Optimization and approximation on systems of geometric objects

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Part II

Approximating Optimization Problems on Geometric Intersection Graphs

Overview

Wireless communication networks increase their influence on human interaction on a daily basis. Many people do away with their fixed landline phone and only use a cell phone, wireless Internet allows one to check e-mail everywhere, etcetera. Making such networks work has led to a lot of new problems and challenges, ranging from practical questions to theoretical conundrums.

The purpose of this part of the thesis is to advance insight into some of these theoretical problems. We address them by considering a model that is commonly used for wireless networks, the geometric intersection graph model, and in particular the (unit) disk graph model. We then study several optimization problems on wireless networks, which translate to well-known graph optimization problems (such as Maximum Independent Set and Minimum Connected Dominating Set) on geometric intersection graphs. As geometric intersection graphs have a geometric representation, we can show that these problems can be better solved or approximated on such graphs than on general graphs.

Problems

We consider various optimization problems on graphs that are relevant to geometric intersection graph models and specifically to (unit) disk graphs models of wireless communication networks.

Definition II.1 *Let G be a graph. A set $S \subseteq V(G)$ is an* independent set *if there are no $u, v \in S$ such that $(u, v) \in E(G)$. A set $S \subseteq V(G)$ is a* vertex cover *if for each $(u, v) \in E(G)$ it holds that $u \in S$ or $v \in S$.*

Observe that an independent set is the complement of a vertex cover (and vice versa) [115]. Furthermore, we are usually looking for a maximum independent set and a minimum vertex cover. An independent set is *maximum* if there is no independent set of greater cardinality. A vertex cover is *minimum* if there is no vertex cover of smaller cardinality. In the context of wireless communication networks, an independent set of a (unit) disk graph can be seen as a set of nodes that can transmit simultaneously without signal interferences. Vertex covers are mostly interesting from a theoretical point of view in this context.

Definition II.2 *Let G be a graph. A set $S \subseteq V(G)$ is a* dominating set *if for each vertex $v \in V$, $v \in S$ or there is a vertex $u \in S$ for which $(u, v) \in E(G)$.*

Definition II.3 *Let G be a graph. A set $S \subseteq V(G)$ is a* connected dominating set *if S is a dominating set and the subgraph of G induced by S is connected.*

A dominating set in a wireless communication network can be seen as a set of emergency transmitters capable of reaching every node in the network, or as central nodes in node clusters. A connected dominating set can be used as a backbone for easier and faster communications. The problem is to find a minimum (connected) dominating set, i.e. we look for (connected) dominating sets of minimum cardinality.

Previous Work

All problems mentioned above are NP-complete on general graphs (see Garey and Johnson [115]). Since (unit) disk graphs are a restricted class of graphs with a nice geometric interpretation, one might hope that these problems are better solvable. Unfortunately, these problems are NP-complete on unit disk graphs and other classes of intersection graphs of planar objects as well [267, 17, 67]. The NP-hardness holds even if the degree is at most 3 and (except for Maximum Independent Set and Minimum Vertex Cover) the graph is bipartite [65]. Therefore most research has focused on approximation algorithms.

We survey the previous work in this area, with strong emphasis on geometric intersection graphs and (unit) disk graphs in particular. Work specific to a particular chapter will be surveyed there. A detailed survey of available approximation algorithms and inapproximability results on general graphs can be found (for example) in the compendium of Ausiello et al. [19].

Maximum Independent Set

On general n-vertex graphs, Maximum Independent Set has a polynomial time $O(n/\log^2 n)$-approximation algorithm [39] and is not approximable within $O(n^{1-\epsilon})$ for any $\epsilon > 0$, unless NP=ZPP [142]. On geometric intersection graphs however, it is easy to give a constant-factor approximation algorithm. Many geometric intersection graphs have no $K_{1,m}$ induced subgraph for some $m > 1$. Unit disk graphs have no $K_{1,6}$ induced subgraph for example. In general, Maximum Independent Set has a polynomial-time $m/2$-approximation algorithm on graphs with no $K_{1,m}$ induced subgraph, even in the weighted case [137, 276, 21, 31]. This immediately gives a 3-approximation for Maximum (Weighted) Independent Set on unit disk graphs (see for example Yu, Kouvelis, and Luo [277]).

