Optimization and approximation on systems of geometric objects

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Chapter 3

Guide to
Geometric Intersection Graphs

The introduction of this thesis presented geometric intersection graphs in the context of their many application areas, including wireless networks, computational biology, and map labeling. This chapter aims to provide a more formal and thorough introduction to these interesting graph classes. In particular, we will be concerned with the structure of (geometric) intersection graphs and the complexity of recognizing such graphs.

First however, we give a formal definition of intersection graphs and geometric intersection graphs. These are supported by examples of classes of (geometric) intersection graphs being studied in the literature. We describe the inclusion relations of these classes. We give special attention to some classes that are being studied in relation to wireless communication networks. We also expose connections between geometric intersection graphs and both general graphs and planar graphs. General graphs are shown to be intersection graphs of convex objects in $\mathbb{R}^3$ [268] and planar graphs are the intersection graphs of internally disjoint boxes in $\mathbb{R}^3$ [251] and of internally disjoint triangles [80], disks [169], or smooth convex objects in $\mathbb{R}^2$ [237].

We then consider questions related to recognizing geometric intersection graphs. We present several positive results on the recognition of certain classes of geometric intersection graphs, such as interval graphs, which can be recognized in polynomial time [38]. Most of these results however are on intersection graphs of one-dimensional objects. Recognizing intersection graphs of two-dimensional objects, such as intersection graphs of disks or polygons, is often NP-hard [41, 174, 42]. In fact, for intersection graphs of disks of equal radius, one can even give an approximation hardness result [184]. Several recognition problems are NP-hard and not NP-complete, as membership of NP is not known for these problems. A full discussion of this particular issue is deferred to Chapter 4.

3.1 Intersection Graphs

We define intersection graphs as follows. All considered graphs are simple, finite, and undirected, unless otherwise stated.

Definition 3.1.1 *Given a universe $\mathbb{U}$ and some finite collection $\mathcal{S}$ of subsets of $\mathbb{U}$, the* intersection graph *$G = G[\mathcal{S}]$ of $\mathcal{S}$ is the graph where each $\mathcal{S}(u) \in \mathcal{S}$ corresponds to a unique vertex $u \in V(G)$ and there is an edge between two vertices if and only if the corresponding subsets intersect (i.e. $(u, v) \in E(G)$ if and only if $\mathcal{S}(u) \cap \mathcal{S}(v) \neq \emptyset$).*

If a graph G is (isomorphic to) the intersection graph of some set system $(\mathbb{U}, \mathcal{S})$, then this set system is a *representation* of G, whereas G is said to be *induced* by $(\mathbb{U}, \mathcal{S})$. Usually, we will not distinguish between the sets of the set system and the vertices they correspond to.

It is easy to prove that any graph is an intersection graph of some set system [248]. Given a graph G, take $\mathbb{U} = E(G)$ and let $\mathcal{S} = \{S_v \mid v \in V(G)\}$, where $S_v = \{e = (u, v) \in E(G)\}$. One can however prove more interesting theorems. For instance, *chordal graphs* (the class of graphs having no induced cycle of length greater than three) are precisely the intersection graphs of subtrees of a fixed tree (see e.g. [135, 46, 118, 265]). A nice overview of these and other results can be found in the books by McKee and McMorris [208] and Spinrad [246] and the survey by Kozyrev and Yushmanov [171].

It is not easy to give a precise definition of geometric intersection graphs, as 'geometry' is not easily defined. For the purpose of this thesis however, the following definition is used.

Definition 3.1.2 *Given some finite collection $\mathcal{S}$ of subsets of $\mathbb{R}^d$ for some $d \geq 0$, the intersection graph $G[\mathcal{S}]$ is called a* geometric intersection graph.

Commonly, there is a restriction on the nature of the sets as well. This is expressed in the following definition.

Definition 3.1.3 *Let $\mathcal{A}$ be a set of subsets of $\mathbb{R}^d$ for some $d > 0$. Then G is an $\mathcal{A}$-intersection graph if it is (isomorphic to) the intersection graph $G[\mathcal{S}]$ for some collection $\mathcal{S}$ of translated copies of objects in $\mathcal{A}$.*

It is these restrictions that we will be most interested in. We note here that we only consider sets $\mathcal{A}$ of objects that are either all closed sets or all open sets. We prohibit mixing open and closed sets to simplify the presentation.

