APPENDIX A

SQUEEZED STATES AND NEGATIVE ENERGY DENSITY

The horizon vacuum in Hawking calculation belongs to a special kind of states called "squeezed states" [42]. Their importance for us resides in that they may exhibit negative energy densities, which is important for black hole evaporation. In the following, we are going to review some of their properties that we will be needing in the bulk of the thesis. For more detailed study of their properties see [180] for example.

A.1 SQUEEZED STATES

In the following, we are going to restrict ourselves to a single mode. A general squeezed state takes the form [181, 182, 183]

\[ |z, \xi\rangle = D(z) S(\xi) |0\rangle , \]

(A.1)

where \( D(z) \) is the displacement operator

\[ D(z) = \exp(z a^\dagger - z^* a) = e^{-|z|^2/2} e^{za^\dagger} e^{-z^* a} = e^{z a^\dagger} e^{-z^* a} e^{za^\dagger} , \]

(A.2)

and \( S(\xi) \) is the squeeze operator

\[ S(\zeta) = \exp \left[ \frac{1}{2} (\zeta^* a^2 - \zeta (a^\dagger)^2) \right] . \]

(A.3)
Appendix A - Squeezed States and Negative Energy Density

It is easy to prove that $D(z)$ and $S(\xi)$ satisfy

$$
D^\dagger(z) a D(z) = a + z , \quad (A.4)
$$

$$
D^\dagger(z) a^\dagger D(z) = a^\dagger + z^* , \quad (A.5)
$$

$$
S^\dagger(\zeta) a S(\zeta) = a \cosh |\zeta| - a^\dagger e^{i\theta} \sinh |\zeta| , \quad (A.6)
$$

$$
S^\dagger(\zeta) a^\dagger S(\zeta) = a^\dagger \cosh |\zeta| - a e^{-i\theta} \sinh |\zeta| , \quad (A.7)
$$

where $\zeta = |\zeta| e^{i\theta}$. When $\zeta = 0$, we have the familiar coherent state, so in a way, the squeezed states are a generalization of them. Actually for $\zeta$ real ($\theta = 0$), they saturate the uncertainty inequality

$$
(\Delta x)^2 (\Delta p)^2 = |1 + i \sin \theta \sinh 2|\zeta||^2 . \quad (A.8)
$$

A.2 Negative Energy Density

The other extreme case occurs when $z = 0$, and is called the “squeezed vacuum”. This is precisely the state of the Hawking radiation field (section 1.3.2). It turns out that these states always have a negative energy density somewhere in spacetime. In flat spacetime one finds for real $\zeta$

$$
\mathcal{H} = \frac{1}{2} \omega + \sinh |\zeta| \left( \sinh |\zeta| - \cosh |\zeta| [2\omega x^2 - 1] \right) , \quad (A.9)
$$

which is clearly negative for large enough $x$

$$
\mathcal{H} < \frac{1}{2} \omega \left( 1 - 4 \sinh^2 |\zeta| x^2 \right) \quad \text{For } x^2 > \frac{1}{2\omega} . \quad (A.10)
$$

However, when integrated over the whole space one finds that

$$
\langle H \rangle = \frac{1}{2} \omega + \sinh^2 |\zeta| . \quad (A.11)
$$
In this appendix, we will collect some facts about ten dimensional $\mathcal{N} = 2$ supergravity theories and some of their important properties. We start by reviewing the action of the two types, type-IIA and type-IIB. Then the solution describing a D-brane will be relayed. At the end, some formulas describing the action of T and S-duality on background fields will be given.

This appendix should not be taken as an introduction of any sort to the above mentioned subjects. The reader is already assumed to have some elementary knowledge of string theory. For background material one can consult standard books on string theory e.g. [184, 185, 186].

### B.1 Ten-Dimensional Supergravity

There are two types of $\mathcal{N} = 2$ supergravity in ten dimensions that describe the low energy effective action of type-II string theories. The field content of these supergravity theories is the massless spectrum of the associated superstring theory. We summarize the field content in table B.1 below to set up notation.
Appendix B - A Quick Trip in Ten Dimensions

<table>
<thead>
<tr>
<th>Type</th>
<th>Neveu-Schwarz (NS-NS)</th>
<th>Ramond-Ramond (R-R)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type-II.A</td>
<td>( G_{\mu\nu} ) (graviton), ( \phi ) (dilaton), ( B_{\mu\nu} )</td>
<td>( C_\mu, C_{\mu\nu}; H^{(0)} ) (non propagating)</td>
</tr>
<tr>
<td>Type-II.B</td>
<td>( G_{\mu\nu} ) (graviton), ( \phi ) (dilaton), ( B_{\mu\nu} )</td>
<td>( C, C_{\mu\nu}, C_{\mu\nu\rho\tau} )</td>
</tr>
</tbody>
</table>

Table B.1: The massless spectrum of type-II string theories. The fields \( C^{(n)} \) are form field potentials. NS-NS and R-R stand for two of the four possible sectors in closed superstrings.

## B.1.1 Type-IIA Supergravity

The massless spectrum of type-IIA superstring contains the graviton and its two superpartners that have opposite chiralities. In addition to these fields, the bosonic fields include also the dilaton and the B-field coming from the NS-NS sector, the different p-forms coming from the R-R sector, namely \( C^{(1)} \) and \( C^{(3)} \) (see the table B.1). The effective action has the following bosonic part (the wedge product is understood):

\[
S_{II.A} = S_{NS} + S_R + S_{CS},
\]

\[
S_{NS} = \frac{1}{2\kappa_0^2} \int d^{10}x \left( -G^{1/2} e^{-2\phi} \left[ R + 4(\nabla\phi)^2 - \frac{1}{2} (H^{(3)})^2 \right] \right)
\]

\[
S_R = -\frac{1}{4\kappa_0^2} \int d^{10}x \left( -G^{1/2} \left[ (G^{(2)})^2 + (G^{(4)})^2 \right] \right)
\]

\[
S_{CS} = -\frac{1}{4\kappa_0^2} \int B^{(2)} \, dC^{(3)} \, dC^{(3)}.
\]

where \( S_{NS} \) is the contribution of the NS-NS sector, \( S_R \) comes from the R-R sector, \( S_{CS} \) is a Chern-Simons term, \( G_{\mu\nu} \) is the metric, \( \phi \) is the dilaton, \( H^{(3)} = dB^{(2)} \) is the field strength of the NS-NS 2-form, while the R-R field strengths are \( G^{(2)} = dC^{(1)} \), and \( G^{(4)} = dC^{(3)} + H^{(3)} \wedge C^{(1)} \).