A similar result can be obtained by a greedy algorithm based on the representation of the unit disk graph. Marathe et al. [201] showed that greedily choosing the vertex corresponding to the leftmost disk gives a 3-approximation algorithm for the unweighted case. On general disk graphs, which can have a $K_{1,m}$ induced subgraph for any $m \geq 1$, greedily choosing the vertex with the smallest disk radius results in a 5-approximation algorithm [201].

Agarwal and Mustafa [2] provide a more general approach and consider the intersection graph of a family $\mathcal{S}$ of convex two-dimensional objects. If α is the cardinality of the maximum independent set of this graph, their algorithm returns an independent set of cardinality $(\alpha/(2\log(2n/\alpha)))^{\frac{1}{3}}$ in $O(n^3 + \tau(\mathcal{S}))$ time, where $\tau(\mathcal{S})$ is the time needed to compute the left- and rightmost point of each object and test which objects intersect. A set of disks clearly is a set of convex two-dimensional objects and hence the algorithm of Agarwal and Mustafa also applies to (unit) disk graphs.

Maximum Independent Set on (unit) disk graphs also has a polynomial-time approximation scheme (ptas) using the so-called *shifting technique*. This is a general technique, independently discovered by Baker [22] and Hochbaum and Maass [150]. The basic idea is the following. A set of regularly spaced separators is used to decompose the problem into smaller, easier solvable subproblems. The solutions of the subproblems are merged to form a solution to the global problem. This is repeated for several placements of the separator set. The best solution over these placements is then selected as an approximation of the optimum. Moving the separator set can be regarded as shifting the set through the problem. Hence the name 'shifting technique'.

Since its discovery, the shifting technique has been used to solve various problems [4, 154, 103, 272]. In the context of (unit) disk graphs, Matsui [206] and Hunt et al. [154] both presented a ptas using the shifting technique, employing different proof ideas. Nieberg, Hurink, and Kern [219] give a (robust) ptas for Maximum Independent Set on unit disk graphs for which no disk representation is given. The ptas extends to graphs of polynomially bounded growth. A *robust algorithm* on unit disk graphs solves the problem correctly for every unit disk graph. For graphs that are not unit disk graphs, the algorithm may either produce the correct output for the problem, or provide a certificate that the input is not a unit disk graph.

Hunt et al. [154] also consider unit disk graphs with a representation where the disk centers are at least λ apart, so-called *λ-precision unit disk graphs*. Using the shifting technique, they give an eptas for Maximum Independent Set on unit disk graphs of constant precision.

Erlebach, Jansen, and Seidel [103] generalize the shifting technique to give a ptas for Maximum Independent Set on general disk graphs, which extends to intersection graphs of fat objects. Li and Wang [192] extend these ideas to several other disk models for wireless networks, such as the ones discussed in Section 3.2.1. Chan [57] presents a ptas for Maximum Independent Set on the intersection graph of a set of fat objects. The scheme uses polynomial space. Under the used definition, a set of disks is fat. Hence the presented scheme is a ptas for Maximum Independent Set on disk graphs. The above schemes extend to the weighted case. Chan and Har-Peled [59] generalize in a different direction and give a ptas for Maximum Independent Set on intersection graphs of pseudo-disks. For the weighted case, they give a constant-factor approximation algorithm.

