Optimization and approximation on systems of geometric objects

van Leeuwen, E.J.

Chapter 7

Better Approximation Schemes on Disk Graphs

In the previous chapter, we considered unit disk graphs of bounded density, leading to new approximation schemes for several optimization problems. Here we extend these ideas to disk graphs and introduce the notion of bounded level density. We give an eptas for Maximum Independent Set on disk graphs of bounded level density, which is also a ptas on arbitrary disk graphs. Furthermore, we show that there is an eptas for Minimum Vertex Cover on arbitrary disk graphs, improving results of Erlebach, Jansen, and Seidel [103]. The given description of these schemes also establishes a general framework, making it easier to obtain efficient approximation schemes for other problems. We will in fact see further applications of this framework in later chapters.

These results all form a geometric generalization to the schemes for planar graphs obtained by Baker [22], because each planar graph is a disk graph of ply 1 [169, 210], and thus a disk graph of bounded level density as well.

7.1 The Ply of Disk Graphs

Let $\mathcal{D} = \{D_i \mid i = 1, \ldots, n\}$ be a set of disks in the plane and $G = (V, E)$ the corresponding disk graph. Scale the disks by a factor 2^w for some integer w, such that each disk has radius at least $\frac{1}{2}$. In the following, we will not distinguish between the disks in $\mathcal{D}$ and the vertices of the graph they induce.

Previously, we showed that an eptas exists for Maximum Independent Set, Minimum Vertex Cover, and Minimum (Connected) Dominating Set on unit disk graphs of bounded density. The density of a unit disk graph is (informally) the maximum number of disk centers in any 1×1 box. A careful examination of the proof of these schemes showed that they can be extended to disk graphs of bounded density and constant maximum radius, but do not generalize to disk graphs of arbitrary density and radius. Hence another approach is needed.

The *ply* of a point p in the plane with respect to $\mathcal{D}$ is the number of disks of $\mathcal{D}$ strictly containing p (i.e. having p strictly inside the disk). Then the *ply* of $\mathcal{D}$ is the maximum ply of any point in the plane [210]. Observe that disk graphs of bounded ply are more general than disk graphs of bounded density and bounded maximum radius. Hence an eptas on disk graphs of bounded ply would generalize previous results. Below we give such an approximation scheme for Minimum Vertex Cover. The analysis relies heavily on the following properties of disk graphs of bounded ply.

Lemma 7.1.1 *Given a set $\mathcal{D}$ of disks of ply γ, the number of disks of radius at least r intersecting*

- *a line of length k is at most $\frac{4}{r\pi}(k+4r)\gamma$,*
- *the boundary of a $k \times k$ square ($k \geq 4r$) is at most $\frac{16}{r\pi}k\gamma$,*
- *a $k \times k$ square is at most $\frac{(k+4r)^2}{r^2\pi}\gamma$,*
- *two perpendicular, intersecting lines of length k is at most $\frac{8}{r\pi}(k+2r)\gamma$.*

Proof (Sketch): Consider a line of length k. Replace each disk D of radius at least r intersecting the line by a canonical disk D' of radius precisely r, such that D' intersects the line and $D' \subseteq D$. Any such canonical disk is contained in a size $(4r) \times (k+4r)$ rectangle centered over the line. As the canonical disks have ply at most γ and each has area $r^2\pi$, one can readily see that at most $\frac{4}{r\pi}(k+4r)\gamma$ disks intersect the line. The other bounds follow similarly. □

Lemma 7.1.2 *A set $\mathcal{D}$ of disks of ply γ, radius at least r and at most r', and intersecting a $k \times k$ square, has a path decomposition of width at most $\frac{4}{r\pi}(k+2r'+4r)\gamma - 1$ and consisting of at most $\frac{(k+4r)^2}{r^2\pi}\gamma$ bags.*

Proof (Sketch): Sweep a vertical line of length $k + 2r'$ through the square from left to right. At any position of the line, place the disks intersecting the line in a bag. This yields a valid path decomposition. Moreover, one can find such a decomposition in $O(|\mathcal{D}| \log |\mathcal{D}|)$ time [258]. The bounds follow from the previous lemma. □

7.2 Approximating Minimum Vertex Cover

To approximate the minimum vertex cover problem, we use the *(geometric) shifting technique* introduced by Hochbaum and Maass [150]. To apply this technique, a decomposition of the minimum vertex cover problem into smaller subproblems is needed. Here we use a decomposition of the disks similar to the ones proposed by Hochbaum and Maass [150], Erlebach, Jansen, and Seidel [103], and Chan [57]. Combining the shifting technique with this decomposition yields the desired approximation factor (see Section 7.2.4).

First partition the disks into levels. A disk has *level* $j \in \mathbb{Z}_{\geq 0}$ if its radius r satisfies $2^{j-1} \leq r < 2^j$. Since all disks have radius at least $\frac{1}{2}$, each disk is indeed assigned a level. The level of the largest disk is denoted by l. For a set of disks $\mathcal{D}$, let $\mathcal{D}_{=j}$ denote the set of disks in $\mathcal{D}$ of level j. Similarly, we define $\mathcal{D}_{\geq j}$ as the set of disks of level at least j, and so on. Finally, $\mathcal{D}_{>j,<j'}$ is the set of disks of level greater than j, but less than j'.

Now let $k \geq 5$ be an odd positive integer (whose precise value is determined later). For each level j, we decompose the plane into squares of size $k2^j \times k2^j$ such that these squares induce a quadtree. Formally, for each level j, we consider the horizontal lines $y = hk2^j$ and vertical lines $x = vk2^j$ $(h, v \in \mathbb{Z})$. The squares induced by these lines are called *level j squares*, or put simply, *j-squares*.

Note that each j-square is completely contained in some $(j+1)$-square. Conversely, each $(j+1)$-square S contains exactly four j-squares, denoted by S_1 through S_4. The squares $S_1, \ldots, S_4$ are *siblings* of each other. We let $\mathcal{D}^S$ denote the set of disks intersecting S and $\mathcal{D}^{\mathrm{b}(S)}$ denotes the set of disks which intersect the boundary of S. Furthermore, we define $\mathcal{D}^{\mathrm{i}(S)} = \mathcal{D}^S - \mathcal{D}^{\mathrm{b}(S)}$ (i.e. the set of disks fully contained in the interior of S) and let $\mathcal{D}^{+(S)} = \mathcal{D}^{\mathrm{i}(S)} - \bigcup_{i=1}^{4} \mathcal{D}^{\mathrm{i}(S_i)} = \bigcup_{i=1}^{4} \mathcal{D}^{\mathrm{b}(S_i)} - \mathcal{D}^{\mathrm{b}(S)}$ (i.e. the set of disks intersecting the boundary of at least one of the four children of S, but not the boundary of S itself). The meaning of combinations like $\mathcal{D}^{\mathrm{b}(S)}_{\leq j}$ should be self-explaining. We use $j(S)$ to denote the level of a square S.

7.2.1 A Close to Optimal Vertex Cover

We prove the following theorem, which will be auxiliary to the main theorem.