## B.1.2 Type-IIB Supergravity

This theory is chiral as the two gravitinos, the super-partners of the graviton, have the same chirality. The bosonic field content of this theory differs from the previous one in the R-R sector. Here, one has the following p-form potentials: a scalar \( C^{(0)} \), \( C^{(2)} \), and \( C^{(4)} \) (see table B.1). Strictly speaking, we do not have a satisfactory action due to the self-duality of the field strength of \( C^{(4)} \). We are going to implement this condition at the level of the equations of motion as an extra constraint. In this case,
the action reads:

\[ S_{11B} = S_{NS} + S_R + S_{CS} \]

\[ S_{NS} = \frac{1}{2\kappa_0^2} \int d^{10}x \ (-G)^{1/2} e^{-2\phi} \left[ R + 4(\nabla \phi)^2 - \frac{1}{2}(H^{(3)})^2 \right] \]

\[ S_R = -\frac{1}{4\kappa^2} \int d^{10}x \ (-G)^{1/2} \left\{ \frac{1}{12}(G^{(3)} - C^{(0)} H^{(3)})^2 + (dC^{(0)})^2 + \frac{1}{2}(G^{(5)})^2 \right\} \]

\[ S_{CS} = -\frac{1}{4\kappa^2} \int C^{(4)} H^{(3)} G^{(3)}. \]  

(B.2)

Now, \( G^{(3)} = dC^{(2)} \) and \( G^{(5)} = dC^{(4)} + \frac{1}{2} H^{(3)} C^{(2)} + \frac{1}{2} B^{(2)} G^{(3)}. \) Remember that we are imposing the self-duality condition on \( G^{(5)} \) by hand \( G^{(5)} = \ast G^{(5)}. \)

A point worth mentioning here is that, in the actions above the gravity part (the term proportional to the Ricci scalar \( R \)) is not of the canonical form; there is an extra \( \exp(-2\phi) \) multiplying it. The frame these actions are written in is called the “string frame”. One can go to the “Einstein frame” with the canonical action for gravity by rescaling the metric by an appropriate power of \( \exp(\phi) \) \( (\exp(-\phi/2) \) in ten dimensions).

### B.2 D-BRANES IN SUPERGRAVITY

D-brane can be seen as extended hypersurfaces where open strings can end. We will be calling them, as is custom in string theory, Dp-branes where \( p \) stands for the spatial extension of the D-brane. Type-IIA (IIB) string theory include in its spectrum D0, D2, D4, D6, D8 (respectively, D(-1), D1, D3, D5, D7, D9) branes.

A Dp-brane is massive and charged under the R-R \( (p+1) \)-form \( C^{(p+1)} \). Due to these properties, the presence of a D-brane will generate a non-trivial geometry. Taking into account such modification in the geometry is called “backreacting” the D-brane. There is another important property that these D-branes enjoy. In weakly coupled string theory, the so called “probe approximation”, it can be checked that straight D-branes preserve half of the total 32 possible supersymmetries in ten dimensions i.e. they are \( 1/2 \)-BPS solutions. Since supersymmetry is robust, they should, and indeed they do, preserve the same amount of supersymmetry when the string coupling constant is increased \[187\].
B.2.1 ELECTRIC AND MAGNETIC D-BRANES

Before spelling out the supergravity solution describing a Dp-brane, there is a small subtlety that we should take care of first. In the following, we set $B = 0$. From the point of view of the field strengths $G$ of the R-R forms $C$ described in table B.1, there are two kinds of D-branes: “electrically” charged and “magnetically” charged D-branes. This choice of naming will be clear in a moment. The R-R forms $C^{(n)}$ can be seen as a higher dimensional generalization of the Maxwell field $A_\mu$. Let us revisit the four dimensional Maxwell theory in the language of forms. The field strength $F = dA$ satisfies the following equation:

$$dF = 0, \quad d * F = * j$$

(B.3)

where $*$ is the Hodge dual, and $j$ is a current due to charged particles under $A$. Let us call these charges electrically charged particles. An example is an electron. In the vacuum $j = 0$, the field $\tilde{F} = *F$ satisfies the same equations as $F$. One can wonder if there are charged particles under a “dual” gauge field $\tilde{A}$ defined such that $d\tilde{A} = \tilde{F}$. If such a particle exists then, $\tilde{F}$ will satisfy the same equations as $F$ in (B.3). These new particles are described as being magnetically charged under the original field $A$. An example is the hypothetical monopole.

This story generalizes to the R-R forms. In this case, for the R-R form $C^{(p+1)}$, the magnetic D-brane is a D$(8-p)$-brane. In the following, when dealing with only the magnetic D$(8-p)$-brane, we will use the dual field $C^{(9-p)}$ instead of the original one $C^{(p+1)}$ in the action (B.1.1) or (B.1.2).

B.2.2 THE BACKREACTED Dp-BRANE

In looking for the solution corresponding to a Dp-brane, the only surviving part of the actions (B.1) and (B.2) is the NS-NS common part without the B-field and the kinetic term of the associated $C^{(p+1)}$ form:

$$S = \frac{1}{2\kappa_0^2} \int d^{10}x \left( -G \right)^{1/2} \left( e^{-2\phi} \left[ R + 4(\nabla \phi)^2 \right] - \frac{1}{2} dC^{(p+1)} \wedge *dC^{(p+1)} \right)$$

(B.4)

Calling the coordinates along the brane $x^\mu$ and the ones transverse to it $y^\alpha$, the solution describing the Dp-branes turns out to be:

$$ds^2 = H_p^{-1/2}(y) \eta_{\mu\nu} dx^\mu dx^\nu + H_p^{1/2}(y) \delta_{\alpha\beta} dy^\alpha dy^\beta$$

$$e^{2\phi} = g_s^2 H_p^{(3-p)/2}(y), \quad C^{(p+1)} = \frac{1 - H_p(y)}{g_s} dvol$$

(B.5)

where $H_p(y)$ is a harmonic function in the transverse space.
Appendix B - A Quick Trip in Ten Dimensions

B.3 T AND S DUALITIES

In the following we are going to touch upon two important dualities that will be of use in the third chapter of the thesis; T and S dualities. The first one relates the two types of closed string theory. The other, however, is a relation between strongly and weakly coupled type-IIB string theory.

B.3.1 T-DUALITY AND BUSCHER RULES

Studying the spectrum of perturbative bosonic closed string theory on $\mathbb{R}^{1,8} \times S^1$ reveals that it is invariant under sending the radius of $S^1$ to its inverse. Such a duality is called “T-duality” and generalizes to closed superstrings with a surprise: type-IIA is mapped to type-IIB and vice versa.

On the level of supergravity, T-duality requires a $U(1)$ isometry direction on which it is performed. Its action on the background fields is given by “Buscher rules” [188, 189, 190, 191]

\[\tilde{g}_{\mu\nu} = g_{\mu\nu} - \frac{g_{\mu\nu} g_{\mu\nu} - b_{\mu\nu} b_{\mu\nu}}{g_{yy}}, \quad \tilde{b}_{\mu\nu} = b_{\mu\nu} - \frac{b_{\mu\nu} g_{\mu\nu} - g_{yy} b_{\mu\nu}}{g_{yy}}\]

\[\tilde{C}^{(n)}_{\mu...\nu\alpha\beta} = C^{(n-1)}_{\mu...\nu\alpha\beta} - (n-1) C^{(n-1)}_{[\mu...\nu\alpha]} b_{\beta]\gamma} - n(n-1) \frac{C^{(n-1)}_{[\mu...\nu\alpha]} b_{[\gamma]y} b_{\beta]\gamma]}{g_{yy}}\]

where $y$ is the U(1) isometry direction, the tilde stand for the T-dual fields, $g_{\mu\nu}$ is the metric, $b_{\mu\nu}$ is the b-field, $\phi$ is the dilaton, and $C^{(n)}_{\mu...\nu}$ is a R-R $n$-form.