3.1.1   Interval Graphs and Generalizations

Perhaps the first and most frequently studied class of geometric intersection graphs are *interval graphs*. Here the universe is $\mathbb{R}^1$ and the sets are *intervals* or *segments* of the real line (i.e. connected subsets of $\mathbb{R}^1$). Sometimes these intervals are forced to have unit length (i.e. equal length, usually assumed to be 1), leading to *unit interval graphs* (or *indifference graphs*).

Several characterizations of interval graphs are known. Lekkerkerker and Boland [189] proved that G is an interval graph if and only if it is a chordal graph and has no asteroidal triple (a set of three vertices of the graph, any two

of which are connected by a path containing no vertex of the neighborhood of the third vertex). Gilmore and Hoffman [123] proved that G is an interval graph if and only if it is a chordal graph and a cocomparability graph (a graph with a partial order on the vertices where two vertices are adjacent if and only if they are not related in the ordering). Unit interval graphs are precisely the interval graphs with no $K_{1,3}$ induced subgraph [268].

Interval graphs are recognizable in linear time [38]. Several optimization problems that are NP-hard on general graphs are easily polynomial-time solvable on interval graphs, such as Maximum Clique and Minimum Vertex Cover. An important structural property of interval graphs is that interval graphs are perfect graphs. Further properties and characterizations of (unit) interval graphs may be found in the books by Golumbic [126] and McKee and McMorris [208], and the paper by Kozyrev and Yushmanov [171].

Most classes of geometric intersection graphs can be viewed as generalizations of interval graphs, either because they stick to the idea of segments or because the considered geometric objects are intervals when restricted to $\mathbb{R}^1$. We first consider other classes of intersection graphs of segments.

In *multi-interval graphs*, we allow the sets $\mathcal{S}(u)$ to consist of multiple intervals on the real line. Clearly, any graph is a multi-interval graph by taking sufficiently many intervals for each vertex. However, determining whether the minimum number of intervals needed per vertex (the *interval number*) is at most k is an NP-complete problem for any fixed integer $k \geq 2$ [269].

Tolerance (interval) graphs place a restriction on the nature of the intersection when determining whether two vertices are adjacent. Each vertex is assigned a positive value (its *tolerance*) and two vertices are adjacent if the value of some function of the intersection of their intervals is at least the value of some function of their tolerances. For instance, in a *max-tolerance graph*, the length of the intersection should be at least the maximum of the tolerances. Note that the idea behind tolerance graphs is not necessarily restricted to interval graphs, but can be applied to any intersection graph [208]. The book by Golumbic and Trenk [127] provides a good overview of this subject. Here however we consider only tolerance interval graphs.

If instead of intervals on the real line, one considers intervals (arcs) on a circle, *circular-arc graphs* are obtained. Even though they are only a slight generalization of interval graphs, their structure is fundamentally different. Circular-arc graphs are not necessarily chordal or perfect, as they can contain induced cycles of any length. In fact, no characterization in terms of forbidden subgraphs is known [208]. They can however be recognized in linear time [207]. If all intervals have equal length (*unit circular-arc graphs*), a structural characterization does exist [253] and recognition is possible in linear time [195].

If we stick to line segments, but in $\mathbb{R}^2$, we arrive at *k-DIR graphs*, which are intersection graphs of line segments that can point in one of k directions. Recognizing graphs in this class is NP-complete for any fixed $k \geq 2$ [173]. Intersection graphs of piecewise linear curves consisting of at most k line segments (*k-segment intersection graphs*) are NP-hard to recognize for fixed $k \geq 2$ [179].

It is not known whether the recognition problem is in NP. Demonstrating membership of NP seems hard, as there are 1-segment intersection graphs where the coordinates of the endpoints of the segments must be double exponential integers [179].

Generalizing further, we have intersection graphs of arbitrary simple curves (*string graphs*). String graphs are NP-hard to recognize [172]. Recognizing string graphs surprisingly is in NP [233] and thus NP-complete. Note that every graph is the intersection graph of simple curves in $\mathbb{R}^3$. This is not true however in $\mathbb{R}^2$ (i.e. not every graph is a string graph) [241, 242, 98].