Theorem 7.2.1 *Let $\mathcal{D}$ be a set of n disks of ply γ and $k \geq 5$ an odd positive integer. Then in time $\mathrm{O}(k^2 n^2 \gamma^{64k/\pi})$, one can find a vertex cover VC of $\mathcal{D}$ such that $|VC| \leq \sum_S \left| OPT^S_{=j(S)} \right|$, where the sum is over all squares S and OPT is any minimum vertex cover of $\mathcal{D}$.*

We can obtain a vertex cover of the required cardinality by applying bottom-up dynamic programming to the j-squares. Roughly speaking, for each j-square S, we consider all subsets of $\mathcal{D}^{\mathrm{b}(S)}_{>j}$ (the disks of level greater than j intersecting the boundary of S). For each such subset, we compute a close to optimal vertex cover for $\mathcal{D}^S$ containing this subset. Formally, we define for each j-square S and each $W \subseteq \mathcal{D}^{\mathrm{b}(S)}_{>j}$ a function $\mathsf{size}(S, W)$. The function is defined recursively on j.

$$\mathsf{size}(S,W) = \begin{cases} \min\left\{ |T| \;\middle|\; T \subseteq \mathcal{D}^S_{=j} \cup \mathcal{D}^{\mathrm{i}(S)}_{>j};\ T \cup W \text{ covers } \mathcal{D}^S \right\} & \text{if } j = 0; \\ \min\limits_{U \subseteq \mathcal{D}^{+(S)}_{\geq j} \cup \mathcal{D}^{\mathrm{b}(S)}_{=j}} \left\{ |U| + \sum\limits_{i=1}^{4} \mathsf{size}\big(S_i, (U \cup W)^{\mathrm{b}(S_i)}\big) \right\} & \text{if } j > 0. \end{cases}$$

Here we define the minimum over an empty set to be ∞. Observe that W must be a vertex cover of $\mathcal{D}^{\mathrm{b}(S)}_{>j}$ and U must be a vertex cover of $\mathcal{D}^{+(S)}_{\geq j} \cup \mathcal{D}^{\mathrm{b}(S)}_{=j}$. Let $\mathsf{sol}(S, W)$ be the subfamily of $\mathcal{D}$ attaining $\mathsf{size}(S, W)$, or $\emptyset$ if $\mathsf{size}(S, W)$ is ∞.

7.2.2 Properties of the size- and sol-Functions

We first show that the sum of $\mathsf{size}(S, \emptyset)$ over all level l squares S attains the value stated in Theorem 7.2.1. In fact, we prove a slightly more general result.

Let $\mathcal{C}$ be any vertex cover for $\mathcal{D}$.

Lemma 7.2.2 $\sum_{S;\, j(S)=l} \mathsf{size}(S,\emptyset) \leq \sum_S \left|\mathcal{C}^S_{=j(S)}\right|$.

Proof: Apply induction on j. We prove that the following invariant holds:

$$\mathsf{size}\left(S, \mathcal{C}^{\mathrm{b}(S)}_{>j}\right) \leq \left|\mathcal{C}^{\mathrm{i}(S)}_{>j}\right| + \sum_{S' \subseteq S} \left|\mathcal{C}^{S'}_{=j(S')}\right|.$$

Here S is some j-square. For $j = 0$, the correctness of the invariant follows from the definition of size. So assume that $j > 0$ and that the invariant holds for all j'-squares with $j' < j$. Note that

$$\sum_{i=1}^{4} \left|\mathcal{C}^{\mathrm{i}(S_i)}_{>j-1}\right| = \left|\mathcal{C}^{\mathrm{i}(S)}_{>j}\right| + \left|\mathcal{C}^{\mathrm{i}(S)}_{=j}\right| - \left|\mathcal{C}^{+(S)}_{\geq j}\right|.$$

Then from the description of size and by applying induction,

$$\begin{aligned}
&\mathsf{size}\left(S, \mathcal{C}^{\mathrm{b}(S)}_{>j}\right) \\
&\leq \left|\mathcal{C}^{+(S)}_{\geq j}\right| + \left|\mathcal{C}^{\mathrm{b}(S)}_{=j}\right| + \sum_{i=1}^{4} \mathsf{size}\left(S_i, \left(\mathcal{C}^{\mathrm{b}(S)}_{\geq j} \cup \mathcal{C}^{+(S)}_{\geq j}\right)^{\mathrm{b}(S_i)}\right) \\
&= \left|\mathcal{C}^{+(S)}_{\geq j}\right| + \left|\mathcal{C}^{\mathrm{b}(S)}_{=j}\right| + \sum_{i=1}^{4} \mathsf{size}\left(S_i, \mathcal{C}^{\mathrm{b}(S_i)}_{>j-1}\right) \\
&\leq \left|\mathcal{C}^{+(S)}_{\geq j}\right| + \left|\mathcal{C}^{\mathrm{b}(S)}_{=j}\right| + \sum_{i=1}^{4} \left|\mathcal{C}^{\mathrm{i}(S_i)}_{>j-1}\right| + \sum_{i=1}^{4} \sum_{S'_i \subseteq S_i} \left|\mathcal{C}^{S'_i}_{=j(S'_i)}\right| \\
&= \left|\mathcal{C}^{\mathrm{i}(S)}_{>j}\right| + \left|\mathcal{C}^{S}_{=j}\right| + \sum_{i=1}^{4} \sum_{S'_i \subseteq S_i} \left|\mathcal{C}^{S'_i}_{=j(S'_i)}\right| \\
&= \left|\mathcal{C}^{\mathrm{i}(S)}_{>j}\right| + \sum_{S' \subseteq S} \left|\mathcal{C}^{S'}_{=j(S')}\right|.
\end{aligned}$$

Since l is the level of the largest disk, $\mathcal{C}^{\mathrm{i}(S)}_{>j} = \emptyset$ and $\mathcal{C}^{\mathrm{b}(S)}_{>j} = \emptyset$ for all j-squares S with $j \geq l$. Hence

$$\sum_{S;\, j(S)=l} \mathsf{size}(S,\emptyset) \leq \sum_{S;\, j(S)=l} \sum_{S' \subseteq S} \left|\mathcal{C}^{S'}_{=j(S')}\right| = \sum_S \left|\mathcal{C}^S_{=j(S)}\right|.$$

This proves the lemma. $\square$

The lemma implies that $\sum_{S;\, j(S)=l} \mathsf{size}(S,\emptyset) \leq \sum_S \left|OPT^S_{=j(S)}\right|$, where OPT is a minimum vertex cover of $\mathcal{D}$. We now prove that the union of $\mathsf{sol}(S,\emptyset)$ over all level l squares S is a vertex cover of $\mathcal{D}$.

Lemma 7.2.3 $\bigcup_{S;\, j(S)=l} \mathsf{sol}(S, \emptyset)$ *is a vertex cover of* $\mathcal{D}$.

Proof: For any level j_0 and any collection of sets $\{W_S \subseteq \mathcal{D}^{\mathrm{b}(S)}_{>j_0} \mid j(S) = j_0\}$, we prove the following claim:

$$\bigcup_{S;j(S)=j_0} \mathsf{sol}(S, W_S) \cup W_S \text{ covers } \bigcup_{S;j(S)=j_0} \mathcal{D}^S \text{ if } \sum_{S;j(S)=j_0} \mathsf{size}(S, W_S) \neq \infty.$$

Apply induction on j_0. For $j_0 = 0$, this follows trivially from the definition of size and sol. So assume that $j_0 > 0$ and that the claim holds for all $j_0' < j_0$.

Suppose $\sum_{S;j(S)=j_0} \mathsf{size}(S, W_S) \neq \infty$ is true for some collection of sets $\{W_S \subseteq \mathcal{D}^{\mathrm{b}(S)}_{>j_0} \mid j(S) = j_0\}$. For any j_0-square S, let

$$U_S^* = \arg \min_{U \subseteq \mathcal{D}^{+(S)}_{\geq j_0} \cup \mathcal{D}^{\mathrm{b}(S)}_{=j_0}} \left\{ |U| + \sum_{i=1}^{4} \mathsf{size}\left(S_i, (U \cup W_S)^{\mathrm{b}(S_i)}\right) \right\}.$$

As $\mathsf{size}(S, W_S) \neq \infty$, it must be that $\mathsf{size}(S_i, (U_S^* \cup W_S)^{\mathrm{b}(S_i)}) \neq \infty$ for $i = 1, \ldots, 4$ as well. For any S' where $S' = S_i$ for some j_0-square S and $i \in \{1, \ldots, 4\}$, let $W_{S'} = (U_S^* \cup W_S)^{\mathrm{b}(S')}$. It follows that

$$\sum_{S';j(S')=j_0-1} \mathsf{size}(S', W_{S'}) \neq \infty.$$

Then by induction, $\bigcup_{S';j(S')=j_0-1} \mathsf{sol}(S', W_{S'}) \cup W_{S'}$ covers $\bigcup_{S';j(S')=j_0-1} \mathcal{D}^{S'}$. Observe that for any j_0-square S

$$\begin{aligned} W_S \cup \mathsf{sol}(S, W_S) &= W_S \cup U_S^* \cup \bigcup_{i=1}^{4} \mathsf{sol}(S_i, (U_S^* \cup W_S)^{\mathrm{b}(S_i)}) \\ &= \bigcup_{i=1}^{4} \left((U_{S_i}^* \cup W_{S_i})^{\mathrm{b}(S_i)} \cup \mathsf{sol}\left(S_i, (U_{S_i}^* \cup W_{S_i})^{\mathrm{b}(S_i)}\right)\right) \\ &= \bigcup_{\substack{S'=S_i \\ i=1,\ldots,4}} W_{S'} \cup \mathsf{sol}(S', W_{S'}). \end{aligned}$$

As $\bigcup_{S';j(S')=j_0-1} \mathcal{D}^{S'} = \bigcup_{S;j(S)=j_0} \mathcal{D}^S$, we have $\bigcup_{S;j(S)=j_0} \mathsf{sol}(S, W_S) \cup W_S$ covers $\bigcup_{S;j(S)=j_0} \mathcal{D}^S$.