B.3.2 TYPE-IIB AND S-DUALITY

Looking back at the massless spectrum of type-IIB string theory (table B.1), we see a striking resemblance between the NS-NS and R-R fields: both of them contain a scalar (dilaton $\phi$ vs R-R zero-form $C^{(0)}$), and a two-form field (B-field $b_{\mu\nu}$ vs R-R two-form $C^{(2)}$). This observation raises the following question “Can we map these fields between themselves or is it just a coincidence?” It turns out that the answer to this question is yes, and this is due to an SL(2,$\mathbb{Z}$) symmetry that type-IIB string theory enjoys [192, 193]. Actually an SL(2,\mathbb{R}) symmetry is already visible in the classical
type-IIB supergravity low energy action (B.2), which is believed to be broken to its discrete version \( \text{SL}(2,\mathbb{Z}) \) once quantum and non-perturbative effects are included. The type-IIB effective action can be recast in an \( \text{SL}(2,\mathbb{R}) \) duality invariant way using the following field redefinition

\[
\tau = C^{(0)} + ie^{-\phi}
\]

and combining the two two-forms \( b_{\mu\nu} \) and \( C^{(2)} \) in an \( \text{SL}(2,\mathbb{Z}) \) doublet. The new field \( \tau \) then transforms in a fractional way,

\[
\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad \text{with } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2,\mathbb{Z})
\]

What interests us is a special duality transformation that goes under the name of “S-duality”. Its effect on type-IIB fields is the following

<table>
<thead>
<tr>
<th>Field Transformations</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau \to -1/\tau )</td>
<td>mixing of the dilaton ( \phi ) and the zero-form ( C^{(0)} )</td>
</tr>
<tr>
<td>( b_{\mu\nu} \to -C^{(2)} )</td>
<td>{\ F1\text{-string} \leftrightarrow D1\text{-brane} }</td>
</tr>
<tr>
<td>( C^{(2)} \to b_{\mu\nu} )</td>
<td>{\ NS5\text{-brane} \leftrightarrow D5\text{-brane} }</td>
</tr>
<tr>
<td>( C^{(4)} \to C^{(4)} )</td>
<td>D3-brane changes into itself.</td>
</tr>
</tbody>
</table>

In the case where there is no D(-1)-brane (\( C^{(0)} = 0 \)), the first part of the transformation becomes \( \phi \to -\phi \), which amounts to taking the string coupling constant \( g_s = e^\phi \) to its inverse. In this case the S-duality is just a weak/strong coupling duality.
APPENDIX C

THE D1-D5 GENERATING FUNCTION

When evaluating the different effective geometries (3.8) in the fourth chapter, one is led to computing the functions in (4.17). The latter can be evaluated once the following generating function is known

\[ f_v = \frac{Q_5}{4\pi^2L} \int d^4 u \int_0^L ds \int_{d,d} f(d,d) \frac{e^{i\alpha} e^{iu(x-F(s)) + ivF'(s)}}{|u|^2}, \]  

(C.1)

where the constant \( e^{\alpha} \) is due to normal ordering (section 4.2.2). To evaluate \( \alpha \) one starts with the expression of \( F(s) \) in terms of oscillators (3.19), then uses the identity

\[ e^{\alpha c^\dagger} e^{\beta c^\dagger} = e^{-\alpha\beta} e^{\beta c^\dagger} e^{\alpha c^\dagger}, \]  

(C.2)

valid for any operators \( c \) and \( c^\dagger \) satisfying \([c, c^\dagger] = I\). One finds that

\[ \alpha = \sum_k \left( \frac{|u|^2 \mu^2}{2k} + \frac{2\pi^2 \mu^2 |v|^2}{L^2} \right). \]  

(C.3)

The relation between the functions in (4.17) and the generating function (C.1) is easily found to be

\[ f_v(x) = 1 + f_v(x)|_{u=0}, \quad f_1(x) = 1 - \partial_\nu \partial_\nu f_v(x)|_{\nu=0}, \quad A_i(x) = -i \partial_\nu f_v(x)|_{\nu=0}. \]  

(C.4)

In the following, we are going to evaluate \( f_v \) in two special cases. The first case is a pure state describing an excited single rotating frequency mode, useful for the geometries discussed in sections 4.2.2 and 4.6. The second case is the generic thermodynamical ensemble, relevant for sections 4.3 and 4.4.
Appendix C - The D1-D5 Generating Function

C.1 Simple Phase Space Densities

It turns out that all our phase space densities \( f_{d,\bar{d}} \) that we will be dealing with in the fourth chapter take the following simple form

\[
f_{d,\bar{d}} = \prod_{k,\pm} f^{(k^\pm)}(|d_k^\pm|^2), \tag{C.5}
\]

where \( d_k^\pm = (d_k^1 \pm i d_k^2)/\sqrt{2} \). There is another contribution coming from the 34-plane which looks exactly the same as the expression above. Such contribution will be implicit in the following as we can always reconstruct it given the 12-plane contribution. This form of the phase space density allows us to simplify the expression of \( f_v \) in (C.1) as follows. First, one Fourier transforms the \( x \) dependence using

\[
\frac{1}{|x|^2} = \frac{1}{4\pi^2} \int d^4 u \frac{e^{i u \cdot x}}{|u|^2}. \tag{C.6}
\]

Then, using that the operators \( c_k^\dagger \) commute with the operators \( c_l \) for different modes \( (k \neq l) \), one can rewrite (C.1) as

\[
f_v = \frac{Q_5}{4\pi^2 L} \int \frac{d^4 u}{|u|^2} e^{i(u \cdot x)} \int_0^L ds \sum_k e^{\alpha_k} \prod_{\pm} \int_{d_k,\bar{d}_k} f^{(k^\pm)}(|d_k^\pm|^2) e^{-i u \cdot x} \gamma_k (d_k^\pm e^{i \frac{2\pi s}{L}} + \bar{d}_k e^{-i \frac{2\pi s}{L}}) - v^\pm \lambda_k (d_k^\pm e^{i \frac{2\pi s}{L}} - \bar{d}_k e^{-i \frac{2\pi s}{L}}), \tag{C.7}
\]

where \( \alpha_k \) is the restriction of \( \alpha \) (C.3) to the \( k \)th oscillator, \( a^\pm = (a^1 \pm i a^2)/\sqrt{2} \), and:

\[
\gamma_k = \frac{\mu}{\sqrt{2} k}, \quad \lambda_k = \frac{\pi \mu}{L} \sqrt{2 k}. \tag{C.8}
\]

Next, integrating over all possible values of the complex numbers \( d_k^\pm \) allows us to absorb the \( s \)-dependent phase, in the expression above, in the definition of \( d_k^\pm \). By doing so, the integration over \( s \) is easily performed leaving

\[
f_v = \frac{Q_5}{4\pi^2} \int d^4 u \frac{e^{i(u \cdot x)}}{|u|^2} \prod_k e^{\alpha_k} \int_{d_k,\bar{d}_k} f^{(k^\pm)}(|d_k^\pm|^2) e^{-\sigma_k^\pm d_k^\pm + \bar{\sigma}_k^\pm \bar{d}_k^\pm}, \tag{C.9}
\]

where we introduced the following quantities:

\[
\sigma_k^\pm = \lambda_k v^\pm + i \gamma_k u^\pm, \quad \bar{\sigma}_k^\pm = \lambda_k v^\mp - i \gamma_k u^\mp. \tag{C.10}
\]

Once again, by redefining the phase of \( d_k^\pm \) appropriately, we can integrate over them leaving us with the expression:

\[
f_v = \frac{Q_5}{2\pi^2} \int d^4 u \frac{e^{i(u \cdot x)}}{|u|^2} \prod_k e^{\alpha_k} \int |d_k^\pm| |d_k^\mp| f^{(k^\pm)}(|d_k^\pm|^2) J_0(2 |d_k^\pm| |\sigma_k^\pm|), \tag{C.11}
\]
where $J_0(x)$ is the Bessel function of the first kind, and

$$2 \alpha_k = \gamma_k^2 |u_{12}|^2 + \lambda_k^2 |v_{12}|^2, \quad (C.12)$$

$$|\sigma_k^\pm|^2 = \alpha_k \pm \frac{1}{2} \left( \frac{2\pi}{L} \right)^2 \mu^2 (u^1 v^2 - u^2 v^1). \quad (C.13)$$

This is as far as we can get for these class of phase space densities. Let us now discuss our specific examples.