A relatively recent overview of the results in this area and a description of further classes can be found in the course notes of Kratochvíl [175].

3.1.2 Intersection Graphs of Higher Dimensional Objects

Another way to generalize interval graphs, more in line with further topics of this thesis, is to consider d-dimensional objects that are intervals if $d = 1$. A good example are *intersection graphs of d-dimensional axis-parallel boxes*. A d-dimensional axis-parallel box is simply the Cartesian product of d orthogonal intervals, e.g. $[a_1, b_1] \times \cdots \times [a_d, b_d]$ for numbers $a_i < b_i$. For the case $d = 2$, these are also known as *rectangle intersection graphs*.

An important related property of general graphs is their *boxicity*, the minimum number d such that the given graph is isomorphic to an intersection graph of d-dimensional axis-parallel boxes. Roberts [228] (who was the first to study boxicity) proved that any n-vertex graph has boxicity at most $\lfloor n/2 \rfloor$. Determining the boxicity of a graph is NP-hard [71]. Recognizing whether the boxicity is at most d is NP-complete for any fixed $d \geq 2$ [173, 273, 196]. Note that for $d = 1$, the recognition problem is equal to the problem of recognizing interval graphs, which is in P.

Given the above, it is not hard to imagine a natural generalization of unit interval graphs. This leads to *intersection graphs of d-dimensional axis-parallel unit cubes*, which are d-dimensional axis-parallel boxes with side length equal to one. For $d = 2$, these are called *unit square intersection graphs*, or simply *unit square graphs*. The notion of *cubicity* can be defined analogously to boxicity. Roberts [228] bounded the cubicity of n-vertex graphs by $\lfloor 2n/3 \rfloor$, the tight example being complete k-partite graphs with $k = \lfloor n/3 \rfloor$. Recognizing graphs of cubicity d is NP-complete [273, 40, 70, 73] for any fixed $d \geq 2$. The case $d = 1$ is in P, as these are precisely the unit interval graphs.

As an intermediate step between unit cubes and rectangles, one could consider axis-parallel cubes of arbitrary side length. In two dimensions, these are *square (intersection) graphs*. There seem to be no results on the minimum dimension d needed for a graph to be isomorphic to an intersection graph of d-dimensional axis-parallel cubes. Breu [40] showed that testing whether a graph is an intersection graph of squares where the ratio between the size of the largest and of the smallest square is some fixed constant $\rho \geq 1$, is NP-hard. The recognition problem of intersection graphs of arbitrary-sized squares is

NP-hard as well [182].

A further relevant class of geometric intersection graphs are *triangle intersection graphs*. It is known that any planar graph is an intersection graph of internally disjoint triangles [80] and conjectured that any planar graph is an intersection graph of homothetic triangles [176]. The class of intersection graphs of isosceles right triangles is also interesting, as this class was shown to be equivalent to max-tolerance interval graphs [160].

The definitions of box, cube, and triangle intersection graphs invite to generalizations to intersection graphs of other geometric objects. One could for instance consider *intersection graphs of convex objects*. Again, for $d = 1$, this corresponds to interval graphs and hence they can be recognized in linear time. For $d = 2$, recognition is NP-hard [178]. Proving membership of NP is likely to be very difficult, as Pergel (see [176]) proved that convex object intersection graphs exist for which any integer representation requires double exponential integers. However, recognizing the intersection graph of scaled and translated copies of a fixed convex polygon is both NP-hard [182] and in NP [261], and thus NP-complete. The recognition problem for higher dimensions is trivial, because as shown later in Theorem 3.3.1, any graph is the intersection graph of three-dimensional convex objects.

Finally, we treat a subclass of planar convex object intersection graphs, namely the class of *polygon-circle graphs*. These are the intersection graphs of polygons inscribed in a circle, i.e. all corners of the polygons should lie on one given circle. Recognizing these graphs is NP-complete [180, 224], but polynomial if the girth is greater than four [181]. It can be proved by a simple argument that chordal graphs are polygon-circle graphs (see Corollary 3.3.3).

3.2 Disk Graphs and Ball Graphs

A different generalization of interval graphs are *ball graphs*, intersection graphs of d-dimensional balls. A d-dimensional ball is given by its center and consists of all points within a certain distance. For $d = 2$, these are the well-known disk graphs.