From the previous lemma, we know that $\sum_{S;j(S)=l} \mathsf{size}(S, \emptyset) \neq \infty$. Because each edge is induced by $\mathcal{D}^S$ for some l-square S, $\bigcup_{S;\, j(S)=l} \mathsf{sol}(S, \emptyset)$ is a vertex cover of $\mathcal{D}$. $\square$

7.2.3 Computing the size- and sol-Functions

We show that it is sufficient to compute size and sol for a limited number of j-squares. This can be done in the time stated in Theorem 7.2.1.

Call a j-square *nonempty* if it is intersected by a level j disk and *empty* otherwise. A j-square S is *relevant* if one of its three siblings is nonempty or there is a nonempty square S' containing S, such that S' has level at most $j + \lceil \log k \rceil$ (so each nonempty j-square is relevant). Note that this definition induces $\mathrm{O}(k^2 n)$ relevant squares. A relevant square S is said to be a *relevant child* of another relevant square S' if $S \subset S'$ and there is no third relevant square S'', such that $S \subset S'' \subset S'$. Conversely, if S is a relevant child of S', S' is a *relevant parent* of S.

Lemma 7.2.4 *For each relevant* 0*-square* S*, all* size*- and* sol*-values for* S *can be computed in* $\mathrm{O}\left(nk^3\gamma\,\gamma^{(24k+8)/\pi}\right)$ *time.*

Proof: From Lemma 7.1.1, $\left|\mathcal{D}^{\mathrm{b}(S)}_{>0}\right|$ is bounded by $16k\gamma/\pi$. As an independent set has $\gamma = 1$, all independent sets and hence all vertex covers of $\mathcal{D}^{\mathrm{b}(S)}_{>0}$ can be enumerated in $\mathrm{O}(k\,\gamma^{16k/\pi})$ time using Lemma 6.3.2. For a fixed set W, $\mathsf{size}(S, W)$ is defined as the cardinality of a minimum subset of $\mathcal{D}^{S}_{=0} \cup \mathcal{D}^{\mathrm{i}(S)}_{>0}$, such that this subset and W cover $\mathcal{D}^S$. We may assume that W covers $\mathcal{D}^{\mathrm{b}(S)}_{>0}$, otherwise such a subset does not exist and $\mathsf{size}(S, W)$ is ∞. Then the requested subset is a minimum vertex cover for $\mathcal{D}^S - W$. Similar to Lemma 7.1.2, one can show that $\mathcal{D}^S$ has a path decomposition of width at most $\frac{8}{\pi}(k + 4)\gamma$ and $\mathrm{O}(|\mathcal{D}^S|)$ bags. Moreover, these path decompositions can be precomputed for all level 0 squares in $\mathrm{O}(n \log n)$ time. Adapting the algorithm of Lemma 5.3.4 and using Lemma 6.3.2, the cover can be computed in $\mathrm{O}(|\mathcal{D}^S|k^2\gamma\,\gamma^{8(k+4)/\pi})$ time. Therefore one can compute all size- and sol-values for S in $\mathrm{O}\left(nk^3\gamma\,\gamma^{(24k+8)/\pi}\right)$ time. $\square$

Assume that the size- and sol-values of all relevant children of S are known.

Lemma 7.2.5 *For each relevant* j*-square* S $(j > 0)$ *with relevant* $(j-1)$*-square children, all* size*- and* sol*-values for* S *can be computed in* $\mathrm{O}(k\,\gamma^{64k/\pi})$ *time.*

Proof: If one of the children $S_1, \ldots, S_4$ of S is relevant, then, by the definition of relevant, all children of S must be relevant. Following the definition of size, we enumerate all vertex covers W of $\mathcal{D}^{\mathrm{b}(S)}_{>j}$ and for each such W all vertex covers U of $\mathcal{D}^{+(S)}_{\geq j} \cup \mathcal{D}^{\mathrm{b}(S)}_{=j}$. Using the ideas of Lemma 7.1.1, we can show that $\left|\mathcal{D}^{\mathrm{b}(S)}_{>j}\right| \leq 16k\gamma/\pi$, $\left|\mathcal{D}^{+(S)}_{\geq j} \cup \mathcal{D}^{\mathrm{b}(S)}_{=j}\right| \leq 48k\gamma/\pi$, and $\left|\mathcal{D}^{\mathrm{b}(S)}_{\geq j} \cup \mathcal{D}^{+(S)}_{\geq j}\right| \leq 48k\gamma/\pi$. Then all independent sets, and hence all vertex covers, of $\mathcal{D}^{\mathrm{b}(S)}_{>j}$ and of $\mathcal{D}^{+(S)}_{\geq j} \cup \mathcal{D}^{\mathrm{b}(S)}_{=j}$ can be enumerated in $\mathrm{O}(k\,\gamma^{64k/\pi})$ time by applying Lemma 6.3.2. Since size and sol of all relevant children of S are known and assuming that for a given W and U we can compute $|U| + \sum_{i=1}^{4} \mathsf{size}(S_i, (U \cup W)^{\mathrm{b}(S_i)})$ in constant time, the running time of $\mathrm{O}(k\,\gamma^{64k/\pi})$ follows immediately. $\square$

Lemma 7.2.6 *For each relevant* j*-square* S $(j > 0)$ *with no relevant children of level* $j-1$*, all* size*- and* sol*-values for* S *can be computed in* $\mathrm{O}(n\,\gamma^{32/\pi})$ *time.*

Proof: We start with two simple observations. The first is that S must be empty, because S has no relevant children of level $j-1$. Secondly, one notes that by the definition of relevant, the nearest nonempty square containing S (if it exists) has level at least $j + \lceil \log k \rceil$. Hence $\mathcal{D}^{\mathrm{b}(S)}_{>j} = \mathcal{D}^{\mathrm{b}(S)}_{\geq j+\lceil \log k \rceil}$.

Now consider any j'-square $S' \subseteq S$ for which there is no relevant square S'' such that $S' \subseteq S'' \subset S$. Then the nearest nonempty square containing S' (if it exists) has level at least $j + \lceil \log k \rceil$. Hence any disk of level at least j' intersecting S' has level at least $j + \lceil \log k \rceil$. This implies that $\mathcal{D}^{\mathrm{b}(S'_i)}_{>j'-1} = \mathcal{D}^{\mathrm{b}(S')}_{>j'}$ for any $i = 1, \ldots, 4$ and that $\mathcal{D}^{\mathrm{b}(S')}_{=j'} = \emptyset$. Since S' is empty, it also follows that $\mathcal{D}^{\mathrm{i}(S')}_{\geq j'} = \emptyset$ and, if $j' > 0$, $\mathcal{D}^{+(S')}_{\geq j'} = \emptyset$ as well.