## C.2 The Monochromatic State

In the following, we restrict ourselves to the 12-plane as the contribution from the other directions is trivial. The simplest state that describes a rotation (3.11) is

$$|\psi\rangle = [(a_{q^+})^\dagger]^J |0\rangle. \quad (C.14)$$

The state so constructed describes a circular profile following (4.26). Its associated phase space density $f_{d,\bar{d}}$ can easily be evaluated to be (see (4.16, 4.14) and (4.34))

$$f(d, \bar{d}) = e^{-\frac{1}{2}d_+^2} \frac{d_+^{2J}}{J!} \prod_{k \neq q^+} e^{-\frac{1}{2}d_k^2}. \quad (C.15)$$

We have dropped the delta function (4.21) here and expect (C.15) to be valid for large values of $J$. It is therefore better thought of as a semiclassical profile rather than the full quantum profile.

In evaluating $f_v$ (C.11), we distinguish two cases

**In the case** $k \neq q^+$ **In this case**, $f^{(k)}_{d, \bar{d}}$ is given by $\exp(-|d_k^\pm|^2)$ which leads upon integration over $|d_k^\pm|$ to

$$I_k = \frac{1}{2} e^{-|\sigma^\pm_k|^2}. \quad (C.16)$$

**In the case** $k = q^+$ **In this case**, the phase space density is given by

$$e^{-|d_+^\pm|^2} \frac{|d_+^\pm|^{2J}}{J!},$$

which leads, upon integration over $|d_+^\pm|$ using (4.36), to

$$I_{q^+} = \frac{1}{2} e^{-|\sigma_-^q|^2} L_J(|\sigma_+^q|^2). \quad (C.17)$$
where $L_J(x)$ is the Laguerre polynomial of order $J$.

Putting everything together we end up with

$$f_v(x) = \frac{Q_5}{4\pi^2 L} \int d^4u \frac{e^{i(u \cdot x + \bar{u} \cdot \bar{x})}}{|u|^2} L_J \left( \frac{\mu^2}{2k} \left[ \left( \frac{2\pi}{L} k v^1 + u^1 \right)^2 + \left( \frac{2\pi}{L} k v^2 - u^1 \right)^2 \right] \right)$$

$$= Q_5 L_J \left( \frac{\mu^2}{2k} \left[ \left( \frac{2\pi}{L} k v^1 - i\partial_2 \right)^2 + \left( \frac{2\pi}{L} k v^2 + i\partial_1 \right)^2 \right] \right) \frac{1}{|x|^2}, \quad (C.18)$$

where the last equation is seen as a formal expression.

### C.3 The Generic Thermodynamical Ensemble

Another important class of examples is the generic thermodynamical ensemble discussed in section 4.4. In this case, the phase space density reads (4.52), restricting once again to the 12-plane,

$$f_k(d, \bar{d}) = (1 - e^{-\beta^d_k}) (1 - e^{-\beta_d^\bar{k}}) e^{-(1-e^{-\beta^d_k})|d^k|^2 - (1-e^{-\beta_d^\bar{k}})|d^\bar{k}|^2}. \quad (C.19)$$

Plugging this expression in (C.11), then performing the $|d^\pm_k|$ integral gives

$$f_v = \frac{Q_5}{4\pi^2} \int d^4u \frac{e^{iux}}{|u|^2} \exp \left( -\frac{\mu^2}{8} \left[ D |u|^2 + \left( \frac{2\pi}{L} \right)^2 N |v|^2 \right] \right)$$

$$\exp \left( \frac{\mu^2}{2} \left( \frac{2\pi}{L} J \left[ u^1 v^2 - u^2 v^1 + u^3 v^4 - u^4 v^3 \right] \right) \right), \quad (C.20)$$

where:

$$N = 2 \sum_k k \left( \frac{e^{-\beta^d_k}}{1 - e^{-\beta^d_k}} + \frac{e^{-\beta_d^\bar{k}}}{1 - e^{-\beta_d^\bar{k}}} \right), \quad (C.21)$$

$$J = j_{12} = j_{34} = \sum_k \left( \frac{e^{-\beta^d_k}}{1 - e^{-\beta^d_k}} - \frac{e^{-\beta_d^\bar{k}}}{1 - e^{-\beta_d^\bar{k}}} \right), \quad (C.22)$$

$$D = 2 \sum_k \frac{1}{k} \left( \frac{e^{-\beta^d_k}}{1 - e^{-\beta^d_k}} + \frac{e^{-\beta_d^\bar{k}}}{1 - e^{-\beta_d^\bar{k}}} \right). \quad (C.23)$$

One can rewrite the expression (C.20) in a formal way as

$$f_v = Q_5 e^{-\frac{\mu^2}{8} \left( \frac{2\pi}{L} \right)^2 N |v|^2} e^{-\frac{\mu^2}{8} \left( \frac{2\pi}{L} \right) J \left[ v^2 \partial_1 - v^1 \partial_2 + v^4 \partial_3 - v^3 \partial_4 \right]} \frac{1 - e^{-\frac{\mu^2}{8} |x|^2}}{|x|^2}. \quad (C.24)$$
APPENDIX D

CALABI-YAU MANIFOLDS AND STRING COMPACTIFICATION

In this appendix, we are going to describe Calabi-Yau manifolds and the compactification on them in the case they are six dimensional. We will refer to the latter as Calabi-yau threefolds. The emphasis will be on the low energy field content while the construction of the resulting effective action will be carried out in the fifth chapter (section 5.1).

D.1 FROM KÄHLER TO CALABI-YAU MANIFOLDS

Before being able to define a Calabi-Yau manifold, we need first to understand what a Kähler manifold is? the answer to this question requires the notion of complex structure. The material discussed here can be found in many places, see for example [194, 163, 195, 196].