Definition 3.2.1 *A graph isomorphic to an intersection graph of two-dimensional balls (i.e. disks) is called a* disk graph.

We emphasize disk graphs as they motivated most of the research in this thesis.

Recognizing disk graphs is NP-hard [174], even if the ratio between the radii of the largest and smallest disk is bounded by a constant [41]. The complexity of recognizing intersection graphs of higher dimensional balls is unknown, but expected to be NP-hard [42].

When generalizing unit interval graphs, we get *unit ball graphs*, i.e. intersection graphs of d-dimensional balls of equal radius.

Definition 3.2.2 *A graph isomorphic to an intersection graph of two-dimensional balls (i.e. disks) of radius $1/2$ is called a* unit disk graph.

Although we define unit disk graphs to have a representation with disks of radius $1/2$, this number is mostly chosen for convenience. The most important property is that all disks have equal radius. By scaling, one can always assume this common radius to be $1/2$.

We should note here that disk graphs and unit disk graphs are really different graph classes. For instance, unit disk graphs cannot have a $K_{1,6}$ or $K_{2,3}$ induced subgraph [257], whereas these graphs are disk graphs. An example of a graph that is not a disk graph is $K_{3,3}$. In fact, all triangle-free disk graphs must be planar [199].

Instead of focusing on intersections of d-dimensional balls, another way to define unit ball graphs is to place n points in $\mathbb{R}^d$ and say that two vertices are adjacent if and only if they are at distance at most 1. If $d = 1$, this definition corresponds to the definition of indifference graphs.

Unit disk graphs are sometimes also called *geometric graphs* (see for example DeWitt and Krieger [86]). This should not be confused with the currently used definition of geometric graphs, namely graphs where each vertex is assigned a point in $\mathbb{R}^d$ and edges are drawn as straight lines between the points. Although under this modern definition, a representation of a unit disk/ball graph induces a geometric graph, it can be readily observed that not every geometric graph is a unit disk/ball graph.

The *(unit) sphericity* of a graph is the minimum d such that the graph is isomorphic to an intersection graph of d-dimensional unit balls [144, 145, 109]. Maehara [198] proved that the sphericity of any n-vertex graph G is at most $n - \omega(G)$, where $\omega(G)$ denotes the size of the largest clique of G. Moreover, this bound is essentially tight. Recognizing unit disk graphs (i.e. graphs of sphericity two) is NP-hard, even if the graph is planar or has sphericity at most three [42]. Recognizing graphs of sphericity at most three is also NP-hard, but the complexity for higher constants is unknown (though conjectured to be NP-hard) [42].

Kuhn, Moscibroda, and Wattenhofer [184] provide a strengthening of the NP-hardness result in two dimensions. Define

$$q(G, c) = \frac{\max_{(u,v) \in E(G)} \|c_u - c_v\|}{\min_{(u,v) \notin E(G)} \|c_u - c_v\|}$$

as the *quality* of a mapping $c : V(G) \to \mathbb{R}^2$ of a unit disk graph G, where c_u (c_v) denotes the location of the center of the disk corresponding to u (v). Note that any (nontrivial) unit disk graph by definition has a mapping of quality less than one. Kuhn, Moscibroda, and Wattenhofer [184] show however that it is NP-hard to decide if a mapping of quality at most $\sqrt{3/2} - \epsilon$ exists, where ϵ tends to 0 as the number of vertices of the graph approaches infinity. On the positive side, Moscibroda et al. [215] give a polynomial-time algorithm yielding a mapping of quality $O((\log^{5/2} n) \cdot \sqrt{\log \log n})$. This clearly leaves a large gap and a major open question.

An important special case of (unit) ball graphs are intersection graphs of internally disjoint (unit) balls, called *(unit) ball touching graphs* or *(unit)*

ball contact graphs. In two dimensions, ball contact graphs are also called *disk contact graphs* or *coin graphs* [231]. Coin graphs are interesting, as they coincide with the class of all planar graphs [169] (see also Section 3.3.1). Hence these graphs are recognizable in linear time [152]. If however the ratio of the radii of the largest and smallest disk is any fixed constant, then the recognition problem becomes NP-hard [41]. The complexity for recognition of ball contact graphs in higher dimensions is open. For $d = 1$, (unit) ball contact graphs are disjoint unions of paths and are thus recognizable in linear time.