For rectangle intersection graphs where the rectangles are noncrossing, Agarwal and Mustafa [2] give a constant-factor approximation algorithm. For intersection graphs of rectangles that have unit height, Agarwal, van Kreveld, and Suri [4] and Chan [58] give a ptas. If the rectangles have arbitrary height and are d-dimensional, a $O(\log_k^d n)$-approximation algorithm can be given that runs in $O(n^{O(k)})$ time [164, 32, 58] for any $k \geq 2$. Chalermsook and Chuzhoy [54] recently improved on this by presenting a polynomial-time $O(\log^{d-2} n \log\log n)$-approximation algorithm. This algorithm does not apply to the weighted case. For this case however, Chan and Har-Peled [59] give a $O(\log n/\log\log n)$-approximation algorithm. Chlebík and Chlebíková [65] prove that Maximum Independent Set is APX-hard on intersection graphs of d-dimensional axis-parallel boxes for any $d \geq 3$.

Minimum Vertex Cover

For general n-vertex graphs, Minimum Vertex Cover can be approximated in polynomial time within $2 - \frac{\log\log n}{2\log n}$ [214, 25] or (for dense graphs) within $2 - \frac{2\ln\ln n}{\ln n}$ [138], and cannot be approximated within 1.3606, unless P=NP [87].

On unit disk graphs, one can give a polynomial-time 3/2-approximation algorithm [201]. Malesińska [199] gives a constant-factor approximation algorithm for Minimum Vertex Cover on general disk graphs. More important however is the existence of a ptas. Hunt et al. [154] prove that Minimum Vertex Cover has a ptas on unit disk graphs, again using the shifting technique, and an eptas on constant-precision unit disk graphs. Marx [202] even managed to show that an eptas exists on arbitrary unit disk graphs. Nieberg, Hurink, and Kern [219] give a (robust) ptas for Minimum Vertex Cover on unit disk graphs for which no disk representation is given. The idea behind the scheme also works on graphs of polynomially bounded growth.

Erlebach, Jansen, and Seidel [103] use their multi-level shifting technique to give a ptas for Minimum Vertex Cover on general disk graphs, which extends to intersection graphs of fat objects. Li and Wang [192] give a ptas for other disk models, such as given in Section 3.2.1. It is interesting to note that the schemes also apply to the weighted case, but the eptas does not seem to do so.

Chlebík and Chlebíková [65] demonstrate that Minimum Vertex Cover is APX-hard on intersection graphs of d-dimensional axis-parallel boxes for $d \geq 3$.

Minimum (Connected) Dominating Set

Given any n-vertex graph, Minimum Dominating Set can be approximated within $1 + \ln n$ in polynomial time [156, 197, 66], but this problem has no polynomial-time algorithm achieving ratio $(1-\epsilon)\ln n$ for any $\epsilon > 0$, unless $NP \subset DTIME(n^{O(\log\log n)})$ [108]. Minimum Connected Dominating Set has a polynomial time $(3 + \ln n)$-approximation algorithm, and similar to Minimum Dominating Set cannot be approximated within $(1-\epsilon)\ln n$ for any $\epsilon > 0$, unless $NP \subset DTIME(n^{O(\log\log n)})$ [132]. Better results can be proved if one assumes that the maximum vertex degree is bounded by Δ, giving a ratio of $1 + \ln\Delta$ and $3 + \ln\Delta$ respectively.

Marathe et al. [201] propose constant-factor approximation algorithms for Minimum (Connected) Dominating Set on graphs without a $K_{1,m}$ induced subgraph, yielding approximation ratios of $m-1$ and $2(m-1)$ respectively. This gives a ratio of 5 and 10 respectively on unit disk graphs. Hunt et al. [154] give a ptas for Minimum Dominating Set on unit disk graphs and an eptas on constant-precision unit disk graphs. Nieberg, Hurink, and Kern [219] give a (robust) ptas for Minimum Dominating Set on unit disk graphs for which no disk representation is given. The idea behind the scheme also works on graphs of polynomially bounded growth.

A ptas for Minimum Connected Dominating Set on unit disk graphs was discovered by Cheng et al. [64]. It was recently improved by Zhang et al. [278]. Zhang et al. also give a ptas for Minimum Connected Dominating Set on three-dimensional unit ball graphs.

Chlebík and Chlebíková [65] show that Minimum (Connected) Dominating Set is APX-hard on intersection graphs of d-dimensional axis-parallel boxes for any $d \geq 3$.