**Complex Structure** Given a \(2m\)-dimensional manifold \(M\), a complex structure is an endomorphism of the tangent bundle \(J : TM \rightarrow TM\), that squares to \(J^2 = -i_{2m \times 2m}\), and whose Nijenhuis tensor \(N : TM \times TM \rightarrow TM\), defined by:

\[
N[X,Y] = [X,Y] + J[JX,Y] + J[X,Y] - [JX,JY],
\]

where \([\cdot,\cdot]\) is the Lie bracket, vanishes. The existence of a complex structure allows us to introduce complex coordinate \(z^i,\bar{z}^i\), where locally \(J\) takes the form

\[
J = -i dz^i \otimes \frac{\partial}{\partial z^i} + i d\bar{z}^i \otimes \frac{\partial}{\partial \bar{z}^i} \equiv -i dz^i \otimes \partial_i + i d\bar{z}^i \otimes \bar{\partial}_i.
\]
Hermitian Metric Given a complex manifold with complex structure $J$, a Hermitian metric $g$ is a metric that satisfies

$$g(JX, JY) = g(X, Y).$$  \hfill (D.3)

Locally, it can be written as

$$g = g_{i\bar{j}} dz^i \otimes d\bar{z}^j + g_{\bar{i}j} d\bar{z}^\bar{i} \otimes dz^j. \hfill (D.4)$$

Kähler Form Given a Hermitian metric $g$, we can associate to it a Kähler form $\omega$ defined locally by

$$\omega = \frac{i}{2} g_{i\bar{j}} dz^i \wedge d\bar{z}^j. \hfill (D.5)$$

Kähler Manifold A complex manifold is Kähler if the Kähler form $\omega$ is closed $d\omega = 0$, which implies that $\omega$ is harmonic. In this situation the metric, called the “Kähler metric”, takes the following local form

$$g_{i\bar{j}} = \partial_i \bar{\partial}_j \mathcal{K}. \hfill (D.6)$$

$\mathcal{K}$ is called the “Kähler potential”, and it is not unique because the equation above is invariant under

$$\mathcal{K} \longrightarrow \mathcal{K} + f(z) + \bar{f}(\bar{z}). \hfill (D.7)$$

Ricci Form In the case of a Kähler manifold, the only non-vanishing component of the Ricci tensor is $R_{i\bar{j}}$ given by

$$R_{i\bar{j}} = -\partial_i \bar{\partial}_j \ln \sqrt{g}, \hfill (D.8)$$

where $g = \det g_{i\bar{j}}$. This allows us to define a closed real two-form, the “Ricci form”, as

$$\mathcal{R} = i R_{i\bar{j}} dz^i \wedge d\bar{z}^j. \hfill (D.9)$$

One can check that the Ricci form defined above is closed. However it is not exact, which implies that it defines a non-trivial class called the “first Chern class”. The latter is invariant under smooth changes of the metric $g \rightarrow g + \delta g$.

Calabi-Yau Manifold A Calabi-Yau manifold is a compact Ricci flat Kähler manifold. Ricci flat means that the associated Ricci tensor vanishes. This turns out to be equivalent to either of the two requirements:

- There exists a nowhere vanishing holomorphic $(n, 0)$-form $\Omega$.
- The holonomy of the Kähler manifold is a subgroup of SU(n) where $n$ is the complex dimension of our Calabi-Yau.
D.2 COHOMOLOGY OF CALABI-YAU MANIFOLDS

The existence of a complex structure allows us to introduce complex coordinates, as well as a refinement of the notion of p-forms. In this case, we have a double grading of forms; the \( (p, q) \)-forms \( \alpha \). Locally, they can be written as:

\[
\alpha^{(p,q)} \sim \alpha_{i_1 \ldots i_p \bar{j}_1 \ldots \bar{j}_q} dz^{i_1} \wedge \ldots \wedge dz^{i_p} \wedge d\bar{z}^{\bar{j}_1} \wedge \ldots \wedge d\bar{z}^{\bar{j}_q}.
\]

Such a refinement allows for the study of “Dolbeault cohomology.” Remember that “de Rham cohomology” is defined using the exterior derivative \( d \), that maps a \( p \)-form to a \((p+1)\)-form, as follows. The \( p^\text{th} \) cohomology group is the set of closed \( p \)-forms \( \alpha \) (i.e. \( d\alpha = 0 \)) modulo exact ones (i.e. \( \alpha = d\beta \)). In the same spirit, one defines now an exterior derivative \( \bar{\partial} \) that maps a \((p, q)\)-form to a \((p, q+1)\)-form. The Dolbeault \((p, q)\)-cohomology is defined similar to the de Rham one except that one uses \( \bar{\partial} \) instead of \( d \).

In deriving the four-dimensional effective action, the dimensions \( h^{p,q} \) of \((p, q)\)-cohomologies fix the four dimensional field content (see section D.3.2 below). On top of specifying the moduli space of Calabi-Yau deformations, they also encode information about the different form fields (RR and the NS B-field) after reduction. In the case of a simply connected Calabi-Yau threefold, cohomology dimensions can be collected in the following Hodge diamond.

\[
\begin{array}{cccccc}
& & & & & 1 \\
& & & & 0 & 0 \\
& & & h^0,0 & h^0,1 & 0 \\
& & h^1,0 & h^1,1 & h^0,1 & 0 \\
h^2,0 & h^2,1 & h^1,2 & h^0,2 & 1 & h^1,1 \\
h^3,0 & h^3,1 & h^2,2 & h^1,3 & h^0,3 & h^1,2 \\
h^3,2 & h^3,3 & h^2,3 & h^1,4 & 1 & 0 \\
& & & & & 1 \\
\end{array}
\]

\[\text{(D.11)}\]

D.3 COMPACTIFICATION

We have seen that consistency of perturbative superstrings fixes the spacetime dimension to be ten. However, we can do physics in lower dimensions by compactifying the unwanted dimensions [197, 198]. We will restrict ourselves to the compactification of type-IIA supergravity on Calabi-Yau threefolds. For general cases see for example [199, 200], and also [201] for quick ideas about the procedure. For other approaches to get lower dimensional physics see e.g. [202].
D.3.1 SOME GENERAL REMARKS ON COMPACTIFICATION

Before getting down to the actual Calabi-Yau compactification, let us pause for a moment and discuss some general properties of compactification.

THE KALUZA-KLEIN TOWER

To get acquainted with compactification, let us take the simple theory of a massless scalar field $\Phi$ on spacetime that is a direct product of a circle $S^1$ of radius $R$ with Minkowski spacetime $M$ that we are living in. Let us denote by $y$ the coordinate on $S^1$, and by $x^\mu$ the coordinates on $M$. We start by decomposing $\Phi$ in a complete basis of functions on $S^1$. In this case one gets

$$\Phi(x, y) = \sum_{n \in \mathbb{Z}} \phi_n(x) e^{in y/(2\pi R)} \quad (D.12)$$

Using this expansion, the field equation $\Box \Phi = 0$ can equivalently be rewritten as a collection of infinitely many equations of the form

$$\left(\Box_M - \mu_n^2 \right) \phi_n(x) = 0; \quad \mu_n = n/(2\pi R) \quad (D.13)$$

One recognizes these equations as the field equations of massive scalar fields in $M$ with masses $\mu_n$. At low energies and for small enough $R \ll 1$, we can forget about the massive tower of fields, called the “Kaluza-Klein (KK in short) tower”, and study the effective action of the resultant massless field $\phi_0$.

In general, every massless field will be accompanied by its own KK-tower, where the role of $R$ will be played by some typical scale in the compact space. We will be dealing with small compact spaces to suppress the contributions of the massive KK-tower to the effective action of the massless fields.