Using these observations, we can simplify the definition of size considerably for such S'. For any set $W' \subseteq \mathcal{D}^{\mathrm{b}(S')}_{>j'}$,

$$\mathsf{size}(S', W') = \begin{cases} 0 & \text{if } j' = 0,\ W' \text{ covers } \mathcal{D}^{\mathrm{b}(S')}_{>j'}; \\ \infty & \text{if } j' = 0,\ W' \text{ not covers } \mathcal{D}^{\mathrm{b}(S')}_{>j'} \\ \sum_{i=1}^{4} \mathsf{size}\left(S'_i, W'^{\mathrm{b}(S'_i)}\right) & \text{if } j' > 0. \end{cases}$$

Applying this simplification repeatedly, it can be seen that for any $W \subseteq \mathcal{D}^{\mathrm{b}(S)}_{>j}$,

$$\mathsf{size}(S, W) = \begin{cases} 0 & \text{if } S \text{ has no relevant children and} \\ & \qquad W \text{ covers } \mathcal{D}^{\mathrm{b}(S)}_{>j}; \\ \infty & \text{if } S \text{ has no relevant children and} \\ & \qquad W \text{ doesn't cover } \mathcal{D}^{\mathrm{b}(S)}_{>j}; \\ \sum_{S''} \mathsf{size}\left(S'', W^{\mathrm{b}(S'')}\right) & \text{otherwise,} \end{cases}$$

where the sum is over all relevant children S'' of S.

Any relevant child of S is either nonempty, or the sibling of a nonempty square. As the number of nonempty squares is $\mathrm{O}(n)$ and a square has three siblings, the number of relevant children of S is $\mathrm{O}(n)$. So for fixed W, it takes $\mathrm{O}(n)$ time to compute $\mathsf{size}(S, W)$. As $\mathcal{D}^{\mathrm{b}(S)}_{>j} = \mathcal{D}^{\mathrm{b}(S)}_{\geq j+\lceil \log k \rceil}$, we know from Lemma 7.1.1 and Lemma 6.3.2 that all vertex covers $W \subseteq \mathcal{D}^{\mathrm{b}(S)}_{>j} = \mathcal{D}^{\mathrm{b}(S)}_{\geq j+\lceil \log k \rceil}$ can be enumerated in $\mathrm{O}(\gamma^{32/\pi})$ time. $\square$

Lemma 7.2.7 *$\sum_{S;\, j(S)=l} \mathsf{size}(S, \emptyset)$ can be computed in $\mathrm{O}(k^2 n^2\, \gamma^{64k/\pi})$ time.*

Proof: Recall that there are $\mathrm{O}(k^2 n)$ relevant squares. Let S be a relevant j-square without a relevant parent. Following Lemmas 7.2.4, 7.2.5, and 7.2.6, we can compute $\mathsf{size}(S, \emptyset)$ for all such squares S in $\mathrm{O}(k^2 n^2\, \gamma^{64k/\pi})$ time.

Now consider any level l square S. If S is relevant, then it cannot have a relevant parent. Hence by the preceding argument, $\mathsf{size}(S, \emptyset)$ is known. If S is not relevant, then we can use the same arguments as in Lemma 7.2.6 to show that $\mathsf{size}(S,\emptyset) = \sum_{S''} \mathsf{size}(S'',\emptyset)$, where the sum is over all relevant j''-squares $S'' \subset S$ without a relevant parent. It follows that $\sum_{S;\, j(S)=l} \mathsf{size}(S,\emptyset)$ can be computed in $\mathrm{O}(k^2 n^2 \gamma^{64k/\pi})$ time. $\square$

Proof of Theorem 7.2.1: Follows by Lemmas 7.2.2, 7.2.3, and 7.2.7. $\square$

7.2.4 An eptas for Minimum Vertex Cover

We now apply the shifting technique to obtain a $(1+\epsilon)$ approximation of the optimum. For some integer a $(0 \le a \le k-1)$, define the decomposition as follows. We call a line of level j *active* if it is of the form $y = (hk + a2^{l-j})2^j$ or $x = (vk + a2^{l-j})2^j$ $(h, v \in \mathbb{Z})$. The active lines partition the plane into j-squares as before, except that they are now shifted by the shifting parameter a. The structure however remains the same, and thus we can apply Theorem 7.2.1 to compute a close to optimal vertex cover.

Let VC_a denote the set returned by the algorithm for some value of a $(0 \le a \le k-1)$ and let $VC_{\min}$ be a smallest such set.

Lemma 7.2.8 $|VC_{\min}| \le (1 + \frac{12}{k})\,|OPT|$.

Proof: We claim a line of level j (i.e. of the form $y = h'2^j$ or $x = v'2^j$) is active for precisely one value of a. A horizontal line $y = h'2^j$ is active if $h' = hk + a2^{l-j}$ for some h and a, i.e. if $h' \equiv a2^{l-j} \bmod k$. As $\gcd(k, 2^{l-j}) = 1$, such a value of a exists. Hence the line is active for at least one value of a.

Suppose that a horizontal line of level j is active for two values of a. Then $hk + a2^{l-j} = h'k + a'2^{l-j}$ for some choice of h, h', a, and a'. Simplifying gives $(h-h')k = (a'-a)2^{l-j}$, or $k|(a'-a)2^{l-j}$. Since k is odd, $k|(a'-a)$, which is impossible as $1 \le |a'-a| \le k-1$. Hence each horizontal line of level j is active for precisely one value of a. The same arguments hold for vertical lines of level j.

Define $\mathcal{D}_a^b$ as the set of disks intersecting the boundary of a j-square S at their level, i.e. $\mathcal{D}_a^b = \bigcup_S \mathcal{D}_{=j(S)}^{\mathrm{b}(S)}$. A level j disk is in $\mathcal{D}_a^b$ if and only if it intersects an active line of level j. It can be in $\mathcal{D}_a^b$ for at most four different values of a, intersecting both a horizontal and a vertical active line at most twice, because a line of level j is active for exactly one value of a, the distance between consecutive lines is 2^j, and disks of level j have radius less than 2^j. Hence there is a value of a (say a^*) for which $\left|OPT \cap \mathcal{D}_{a^*}^b\right| \le \frac{4}{k}\,|OPT|$.

From Lemma 7.2.2, we know that $|VC_{a^*}| \le \sum_S |OPT_{=j(S)}^S|$. Observe that for a fixed value of a, any disk can intersect at most four squares at its level. Then

$$\begin{aligned}
|VC_{a^*}| &\le \sum_S \left|OPT_{=j(S)}^S\right| \\
&= \sum_S \left|OPT_{=j(S)}^S - OPT_{=j(S)}^{\mathrm{b}(S)}\right| + \sum_S \left|OPT_{=j(S)}^{\mathrm{b}(S)}\right|
\end{aligned}$$

$$\leq \quad |OPT| - \left|OPT \cap \mathcal{D}^b_{a^*}\right| + 4\left|OPT \cap \mathcal{D}^b_{a^*}\right|$$
$$\leq \quad |OPT| + \tfrac{12}{k}\,|OPT|\,.$$

Hence $|VC_{\min}| \leq |VC_{a^*}| \leq (1+\frac{12}{k})\,|OPT|$ and the lemma follows. $\square$

Combining Theorem 7.2.1 and Lemma 7.2.8, we obtain the following result.

Theorem 7.2.9 *There is an eptas for Minimum Vertex Cover on disk graphs.*

Proof: The idea is similar to Lemma 6.3.14 and Theorem 6.3.15, except we have no polynomial-time algorithm to find a maximum clique in a disk graph. However, it suffices to reduce the ply. Consider a point p in the plane of ply more than $\frac{1}{\epsilon}$. Note that the set of disks $\mathcal{D}_p$ containing p form a clique. Marx [202] observed that $\mathcal{D}_p$ is actually a $(1+\epsilon)$-approximation of a minimum vertex cover for $\mathcal{D}_p$. Hence we remove $\mathcal{D}_p$ from $\mathcal{D}$ and repeat until the ply is bounded by $\frac{1}{\epsilon}$. Using the algorithm by Eppstein, Miller, and Teng [100] to determine the ply of a set of disks, this can be done in $\mathrm{O}(n^3 \log n)$ time.

