-trivial structure. If the charges, $\Gamma_a$, do not carry intrinsic entropy (e.g. D6's with Abelian flux) then the five-dimensional uplifts of these solutions will yield smooth solutions in some duality frame and the throat will not end in a horizon but will be everywhere smooth, even at the bottom of the throat. Outside the throat, however, such solutions are essentially indistinguishable from single center black holes. Thus, such solutions have been argued to be ideal candidate black hole states geometries corresponding to single center black holes.

**Barely Bound Centers**

For certain values of charges and moduli it is possible for some centers to move off to infinity. Although this would normally signal the decay of any associated states (as happens, for instance, for two centers at a wall of marginal stability [122]) one can argue that this is not always the case [56]. In particular, it is important to distinguish between cases when centers are forced to infinity (marginal stability) versus those where there is simply an infinite (flat) direction in the solution space (threshold stability; see [64, Appendix B]). Although the first case clearly signals the decay of a state, in the second case, when centers move off to infinity along one direction of the solution space but may also stay within a finite distance in other regions of the solution space, it is still possible to have bound states. Quite essential to this argument is the fact that in some cases, although the solution space may seem naively non-compact (in the standard metric on $\mathbb{R}^{2N-2}$), its symplectic volume is actually finite and it admits normalizable wave-functions whose expectation values can be argued to be finite [56, section 7]. There are also cases with unbound centers.
where the symplectic form on the solution space is degenerate and, in such cases, it is not clear if there is a bound state. See section 6.1.1 for a thorough discussion on this point.

**Quantization**

Following the discussion above (see also appendix F), we parametrize the three-center solution space by the coordinates \(j, \sigma, \theta\) and \(\phi\). However, for the purpose of deriving the symplectic form, it turns out to be more convenient to work momentarily with the variables \(J^i\) and \(\sigma\), where \(\sigma\) represents an angular coordinate for rotations around the \(J\)-axis. Obviously, \(\sigma\) does not correspond to a globally well-defined coordinate, but rather should be viewed as a local coordinate on an \(S^1\)-bundle over the space of allowed angular momenta. Ignoring this fact for now, the rotation \(\delta x_p^i = \varepsilon_{ab} n^a x_p^b\) that we used in (6.4) corresponds to the vector field

\[
X_n = \frac{n^i J^i}{|J|} \frac{\partial}{\partial \sigma} + \varepsilon^{ijk} n^j J^k \frac{\partial}{\partial J^i}.
\]

(6.40)

The second term is obvious, as \(J\) is rotated in the same way as the \(x_p\). The first term merely states that there is also a rotation around the \(J\)-axis given by the component of \(n\) in the \(J\)-direction. The final result in (6.4) therefore states that

\[
\omega(X_n, m^i \frac{\partial}{\partial J^i}) = n^i m^i, \quad \omega(X_n, \frac{\partial}{\partial \sigma}) = 0.
\]

(6.41)

It is now easy to determine that

\[
\omega\left(\frac{\partial}{\partial J^i}, \frac{\partial}{\partial J^j}\right) = \varepsilon^{ijk} J^k |J|^2, \quad \omega\left(\frac{\partial}{\partial J^i}, \frac{\partial}{\partial \sigma}\right) = -\frac{J^i}{|J|}.
\]

(6.42)

Denoting \(\vec{J}\) as \(j\), and parametrizing \(J^i\) in terms of \(j\) and the standard spherical coordinates \(\theta, \phi\), the symplectic form defined by (6.42) becomes

\[
\omega = j \sin \theta \, d\theta \wedge d\phi - dj \wedge d\sigma.
\]

(6.43)

However, we clearly made a mistake since this two-form is not closed. The mistake was that \(\sigma\) was not a well-defined global coordinate but rather a coordinate on an \(S^1\)-bundle. We can take this into account by including a parallel transport in \(\sigma\) when we change \(J^i\). The result at the end of the day is that the symplectic form is modified to

\[
\omega = -d(j \cos \theta) \wedge d\phi - dj \wedge d\sigma.
\]

(6.44)

This very simple form of the symplectic form explains why it is more natural to work with the coordinates \(j, \sigma, \theta\) and \(\phi\). However, in order to quantize the solution space,
we have to understand what the range of the variables is. Since \( \theta, \phi \) are standard spherical coordinates on \( S^2 \), \( \phi \) is a good coordinate but degenerates at \( \theta = 0, \pi \). The magnitude of the angular momentum vector \( j \) is bounded as can be seen from (F.2). By carefully examining the various possibilities in the three-center case (see appendix F), one finds that generically \( j \) takes values in an interval \( j \in [j_-, j_+] \), where \( j = j_- \) or \( j = j_+ \), only if the three centers lie on a straight line. An exceptional case is if \( j_- = 0 \), implying that the three centers can sit arbitrarily close to each other as seen in appendix F. Note that this latter case corresponds exactly to the scaling solutions described above.

As we mentioned before, at \( j = j_- \) and \( j = j_+ \) the centers align, and rotations around the \( J \)-axis act trivially. In other words, at \( j = j_\pm \) the circle parametrized by \( \sigma \) degenerates. Actually, we have to be quite careful in determining exactly which \( U(1) \) degenerates where. Fortunately, what we have here is a toric Kähler manifold, with the two \( U(1) \) actions given by translations in \( \phi \) and \( \sigma \), and we can use results in the theory of toric Kähler manifolds from section 6.2.1 to describe the quantization of this space. We have to distinguish two cases: \( j_- = 0 \) which corresponds to a scaling point inside the solution space, and \( j_- > 0 \) where the scaling point is absent.

\[
\begin{align*}
  y &= j \cos \theta \\
  x &= j \\
  j_+ &\quad j_-
\end{align*}
\]

**Figure 6.1:** (Left) The polytope for \( j_- = 0 \). (Right) The polytope for \( j_- > 0 \).

**The Non Scaling Case**  We start by defining \( x = j \), and \( y = j \cos \theta \), to be two coordinates on the plane. Then the ranges of the variables \( x \) and \( y \) are given by

\[
 x - j_- \geq 0, \quad j_+ - x \geq 0, \quad x - y \geq 0, \quad x + y \geq 0.
\]  \hspace{1cm} (6.45)
Together, these four inequalities define a Delzant polytope in $\mathbb{R}^2$ which completely specifies the toric manifold (see section 6.2.1). At the edges a $U(1)$ degenerates, at the vertices both $U(1)$'s degenerate. The geometry and quantization of the solution space can be done purely in terms of the combinatorial data of the polytope (see figure 6.1). To proceed we assume that all three centers carry different charges; if two centers carry identical charges, one needs to take into account their indistinguishability, quantum mechanically, and take a quotient of the corresponding solution space. We will not consider this possibility in the remainder.