THE METRIC REDUCTION AND NON-ABELIAN GAUGE FIELDS

In the following, we are going to study the reduction of the metric $g_{MN}$ on a smooth compact manifold. To keep the discussion as general as possible, let us call $y^\alpha$ the coordinates of the compact manifold $X$, and $x^\mu$ the coordinates of the non-compact part $M$. Let us further suppose that the compact space $M$ has an isometry group $G$. Such isometries are characterized by Killing vectors $\xi^a_\alpha(y)$, where $a$ is the adjoint index of the group $G$. Under these assumptions, the most general form of the metric $g_{MN}$ takes the form:

$$ds^2 = h_{\alpha\beta}(x) \left( dy^\alpha - A^a_\alpha(x) \xi^a_\alpha(y) dx^\mu \right) \left( dy^\beta - A^a_\beta(x) \xi^a_\beta(y) dx^\nu \right) + g_{\mu\nu}(x) dx^\mu dx^\nu. \quad (D.14)$$
After reduction, the following fields emerge

- A metric $g_{\mu\nu}$ which gives rise to gravity.
- Several scalars coming from $h_{\alpha\beta}(x)$. Knowing the number of these scalars is the non-trivial part in this reduction.
- (Non-)Abelian gauge fields $A_a^\mu$ with gauge group $G$, the latter is equal to the isometry group of the compact manifold.

We are interested in Calabi-Yau threefold compactification. Let us investigate its isometries. Remember that a Killing vector $\kappa$ satisfies

$$\nabla_\alpha \kappa_\beta + \nabla_\beta \kappa_\alpha = 0 ,$$  

(D.15)

where $\nabla_\alpha$ is the covariant derivative of our Calabi-Yau $X$. Hitting both sides of the equality by $\nabla^\alpha$, then using the Ricci flatness property of Calabi-Yau’s leads to

$$\int \sqrt{g} \kappa_\beta \nabla^\alpha \nabla_\alpha \kappa_\beta = 0 ,$$  

(D.16)

where $g$ is the determinant of the Calabi-Yau metric $g_{\alpha\beta}$. This implies that $\kappa$ is a covariantly constant vector, hence a singlet under the holonomy group. Such a vector does not exist in the case where the holonomy group of the Calabi-Yau is exactly SU(3): the 6 of SO(6) (vectors) transform as $3 + \bar{3}$ of SU(3). In the following, we will be working with simply connected Calabi-Yau threefolds. The holonomy group of the latter is exactly SU(3). This means that the ten-dimensional metric will not give rise to gauge fields after reduction.

**Forms and (Co)Homology**

Until now, we have dealt with the metric and scalar fields. The remaining bosonic fields are form-fields. In the absence of fluxes, every $n$-form $C^{(n)}$ can always be written globally as

$$C^{(n)} = \bigoplus_{0 \leq p \leq n} C_X^{(p)}(y) C_M^{(n-p)}(x) ,$$  

(D.17)

where the subscript stands for the space in which the components of the form live in. Using this decomposition, the field equation $\triangle C^{(n)} = 0$ reduces to

$$\triangle_X C^{(p)} = 0 , \quad \triangle_M C^{(n-p)} = 0 , \quad p = 0, 1, ..., n .$$  

(D.18)

In other words we get $(n - p)$-forms living in $M$, with degeneracy given by the number of independent solutions of the first equation above, $\triangle_X C^{(p)} = 0$. In differential geometry, this equation specifies the $p$-cohomology of the manifold $X$ (See for example [194]).
Killing Spinors and the Survival of Supersymmetry

Since we will be dealing with bosonic solutions, preserving supersymmetry amounts to the existence of a Killing spinor. Such spinors guarantee the consistency of setting the fermions to zero with supersymmetry transformations. The defining equation of a Killing spinor takes the following schematic form

\[ \delta \psi = (\Gamma \cdot \nabla + \Gamma \cdot \text{fluxes}) \epsilon = 0 \]  

(D.19)

where \( \nabla_\alpha \) is the covariant derivative and fluxes stand for possible form fluxes. Using the ansatz \( \epsilon = \xi \otimes \varepsilon \) where \( \xi \) lives in the internal part of the geometry and satisfies –due to (D.19)–

\[ (\nabla_\alpha + \Gamma \text{fluxes}|_{\text{int},\alpha}) \xi \equiv O_\alpha \xi = 0 \]

(D.20)

Such an equation admits a solution if

\[ [O_\alpha, O_\beta] \xi = (R_{\alpha\beta\gamma\delta} \Gamma^{\gamma\delta} + [\Gamma \cdot \text{fluxes}]_{\rho\sigma}) \xi = 0 \]

(D.21)

which can be translated to constraints on the holonomy of the internal space. This is a necessary condition to have some unbroken supersymmetry after compactification. In the case of Calabi-Yau threefolds, only one quarter of the original supersymmetry survives the compactification, leading to an effective action with \( N = 2 \) supersymmetry in four dimensions.

D.3.2 Type-IIA on a Calabi-Yau

It is time to discuss our compactification. First, let us remind ourselves about the field content of type-IIA ten-dimensional supergravity. We will restrict ourselves to the bosonic part as the fermions will be added by the requirement of supersymmetry. The ten-dimensional fields are: the NS-NS fields (graviton \( g_{MN} \), b-field \( b_{MN} \) and the dilaton \( \phi \)), and the R-R forms \( (C_M \text{ and } C_{MNP}) \). The reduction of the dilaton and forms is already done, we need just to select the appropriate information from (D.11). The only remaining task is the metric reduction. We have already done half of the work (metric + no vectors). Let us then deal with the scalar part. These are the deformations of the Calabi-Yau metric that preserve, to first order, the Ricci flatness condition. The most general metric perturbation reads

\[ \delta g = \delta g_{ij} dz^i dz^j + \delta g_{i\bar{i}} dz^i d\bar{z}^\bar{i} + c.c., \]

(D.22)

where c.c means complex conjugate terms. The first perturbation \( \delta g_{ij} \) does not respect the \( (1, 1) \) decomposition of the metric (D.4), and turns out to describe deformations of the complex structure. Using the unique \( (3, 0) \)-form \( \Omega \) and the inverse of
the metric $g^{ij}$, the Ricci flatness condition implies that the form $\xi_{ijk} = \Omega_{ijk} g^{lk} \delta g_{kj}$ is harmonic. This means that the complex structure deformations give rise to $h^{1,2}$ complex scalar fields. The second perturbation $\delta g_{ij}$, on the contrary, respects the $(1,1)$ decomposition of the metric (D.4). It turns out that they describe the deformation of the Kähler form. Once again, Ricci flatness implies that $\delta g_{ij}$ is harmonic. This means that the Kähler deformations lead to $h^{1,1}$ real scalars.

In the language of $\mathcal{N} = 2$ multiplets and concentrating on the bosonic content only, the reduction of type-IIA on a Calabi-Yau threefold gives rise to

- One supergravity mutiplet: $g_{\mu\nu}$ and one particular linear combination of $C_\mu$ and $C_{\mu ij}$.
- One universal hypermultiplet: $C_{ijk}, \phi$ and $b_{\mu\nu}$.
- $h^{1,1}$ vectormultiplets: $(b_{ij} + i \delta g_{ij})$ and a linear combination of the one-forms $C_\mu$ and $C_{\mu ij}$ except the one that belongs to the supergravity multiplet.
- $h^{1,2}$ hypermultiplets: $\delta g_{ij}$ and $C_{ijk}$.
APPENDIX E

ADDING FERMIONS

The aim of this appendix is to study the structure of \((\det \partial_i \partial_j \mathcal{K})^{-1/2}\) that enters in the measure (6.25), where \(\mathcal{K}\) is given by (6.20). Already a simplification emerges due to (6.21). A careful look at this formula reveals that the only non trivial piece is \(\sqrt{\partial_i \partial_j g}\) where \(g\) is given by (6.18). Our main result is

\[
\det \partial_i \partial_j g = \left( \prod_{j=1}^{m} \frac{1}{l_j} \right) A(l) ,
\]

where \(A(l)\) is a homogeneous polynomial of order \(m - n\) in the \(l_a\) (given by (6.17)) with coefficients such that for no Delzant polytopes it will contain an overall \(l_a\) factor. We will prove this in two steps. First, we will evaluate the relevant determinant to show the form (E.1) explicitly. In the second step, we explain the properties of \(A(l)\), namely, that it has no poles nor contains an overall \(l_j\) factor.

