Optimization and approximation on systems of geometric objects

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Chapter 1

Introduction

Can a combinatorial optimization problem be approximated better if it is determined by a system of geometric objects? Many combinatorial optimization problems are NP-hard to solve and frequently even hard to approximate. On systems of geometric objects however, these same problems can usually be solved efficiently or approximated well.

The computational complexity of such geometric optimization problems highly depends on the underlying system of objects. We consider objects that arise naturally in diverse applications. For instance, the broadcasting range of a wireless device can be seen as a disk, an atom is just a sphere, and a label on a map is a rectangle. In this thesis, we consider hard combinatorial optimization problems on systems of geometric objects, motivated by applications in for example wireless networks, computational biology, and map labeling.

1.1 Optimization Problems and Systems of Geometric Objects

Given a combinatorial optimization problem and its objective function, one preferably is able to compute the optimum objective value and a corresponding solution in an efficient way. Unfortunately, depending on one's notion of efficient, this is not always possible. The measure we use to decide whether an algorithm is efficient is polynomial running time. That is, whether the algorithm returns the optimum objective value (and possibly a corresponding solution) using a number of 'basic steps' that is polynomial in the size of the problem instance. An optimization problem having such an algorithm is called *polynomial-time solvable*.

Many optimization problems are solvable in polynomial time. However, there are also many optimization problems that are not known to be solvable in polynomial time. In fact, there is very strong belief that these problems cannot be solved in polynomial time. To support this idea, a class of hard optimization problems has been identified, namely the class of *NP-hard* optimization problems.

The most sensible way to work around NP-hardness while still restricting to polynomial-time algorithms is to try and find approximate solutions to such optimization problems. An *approximation algorithm* returns an objective value (and possibly a corresponding solution) within a certain (additive or multiplicative) factor of the optimum objective value. Approximation algorithms have been developed for many different optimization problems and are widely used in practice. We should note however that sometimes one can even show that approximating a problem within a certain factor is NP-hard.

In this field of hard optimization problems, we try to ascertain the influence on the computational complexity of these problems when using a system of geometric objects as the underlying combinatorial structure. Since many optimization problems use graphs as their underlying structure, it might be no surprise that one of the central notions of this thesis is a graph induced by a system of geometric objects: a *geometric intersection graph*.

Given a system of geometric objects, the vertices of the induced graph correspond to the given objects and there is an edge between two vertices if the corresponding objects intersect. There are many famous examples of geometric intersection graphs, for instance the classic *interval graph* is an intersection graph of intervals on the real line. For this thesis however, we are concerned with intersection graphs of objects in two- and higher-dimensional space, such as *(unit) disk graphs*, which are intersection graphs of (equally-sized) disks.

Most classical optimization problems are still NP-hard on (intersection graphs of) systems of two-dimensional geometric objects [17, 67, 110, 157, 267]. However, there is considerable evidence (e.g. [4, 68, 74, 103, 150, 154, 201, 279]) to support the claim that *approximating* these problems is easier, meaning that better approximation algorithms exist, on systems of geometric objects than in general. The goal of this thesis is to further investigate this claim.

> **In this thesis we study the approximability of hard combinatorial optimization problems on (intersection graphs of) systems of geometric objects.**

As part of this study, we give both approximation algorithms and inapproximability results. In the process, we show that the approximability highly depends on the shape and size of the geometric objects under consideration.

1.2 Application Areas

It turns out that optimization problems on systems of geometric objects, and particularly on geometric intersection graphs, occur in many application areas. We consider some of the foremost application areas and associated problems and systems of objects below.

1.2.1 Wireless Networks

It is very common for wireless networks to be modeled as geometric intersection graphs. In 1961 already, Gilbert [122] used the following idea to model wireless networks. Each wireless device is assumed to be a point in the plane and an edge is drawn if the distance between two points is at most some constant δ.

In other words, the network is modeled by the intersection graph of a set of disks of radius $\delta/2$. This graph would later be called a *unit disk graph* [136].

By now, unit disk graphs have easily become the prevalent model for wireless networks (although more advanced models exist, see Section 3.2.1). The model implicitly assumes that all devices in the network have the same broadcasting range. This is a perfect fit to an increasingly popular network type, called a *(mobile) ad hoc network* [82, 166, 225, 232, 252]. A mobile ad hoc network is an autonomous collection of mobile devices that communicate over wireless channels. The network is self-organizing and self-reliant. In contrast to classic wireless networks, no fixed infrastructure in the form of base stations is present and messages are routed from source to destination through multiple hops. Mobile ad hoc networks already exist since the late 1970s [1]. With the ongoing miniaturization of chips and wireless transceivers, these networks attained renewed interest in the last few years. This led for instance to the advent of *wireless sensor networks* [5, 121, 133, 274].

Several well-known optimization problems are relevant to (ad hoc) wireless networks, and have consequently been studied on unit disk graphs and its generalizations. A *maximum independent set* in the graph corresponds to a largest set of devices that can transmit simultaneously without causing interference. The *minimum dominating set* problem has been studied to find a collection of emergency transmitters, capable of reaching every device in the network. Alternatively, it can be used to construct a routing backbone. This is also where a *minimum connected dominating set* comes in.