In order to quantize our space of solution, we start by the construction of complex coordinates following (6.19). Up to irrelevant numerical factors, they are given by

$$z^2 = j^2 \sin^2 \theta \frac{j-j}{j^*} e^{2i\sigma},$$

$$w^2 = \frac{1 + \cos \theta}{1 - \cos \theta} e^{2i\phi} \tag{6.46}$$

and the Kähler potential ends up being equal to

$$K = j_- \log(j-j_-) - j_+ \log(j_+ - j) + 2j. \tag{6.47}$$

A basis for the Hilbert space is given by wave functions $\psi_{m,n} = z^m w^n$. To find the range of $n, m$ we look at the norm (6.25)

$$\int e^{-2n} \frac{(j_+ - r)^{j_+ - 1} (r^2 \sin^2 \theta \frac{r-j_-}{j_+ - r})^n}{(r-j_-)^{j_- - 1}} \left( \frac{1 + \cos \theta}{1 - \cos \theta} \right)^{m+1/2} r \, dr \, d\cos \theta,$$  

where we disregarded an overall non-vanishing smooth function. This norm is finite if, $j_- \leq n < j_+$, and, $-n \leq m + \frac{1}{2} \leq n$, leading to the following number of states

$$\mathcal{N} = (j_+ - j)(j_+ + j_-). \tag{6.49}$$

which is in agreement with the wall crossing formula as we will show below in section 6.3.3. One can check easily that the same result is obtained using (6.29).

The Scaling Case As was done in the non-scaling case, we must first construct the appropriate polytope for these solutions (see figure 6.1). The only property that differentiates these solution spaces is that $j_- = 0$ (this is the scaling point). As a result, the associated polytope differs slightly from the non-scaling one; for instance, the first inequality in (6.45) is redundant. This may seem to be a small modification, but, it actually changes the topology of the solution space as follows. Remember that, for fixed $j$ and $\sigma$ there is an $S^2$ that we parametrized by $\theta$ and $\phi$. Taking the limit from non-scaling to scaling corresponds to the limit $j \to 0$, which leads to the vanishing of this $S^2$. 

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Using the coordinates $x = j$ and $y = j \cos \theta$ as before, the scaling solution’s polytope is defined by

$$j_+ - x \geq 0, \quad x + y \geq 0, \quad x - y \geq 0.$$  \hfill (6.50)

The construction of the complex coordinates is achieved through the function $g$ (6.18). They turn out to be given by

$$z_1^2 = \frac{j^2 \sin^2 \theta}{j_+ - j} e^{2i\sigma}, \quad z_2^2 = \frac{1 + \cos \theta}{1 - \cos \theta} e^{2i\phi}.$$  \hfill (6.51)

Note that the complex variable $z_1$ in this case is not the naive $j_+ \to 0$ limit of the non-scaling complex variable counterpart $z$ given by the first equation in (6.46). The wave functions that belong to the Hilbert space are the ones that have a finite norm (6.27), which in this case reads

$$|\psi_{n,m}|^2 \sim \int e^{-\frac{j(j_+ - x)}{2} - \frac{1}{2} - n} \, j^{2n+1}$$

$$(1 + \cos \theta)^{n+(m+1/2)} (1 - \cos \theta)^{n-(m+1/2)} \, dj \, d\cos \theta.$$  \hfill (6.52)

Requiring that the norm is finite imposes the following restrictions

$$0 \leq n \leq j_+ - 1, \quad -n \leq m + 1/2 \leq n,$$  \hfill (6.53)

which can be reproduced using (6.29). So the number of states is given by

$$\mathcal{N} = j_+^2.$$  

Unfortunately, we cannot compare this prediction to the result obtained from the wall-crossing formula, because it is not clear how to treat scaling solutions within the framework of the attractor flow conjecture [30]. On the other hand, this proves the usefulness of the tools we developed in this chapter as they provide the only known way to compute the number of BPS states for scaling solutions.

Another important property that is worth mentioning is that the probability density, given by the integrand of (6.52), vanishes at $j = 0$. This suggests that, although classically the coordinate locations of the centers can be arbitrarily close together, quantum mechanically this is not true anymore. The probability that the centers sit on top of each other is zero, which implies that there is a minimum non-vanishing expected inter-center distance. Since the depth of the throat is related to the coordinate distance between the centers, it follows that the throat will be effectively capped off once quantum effects are taken into account [56].

### 6.3.3 Comparison to the Split Attractor Flow Picture

In the previous subsections we computed the number of states corresponding to the position degrees of freedom of a given set of bound black hole centers. The approach
we developed amounts essentially to calculating the appropriate symplectic volume of the solution space. To count the total number of BPS states of a given total charge, one needs to take into account the fact that the different black hole centers may themselves carry internal degrees of freedom and that there may be many multi-center realizations of the same total charge. In the special case, however, when all the centers have no internal degrees of freedom, the only states that one can get are position degrees of freedom. In this case it is interesting to compare the number of states obtained in our approach, using geometric quantization, with the number obtained by considering jumps at walls of marginal stability as in [30] (see also section 5.4.2).

To make this comparison, we use the attractor flow conjecture which states that, to each component of solution space there corresponds a unique attractor flow tree. Given a component of solution space, we can calculate its symplectic volume and hence the number of states. Given the corresponding attractor flow tree, we can calculate the degeneracy using the wall crossing formula of [30] (equation (6.54) below).

As mentioned before, in the two-center case we get a perfect agreement between the two calculations. This is not so surprising because both approaches are, in fact, counting the number of states in an angular momentum multiplet with $j = \frac{1}{2}(\Gamma_1, \Gamma_2) - \frac{1}{2}$. Furthermore, there is no ambiguity in specifying the split attractor tree. Things become more interesting in the three-center case where there are now naively three attractor trees for a given set of centers. According to the attractor flow conjecture, only one tree should correspond to any given solution space.

Let us consider the three-center attractor flow tree depicted in figure 5.1. For the given charges, $\Gamma_1$, $\Gamma_2$, and $\Gamma_3$, there are three possible trees, but in terms of determining the relevant number of states, the only thing that matters is the branching order. In figure 5.1 the first branching is into charges $\Gamma_3$ and $\Gamma_4 = \Gamma_1 + \Gamma_2$, so the degeneracy associated with this split is $|\langle \Gamma_4, \Gamma_3 \rangle|$, and the degeneracy of the second split is $|\langle \Gamma_1, \Gamma_2 \rangle|$, giving a total number of states

$$N_{\text{tree}} = |\Gamma_{12}| |(\Gamma_{13} + \Gamma_{23})|,$$

where we have adopted an abbreviated notation, $\Gamma_{ij} = \langle \Gamma_i, \Gamma_j \rangle$.