Let $\mathcal{D}_0$ denote the remaining set of disks. Choose k as the smallest odd integer larger than $\frac{12}{\epsilon}$. Compute and output $VC_{\min}$ in $\mathrm{O}(k^3 n^2\, k^{64k/\pi})$ time using Theorem 7.2.1. Following Lemma 7.2.8 and the choice of k, this results in a $(1+\epsilon)$-approximation of a minimum vertex cover of $\mathcal{D}_0$. Combining the different approximations gives a $(1+\epsilon)$-approximation of a minimum vertex cover of $\mathcal{D}$. This gives the eptas. $\square$

This result improves the $n^{\mathrm{O}(\epsilon^{-2})}$-time ptas for Minimum Vertex Cover on disk graphs by Erlebach, Jansen, and Seidel [103].

7.3 Approximating Maximum Independent Set

The maximum independent set problem can also be approximated well using the ideas of the previous sections. We can show that it has an eptas on disk graphs of bounded ply. In fact, we can prove a more general result, namely that Maximum Independent Set has an eptas on disk graphs of bounded level density. This notion is defined as follows. Partition the disks into levels as before (i.e. a disk has level j if its radius is in $[2^{j-1}, 2^j)$). For each level j, let d_j denote the maximum number of level j disks in any $2^j \times 2^j$ box. Then the *level density*, denoted by d, is the maximum d_j over all levels j. Scaling a set of disks by a constant factor can reduce the level density by a factor of 2, but this is of little consequence to the analysis of the algorithm below.

Disk graphs of bounded level density are more general than disk graphs of bounded ply, as a disk graph of ply γ has level density at most 4γ. However, a disk graph of bounded level density can contain overlapping disks from an arbitrary number of levels, giving it arbitrarily large ply.

Consider a set of disks $\mathcal{D}$ of level density d. Let $k \geq 5$ be an odd positive integer to be determined later and let the plane be partitioned into j-squares as before. We prove the following auxiliary theorem.

Theorem 7.3.1 *Let $\mathcal{D}$ be a set of disks of level density d and $k \geq 5$ an odd positive integer. Then one can find in $\mathrm{O}(k^3 n^9 (2ed)^{32k/\pi})$ time an independent set IS of $\bigcup_S \mathcal{D}^{\mathrm{i}(S)}_{=j(S)}$ such that $|IS| \geq \sum_S \left|OPT^{\mathrm{i}(S)}_{=j(S)}\right|$, where sum and union are over all squares S and OPT is any maximum independent set of $\mathcal{D}$.*

We employ a similar approach as with Minimum Vertex Cover. For any j-square S and any independent set $W \subseteq \mathcal{D}^{\mathrm{b}(S)}_{>j}$, we compute (the cardinality of) a close to maximum independent set of $\mathcal{D}^{\mathrm{i}(S)}_{>j} \cup \bigcup_{S' \subseteq S} \mathcal{D}^{\mathrm{i}(S')}_{=j(S')}$ that is independent of W. For each j-square S and each independent set $W \subseteq \mathcal{D}^{\mathrm{b}(S)}_{>j}$,

$$\mathsf{size}(S,W) = \begin{cases} \max\left\{ |T| \;\middle|\; T \subseteq \mathcal{D}^{\mathrm{i}(S)};\ T \cup W \text{ independent}\right\} & \text{if } j = 0; \\ \max\limits_{U \subseteq \mathcal{D}^{+(S)}_{\geq j}} \left\{ |U| + \sum\limits_{i=1}^{4} \mathsf{size}\Big(S_i, (U \cup W)^{\mathrm{b}(S_i)}\Big)\right\} & \text{if } j > 0. \end{cases}$$

Let $\mathsf{sol}(S,W)$ be the subset of $\mathcal{D}$ attaining $\mathsf{size}(S,W)$.

Lemma 7.3.2 $\sum_{S;\, j(S)=l} \mathsf{size}(S,\emptyset) \geq \sum_S \left|\mathcal{I}^{\mathrm{i}(S)}_{=j(S)}\right|$ *for any independent set $\mathcal{I}$.*

Proof: Define

$$up(S) = \bigcup_{S' \supset S} \mathcal{I}^{\mathrm{i}(S')}_{=j(S')}.$$

We use induction on j to prove the following invariant for any j-square S:

$$\mathsf{size}\left(S, (up(S))^{\mathrm{b}(S)}\right) \geq \left|\mathcal{I}^{\mathrm{i}(S)}_{>j}\right| + \sum_{S' \subseteq S} \left|\mathcal{I}^{\mathrm{i}(S')}_{=j(S')}\right|.$$

It follows immediately from the definition of size that the invariant is true for $j = 0$. So consider a $j > 0$ and assume the invariant holds for all $j' < j$. Then

$$\begin{aligned} &\mathsf{size}\left(S, (up(S))^{\mathrm{b}(S)}\right) \\ &\qquad \geq \left|\mathcal{I}^{+(S)}_{\geq j}\right| + \sum_{i=1}^{4} \mathsf{size}\left(S_i, \left(\mathcal{I}^{+(S)}_{\geq j} \cup (up(S))^{\mathrm{b}(S)}\right)^{\mathrm{b}(S_i)}\right) \\ &\qquad = \left|\mathcal{I}^{+(S)}_{\geq j}\right| + \sum_{i=1}^{4} \mathsf{size}\left(S_i, (up(S_i))^{\mathrm{b}(S_i)}\right) \\ &\qquad \geq \left|\mathcal{I}^{+(S)}_{\geq j}\right| + \sum_{i=1}^{4} \left|\mathcal{I}^{\mathrm{i}(S_i)}_{>j-1}\right| + \sum_{i=1}^{4} \sum_{S'_i \subseteq S_i} \left|\mathcal{I}^{\mathrm{i}(S'_i)}_{=j(S'_i)}\right| \\ &\qquad = \left|\mathcal{I}^{\mathrm{i}(S)}_{>j}\right| + \sum_{S' \subseteq S} \left|\mathcal{I}^{\mathrm{i}(S')}_{=j(S')}\right|, \end{aligned} \tag{7.1}$$

where the equality

$$\left(\mathcal{I}_{\geq j}^{+(S)} \cup (up(S))^{\mathrm{b}(S)}\right)^{\mathrm{b}(S_i)} = (up(S_i))^{\mathrm{b}(S_i)}$$

holds, because by definition

$$\left(\mathcal{I}_{\geq j}^{+(S)}\right)^{\mathrm{b}(S_i)} = \left(\mathcal{I}_{\geq j}^{\mathrm{i}(S)}\right)^{\mathrm{b}(S_i)} = \left(\left(\bigcup_{S' \supseteq S} \mathcal{I}_{=j(S')}^{\mathrm{i}(S')}\right)^{\mathrm{i}(S)}\right)^{\mathrm{b}(S_i)}$$

and $\left(\mathcal{I}_{=j(S)}^{\mathrm{i}(S)}\right)^{\mathrm{b}(S)} = \emptyset$. Then

$$\begin{aligned}
&\left(\mathcal{I}_{\geq j}^{+(S)} \cup \left(\bigcup_{S' \supset S} \mathcal{I}_{=j(S')}^{\mathrm{i}(S')}\right)^{\mathrm{b}(S)}\right)^{\mathrm{b}(S_i)} \\
&\quad = \left(\left(\bigcup_{S' \supseteq S} \mathcal{I}_{=j(S')}^{\mathrm{i}(S')}\right)^{\mathrm{i}(S)} \cup \left(\bigcup_{S' \supseteq S} \mathcal{I}_{=j(S')}^{\mathrm{i}(S')}\right)^{\mathrm{b}(S)}\right)^{\mathrm{b}(S_i)} \\
&\quad = \left(\bigcup_{S' \supseteq S} \mathcal{I}_{=j(S')}^{\mathrm{i}(S')}\right)^{\mathrm{b}(S_i)} \\
&\quad = (up(S_i))^{\mathrm{b}(S_i)}.
\end{aligned}$$

Returning to Equation 7.1, as l is the level of the largest disk, $up(S) = \emptyset$ and $\mathcal{I}_{>j}^{\mathrm{i}(S)} = \emptyset$ for any square S of level at least l. Then

$$\sum_{S;\, j(S)=l} \mathsf{size}(S, \emptyset) \geq \sum_{S;\, j(S)=l} \sum_{S' \subseteq S} \left|\mathcal{I}_{=j}^{\mathrm{i}(S')}\right| = \sum_{S} \left|\mathcal{I}_{=j(S)}^{\mathrm{i}(S)}\right|.$$

The lemma follows. $\square$

Clearly, the lemma implies that $\sum_{S;\, j(S)=l} \mathsf{size}(S, \emptyset) \geq \sum_S \left|OPT_{=j(S)}^{\mathrm{i}(S)}\right|$, where OPT is a maximum independent set.