In the case of unit ball contact graphs, we know a bit more. Any n-vertex graph is a unit ball contact graph in dimension $n - 1$ [147]. Recognizing unit ball contact graphs is known to be NP-hard for dimension 2 [42], 3, 4, 5 [146], 8, 9, 24, and 25 [147]. The hardness proofs for dimensions 5, 9, and 25 follow from a construction of Kirkpatrick and Rote (see Hliněný [146]) who showed that a graph G is isomorphic to a unit d-ball contact graph if and only if $G \oplus K_2$ is isomorphic to a unit $(d + 1)$-ball contact graph, where $G \oplus K_2$ is obtained from the disjoint union of G and K_2 by adding all edges between the vertices of the summands.

Another generalization of disk graphs are *intersection graphs of noncrossing arc-connected sets* [174]. A set is *arc-connected* if between any two points of the set an arc can be drawn containing only points of the set. The class of intersection graphs of arc-connected sets in the plane coincides with the class of string graphs [174]. Two arc-connected sets X and Y are said to be *noncrossing* if both $X - Y$ and $Y - X$ are arc-connected. Intersection graphs of noncrossing arc-connected sets are not equivalent to string graphs, as $K_{3,3}$, which is a string graph, is not an intersection graph of noncrossing arc-connected sets [174]. Recognizing intersection graphs of noncrossing arc-connected sets in the plane is NP-hard [174]. However, each graph is an intersection graph of three-dimensional noncrossing arc-connected sets.

Noncrossing arc-connected sets in the plane are essentially *k-admissible regions* for some even integer k, which are a collection of noncrossing arc-connected sets each bounded by a simple closed Jordan curve, such that each pair of curves intersects at most k' times, for some even $k' \leq k$ [226]. We call a collection of 2-admissible regions a collection of *pseudo-disks*.

We should note that several of the NP-hard recognition problems described in this chapter that are not in P, such as the problem of recognizing disk graphs, are not known to be in NP. In particular, we know of no polynomially-sized representation for these graph classes. However, one can prove that the recognition problems are in PSPACE [179, 147, 51].

3.2.1 Models for Wireless Networks

In the introduction (Chapter 1), we mentioned wireless networks as one of the main application areas of geometric intersection graphs. Particularly (unit) disk graphs are frequently used as a model in this setting. To bring these models even closer to the situations encountered in practice, several more

sophisticated graph models have been proposed. We describe some of them. Further models can be found in the survey by Schmid and Wattenhofer [236].

A restriction of disk graphs is the following. Suppose that the radii of the disks model broadcasting ranges. Then u can hear v if and only if u is within v's broadcasting range, i.e. if u lies within the disk centered on v. Use this to determine the adjacency of vertices in the graph (so that u and v are adjacent if and only if u lies within the disk centered on v or v lies within the disk centered on u) and one obtains a *containment disk graph* [129, 199]. Malesińska [199] proved that the class of containment disk graphs is not contained in the class of disk graphs, as $K_{3,3}$ is a containment disk graph, but not a disk graph (recall that triangle-free disk graphs are planar). It is unclear whether a disk graph exists that is not a containment disk graph. However, any unit disk graph is a containment disk graph.

We presented the containment disk graph as an undirected graph. Given its motivation however, it makes more sense to define it as a directed graph. In this case, there is a directed edge from v to u if and only if u is contained in the disk centered on the location of v. This graph is called a *directed disk graph* [93], but in many cases it is also referred to as a *(directed) geometric radio network* [69].

A graph class that generalizes both disk graphs and containment disk graphs is the class of *double disk graphs* [129, 199, 94]. As the name implies, centered on the location of a vertex are two disks, s and b, such that the radius of b is at least the radius of s. Then two vertices u and v are adjacent if and only if $s(u)$ and $b(v)$ intersect or $s(v)$ and $b(u)$ intersect. The idea behind this graph model is that any wireless device has a range within which it can communicate with other devices and a larger range within which its signal interferes with the signals of other devices.