To compare this with the number of states arising from geometric quantization of the solution space, (6.49), we need to determine $j_+$ and $j_-$. As described in Appendix F, $j_+$ and $j_-$ correspond to two different collinear arrangements of the centers and, in a connected solution space, there can be only two such configurations. To relate this to a given attractor flow tree, we will assume that we can tune the moduli to force the centers into two clusters as dictated by the tree. For the configuration in figure 5.1, for instance, this implies we can move the moduli at infinity close to the first
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wall of marginal stability (the horizon dark blue line) which will force $\Gamma_3$ very far apart from $\Gamma_1$ and $\Gamma_2$. In this regime, it is clear that the only collinear configurations are $\Gamma_1\cdot\Gamma_2\cdot\Gamma_3$ and $\Gamma_2\cdot\Gamma_1\cdot\Gamma_3$; it is not possible to have $\Gamma_3$ in between the other two charges. Since $j_+$ and $j_-$ always correspond to collinear configurations, they must, up to signs, each be one of

\[ j_1 = \frac{1}{2}(\Gamma_{12} + \Gamma_{13} + \Gamma_{23}) , \]
\[ j_2 = \frac{1}{2}(\Gamma_{12} - \Gamma_{13} + \Gamma_{23}) . \]

$j_+$ will correspond to the larger of $j_1$ and $j_2$, and $j_-$ to the smaller but, from the form of (6.49), we see that this does not effect $\mathcal{N}$ as it depends only on $\lvert j_1^2 - j_2^2 \rvert$. Thus

\[ \mathcal{N} = \lvert (j_1 - j_2)(j_1 + j_2) \rvert = \lvert \Gamma_{12} \rvert \lvert (\Gamma_{13} + \Gamma_{23}) \rvert , \]

which nicely matches (6.54).

Let us make some further remarks on the results derived here. The scaling solutions corresponding to $\lambda \to 0$ have $j_- = 0$ even if the centers do not align at this point. Therefore, the connection to the wall crossing formula breaks down. The procedure of geometric quantization itself, however, is well defined for these solutions. The curvature scales always stay small allowing us to trust the supergravity solutions. Thus, one can see the resulting degeneracy as a reliable prediction.

6.4 Dipole Halos

Although the three-center case showed various interesting features, however, one would like to find other systems for which the quantization described in section 6.2 is applicable. After all, our main motivation was the study of systems which can be possibly related to black hole states. A first step in this direction would be to find quantizable systems with an exponentially large number of states. Luckily, there is a multi-center solution that, in some regime of charges, can be argued to be close enough to a single centered black hole [116, 117, 118]. These are a special class of what was called “dipole halo” solutions in [56]. The latter correspond to a purely fluxed D6-D6 pair (hence the name dipole) bound with an arbitrary number of anti-D0’s (which explains the halo appelation). Depending on the sign of the B-field, these D0’s bind to the D6 or anti-D6 respectively. When we take the B-field to be zero at infinity, the system is at the wall of threshold stability and the D0’s are free to move in the equidistant plane between the D6 and anti-D6. This system and its behavior under variations of the asymptotic moduli was studied in detail in appendix B of [64].
These dipole halo systems come in two variations, scaling and non-scaling dipole halos, depending on the regime of their charges. As we will argue later on, the scaling regime corresponds to \( j \to 0 \). By maximizing \( \cos \theta_a \)'s in (6.62) below, it is clear that the scaling behavior can only be present if the total D0 charge \( N \) satisfies \( N \geq I/2 \), where \( I = -\langle \Gamma_6 \Gamma_\bar{b} \rangle \).

Before discussing the interesting scaling dipole solutions, we first quantize the non-scaling ones. Our aim in studying such solution spaces is twofold. On one hand, it will be a nice opportunity to deal with orbifold toric geometry and check the validity of (6.29). On the other hand, since the solution space does not develop a scaling behavior, we can compare the degeneracy we get to the one expected from the wall crossing formula.

### 6.4.1 Meet the Dipole Halo

As the number of states will be independent of the asymptotic moduli, as long as we don't cross a wall of marginal stability, we are free to choose them such that the solution space has its simplest form. The symplectic form on the solution space is most easily calculated at the line of threshold stability discussed in the subsection “barely bound centers” of section 6.3.2. In our example, this corresponds to \( B|_\infty = 0 \) [64]. The dipole halo system is comprised of:

- **The dipole part**: this is a pair of purely fluxed D6 and \( \overline{D6} \) with charges

  \[
  \Gamma_6 = e^{\frac{1}{\hat{p}} A \alpha A} = \alpha_0 + \frac{1}{2} p^A \alpha_A + \frac{1}{8} D_{ABC} p^A p^B \alpha_C + \frac{1}{48} D_{ABC} p^A p^B p^C, \\
  \Gamma_\bar{6} = -e^{\frac{1}{\hat{p}} \bar{A} \alpha A} = -\alpha_0 + \frac{1}{2} \bar{p}^\Lambda \alpha_\Lambda - \frac{1}{8} D_{ABC} \bar{p}^A \bar{p}^B \alpha_C + \frac{1}{48} D_{ABC} \bar{p}^A \bar{p}^B \bar{p}^C,
  \]

  (6.58)

  (6.59)

  where we used the Harmonic basis \( \alpha_\Lambda, \alpha_\Lambda; \Lambda = 0, 1, \ldots, h^{1,1} \) of \( H^* \), the even-cohomology of the Calabi-Yau threefold (see section 5.1.2), and \( D_{ABC} = \int_{CY} \alpha_A \alpha_B \alpha_C \). In the following, we will use the shorthand notation \( \Gamma_6 = (1, \frac{\hat{p}^A}{7}, \frac{\hat{p}^A}{8}, \frac{\hat{p}^A}{38}) \), and \( \Gamma_\bar{6} = (-1, \frac{\bar{p}^\Lambda}{7}, \frac{-\bar{p}^\Lambda}{8}, \frac{-\bar{p}^\Lambda}{38}) \) for the purely fluxed D6 and \( \overline{D6} \) charges. It is clear from the choice of the flux of the D6 and \( \overline{D6} \) that, the total D6 charge vanishes whereas the total D4 charge is \( p^A; A = 1, \ldots, h^{1,1} \). That is the reason behind using dipole in the name of such system.

- **The halo part**: this corresponds to a set of D0’s with charges \( \Gamma_a = (0, 0, 0, -q_a) \) with all the \( q_a \) positive and \( \sum_a q_a = N \). These D0’s do not talk to each other, hence the name halo. Their positions are constrained through their interaction with the D6 and \( \overline{D6} \).
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The positions of D0’s, D6, and D6̄ are constrained by (5.41), which reduces in the present case to:
\[ -\frac{q_a}{r_{6a}} + \frac{q_a}{r_{6a}} = 0, \]  
\[ -\frac{I}{r_{6\bar{6}}} + \sum_a \frac{q_a}{r_{6a}} = -\beta. \]  

Here \( I = -\langle \Gamma_6, \Gamma_{\bar{6}} \rangle = \frac{e^2}{\pi} \) is given in terms of the total D4-charge \( p \) of the system, and \( \beta = \langle \Gamma_6, h \rangle \) with \( I, \beta > 0 \). From the first line we indeed see that the D0’s lie in the plane equidistant from the D6 and D6̄, as we are at the line of threshold stability, and so we can simply write \( r_a := r_{6a} = r_{6a}^\chi \).