Lemma 7.3.3 $\bigcup_{S;\, j(S)=l} \mathsf{sol}(S, \emptyset)$ *is an independent set of* $\bigcup_S \mathcal{D}_{=j(S)}^{\mathrm{i}(S)}$.

This lemma follows straightforwardly from the definitions of size and sol in a similar way as in Lemma 7.2.3.

To compute $\sum_{S;\, j(S)=l} \mathsf{size}(S, \emptyset)$, it is again sufficient to consider only relevant j-squares, where the definition of relevant is the same as before. As was observed earlier, we need only to consider independent sets W, T, and U in the definition of size, as size will be $-\infty$ otherwise. Crucial in the analysis of the algorithm will therefore be bounds on the maximum cardinality of certain independent sets. In particular, we apply the following theorem.

Theorem 7.3.4 *The maximum number of disjoint disks of radius r intersecting a square of size $2r \times 2r$ is 7.*

The (lengthy) proof is detailed in Section 7.5.

Lemma 7.3.5 *For each relevant* 0*-square* S*, all* size- *and* sol*-values can be computed in* $\mathrm{O}(k^2 n^8 d\,(2ed)^{24k/\pi})$ *time.*

Proof: As disks in $\mathcal{D}^{\mathrm{b}(S)}_{\geq\lceil \log k\rceil}$ have radius at least $\frac{1}{2}k$, we can use Theorem 7.3.4 to bound the maximum cardinality of any independent set in $\mathcal{D}^{\mathrm{b}(S)}_{\geq\lceil \log k\rceil}$ by 7. Hence all independent sets in $\mathcal{D}^{\mathrm{b}(S)}_{\geq\lceil \log k\rceil}$ can be enumerated in $\mathrm{O}(n^7)$ time.

To enumerate all independent subsets of $\mathcal{D}^{\mathrm{b}(S)}_{>0}$, we should consider independent subsets of $\mathcal{D}^{\mathrm{b}(S)}_{>0,<\lceil \log k\rceil}$ as well. For some j' with $0 < j' < \lceil \log k\rceil$, we can use an area bound to show that $\left|\mathcal{D}^{\mathrm{b}(S)}_{=j'}\right| \leq 4kd2^{j-j'}$. Then

$$\left|\mathcal{D}^{\mathrm{b}(S)}_{>0,<\lceil \log k\rceil}\right| \;\leq\; \sum_{j'=1}^{\lceil \log k\rceil -1} \left(4kd2^{j-j'}\right) \;\leq\; 4kd.$$

As an independent set of disks has ply 1, it follows from Lemma 7.1.1 that any independent subset of $\mathcal{D}^{\mathrm{b}(S)}_{>0,<\lceil \log k\rceil}$ has cardinality at most $16k/\pi$. Then, following Lemma 6.3.2, all independent sets of disks in $\mathcal{D}^{\mathrm{b}(S)}_{>0,<\lceil \log k\rceil}$ can be enumerated in $\mathrm{O}(k\,(ed)^{16k/\pi})$ time. Hence all independent sets $W \subseteq \mathcal{D}^{\mathrm{b}(S)}_{>0}$ can be enumerated in $\mathrm{O}(kn^7\,(ed)^{16k/\pi})$ time.

For fixed $W \subseteq \mathcal{D}^{\mathrm{b}(S)}_{>0}$, it remains to compute a maximum $T \subseteq \mathcal{D}^{\mathrm{i}(S)}_{\geq 0}$ such that $T \cup W$ is an independent set. That is, to compute a maximum independent set of $\mathcal{D}^{\mathrm{i}(S)}_{\geq 0} - N[W]$, where $N[W]$ is the closed neighborhood of W. We use a path decomposition to find this set. First, observe that $\mathcal{D}^{\mathrm{i}(S)}_{\geq\lceil \log k\rceil} = \emptyset$. For any j' with $0 \leq j' < \lceil \log k\rceil$, the number of disks of $\mathcal{D}^{\mathrm{i}(S)}_{=j'}$ intersecting a vertical line of length k in S is bounded by $2\lceil (k-2^{j'})/2^{j'}\rceil d$. Hence the number of disks of $\mathcal{D}^{\mathrm{i}(S)}_{\geq 0}$ intersecting such a line is at most

$$\sum_{j'=0}^{\lceil \log k\rceil -1} \left(2\lceil 2^{-j'}k - 1\rceil d\right) \;\leq\; 4kd.$$

It follows that $\mathcal{D}^{\mathrm{i}(S)}_{\geq 0} - N[W]$ has a path decomposition of width at most $4kd$ and $\mathrm{O}(n)$ bags. Following Lemma 7.1.1, any independent set of disks in $\mathcal{D}^{\mathrm{i}(S)}_{\geq 0} - N[W]$ intersecting a vertical line in S has cardinality at most $8k/\pi$. Adapting the algorithm of Theorem 5.3.2 to only consider independent sets, the maximum T can be found in $\mathrm{O}(k^2 nd\,(2ed)^{8k/\pi})$ time. $\square$

Lemma 7.3.6 *For each relevant* j*-square* S *with relevant* $(j-1)$*-square children, all* size- *and* sol*-values can be computed in* $\mathrm{O}(k^2 n^7\,(2ed)^{32k/\pi})$ *time.*

Proof: Using the same arguments as in the previous lemma, all independent subsets W of $\mathcal{D}^{\mathrm{b}(S)}_{>j}$ can be enumerated in $\mathrm{O}(kn^7\,(ed)^{16k/\pi})$ time. For any j' with $j-1<j'<j+\lceil\log k\rceil$, $\left|\mathcal{D}^{+(S)}_{=j'}\right| \le (4\lceil 2^{j-j'}k-1\rceil-4)d$. Hence

$$\left|\mathcal{D}^{+(S)}_{>j-1}\right| \;\le\; \sum_{j'=j}^{j+\lceil\log k\rceil-1} (4\lceil 2^{j-j'}k-1\rceil-4)d \;\le\; 8kd.$$

Following Lemma 7.1.1, any independent set of disks in $\mathcal{D}^{+(S)}_{\ge j}$ has cardinality at most $16k/\pi$. Then, for any fixed $W\subseteq \mathcal{D}^{\mathrm{b}(S)}_{>j}$, all independent sets $U\subseteq \mathcal{D}^{+(S)}_{\ge j} - N[W]$ can be enumerated in $\mathrm{O}(k\,(2ed)^{16k/\pi})$ time. $\square$

Lemma 7.3.7 *For each relevant j-square S with no relevant $(j-1)$-square children, all* size *and* sol*-values can be computed in $\mathrm{O}(n^8)$ time.*