Another generalization of unit disk graphs is the quasi unit disk graph. In a unit disk graph, two vertices are adjacent if and only if the distance between their locations is at most one. Usually however, the probability of successfully connecting to another device decreases as it is further away from the source of the signal. In a *quasi unit disk graph*, given some $\rho \in [0, 1]$, two vertices are adjacent if they are within distance ρ, can be adjacent if they are within distance more than ρ but at most one, and are not adjacent if their distance is more than one [187]. Note that the behavior is undefined if the distance is in $(\rho, 1]$. This could be determined by an adversary or some probabilistic model. Kuhn, Moscibroda, and Wattenhofer [184] prove that recognizing ρ-quasi unit disk graphs with $\rho \geq \sqrt{1/2}$ is NP-hard.

Both unit disk graphs and quasi unit disk graphs have the property that the size of any independent set of the r-neighborhood of any vertex is polynomial in r, where the r-neighborhood of a vertex u consists of all vertices having a path of length at most r to u. This behavior can be used to define a class of graphs. A *bounded independence graph* or a *graph of polynomially-bounded growth* is a graph where for any r and any vertex u, all independent sets in the r-neighborhood of u have cardinality polynomial in r [219].

3.3 Relation to Other Graph Classes

Geometric intersection graphs have many relations to other well-known graph classes, for which it might be surprising that they are contained in a particular class of geometric intersection graphs. Some of these relations were already discussed in the previous section. Here we expand on these results and (where possible) sketch a proof. In particular, we will discuss the strong connections between geometric intersection graphs and planar graphs.

We begin by showing that any graph is an intersection graph of (internally disjoint) three-dimensional convex polytopes. This result is frequently attributed to Wegner [268]. Wegner himself [268, p. 28] however attributes it to Grünbaum, but this proof seems to be unpublished. Kalinin [158] also gives a proof. Here we follow Wegner's proof [268].

Theorem 3.3.1 *Any graph is the intersection graph of a set of (internally disjoint) three-dimensional convex polytopes.*

Proof: For any integer n, there exists a family $\mathcal{S}$ of n internally disjoint three-dimensional convex polytopes with nonempty interior, any two of which intersect. Moreover, the intersection of polytopes u and v is a (two-dimensional) facet of u or v. Finding such a family is known as Crum's problem [33, 227] and was solved by Besicovitch [33] and Rado [227].

Let G be any n-vertex graph and $\mathcal{S}$ a family as described above. Now let $\mathcal{S}'$ be obtained from $\mathcal{S}$ by taking for each $s \in \mathcal{S}$ a convex subset of the interior of s. Furthermore, for any $(u, v) \in E(G)$, choose a point p_{uv} in the interior of the intersection of $\mathcal{S}(u)$ and $\mathcal{S}(v)$. Let P_u denote the set of such points involving vertex u. Let $\widetilde{\mathcal{S}}$ be the set obtained by taking for each $u \in V(G)$ the convex hull of $\mathcal{S}'(u)$ and P_u. If H is the graph induced by $\widetilde{\mathcal{S}}$, then clearly $E(H) \supseteq E(G)$. Suppose that $(u, v) \in E(H) - E(G)$. Without loss of generality, the intersection of $\mathcal{S}(u)$ and $\mathcal{S}(v)$ is a facet of $\mathcal{S}(u)$. By the choice of P_u and $\mathcal{S}'(u)$, there is a hyperplane separating $\widetilde{\mathcal{S}}(u)$ and this facet. As $\widetilde{\mathcal{S}}(u)$ ($\widetilde{\mathcal{S}}(v)$) is a convex subset of $\mathcal{S}(u)$ ($\mathcal{S}(v)$), $\widetilde{\mathcal{S}}(u)$ and $\widetilde{\mathcal{S}}(v)$ cannot intersect. This contradicts that $(u, v) \in E(H)$. Hence $E(H) = E(G)$.   □

Note that the constructed polytopes in fact have at most one point in common.

So what about intersection graphs of two-dimensional convex polytopes? If the polytopes are internally disjoint, they can be fully characterized (see Theorem 3.3.5). If we allow arbitrary intersections however, no characterization is known. Wegner [268, p. 25] showed that K_5 with each edge bisected is not the intersection graph of convex two-dimensional polytopes, since this would imply a planar drawing of K_5. (This is also implied by a result of Sinden [241, 242] and Ehrlich, Even, and Tarjan [98], who showed that this bisection of K_5 is not a string graph.)

We can give the following positive result. A planar graph is *outerplanar* if it has a planar embedding in which each vertex lies on the boundary of the outer face. We call this an *outerplanar embedding*.