The approximability of the weighted case of Minimum (Connected) Dominating Set on unit disk graphs was a longstanding open problem, until recently, when Ambühl et al. [13] gave a 72- and a 89-approximation algorithm respectively. Huang et al. [153] proposed a polynomial-time $(6+\epsilon)$-approximation algorithm for any (fixed) $\epsilon > 0$ for Minimum-Weight Dominating Set on unit disk graphs. This was subsequently improved upon by Dai and Yu [74], who presented a polynomial-time $(5+\epsilon)$-approximation algorithm for any (fixed) $\epsilon > 0$. Applying the 3.875-approximation algorithm for Node-Weighted Steiner Tree on unit disk graphs by Zou et al. [279], one immediately obtains a $(8.875+\epsilon)$-approximation for Minimum-Weight Connected Dominating Set.

Local (Distributed) Algorithms

Because many of the problems described here are motivated by applications in wireless networks, a large amount of research has focused on distributed algorithms, particularly on so-called *local algorithms*, where the state of a node depends only on the state of nodes at a constant distance.

Kuhn et al. [185] give a local ptas for Maximum Independent Set and Minimum Dominating Set on graphs of polynomially bounded growth using nodes at distance $\mathrm{O}(\log^* n)$. Wiese and Kranakis [270] show that on unit disk graphs, a local ptas exists for Maximum Independent Set and Minimum Vertex Cover which only requires constant distance.

For Minimum Connected Dominating Set the first distributed approximation algorithm, attaining a ratio of 8, was given by Wan, Alzoubi, and Frieder [12, 266]. It has message complexity $\mathrm{O}(n \log n)$ and time complexity $\mathrm{O}(n)$. Czyzowicz et al. [72] presented the first local algorithm for Minimum Connected Dominating Set on unit disk graphs, yielding a $(7.453+\epsilon)$-approximation for any $\epsilon > 0$.

This is only a small portion of the known distributed algorithms for these problems. Since we study centralized algorithms in this thesis, we chose to survey only local algorithms here.

Other Optimization Problems

We briefly survey results on two optimization problems that are frequently studied on geometric intersection graphs, but are not studied in this thesis.

The *maximum clique* problem is to find a largest subset of pairwise connected vertices of a graph. It is the complement of the maximum independent set problem. Interestingly, this problem is polynomial-time solvable on rectangle intersection graphs [155], unit disk graphs [67], and intersection graphs of homothetic triangles [160]. On intersection graphs of two-dimensional convex polygons however, it is NP-hard [23] by reduction from Maximum Independent Set on planar graphs. It is APX-hard on intersection graphs of ellipses of eccentricity $0 < e < 1$ [14]. Intriguingly, the problem is still open on general disk graphs.

Another frequently studied problem on geometric intersection graphs is *Chromatic Number*, where one wants to determine the smallest number of colors needed for which each vertex can be assigned a color such that no two adjacent vertices receive the same color. On unit disk graphs it is NP-hard to decide if k colors are sufficient for any fixed $k \geq 3$ [130]. One can even show that Chromatic Number has no polynomial-time (4/3)-approximation algorithm [67]. Chromatic Number has a 3-approximation algorithm on unit disk graphs and a 5-approximation algorithm on disk graphs [222, 201, 199, 130, 101] by a simple greedy strategy. Kim, Kostochka, and Nakprasit [167] extend this to a 3-approximation algorithm on intersection graphs of translated copies of a fixed convex compact set and a 6-approximation algorithm on intersection graphs of homothetic copies of a fixed convex compact set. It is worth noting that these results follow from an appropriate upper bound on the coloring number in terms of the clique number of the intersection graph.

An interesting related problem is the geometric version of the *conflict-free coloring* problem, where a coloring has to be found such that each point in the covered part of the plane is overlapped by an object with a color that is unique among the colors of all objects overlapping this point. See for example [106, 140, 245] and the references therein.