An explicit expression for the symplectic form (6.3) can be obtained using the following coordinate system. We define an orthonormal frame \((\hat{u}, \hat{v}, \hat{w})\) fixed to the D6-D6̄ pair, such that the D6-D6̄ lie along the \( w \) axis and with the D0’s lying in the \( u-v \) plane. Rotations of the system can then be interpreted as rotations of the \((\hat{u}, \hat{v}, \hat{w})\) frame with respect to a fixed \((\hat{x}, \hat{y}, \hat{z})\) frame. We will parametrize the position of the \( w \) axis (D6-D6̄ line) by two angles, \( (\theta, \phi) \) as shown in figure 6.2. We can furthermore specify the location of the \( a \)’th D0 with respect to D6-D6̄ pair by two additional angles, \( (\theta_a, \phi_a) \). The first angle, \( \theta_a \), is the one between \( \vec{x}_{6\bar{6}} \) and \( \vec{x}_{6a} \), while \( \phi_a \) is a polar angle in the \( u-v \) plane (see figure 6.2). Our \( 2N + 2 \) independent coordinates on solution space are thus \( \{\theta, \phi, \theta_1, \phi_1, \ldots, \theta_N, \phi_N\} \). The angular momentum, \( j(\theta_a, \phi_a) \), is a function of the other coordinates rather than an independent coordinate (when \( N = 1 \), it can be traded for \( \theta_1 \) as demonstrated in the general three center case), and is given by
\[ j = \frac{I}{2} - \sum_a q_a \cos \theta_a. \]  

Using this explicit coordinatization, it is straightforward though tedious to evaluate the symplectic form (6.3). The result is relatively simple:
\[ \omega = -\frac{1}{4} d \left[ 2j \cos \theta \, d\phi + 2 \sum_a q_a \cos \theta_a \, d\sigma_a \right], \]  

with \( d \) denoting the exterior derivative. Note that, as is manifest from our angular coordinatization, we are still in the toric setting with each additional center introducing an additional \( U(1) \) coordinate. Note also that, for a single D0 this reduces to (6.44) when \( \theta_1 \) is traded for \( j \).

In case \( N \geq I/2 \), it is possible to combine a sufficient number of centers and form a scaling throat. We will restrict ourselves here to the non-scaling case \( I/2 > N \), leaving the more interesting scaling regime to the next section.
Figure 6.2: The coordinate system used to derive the D6-D6-N D0 symplectic form. The coordinates \((\theta, \phi)\) define the orientation of the \(\hat{w}\)-axis with respect to the fixed, reference, \((\hat{x}, \hat{y}, \hat{z})\) axis. The D6-D6 lie along the \(\hat{w}\) axis (with the origin between them) and the D0's lie on the \(\hat{u}\-\hat{v}\) plane at an angle \(\phi_a\) from the \(\hat{u}\)-axis. The radial position of each D0 in the \(\hat{u}\-\hat{v}\) plane is encoded in the angle \(\theta_a\) (between \(\vec{x}_a\) and \(\vec{x}_{w_a}\)).
6.4.2 DEGENERACY USING ATTRACTOR TREE

In the following, we are going to combine the wall-crossing formula and solution space considerations to count the number of states of the dipole halo system. The essential idea in this approach is that, by playing with moduli at infinity we can deform our dipole halo tree \( \{ \Gamma_2, \{ \Gamma_1, N \Gamma_\star \} \} \) to a halo one corresponding to the charge \( \Gamma_1 \) surrounded by \( n_a \) centers carrying the charges \( a \Gamma_\star \) plus a far away center \( \Gamma_2 \). The former system \( \{ \Gamma_1, N \Gamma_\star \} \) will be called a "halo" in the following. In doing so, we reduced our task of counting the BPS states to evaluating the number of states \( D_N^{\text{halo}} \) coming from the halo part \( \{ \Gamma_1, N \Gamma_\star \} \). To find this number of states we resort to solution space considerations. By knowing this number, the total degeneracy will be

\[
D_N = (\langle \Gamma_2, \Gamma_1 \rangle + N \langle \Gamma_2, \Gamma_\star \rangle) D_N^{\text{halo}}.
\]

Note that to derive this degeneracy, we use the attractor flow conjecture which only works for non-scaling solutions, so for scaling solutions we will have to resort to other methods. For other values of the moduli, the center \( \Gamma_2 \) will be closer to the halo and deform it. In certain cases, it can even deform so much that the topology of the split changes to \( \{ \Gamma_1, \{ \Gamma_2, N \Gamma_\star \} \} \). Such a change can happen when crossing a wall of threshold stability [64, 30], and in that case the number of states does not change (even though the topology of the tree changes).

The halo configurations are characterized by a split tree of the form \( \{ \Gamma_1, N \Gamma_\star \} \) with \( N = \sum q_a n_a \), where \( n_a \) is the number of centers of charge \( q_a \Gamma_\star \). As all \( q_a \) and \( n_a \) are positive integers, every halo of total charge \( N \Gamma_\star \) thus corresponds to a specific partitioning of \( N \) and vice versa. It follows straightforwardly from the constraint equations (5.41) that all the \( \Gamma_i \) centers orbit \( \Gamma_1 \) at the same distance \( r_{1i} = l \), given by

\[
l = \frac{\langle \Gamma_\star, \Gamma_1 \rangle}{\langle h, \Gamma_\star \rangle}.
\]

Note that this radius is independent of the different \( a_i \). Furthermore, all the centers \( \Gamma_i \) can be placed arbitrarily on this sphere surrounding \( \Gamma_1 \) as they do not interact among each other.

So, for a halo configuration of \( m \) orbiting centers, the solution space is simply the product of \( m \) identical \( S^2 \)'s. When quantizing this system we have to take into account that in case some \( a_i = a_j \), the corresponding centers should be treated as indistinguishable particles and we will have to quotient by the appropriate permutation group. As all centers in the Halo are independent, there is a standard way to get the degeneracy. Let us sketch the idea in the simplest case of a halo consisting of \( N \) equally charged centers, i.e all \( a_i = 1 \). Given that the \( N \) particles are independent and identical, the Hilbert space for all the particles is nothing but the
(anti-)symmetrized product of the one particle Hilbert space, i.e. $\mathcal{H}_N = \mathcal{H}^N/\mathcal{S}_N$. By the definition of $\mathcal{H}_1$, this is the space of sections of $\mathcal{L}^N/\mathcal{S}_N$, where $\mathcal{L}_1$ is the one-particle line bundle. So, we can take the one particle (2-center) solution space and construct a multi-particle wave function on it.

The single particle wave function is a section of a line-bundle $\mathcal{L}$ on $\mathbb{S}^2$, the position space of the electron. The line bundle $\mathcal{L}$ has Euler number $|\langle \Gamma_1, \Gamma_\star \rangle|$. This is because the highest Chern character of a line bundle is the first one. To generalize this to $N$ particles, we have to tensor the bundle $N$ times and (anti-)symmetrize due to indistinguishability. The Euler number of the resulting line bundle then gives the number of sections, i.e. the number of states. It is more convenient to summarize these numbers for different $N$ in terms of a generating function. Such a generating function for Euler numbers of the symmetric product of a line bundle was given in e.g. [170]. The only difference with the discussion there is that, in our case, the line bundles are fermionic in nature. Taking this point into account properly, following e.g. [171], gives the following generating function

$$\sum_N d_N q^N = (1 + q)^{|\langle \Gamma_1, \Gamma_\star \rangle|}, \quad (6.65)$$

where $d_N$ stands for the Euler number of the $N$th symmetric product $\mathcal{L}^N/\mathcal{S}_N$. Newton’s binomial expansion yields

$$d_N = \binom{|\langle \Gamma_1, \Gamma_\star \rangle|}{N}. \quad (6.66)$$

The fact that the different centers in the halo behave like fermions results in an upper bound of $|\langle \Gamma_1, \Gamma_\star \rangle|$ for the number of such centers; this is nothing but Pauli’s exclusion principle at work.