E.1 CALCULATING THE DETERMINANT

It is straightforward to check, using (6.18, 6.17), that

\[
\partial_i \partial_j g = \frac{1}{2} \sum_{a=1}^{m} \frac{m_a c_{ai} m_a c_{aj}}{l_a} = \frac{1}{2} \left( C^T \cdot L^{-1} \cdot C \right)_{ij} ,
\]

with \(C_{ai} = m_a c_{ai}\) an \(m \times n\) matrix, and \(L_{ab} = l_a \delta_{ab}\) an \(m \times m\) matrix, which makes \(C^T L^{-1} C\) a square \(n \times n\) matrix. Since we are interested in the zeros and the pole structure of such a determinant, we are going to neglect over all numerical factors...
Appendix E - Adding Fermions

in the following. The easiest way to evaluate such a determinant is to re-express it in terms of a larger symmetric \((n + m) \times (n + m)\) matrix \(D\) given by

\[
D_{\alpha \beta} = \begin{cases} 
L_{ab} & \text{for } \alpha, \beta = 1, \ldots, m. \\
C_{ai} & \text{for } \alpha = 1, \ldots, m, \beta = m + 1, \ldots, m + n. \\
0 & \text{for } \alpha, \beta = m + 1, \ldots, m + n.
\end{cases} \tag{E.3}
\]

Now using that \(L\) is diagonal and \(\det(\partial_i \partial_j g) = \det D/\det L\), it is easy to show that

\[
\det (C^T L^{-1} C) = \left( \prod_{j=1}^{m} l_j^{-1} \right) \left( \sum_{S} l^S (\det C_S)^2 \right), \tag{E.4}
\]

which has the same structure as (E.1). In the second factor, the sum is over all different subsets \(S \subset \{1, \ldots, m\}\) with \(m-n\) elements, i.e. \(\#S = m-n\). Furthermore, we use the shorthand \(l^S := \prod_{a \in S} l_a\). Finally, there is the definition of the \(n \times n\) matrix \(C_S\). Note that \(C\) was an \(m \times n\) matrix, \(C_S\) is now defined as the matrix \(C\) but with the \(a_1, \ldots, a_{m-n}\)'th rows removed where \(S = \{a_1, \ldots, a_{m-n}\}\).

E.2 Properties of \(A(l)\)

In the following, we are going to show that \(A(l) = \sum_{S} l^S (\det C_S)^2\) has no poles nor does it contain an overall \(l_a\) factor. As is clear from its definition, \(A(l)\) is a homogeneous polynomial of order \(m-n\) in \(x_i\), and hence has no pole in \(x_i\). Let us now turn to the second property. Without loss of generality, let us show the absence of an overall factor \(l_1\). A moment thought translates such property to the non-vanishing of at least one \(\det C_{\tilde{S}}\) with \(1 \notin \tilde{S}\), which can be shown using some basic properties of \(C\) and \(C_{\tilde{S}}\).

By the definition of the \(C_S\), all the \(C_{\tilde{S}}\) include the first row of \(C\), given by \(c_{1i}\). Furthermore, using the geometric interpretation of \(c_{ai}\) below (6.14), the statement \(\exists \tilde{S}_+ \ n \det C_{\tilde{S}_+} \neq 0\) translates to: “there exists a set of \((n-1)\) vectors among the \(m\) different normals \(\vec{c}_a\) that together with \(\vec{c}_1\) form a basis of \(\mathbb{R}^n\)”. We will use the notation \(\vec{c}_a\) for these \(n\) vectors and now show their existence.

Pick one of the vertices that is a corner of the facet orthogonal to \(\vec{c}_1\) and let’s call it \(v_1\). As the polytopes of our interest are Delzant, there are exactly \(n\) edges \(\vec{e}_a\) meeting in the vertex \(v_1\), that furthermore form a basis of \(\mathbb{R}^n\). Now, the different facets meeting in \(v_1\) each lie in a subspace generated by a set of \((n-1)\) of the \(n\) edges \(e_a\). So, we find \(n\) facets that all meet in the vertex \(v_1\). Let us label the \(n\) normals to these facets as \(\vec{e}_a\), by their definition they can be labelled such that they satisfy \(\vec{e}_a \cdot \vec{e}_b \sim \delta_{ab}\). So, we see that the \(\vec{e}_a\) form a basis of \(\mathbb{R}^n\) that includes \(\vec{e}_1\), which concludes the proof i.e. we now know that \(\det C_{\tilde{S}_+} \neq 0\) for \((C_{\tilde{S}_+})_{ai} = c_{ai}\).
In this appendix, we will analyze some properties of the moduli space of three-center solutions. Our starting point will be the set of equations (6.37), which we will rewrite as follows

\[
\begin{align*}
\frac{a}{x} - \frac{b}{y} &= c_2 - c_1, \\
\frac{b}{y} - \frac{c}{z} &= c_3 - c_2, \\
\frac{c}{z} - \frac{a}{x} &= c_1 - c_3.
\end{align*}
\] (F.1)

Here \(a, b, c\) represent the inner products \(\langle \Gamma_a, \Gamma_b \rangle\), \(x, y, z\) are the lengths of the three sides of the triangle spanned by \(\vec{x}_a\), and \(c_2 - c_1 = \langle h, \Gamma_1 \rangle\) etc. The constants \(c_a\) are not uniquely fixed, as we shift them by a fixed amount without modifying the above equations. Still, expressing things in terms of \(c_a\) allows for a somewhat more symmetric treatment.

The first important remark is that up to an \(SO(3)\) rotation, \(x, y, z\) uniquely determine the solution. In other words, the quotient of the solution space by \(SO(3)\) is precisely the set of solutions \(x, y, z\) of (F.1).

Second, we should keep in mind that \(x, y, z\) are the sides of a triangle, i.e. they should be nonnegative numbers that satisfy the triangle inequality \(x + y \geq z\) and its cyclic permutations.

In our discussion of the solution space quantization, the size of angular momentum
will play an important role, as we will use it as a coordinate on the solution space. In terms of the variables used in (F.1), the norm of the angular momentum is given by (see (5.43))

\[ J^2 = \frac{1}{16} \left( x^2(c_2 - c_1)(c_1 - c_3) + y^2(c_3 - c_2)(c_2 - c_1) + z^2(c_1 - c_3)(c_3 - c_2) \right). \]

(F.2)

We would in particular like to know whether \(|J|\) is a good single-valued coordinate on the solution space, and what range of values it takes.