One of the first studies into unit disk graphs was on its *chromatic number* [136]. The minimum number of colors needed is equal to the minimum number of channels needed in the network to communicate without causing interference. Hence solving the coloring problem actually solves the frequency assignment problem.

1.2.2 Wireless Network Planning

Classical wireless networks consist of a number of powerful base stations that provide wireless service to smaller devices. Well-known examples are cellular networks (i.e. GSM networks) and wi-fi networks (i.e. 802.11 networks).

Because these networks rely on base stations, it is imperative to position them carefully. The common model (see e.g. [124]) is to view base stations as disks, where the radius of each disk corresponds to the range of the wireless signal broadcast by the base station, and the smaller wireless devices as points. The question then is how to place as few disks (base stations) as possible, but still cover all points, i.e. provide wireless service to all devices. This is the geometric version of the well-known *minimum set cover* problem. Several variations of this geometric set cover problem exist, depending for instance on whether we are free to choose the location of the disk or should choose from a given set of potential locations, or on how strict one is in insisting that all points are covered. See Part III of this thesis for more variants.

1.2.3 Computational Biology

Optimization problems on systems of geometric objects occur in several biological applications. Armitage [15] described in 1949 the need to find the number and the size of clumps of particles in order to properly count the total number of particles under a microscope. This clearly corresponds to problems on cliques in geometric intersection graphs. Armitage considered several models for the particles, including (unit) disks and rectangles of bounded aspect-ratio.

Kaufmann et al. [160] consider the alignment of DNA-sequences, which can be described by a maximum clique problem on max-tolerance graphs, a generalization of interval graphs. Interestingly, max-tolerance graphs are equivalent to intersection graphs of isosceles right triangles. Xu and Berger [272] use a geometric model to study the problem of attaching or assigning side-chains of a protein to an existing backbone while maximizing system energy.

1.2.4 Map Labeling

In general, map labeling is the problem of placing labels on a map, such that the labeling satisfies certain properties. For example, a label should be close to its corresponding item on the map and the texts of different labels should not overlap. Commonly, the labels are seen as rectangles, the items as points, and the boundary of the rectangle of a certain label should overlap the point modeling the corresponding item. Then one aims to maximize the number of labels that can be placed, without any overlaps among the rectangles. If the positions in which the rectangle may be placed are discretized, this is just the maximum independent set problem on an intersection graph of rectangles [4]. The continuous case is more complex [102].

1.2.5 Further Applications

One of the most commonly cited algorithmic results for optimization problems on systems of geometric objects applies to (among others) a problem in VLSI [150]. In the *geometric packing* problem, one wants to pack the largest number of objects of a certain prescribed shape into a larger object. This corresponds to maximizing yield when cutting chips from a large chip wafer.

A slightly morbid application is bombing. Garwood [117] described in 1947 the problem of minimizing the number of bombs needed to destroy points of interest in a certain area. Assuming that the bombs have a circular area of destruction upon impact, one wants to minimize the number of disks needed to cover certain other geometric objects, e.g. rectangles or points, representing buildings or matériel.

1.3 Thesis Overview

The thesis is comprised of three parts and eleven chapters. Below is an overview of their contents.

Part I: Foundations

We introduce and expand on the basic notions needed to understand the other parts of the thesis.

The topic of this thesis is made from two main ingredients. The first is approximation algorithms for optimization problems. Chapter 2 formally defines the type of optimization problems we consider here and what an approximation algorithm is. We define several classes of optimization problems, each admitting a particular type of approximation algorithm. The algorithms given in this thesis gave rise to new classes of optimization problems. We consider the relation of these classes to classic problem classes. The results of this chapter were obtained in joint work with J. van Leeuwen [263].

The second ingredient of this thesis is systems of geometric objects and geometric intersection graphs. We survey the main results on structural aspects of geometric intersection graphs in Chapter 3. We also present a new way to look at a fundamental question surrounding geometric intersection graphs: do geometric intersection graphs have a representation of polynomial size? In Chapter 4, we prove that this is equivalent to the question whether or not geometric intersection graphs have a representation that is polynomially separated. Chapter 4 is an extended version of joint work with J. van Leeuwen [262].

Part II: Approximating Optimization Problems on Geometric Intersection Graphs

We describe exact algorithms and approximation schemes for optimization problems on geometric intersection graphs. In particular, we consider Maximum Independent Set, Minimum Vertex Cover, and Minimum (Connected) Dominating Set. As most of these problems are motivated by wireless networks, the majority of the graphs studied in this part are disk graphs.

We start by considering unit disk graphs in Chapter 5. If a unit disk graph has a special property called bounded thickness, then many optimization problems (such as Maximum Independent Set and Minimum Connected Dominating Set) can be solved exactly in polynomial time. To this end, we define a new graph decomposition, called a relaxed tree decomposition, which might be interesting on its own. We provide exact algorithms on such decompositions that are applicable to general graphs. If the graph is a unit disk graph, the running time of these algorithms can be expressed in terms of the thickness. Moreover, the bound on the worst-case running time of the algorithm for Minimum Connected Dominating Set is significantly lower on unit disk graphs of bounded thickness than might be expected from the bound in the general case.