Proof: Using the same arguments as in Lemma 7.2.6, for any $W\subseteq \mathcal{D}^{\mathrm{b}(S)}_{>j}$,

$$\mathsf{size}(S,W) = \begin{cases} 0 & \text{if } S \text{ has no relevant children and} \\ & \quad W \text{ is an independent set;} \\ -\infty & \text{if } S \text{ has no relevant children and} \\ & \quad W \text{ is not an independent set;} \\ \sum_{S''} \mathsf{size}\left(S'', W^{\mathrm{b}(S'')}\right) & \text{otherwise,} \end{cases}$$

where the sum is over all relevant children S'' of S. Since the number of relevant children of S is $\mathrm{O}(n)$, for fixed W, it takes $\mathrm{O}(n)$ time to compute $\mathsf{size}(S,W)$. As $\mathcal{D}^{\mathrm{b}(S)}_{>j} = \mathcal{D}^{\mathrm{b}(S)}_{\ge j+\lceil\log k\rceil}$, we know from previous lemmas that all $W\subseteq \mathcal{D}^{\mathrm{b}(S)}_{>j} = \mathcal{D}^{\mathrm{b}(S)}_{\ge j+\lceil\log k\rceil}$ can be enumerated in $\mathrm{O}(n^7)$ time. $\square$

Proof of Theorem 7.3.1: Applying similar ideas as in Lemma 7.2.7, this follows immediately from Lemmas 7.3.2, 7.3.3, 7.3.5, 7.3.6, and 7.3.7. $\square$

Let a $(0\le a\le k-1)$ be an integer. Shift the decomposition as before. Let IS_a be the independent set returned by the algorithm for some value of a $(0\le a\le k-1)$ and let $IS_{\max}$ be a largest such set. Using similar ideas as in Lemma 7.2.8, we obtain the following.

Lemma 7.3.8 $|IS_{\max}| \ge (1-\frac{4}{k})\,|OPT|$.

Proof: Define $\mathcal{D}^b_a$ again as the set of disks intersecting the boundary of a j-square S at their level, i.e. $\mathcal{D}^b_a = \bigcup_S \mathcal{D}^{\mathrm{b}(S)}_{=j(S)}$. Following Lemma 7.2.8, a disk of level j is in $\mathcal{D}^b_a$ for at most 4 different values of a. Hence there is a value of a (say a^*) for which $\left|OPT\cap \mathcal{D}^b_{a^*}\right| \le \frac{4}{k}\,|OPT|$.

From Theorem 7.3.1, we know that $|IS_{a^*}| \ge \sum_S \left|OPT^{\mathrm{i}(S)}_{=j(S)}\right|$.

Then

$$\begin{array}{rcl} |IS_{a^*}| & \geq & \sum_S \left| OPT^{\mathrm{i}(S)}_{=j(S)} \right| \\ & = & |OPT| - \left|(OPT \cap \mathcal{D}^b_{a^*})\right| \\ & \geq & |OPT| - \frac{4}{k}\,|OPT|\,. \end{array}$$

Hence $|IS_{\max}| \geq |IS_{a^*}| \geq (1 - \frac{4}{k})\,|OPT|$ and the lemma follows. □

We can now prove the following.

Theorem 7.3.9 *There is an eptas for Maximum Independent Set on disk graphs of bounded level density, i.e. of level density $d = d(n) = \mathrm{O}(n^{\mathrm{o}(1)})$.*

Proof: Consider any $\epsilon > 0$. Choose k as the largest odd integer such that $(32k/\pi) \cdot \log(2ed) \leq \log n$. If $k < 5$, output any single vertex. Otherwise, using Theorem 7.3.1 and the choice of k, compute and output $IS_{\max}$ in $\mathrm{O}(n^{10} \log^4 n)$ time. Furthermore, if $d = d(n) = \mathrm{O}(n^{\mathrm{o}(1)})$, there is a c_ϵ such that $k \geq 4/\epsilon$ and $k \geq 5$ for all $n \geq c_\epsilon$. Therefore, if $n \geq c_\epsilon$, it follows from Lemma 7.3.8 and the choice of k that $IS_{\max}$ is a $(1-\epsilon)$-approximation of the optimum. Hence there is a fiptas$^\omega$ for Maximum Independent Set on n-vertex unit disk graphs of level bounded density, i.e. of level density $d = d(n) = \mathrm{O}(n^{\mathrm{o}(1)})$. Because the existence of a fiptas$^\omega$ implies the existence of an eptas (see Theorem 2.2.4), the theorem follows. □

Now observe that d is bounded by n. Hence the worst case running time of the scheme is $\mathrm{O}(k^4 n^9 (2en)^{\frac{32}{\pi}k})$.

Theorem 7.3.10 *The above algorithm is a ptas for Maximum Independent Set on disk graphs.*

The ptas given here improves on the $n^{\mathrm{O}(k^2)}$-time ptas by Erlebach, Jansen, and Seidel [103] and matches the $n^{\mathrm{O}(k)}$-time ptas by Chan [57].

7.4 Further Improvements

We gave an eptas for Minimum Vertex Cover on general disk graphs and an eptas for Maximum Independent Set on disk graphs of bounded level density. The latter scheme is also a ptas on general disk graphs. These algorithms extend to any constant dimension. Furthermore, they can be extended to intersection graphs of more general objects than disks, such as squares, triangles, etc., as long as the objects are sufficiently 'disk-like'. In other words, the objects should be *fat*. Many formal definitions of 'fat' exist, but as an example, it is easy to see the algorithms work for α-fat objects (a convex subset s of $\mathbb{R}^2$ is *α-fat* for some $\alpha \geq 1$ if the ratio between the radii of the smallest disk enclosing s and the largest disk inscribed in s is at most α [97]).

We cannot hope for a ptas on intersection graphs of nonfat objects in three dimensions, even if they have ply 1. Theorem 3.3.1 showed that any graph is an intersection graph of a set of three-dimensional convex polytopes of ply 1. Hence Maximum Independent Set and Minimum Vertex Cover are as hard on such intersection graphs as on general graphs.

In the presence of (arbitrary) weights on the vertices of the graph, the presented schemes are extendable to an eptas for Minimum-Weight Vertex Cover and Maximum-Weight Independent Set if the level density is bounded. These schemes are a ptas on disk graphs of arbitrary density. Moreover, they extend to fat objects and to any constant dimension. Unfortunately, the idea of Theorem 7.2.9 that reduces the ply of the disk graph for the minimum vertex cover problem does not seem to carry over to Minimum-Weight Vertex Cover. Therefore the question of the existence of an eptas in the weighted case on disks of arbitrary size remains open.

Beyond these generalizations, an important question is whether one can improve on the algorithms given in this chapter? Here we refer to the results of Section 6.4. Recall that Maximum Independent Set on unit disk graphs of density d cannot have a ptas with running time $2^{\mathrm{O}(\mathrm{poly}(1/\epsilon))} d^{\mathrm{o}(1/\epsilon)} n^{\mathrm{O}(1)}$. Furthermore, there is a constant d_0 such that Minimum Vertex Cover on unit disk graphs of density at most d_0 has no $2^{\mathrm{O}(\mathrm{poly}(1/\epsilon))} n^{\mathrm{O}(1)}$ time eptas. Both results are under the condition that the exponential time hypothesis is true. Because the notions of ply, level density, and density are essentially the same for unit disk graphs, the following result immediately follows from Theorem 6.4.3 and Theorem 6.4.7.

Theorem 7.4.1 *If there exist constants $\delta \geq 1$, $0 < \beta < 1$ such that Maximum Independent Set on disk graphs of level density d has a ptas with running time $2^{\mathrm{O}(1/\epsilon)^{\delta}} d^{\mathrm{O}(1/\epsilon)^{1-\beta}} n^{\mathrm{O}(1)}$, then the exponential time hypothesis is false. If there is a constant $0 < \beta < 1$ such that for any constant γ_0 Minimum Vertex Cover on disk graphs of ply at most γ_0 has a $2^{\mathrm{O}(1/\epsilon)^{1-\beta}} n^{\mathrm{O}(1)}$-time eptas, then the exponential time hypothesis is false.*

The approximation schemes for Maximum Independent Set on disk graphs of level density d and for Minimum Vertex Cover on disk graphs of ply γ described in this chapter are clearly optimal with respect to the above theorem.