It is easy to write down the general solution to (F.1) in terms of a single free parameter \(\lambda\):

\[ x = \frac{a}{\lambda - c_1}, \quad y = \frac{b}{\lambda - c_2}, \quad z = \frac{c}{\lambda - c_3}. \]

(F.3)

This is the general solution if \(a, b, c\) are not equal to zero. If all three are zero, or two out of three are zero, there are either no solutions to (F.1) or the space of solutions is at least two-dimensional. In either case, the symplectic form becomes degenerate and most likely these solution spaces do not give rise to BPS states. Finally, if one of \(a, b, c\) is zero, say \(a = 0\), then either there are no solutions or one finds a fixed value for \(y, z\) from (F.1), while \(x\) is not constrained by (F.1). However, \(x\) is constrained by the triangle inequalities so that the solution space becomes

\[ a = 0, \ b \neq 0, \ c \neq 0, \Rightarrow y, z \text{ fixed}, \ |y - z| \leq x \leq y + z. \]

(F.4)

We now continue with the case where \(a, b, c\) are not equal to zero so that the solutions are of the form (F.3). We again need to distinguish a few cases. The most degenerate case is when \(c_1 = c_2 = c_3\). Then either the moduli space is empty or one-dimensional, but in the latter case, the angular momentum vanishes identically everywhere on the solution space, and thus, the symplectic form is trivially degenerate.

The next case is \(a, b, c\) nonzero and two of the \(c_i\) equal to each other. Using the permutation symmetry of (F.1) and the possibility to simultaneously change the signs of \(a, b, c, c_i, \lambda\), we can distinguish three different cases: (i) \(c_3 > c_1 = c_2\) and \(a, b, c > 0\), (ii) \(c_1 = c_2 > c_3\) and \(a, b, c > 0\), and (iii) \(c_3 > c_1 = c_2, a, b > 0\) and \(c < 0\). Positivity of \(x, y, z\) requires that \(\lambda \in I_1 = (c_3, \infty)\) for cases (i),(ii) and \(\lambda \in I_1 = (c_1, c_3)\) in case (iii). Next we denote by \(I_2\) the set of solutions of the triangle inequalities

\[ \frac{a + b}{\lambda - c_1} > \frac{c}{\lambda - c_3} > \frac{|a - b|}{\lambda - c_1}. \]

(F.5)

It is easy (though somewhat tedious) to see that \(I_1 \cap I_2\) is either empty, an interval of the form \([\lambda_-, \lambda_+]\), an interval of the form \([\lambda_-, \infty)\), an interval of the form \((c_1, \lambda_+)\), or an interval \((c_1, \infty)\). The endpoints \(\lambda_+\) and \(\lambda_-\) always correspond to a point where a triangle inequality is saturated. The interval extends all the way to infinity only...
if $a + b > c > |a - b|$, i.e. when $a, b, c$ satisfy triangle inequalities, which can only happen in case (i) and (ii). In these cases, there is a scaling throat with $x, y, z \to 0$. Actually, this can possibly also happen when $a + b = c$ or $c = |a - b|$. The interval starts at $c_1$ only if we are in case (ii) or (iii) and $a = b$, and in this case the solution space includes configurations where a center can move off to infinity.

From the point of view of angular momentum, the case where one of the centers moves away to infinity (e.g. $x, y \to \infty$) can be viewed as a case where the triangle inequalities $x + z \geq y$ and $y + z \geq x$ are both saturated. Therefore, in all cases we have analyzed so far, the solution space contained just a single connected component described by a single interval of possible values of $\lambda$, and at the endpoints of the interval either one has a scaling solution with vanishing angular momentum, or a solution that saturates at least one triangle inequality. Whenever this happens, we always find that

$$|J|^2 = \frac{1}{4}(\pm a + \pm b + \pm c)^2,$$  

for suitable choices of the signs, as can be seen easily e.g. from (5.42).

It remains to analyze the generic case with all $c_i$ different from each other. Up to an overall sign flip and a permutation, there are two cases, which are (iv) $c_1 < c_2 < c_3$ and $a, b, c > 0$ and (v) $c_1 < c_2 < c_3$ and $a, b > 0, c < 0$. Positivity of $x, y, z$ in case (iv) implies $\lambda > c_3$ and implies $c_2 < \lambda < c_3$ in case (v). The main problem is to analyze the triangle inequalities. They can be analyzed qualitatively by sketching $x + y - z, x - y + z$ and $-x + y + z$ as a function of $\lambda$. We know that each of these functions can have at most two zeroes as a function of $\lambda$, and we know its behavior near the three poles at $\lambda = c_1, c_2, c_3$. We will skip the details, but one finds that the moduli space consists of at most two components, each of which corresponds to a certain interval of possible values of $\lambda$. At the boundaries of each interval a triangle inequality is saturated. Notice that in case (iv) one of the components can be of the form $[\lambda, \infty)$. This is possible whenever $a, b, c$ themselves satisfy triangle inequalities. If this happens, at $\lambda = \infty$ there is a scaling solution.

To summarize, the solution space in all cases consists of at most two components, corresponding to two intervals of possible values of $\lambda$. At the endpoints of the interval some triangle inequality is saturated. This can include configurations where one of the centers moves off to infinity (cases (ii) and (iii) above, with $a = b$), and scaling solutions where $\lambda \to \infty$ (cases (i), (ii) and (iv) with $a, b, c$ obeying triangle inequalities).

Finally, we would like to show that $J^2$ is a good coordinate on each component of the moduli space of solutions to (F.1). In order to do so, we compute $dJ^2/d\lambda$. According to (F.3), $dx/d\lambda = -x^2/a$ and similarly for $y, z$. If we differentiate (F.2), use these relations, and finally replace $c_i$ by the left hand side of the original equations (F.1),
we obtain
\[ 2 \frac{dJ^2}{d\lambda} = \frac{x^3}{a} \left( \frac{a}{x} - \frac{b}{y} \right) \left( \frac{c}{z} - \frac{a}{x} \right) + \frac{y^3}{b} \left( \frac{b}{y} - \frac{c}{z} \right) \left( \frac{c}{z} - \frac{a}{x} \right) + \frac{z^3}{c} \left( \frac{c}{z} - \frac{a}{x} \right) \left( \frac{b}{y} - \frac{c}{z} \right). \]  
(F.7)

We rewrite this as
\[ -2 \frac{abc}{xyz} \frac{dJ^2}{d\lambda} = n_0 a^2 + n_1 a + n_2 = n_0 \left( a + \frac{n_1}{2n_0} \right)^2 + \frac{4n_2 n_0}{4n_0} - n_1^2, \]  
(F.8)

with \( n_0, n_1, n_2 \) certain \( a \)-independent polynomials. The right hand side of (F.8) is positive if \( n_0 \) and \( 4n_2 n_0 - n_1^2 \) are positive. By explicit computation we find
\[ n_0 = \left( z^2 b + \frac{cy}{2z} (x^2 - y^2 - z^2) \right)^2 + \frac{c^2 y^2}{4z^2} \theta, \]
\[ 4n_2 n_0 - n_1^2 = b^2 c^2 x^2 (bz - cy)^2 \theta, \]  
(F.9)

where
\[ \theta = (x + y + z)(x + y - z)(x - y + z)(-x + y + z). \]  
(F.10)

Since \( \theta > 0 \) if all triangle inequalities are satisfied, we have indeed shown that \( J^2 \) is a monotonous function of \( \lambda \) and that \( J^2 \) is a good coordinate on each component of the solution space.