The generating function (6.65) can be generalized to include halos given by an arbitrary partition $\{n_a\}$ of $N$, i.e. $N = \sum_a a n_a$, where $n_a$ is the number of centers carrying the same charge $\Gamma_a = a \Gamma_\star$. It is not hard to see that the generating function including such arbitrary partitions is given by

$$\sum_N D_N^\text{halo} q^N = \prod_{k > 0} (1 + q^k)^{k|\langle \Gamma_1, \Gamma_\star \rangle|}, \quad (6.67)$$

where $D_N$ is the degeneracy of all halos with total charge $N \Gamma_\star$. The degeneracies can be found by expanding the product:

$$D_N = \sum_P \prod_a \binom{a|\langle \Gamma_1, \Gamma_\star \rangle|}{n_a}, \quad (6.68)$$

where the sum is over all possible partitions $P = \{n_a\}$ of $N$ i.e. $N = \sum_a a n_a$. This agrees with the fact that the total degeneracy of a given partition is the product of
the degeneracies of each group of identical terms in the partition. The degeneracies of such a group of identical halo charges was exactly what we calculated in (6.65) to be a binomial coefficient.

We close this subsection by the following remark. The formula (6.67) is similar to (5.6) in [30], but, there are also some obvious differences. The first one comes from the fact that [30] is calculating an index while we are counting the number of states without relative signs. The second difference is that we are neglecting the degeneracies associated to internal degrees of freedom of each individual center.

\section{Degeneracy Using Toric Techniques}

In the following, we are going to use the same toric techniques as in the previous sections to calculate the number of states associated to the non-scaling D6-D6-D0 dipole halo, i.e. those for which $N < I/2$. From the associated symplectic form (6.63), one easily reads the toric coordinates to be

\begin{equation}
\begin{aligned}
y &= j \cos \theta, \\
x_a &= q_a \cos \theta_a \geq 0.
\end{aligned}
\end{equation}

(6.69)

Because some of the centers can be identical, we need to orbifold our polytope by the appropriate symmetric group. Before treating the configuration given by a generic partitioning \( \{ q_a \} \) of \( N \) i.e. \( N = \sum_a n_a q_a \), let us focus on the simple case of \( n \) centers carrying the same D0 charge, \(-q\), so that \( N = q n \). In this case the labelling of the facets (section 6.2.1) turns out to be

- four facets with label 1 given by
  
  i) the facet \( x_1 = 0 \),

  ii) the facet \( x_n = q \),

  iii) the facet \( y = \frac{I}{2} - \sum_a x_a \),

  iv) the facet \( y = -\frac{I}{2} + \sum_a x_a \),

- \((n - 1)\) facets with label 2 given by

  v) the facets \( x_{a+1} - x_a = 0 \), for \( a = 1, \ldots, n - 1 \).
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Given this labelled polytope, we can then again construct the corresponding complex coordinates following the strategy outlined in section 6.2.1. In this case we have

\[ z_0^2 \sim \frac{I/2 + y - \sum a x_a}{I/2 - y - \sum a x_a} e^{2i\sigma_0}, \]

\[ z_1^2 \sim \frac{I/2 + y - \sum a x_a}{I/2 - y - \sum a x_a} \left(\frac{x_1}{x_2 - x_1}\right)^2 e^{2i\sigma_1}, \]

\[ z_n^2 \sim \frac{I/2 + y - \sum a x_a}{I/2 - y - \sum a x_a} \left(\frac{x_n - x_{n-1}}{q - x_n}\right)^2 e^{2i\sigma_n}, \]

(6.70)

\[ z_i^2 \sim \frac{I/2 + y - \sum a x_a}{I/2 - y - \sum a x_a} \left(\frac{x_i - x_{i-1}}{x_{i+1} - x_i}\right)^2 e^{2i\sigma_i}, \quad i = 2, \ldots, n - 1. \]

The next step is to construct the Kähler potential, which turns out to be equal to

\[ K = -\frac{I}{2} \ln \left(\frac{I}{2} + y - \sum a x_a\right) \left(\frac{I}{2} - y - \sum a x_a\right) - 2 \sum a x_a + x_n - x_1 - q \ln(q - x_n). \]

(6.71)

A basis for the Hilbert space is then given by normalizable functions \( \psi_m = \prod_{i=0}^{m_i} z_i^{m_i} \), where the norm is given by (6.25). In addition, \( \det \partial_i \partial_{\bar{j}} g \) turns out to be given by

\[ \det \partial_i \partial_{\bar{j}} g \sim \left(\frac{I}{2} + y - \sum a x_a\right)^{-1} \left(\frac{I}{2} - y - \sum a x_a\right)^{-1} \left(x_1\right)^{-1} (q - x_n)^{-1} \prod_{a=1}^{n-1} (x_{a+1} - x_a)^{-1}. \]

The study of the normalizability of \( \psi_m \) reveals the following constraint on the possible exponents \( m = (m_i) \):

\[ 0 \leq m_1 < m_2 < \ldots < m_n < q, \]

\[- \left(\frac{I-1}{2} - \sum a \left(m_a + \frac{1}{2}\right)\right) \leq m_0 + \frac{1}{2} \leq \left(\frac{I-1}{2} - \sum a \left(m_a + \frac{1}{2}\right)\right), \]

(6.72)

where \( n \) is the number of D0-centers carrying charge \( q \), in perfect agreement with the restriction of (6.29) to this case. The total number of normalizable states is thus

\[ d_{n,q} = \sum_{m=n(n-1)/2}^{(I-n-1)/2} b_n^m(q)(I - n - 2m), \]

(6.73)

where the coefficient \( b_n^m(q) \) is the number of ways to write \( m \) as a sum of \( n \) strictly ordered positive integers all smaller than \( q \).

Let us now generalize the simple example of \( n \) equally charged D0-centers to an arbitrary partition of \( N \). We will label the different groups of equally charged centers by an index \( a \), and the charge of individual centers in this group by \( q_a \) (i.e. \( q_a \neq q_b \).
if and only if $a \neq b$). With $n_a$ we then denote the number of centers with charge $q_a$ so that the total D0-charge $N$ carried by the D0 centers is given by

$$N = \sum_n n_a q_a .$$

Labeling the centers in a given group $a$ by an additional index $i = 1, \cdots, n_a$, we can simply generalize the conditions (6.72) by applying them to each group of equally charged centers separately. The conditions on the powers $m^a = (m^a_i)$ then become

$$0 \leq m^a_1 < m^a_2 < \ldots < m^a_{n_a} < q_a ,$$

$$\left[ \frac{I-1}{2} - \sum_{a,i} (m^a_i + \frac{1}{2}) \right] \leq m_0 + \frac{1}{2} \leq \left[ \frac{I-1}{2} - \sum_{a,i} (m^a_i + \frac{1}{2}) \right] . \quad (6.74)$$