In Chapter 6, we use the algorithms of Chapter 5 to give new, better approximation schemes for these optimization problems on general unit disk graphs by using the so-called shifting technique. The schemes are a ptas on general unit disk graphs and an eptas if the density of the set of disks is bounded. They improve on (the running time of) previous approximation schemes. For Minimum Vertex Cover, we give an improved eptas on arbitrary unit disk graphs. We generalize to intersection graphs of unit fat objects in constant dimension and the weighted case. Furthermore, we prove that the algorithms are optimal (up to constants), unless the exponential time hypothesis fails. The ideas we give for the minimum connected dominating set problem are used to give the first eptas for this problem on apex-minor-free graphs. Chapter 5 and Chapter 6 are based on a revised and extended version of [259].

In Chapter 7 we generalize these ideas to general disk graphs and intersection graphs of fat objects. The crucial idea is to consider systems of fat objects that have bounded ply. We subsequently present new, improved approximation schemes for Minimum Vertex Cover and Maximum Independent Set using the multi-level shifting technique. The approximation scheme for Minimum Vertex Cover is the first eptas for this problem on disk graphs. Most of the results of this chapter were presented in [260].

It seems difficult to generalize the results of Chapter 6 for Minimum (Connected) Dominating Set in the same way. Chapter 8 is devoted to the approximability of these problems on intersection graphs of systems of geometric objects of different sizes. The shifting technique yields a constant-factor approximation algorithm on intersection graphs of any set of fat objects, if the ply is bounded. The foremost innovation however is a general theorem to approximate Minimum Dominating Set on intersection graphs. We apply this to obtain the first constant-factor approximation algorithm for Minimum Dominating Set on intersection graphs of homothetic convex polygons and several other object types. We prove however that these methods cannot extend to intersection graphs of fat objects of arbitrary ply, of convex polygons, or of homothetic polygons by giving a strong approximation hardness result. We also prove APX-hardness on intersection graphs of arbitrary rectangles. This chapter contains an extended version of joint work with T. Erlebach [105].

Part III: Approximating Geometric Coverage Problems

We describe algorithms for the geometric version of (variants of) the minimum set cover problem. That is, the input consists of a set of geometric objects and a set of points, and we are asked to find a subset of the objects covering all points. In Chapter 9 we give the first polynomial-time approximation scheme for this problem on unit squares. We extend this to a ptas for Geometric Budgeted Maximum Coverage on unit squares. Moreover, we show that Geometric Set Cover is (very) hard to approximate on systems of fat objects and APX-hard on several other systems of two-dimensional objects. The chapter is based on yet unpublished joint work with T. Erlebach.

When we no longer insist that all points are covered, we arrive at several interesting new problems, which are variants of the unique coverage problem. In this problem, we are asked to cover a maximum number of points uniquely. Chapter 10 gives the first constant-factor approximation algorithms for this problem on unit disks and on unit squares by combining the shifting technique with complex dynamic programming algorithms. The multi-level shifting technique then generalizes these results to fat objects of bounded ply.

We again present hardness results to prove that the restriction to bounded ply is necessary. We also consider the approximability of the geometric version of a generalization of Minimum Set Cover, called Minimum Membership Set Cover. Chapter 10 is based on joint work with T. Erlebach [104].

Finally, Chapter 11 presents the conclusion and an outlook to future work.

1.3.1 Published Papers

This thesis is partially based on the following four (refereed) papers.

[1] van Leeuwen, E.J., "Approximation Algorithms for Unit Disk Graphs" in Kratsch, D. (ed.) *Graph-Theoretic Concepts in Computer Science, 31st International Workshop, WG 2005, Metz, France, June 23-25, 2005, Revised Selected Papers*, Lecture Notes in Computer Science 3787, Springer-Verlag, Berlin, 2005, pp. 351–361.

[2] van Leeuwen, E.J., "Better Approximation Schemes for Disk Graphs" in Arge, L., Freivalds, R. (eds.) *Algorithm Theory - SWAT 2006, 10th Scandinavian Workshop on Algorithm Theory, Riga, Latvia, July 6-8, 2006, Proceedings*, Lecture Notes in Computer Science 4059, Springer-Verlag, Berlin, 2006, pp. 316–327.

[3] Erlebach, T., van Leeuwen, E.J., "Approximating Geometric Coverage Problems" in Teng, S.H. (ed.) *Proceedings of the Nineteenth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2008, San Francisco, California, USA, January 20-22, 2008*, Association for Computing Machinery, 2008, pp. 1267–1276.

[4] Erlebach, T., van Leeuwen, E.J., "Domination in Geometric Intersection Graphs" in Laber, E.S., Bornstein, C.F., Nogueira, L.T., Faria, L. (eds.) *LATIN 2008: Theoretical Informatics, 8th Latin American Symposium, Búzios, Brazil, April 7-11, 2008, Proceedings*, Lecture Notes in Computer Science 4957, Springer-Verlag, Berlin, 2008, pp. 747–758.