Theorem 3.3.2 *The intersection graph G of a collection $\mathcal{H}$ of connected subgraphs of a fixed outerplanar graph H is a polygon-circle graph.*

Proof: We may assume that H is connected, otherwise we apply the proof given below to each connected component of H. Consider an outerplanar embedding of H. Going along the entire boundary of the outer face, let $v_1, \ldots, v_k$ be the vertices consecutively encountered. This induces an ordering $u_1, \ldots, u_n$ on the vertices of $V(H)$, where $n = |V(H)|$.

Now place n points $p_1, \ldots, p_n$ on (any arc of) a circle and map u_i to p_i. By the definition of the ordering on $V(H)$, this induces an outerplanar embedding of H. For any connected subgraph $\mathcal{H}(w) = \{u_{i_1}, \ldots, u_{i_k}\} \subseteq V(H)$ for $w \in V(G)$, let $C(w)$ be the convex hull of $p_{i_1}, \ldots, p_{i_k}$. One can verify that for $v, w \in V(G)$, $C(v)$ intersects $C(w)$ if and only if $\mathcal{H}(v)$ intersects $\mathcal{H}(w)$.   □

As a corollary, we obtain a result by Duchet [95].

Corollary 3.3.3 *Any chordal graph is a polygon-circle graph.*

Proof: It is known that any chordal graph G is the intersection graph of a family $\mathcal{T}$ of subtrees of a fixed tree T. (This result is attributed to Surányi in [135]. Proofs can be found in [46, 118, 265]). Trees are clearly outerplanar. Now apply Theorem 3.3.2.   □

3.3.1   Relation to Planar Graphs

Most results about the structure of geometric intersection graphs are related to planar graphs. Sometimes one can prove that planar graphs form a subclass of some class of geometric intersection graphs (or vice versa), but often one can give a full characterization. We will see examples of both.

Recall that k-interval graphs are intersection graphs of unions of k intervals. The interval number of a graph G is the minimum number k such that G is isomorphic to a k-interval graph. Scheinerman and West [235] proved that any planar graph has interval number at most three.

For higher dimensions, the following famous result is known.

Theorem 3.3.4 (Koebe [169]) *A graph G is planar if and only if G is isomorphic to a disk contact graph (coin graph).*

This result was rediscovered several times (see Sachs [231] for a history).

The radii of the disks in a disk contact representation of a planar graph are not necessarily polynomially bounded integers. The radii might differ by an exponential factor. Hansen [139] (see Malitz and Papakostas [200]) shows that there exist wheels for which disks of exponentially large radius are needed in any disk contact representation. Also, Breu and Kirkpatrick [41] showed that it is NP-hard to test whether such large disks are necessary.

Moreover, one cannot expect the radii of the disks to be integers. This would imply that any planar graph has a straight line embedding such that all

edges have integer length. Although Geelen, Guo, and McKinnon [119] proved that any planar graph has an integer straight line embedding, Brightwell and Scheinerman [43] demonstrated that there exist planar graphs for which no integer straight line embedding can be induced by a disk contact representation. Otherwise one could trisect an angle of $\pi/3$ using ruler and compass.

An approximation to the coordinates and radii of a disk contact representation can be given though for 3-connected planar graphs. In fact, this holds for an even more general representation. Brightwell and Scheinerman [43] proved that for any n-vertex planar graph G, both G and the dual of G have a disk contact representation, $\mathcal{S}$ and $\mathcal{S}'$ respectively, such that for any edge $e = (u, v) \in E(G)$ and its dual edge $e^* = (f^*, g^*)$, the intersection point of $\mathcal{S}(u)$ and $\mathcal{S}(v)$ coincides with the intersection point of $\mathcal{S}'(f^*)$ and $\mathcal{S}'(g^*)$. Moreover, the line through the centers of $\mathcal{S}(u)$ and $\mathcal{S}(v)$ is perpendicular to the line through the centers of $\mathcal{S}'(f^*)$ and $\mathcal{S}'(g^*)$. Mohar [211, 212] gives an algorithm to determine the centers and radii of such a primal-dual representation to a precision of ϵ in time polynomial in n and $\max\{\log 1/\epsilon, 1\}$. A similar result is proved by Smith [243].