A first step towards counting all possible states with total charge $N$ in D0-centers is to count the degeneracy for a fixed partitioning of $N$. The number of solutions to the constraints (6.74) is given by

$$d_{\{n_a,q_a\}} = \sum_{\text{all allowed } m^a_i} \left[ I - 2 \sum_{a,i} \left( m^a_i + \frac{1}{2} \right) \right] = \prod_a \left( q_a n_a \right) - 2 \sum_{\text{all allowed } m^a_i} \sum_i \left( m^a_i + \frac{1}{2} \right) . \quad (6.75)$$

We can calculate the sum of the last terms by introducing the quantities

$$l^a_i = q_a - 1 - m^a_{n_a-i} ,$$

and noting that then

$$0 \leq l^a_1 < l^a_2 < \ldots < l^a_{n_a} < q_a , \quad (6.76)$$

$$\sum_{\text{all allowed } m^a_i} \sum_{a,i} \left( m^a_i + \frac{1}{2} \right) = N \prod_a \left( \frac{q_a}{n_a} \right) - \sum_{\text{all allowed } l^a_i} \sum_{i,a} \left( l^a_i + \frac{1}{2} \right) , \quad (6.77)$$

where we used that $\sum_a n_a q_a = N$. As $l^a_i$ and $m^a_i$ satisfy the same conditions, equation (6.77) simply implies that

$$\sum_{\text{all possible } m^a_i} \sum_i \left( m^a_i + \frac{1}{2} \right) = \sum_{\text{all possible } l^a_i} \sum_i \left( l^a_i + \frac{1}{2} \right) = \frac{N}{2} \left( \frac{q_a}{n_a} \right) .$$

Using this last equality, we see that the number of states (6.75) is given by

$$d_{\{n_a,q_a\}} = (I - N) \prod_a \left( \frac{q_a}{n_a} \right) = \left( I - \sum_a n_a q_a \right) \prod_a \left( \frac{q_a}{n_a} \right) . \quad (6.78)$$
So, we indeed find back the result we derived using attractor tree arguments: the degeneracy is that of the corresponding halo, multiplied by \( \langle \Gamma_1, \Gamma_2 \rangle + N \langle \Gamma_1, \Gamma_* \rangle = I - N \).

Counting all the different degeneracies for all possible partitions of a given total halo charge \( N \Gamma_* \), gives rise to the following generating function:

\[
Z(q) = \sum_N D_N q^N = \left( \sum_a (I - \sum_n n_a q_a) \prod_n \left( q_a \binom{q}{n_a} q^{n_a q_a} \right) \right) \left( I - q \frac{\partial}{\partial q} \right) \prod_k (1 + q^k)^k.
\]  

Using this generating function, we can estimate the large \( N \) growth of \( D_N \). This turns out to be of the form \( \log D_N \sim N^{2/3} \), modulo logarithmic corrections. Note that, although we find an exponential number of states, the \( N^{2/3} \) scaling of the entropy is far less than the expected horizon area from supergravity for a black hole with charges \( I/4 < N < I/2 \), which is of the order \( N \) (for \( N < I/4 \) no single centered black hole exists with this total charge [30]). It would, however, be extremely interesting to do a similar counting for scaling solutions (\( N \geq I/2 \)), which do admit single center black hole realizations, and compare this to the black hole entropy. This will be the subject of the next section.

### 6.5 Scaling Solutions and Fuzzballs

In the previous section, the solution space associated with a D6-D6 pair (with the intersection product \( I = \langle \Gamma_6, \Gamma_6 \rangle \)) surrounded by a “halo” of \( N \) D0's was quantized in the non-scaling regime, and the entropy was determined to grow as \( S \sim N^{2/3} \). Non-scaling implies that \( N < I/2 \) whereas scaling solutions satisfy \( N > I/2 \).

Earlier arguments in the literature [169, 30] have suggested that scaling solutions carry vastly more entropy and may account for a large fraction of the black hole entropy. Here, we will see that this is not the case, at least for this large class of solutions. Rather, we will see that the (leading) entropy coming from these solutions matches that of free gravitons in AdS$_3$. The change in the leading degeneracy between the non-scaling and scaling regime seems to precisely take into account a bound on possible BPS quantum numbers [17], \( \tilde{h} \leq c/24 \).
6.5.1 D6-\(\overline{D6}\)-D0 Crush and its Quantization

The scaling regime of the D6-\(\overline{D6}\)-D0 system is physically relevant as it is conjectured to correspond to the geometric manifestation of D4-D0 black hole states \[118\]. As discussed in the previous section, for the purpose of counting the degeneracy one can study the system at the wall of threshold stability \[64\]. This can be done thanks to the independence of the scaling solutions from the moduli at infinity. In the following, the threshold point is assumed throughout all the calculations.

The difference between the polytopes associated to the dipole halo in the scaling and non-scaling case resides in two important modifications:

- In the scaling case \( N > I/2 \), the D0’s angles \( \theta_a \) do not span the whole range \([0, \pi/2]\). There is a lower bound on the possible \( \theta_a \)’s as \( j \) defined in (6.62) is positive by definition, which leads to a non-trivial constraint as \( \sum_a q_a = N > I/2 \) in the scaling case.

- For D0 centers that carry a D0 charge \( q_a > I/2 \), the facet \( x_a = q_a \) is not part of the polytope anymore.

Note that \( j \sim 0 \) corresponds to the scaling point. Notice also that, strictly speaking \( j \) never becomes zero as this corresponds to D6 and \( \overline{D6} \) sitting on top of each other. However, we are going to include the point \( j = 0 \) in the following to compactify our polytope.

So, our polytope in the case of scaling dipole halo, neglecting for the moment the possibility of having centers that carry the same D0 charge, is given by

\[-j \leq y \leq j, \quad 0 \leq x_a \leq \min \left\{ q_a, \frac{I}{2} \right\} , \quad (6.80)\]

where \( y \) and \( x_a \) are defined in (6.69). Notice that the constraint:

\[ j = \frac{I}{2} - \sum_a x_a \geq 0 , \quad (6.81)\]

is implied by the first inequality in (6.80). The quantization goes through the same steps as before. Taking the right orbifold version of the naive polytope above, one ends up with the following constraints on (half-)integers \((m, \{m^a_{n_a}\})\) using (6.29)

\[0 \leq m_1^a < m_2^a < \ldots < m_{n_a}^a < q_a ; \forall a , \quad \sum_{a,i} \left( m_i^a + \frac{1}{2} \right) \leq \frac{I-1}{2} \quad (6.82)\]

\[-\left[ \frac{I-1}{2} - \sum_{a,i} \left( m_i^a + \frac{1}{2} \right) \right] \leq m + \frac{1}{2} \leq \left[ \frac{I-1}{2} - \sum_{a,i} \left( m_i^a + \frac{1}{2} \right) \right] \quad (6.83)\]
where \( i_a \) labels the \( n_a \) centers that carry the same charge \( q_a \). We wish to make two observations at this point:

- The constraints on (half-)integers \((m, \{m_a^i\})\) that we got here are similar to the ones in the non-scaling case (6.74) except for an extra condition, which is the second inequality in (6.82). The latter is a consequence of the first inequality in the same equation (6.82). However, since it implies a non-trivial condition, and it is necessary for the consistency of the counting, we should treat it on the same footing as the other constraints. That is why we included it above as part of the constraints.

- The upper bound in the first inequality of (6.82) should be \( \text{Min} \{q_a, \frac{I}{2}\} \), and not \( q_a \). However, a little thought reveals that keeping it as it is written in (6.82) will not alter the counting of states, which will be the subject of the next subsection.

### 6.5.2 Not Enough States

The complication in the scaling regime arises because of the second constraint in equation (6.82). To proceed, let us introduce the quantity

\[
M = \sum_{a,i} \left( m_a^i + \frac{1}{2} \right),
\]

where \( M \) takes both integer and half-integer values. Using that the \( m_a^i \) are the discrete analogues of the classical \( q_a \cos \theta_a \), the interpretation of \( M \) is as the amount of angular momentum carried by the D0 centers (which by the geometry is always opposite in direction to the angular momentum carried by the D6\(_D\) pair):

\[
M = \frac{I - 1}{2} - J.
\]

Such an equality implies that \( M \) is bounded by \( 1/2 \leq M \leq (I - 1)/2 \). It is not hard to find the following approximate generating function

\[
Z(q, y) = \sum_{N,M} d_{N,M} q^N y^M = (I - 2y\partial_y) \prod_{k \geq 1, 0 \leq l \leq k} \left( 1 + q^k y^{l-1/2} \right),
\]

where \( N \) corresponds to the total D0 halo charge, and \( M \) stands for the value of (6.85). The word approximate generating function reflects the fact that the actual degeneracy is given by:

\[
D_{N,I} = \sum_{M=1/2}^{M=(I-1)/2} d_{N,M},
\]
and not $d_{N,I}$ as is familiar from the usual definition of a generating function. Since we are interested in the leading behavior of the entropy, it will be enough to maximize $d_{N,M}$ with respect to $M$. This is like going from the canonical to the microcanonical ensemble, which is a valid transition for large quantum numbers. For a fixed $M$, we get the following entropy for $M,N \gg 1$

$$S(N,M) \sim (\alpha M |N - M|)^{1/3}$$

where $\alpha = \frac{3}{4} \zeta(3)$. Maximizing $S(N,M)$ over $M$ in the range $0 < M < I/2$, we find that

$$S(N) = \begin{cases} 
\left(\frac{\alpha N^2}{4}\right)^{1/3} & \text{if } N \leq I . \\
\left(\alpha \frac{I}{2}(N - \frac{I}{2})\right)^{1/3} & \text{if } I \leq N .
\end{cases}$$

A surprising and physically interesting behavior is that, entropy is dominated by $M \sim I/2$ in the case $N > I$ which corresponds to $j \sim 0$ i.e. the scaling point. This suggests that, for large enough D0 charge, most of the states of the scaling dipole halo correspond to the D0 charges localized deep inside the throat. Such a behavior suggests that, there is a phase transition where for a large enough D0 charge, a single centered black hole dominates over a multi-center solution following our proposal to identify scaling solutions as potential black hole states. It will be very interesting to study this observation in more detail.

Another intriguing observation is that, in the regime of charges where we trust supergravity $N \gg I \gg 1$, the entropy above looks asymptotically the same as the horizon area of the corresponding black hole except that, we have a wrong power: $1/3$ instead of $1/2$. This raises the following question: “do we need to include other solutions with the same asymptotic charges—probably even solutions not belonging to the class of multi-center black holes of $N = 2$ four dimensional supergravity that we are considering—or is this the best that supergravity can do?” To answer this question, we will give below an approximate upper bound on the number of BPS states that we can get from supergravity.

### 6.5.3 Beyond Supergravity?

The approach we will take to get an estimate of the degrees of freedom contained in 'supergravity', is to exploit the fact that both the D4D0 black hole and the D6D6D0 systems and generalizations can be studied in asymptotically AdS space by the decoupling limit of [64]. Here, we will do a counting of states with the same total charge but in the limit of vanishing backreaction. The advantage of this calculation is that, in this limit where the supergravity fields can be treated as free excitations around a fixed $\text{AdS}_3 \times S^2 \times \text{CY}_3$ background, the BPS states arrange themselves as
chiral primary multiplets of the (0,4) superconformal isometry group and hence, allow us to do a precise calculation of all supergravity states with a given total D4-D0 charge. As was shown in detail in e.g. [172], it is most convenient to KK-reduce the eleven-dimensional supergravity fields on the compact $S^2 \times CY_3$ space to fields living on AdS$_3$, where CY$_3$ is the Calabi-Yau threefold. Note that we will assume the size of the CY$_3$ to be much smaller than that of the $S^2$ so that we will only consider the massless spectrum on the Calabi-Yau, while keeping track of the full tower of massive harmonic modes on the sphere. In this case, all states arrange themselves in a set of harmonic towers of chiral primaries, fully determined by the isometries and the original field content, and all BPS states can be enumerated directly using these algebraic constraints.

Following [64], we are looking for the number of states with these CFT quantum numbers

$$L_0 = N, \quad \bar{L}_0 = \frac{I}{2}, \quad J_3 = -J.$$  

(6.90)

One recognizes the states as the Ramond ground states as expected for BPS states. The calculation of the KK-spectrum on AdS$_3$ is however most naturally phrased in the NS sector. The map between the two sectors, R and NS, is called “spectral flow” [173]. After performing the spectral flow in the right-moving sector, the charges (6.90) become (see e.g [64] for some details)

$$L_0 = N, \quad \bar{L}_0 = \frac{I}{2} - J, \quad J_3 = \frac{I}{2} - J = M,$$  

(6.91)

where in the last equation we used (6.85). As expected, our BPS states manifest themselves in the NS sector as chiral primaries, satisfying the condition $L_0 = J_3$.

The well known unitarity bound [173] on the R-charge of chiral primaries translates itself into a bounded range for the four-dimensional angular momentum:

$$0 \leq J \leq \frac{I}{2}$$  

(6.92)

Using the identification of $M$ and $L_0$, the analogue of the generating function (6.86) is

$$Z = Tr_{NS} q^{L_0} y^{\bar{L}_0}.$$  

(6.93)

To calculate the degeneracies we are interested in, we need to enumerate the possible BPS states. As we have only supersymmetry on the right, there are no BPS constraints on the left, and hence, the leftmoving fields can be descendants of any highest weight states. On the right we have $N = 4$ supersymmetry and the BPS states have to be chiral primaries of a given weight. As a consequence, and as was shown in detail in e.g [172], [174], [175], the full spectrum arranges itself in several towers of the form

$$\{s, \tilde{h}_{\text{min}}\} = \oplus_{n \geq 0} \oplus_{\tilde{h} \geq \tilde{h}_{\text{min}}} (L_{-1})^n |\tilde{h} + s\rangle_L \otimes |\tilde{h}\rangle_R,$$  

(6.94)
where $|h\rangle_L$ are highest weight states of weight $h$ of the leftmoving Virasoro algebra, and $|\tilde{h}\rangle_R$ are weight $\tilde{h}$ chiral primaries of the rightmoving $\mathcal{N} = 4$ super-Virasoro algebra. Strictly speaking, we should include also descendants of $|h\rangle_L$ under the global $\mathcal{N} = 4$ superconformal algebra. We are not going to do so in the following, since it will only change the leading behavior of the entropy with an overall numerical factor. We should also include the so-called “singleton representation” but, their contribution is subleading, and hence, we will ignore them also (see e.g. [28]).