A consequence of Koebe's result is that planar graphs are string graphs, where any pair of curves is nowhere tangent and intersects at most twice. Chalopin, Gonçalves, and Ochem [56] showed that in fact one intersection suffices. This was a step forward in proving the following conjecture, sometimes referred to as *Scheinerman's conjecture* [234]: any planar graph is isomorphic to a 1-segment intersection graph. The conjecture was previously shown to hold for triangle-free [77], bipartite [141, 79], and several other types of planar graphs [78]. Recently, Chalopin and Gonçalves [55] managed to prove the conjecture, i.e. any planar graph is indeed isomorphic to a 1-segment intersection graph.

Disk graphs are generally not planar. However, Malesińska [199] proves that triangle-free disk graphs are planar. This also holds for triangle-free intersection graphs of pseudo-disks [174]. As planar graphs are disk intersection graphs, triangle-free (pseudo-)disk graphs are recognizable in polynomial time.

Characterizations similar to Koebe's theorem have been proved for other convex objects. The following result is implied by Koebe's theorem, but has a relatively easy proof due to Wegner [268], which we give below.

Theorem 3.3.5 *A graph is planar if and only if it is isomorphic to the intersection graph of a set of internally disjoint two-dimensional convex polytopes.*

Proof: The if-part is trivial. For the converse, let G be a planar graph. Augment G to an (edge) maximal planar graph G' by adding edges. Add a dummy vertex z in the unbounded face of G' and connect it to all vertices on the unbounded face of G'. Call the resulting graph G''. Since G'' is maximal planar, it is 3-connected and hence its dual is planar and 3-connected as well [213]. Following Stein [247] (see also Tutte [254] and Kelmans [162]), this implies that the dual of G'' has a straight line embedding such that each bounded face

is convex and the face corresponding to z is the unbounded face. Hence we obtain a collection of internally disjoint two-dimensional convex polytopes whose intersection graph is G'. Using the same idea as in the proof of Theorem 3.3.1, we can remove unwanted edges to obtain a representation of G.   □

A result of Thomassen [250] implies that finding such a representation actually takes linear time. Interestingly, Kratochvíl and Kuběna [177] showed that the complement of a planar graph is the intersection graph of a set of two-dimensional convex polytopes as well.

From the above proof, it seems that one might need polytopes with an arbitrary number of corners, but this is not necessarily the case. Some planar graphs are rectangle contact graphs. Thomassen [251] proved that G is a rectangle contact graph if and only if G is a proper subgraph of a 4-connected planar triangulation. Bipartite planar graphs are also rectangle intersection graphs [79, 141]. Generalizing in this direction, Thomassen [251] showed that any planar graph is the intersection graph of internally disjoint three-dimensional axis-parallel boxes.

De Fraysseix, Ossona de Mendez, and Rosenstiehl [80] proved that G is planar if and only if G is a triangle contact graph. Moreover, they gave a polynomial time algorithm to construct a representation by internally disjoint triangles. Although triangles are sufficient, the shapes of the triangles can be very different. Can one prove that they must be similar somehow?

Definition 3.3.6 *We say that two geometric objects are* homothetic *if one can be obtained from the other by only scaling and translating.*

There are planar graphs that are not the intersection graphs of internally disjoint homothetic triangles, although Kratochvíl and Pergel [176] conjecture that planar graphs are homothetic triangle graphs, i.e. without the constraint that the triangles should touch.

Observe that Koebe's result states that any planar graph is an intersection graph of internally disjoint homothetic disks. Schramm [237, 238] generalizes Koebe's theorem to homothetic copies of arbitrary convex planar bodies with smooth boundaries. The result actually is slightly more general.

Theorem 3.3.7 (Schramm [237, 238]) *Let G be any n-vertex planar graph and $\mathcal{A} = \{A_v \mid v \in V\}$ a collection of n planar convex bodies with smooth boundaries. Then G is the intersection graph of $\mathcal{S} = \{S_v \mid v \in V\}$, where S_v is a homothetic copy of A_v. Moreover, the objects in $\mathcal{S}$ are internally disjoint.*

Note that Theorem 3.3.7 requires the convex objects to have smooth boundaries and thus does not contradict the statement that planar graphs are not the intersection graphs of internally disjoint homothetic triangles.