Each field of the five-dimensional supergravity gives rise to such a tower under KK-reduction, where essentially $\tilde{h}$ labels the different spherical harmonics, while $n$ labels momentum excitations in AdS$_3$. It was shown in [172], [174], [175] that given the precise field content of a particular $\mathcal{N} = 1$ supergravity in five dimensions, the reduction on a two-sphere gives the set of towers shown in table 6.1. For each such tower the partition function (6.93) has the following form (the total partition function is the product of those):

$$
Z_{\{s, \tilde{h}_{\text{min}}\}} = \prod_{n \geq 0} \prod_{m \geq 0} \left(1 - y^{m+\tilde{h}_{\text{min}}} q^{n+m+\tilde{h}_{\text{min}}+s} (-1)^{2s+1}\right). \quad (6.95)
$$

Following the same steps as above (section 6.5.2), one gets the following entropy

$$
S \sim (M(N-M))^{1/3}. \quad (6.96)
$$

This has the same wrong exponential as we found before (6.89). This suggests that we need extra degrees of freedom beyond the once obtainable from supergravity. We will further discuss the implication of this estimate in the conclusions.

### 6.6 Large Scale Quantum Effects

Although it has long been understood how to account for the number of black hole microstates in string theory [16], this has generally been done in a dual field theory making it difficult to address some fundamental questions in black hole quantum mechanics, such as information loss via Hawking radiation. For some microscopic black holes (such as those discussed in chapter 3), the ability to dualize to an FP
system has allowed for a more detailed analysis of the structure of the microstates. For these black holes, it has been argued \[33\] that the average microstate is a highly quantum superposition of states with the corresponding spacetime a wildly fluctuating “fuzzball”. The very interesting part of this claim is that, these fluctuations extend over a region of spacetime circumscribed by the putative black hole horizon. The “metrics” corresponding to the states in the superposition are all very different within the region which would be enclosed by a horizon in the naive black hole solution, but, they settle down very quickly to the same metric outside the horizon. Thus, the remarkable claim of \[33\] is that the generic state in the black hole ensemble has quantum fluctuations over a large region of spacetime reaching all the way to the black hole horizon.

Unfortunately, the black hole discussed in \[33\] is microscopic and has no horizon in supergravity (without higher derivative corrections); it would thus be very desirable to be able to demonstrate this type of behavior in a system with a macroscopic black hole. In \[56\] an attempt was made to do exactly this. Scaling multi-center solutions can classically form arbitrarily deep throats that become infinitely deep in the strict $\lambda \rightarrow 0$ limit, where the coordinate separation of the centers vanishes. We expect, however, that quantum effects will prohibit us from localizing the centers arbitrarily close together and will thus, effectively cap off the throat (see picture 6.3). we can trace this back to the fact that the symplectic form, and hence the quantum exclusion principle, is not renormalized as we increase $g_s$ to interpolate between quiver quantum mechanics and gravity. So, even though gravitational effects increase the depth of the throat as it forms, the phase space volume stays very small. Thus, gravitational back-reaction essentially blows up these quantum effects to a macroscopic scale. This is important not only because it is reminiscent of the large scale quantum fluctuations of the D1-D5 black hole, but, also because a smooth geometry with an infinite throat would be hard to understand in the context of AdS/CFT for the following reason. Solution spaces with a scaling point persist and continue to exhibit scaling behavior even after we take a decoupling limit making all the solutions asymptotically AdS$_3 \times S^5$. This is problematic as general arguments suggest that an infinitely deep throat in a smooth geometry that is asymptotically AdS would imply a continuous spectrum in the CFT \[176\]. Thus, it is comforting that the analysis of \[56\] reveals the infinite throat to be an artifact of the classical limit. Indeed, this is precisely the kind of phenomenon that was suggested in \[108\].

Before discussing this phenomena in more detail let us note some caveats. The states defined by quantizing the scaling solutions spaces are not necessarily generic black hole microstates. In fact, the discussion in the previous section suggests that such states require including additional stringy degrees of freedom in the phase space, so they may not reflect the behavior of the actual black hole ensemble. Also, the symplectic form was computed in the gauge theory and extended to gravity
via a supersymmetric non-renormalization theorem; it would be more insightful to have a direct supergravity computation of the symplectic form. These caveats notwithstanding, it is remarkable that these solutions exhibit quantum structure on a large scale even though they are smooth with a small curvature everywhere.

What is actually determined in [56] is the effective minimum distance between the centers at the scaling point. Specifically, a three center solution similar to the one described in section 6.3.2 with a pure fluxed D6-D6 pair and a single D0 with charge “−N” is considered in its state that is localized as much as possible down the throat, and the expectation value of the harmonic $H_0$ and the D6-D0 separation is computed. The latter is shown to be of order $\epsilon \sim N/I \geq 1/2$, implying that the centers cannot be localized arbitrarily close to each other so an infinite throat never forms. Rather, the geometry is effectively capped off at a scale set by the D0-D6 distance (see picture 6.3).

While the computation above is heuristic in many ways it yields two very important qualitative lessons. The first is that quantization of these solution spaces as phase spaces resolves several classical paradoxes such as infinitely deep throats and also clarifies the issue of bound states (see [56] for a discussion of this). More importantly, however, it demonstrates that classical solutions may be invalid even though they do not suffer from large curvature scales or singularities. This is an important
point so let us explore it further.

In this particular system the phase space structure of the supergravity theory can be related to that of quiver quantum mechanics by a non-renormalization theorem. In the latter, the scaling solutions (at weak coupling) are analogous to electron-monopole bound states. Heisenberg uncertainty implies that the minimum inter-center distance is of order $x_{ij} \sim \hbar$. Moreover, because the solution space is a *phase space* rather than a configuration space, the coordinates are conjugate to other coordinates rather than velocities, so it is not possible to localize all coordinate directions with arbitrary precision by constructing delta-function states. Thus this quantity will have a large variance so $\delta x_{ij}/x_{ij} \sim 1$ for very small $x_{ij}$. At weak coupling this is nothing more than the standard uncertainty principle and is not particularly surprising.

What is surprising is that this behavior persists even once gravity becomes strong and the centers backreact stretching the infinitesimal coordinate distance between them to a macroscopic metric distance. Moreover, in this regime, the depth of the throat is extremely sensitive to the precise value of $x_{ij}$ (see [176] for a numerical example), thus, the large relative value of $\delta x_{ij}$ translates into wildly varying depths for the associated throat. The associated expectation values for any component of the metric have an extremely large variance $\delta g/g$ and so cannot possibly correspond to good semi-classical states. It is somewhat unusual to have classical configurations that cannot be well approximated by semi-classical states (i.e. those with low variance) but here, this can be seen to follow from the very small phase space volume this class of classical solutions occupy [108].