Theorem 7.4.2 *Maximum Independent Set on n-vertex disk graphs of level density $d = d(n) = \Omega(n^{\alpha})$ for some constant $0 < \alpha \leq 1$ cannot have an eptas, unless FPT=W[1].*

The bound in this theorem matches the bound in Theorem 7.3.9, where we showed that Maximum Independent Set has an eptas on n-vertex disk graphs of level density $d = d(n) = \mathrm{O}(n^{\mathrm{o}(1)})$.

These results make it very unlikely that one can (significantly) improve on the schemes in this section.

7.5 The Maximum Number of Disjoint Unit Disks Intersecting a Unit Square is 7

We prove Theorem 7.3.4, which basically asks the following. Consider a unit square (i.e. a 1×1 square) and unit disks (i.e. disks of radius $\frac{1}{2}$). Determine the maximum number of nonintersecting unit disks intersecting the unit square. Here touching disks are assumed to intersect.

A trivial lower bound is 7. Placing a central disk in the center of the unit square and 6 disks around it gives a set of 7 nonintersecting disks.

A trivial upper bound is 9. All disks intersecting the unit square are completely contained in a 3×3 square around the unit square. De Groot, Peikert, and Würtz [81, 223] have shown that in the densest packing of 10 nonoverlapping (but possibly touching) disks in a 3×3 square, the disks have radius ≈ 0.444612. Hence a packing with 10 radius $\frac{1}{2}$ disks cannot exist. The upper bound of 9 follows.

We now aim to lower the upper bound. We first prove an upper bound of 8 and then further reduce it to 7, matching the lower bound.

In the following, we assume without loss of generality that the unit square is axis-aligned and that its center lies on the origin. We will not directly prove upper bounds on the number of nonintersecting unit disks intersecting the unit square, but instead focus on the bounding the number of nonintersecting unit disks intersecting the unit square, *but not intersecting the origin*. It can be readily seen that an upper bound of x on the latter number implies an upper bound of $x + 1$ on the former number.

Consider Figure 7.1. The unit square is drawn dashed. The rounded rectangle R around the unit square contains all points at distance exactly $\frac{1}{2}$ from the unit square. Now the center of any unit disk intersecting the unit square, but not intersecting the origin, must lie on or within R, but outside of the unit disk C centered on the origin. This 'allowed' area is shaded in the figure and is denoted by A.

Let c be the center of an arbitrary unit disk with $c \in A$. Consider the line segment from the origin to c and extend this segment until it intersects R (see Figure 7.2). Call this intersection point c^p. We use superscript 'p' to indicate that c^p is the projection of c onto R.

Now let $\mathsf{disk}(x, r)$ denote the disk of radius r centered on point x. If c and c' are the centers of two nonintersecting unit disks, then obviously $\mathsf{disk}(c, \frac{1}{2}) \cap \mathsf{disk}(c', \frac{1}{2}) = \emptyset$. Equivalently, it must be that $c' \notin \mathsf{disk}(c, 1)$ and $c \notin \mathsf{disk}(c', 1)$. We will combine this observation with the following lemma.

Lemma 7.5.1 *Let c and x be arbitrary points in A. If $x \notin \mathsf{disk}(c, 1)$, then $x \notin \mathsf{disk}(c^p, 1)$.*

Proof: It is sufficient to prove that the two intersection points of the boundaries of $\mathsf{disk}(c, 1)$ and $\mathsf{disk}(c^p, 1)$ are not in A. Furthermore, we only need to consider points c on C. Because if c is on C, then the two intersection points

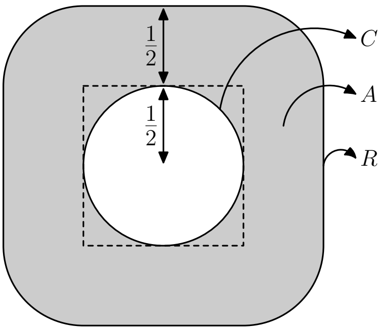

Figure 7.1: The unit disk C, rounded rectangle R at distance $\frac{1}{2}$ from the unit square, and the allowed area A.

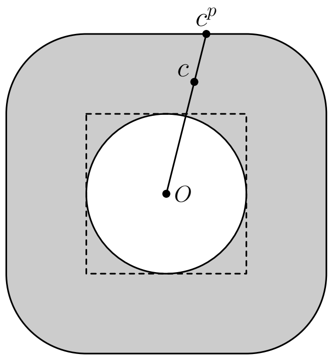

Figure 7.2: The projection point c^p of c on R.

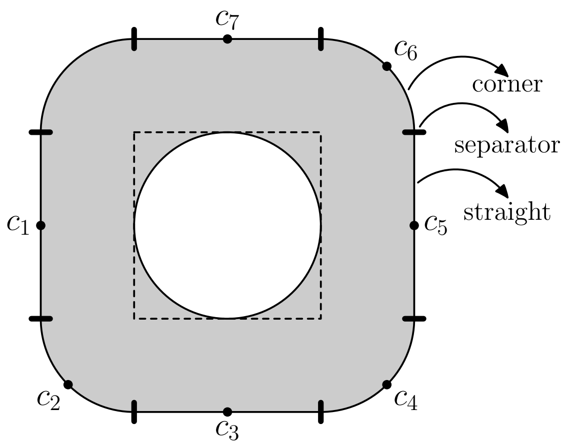

Figure 7.3: Corners, separators, and straights.

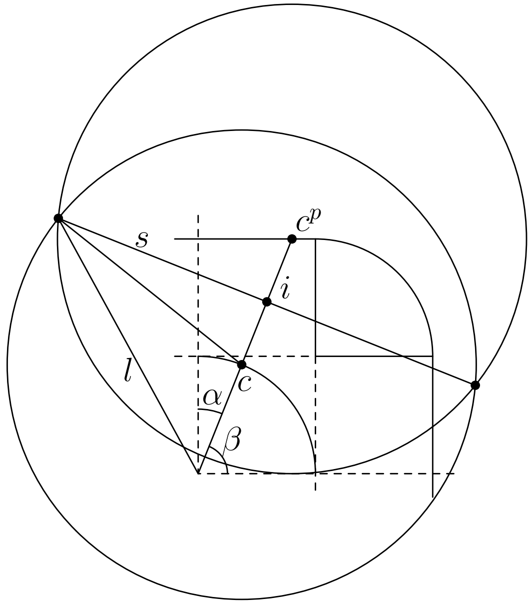

Figure 7.4: The situation if $\frac{1}{2}\pi - \tan^{-1}(\frac{1}{2}) \leq \beta \leq \frac{1}{2}\pi$.

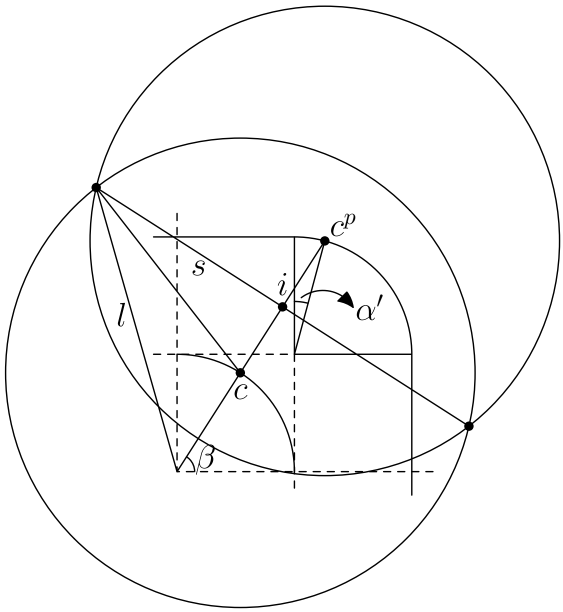

Figure 7.5: The situation if $\frac{1}{4}\pi \leq \beta \leq \frac{1}{2}\pi - \tan^{-1}(\frac{1}{2})$.

of the boundaries of $\mathsf{disk}(c, 1)$ and $\mathsf{disk}(c^p, 1)$ are closer to A than the two intersection points of the boundaries of $\mathsf{disk}(c', 1)$ and $\mathsf{disk}(c'^p, 1)$ for some c' on $\overline{cc^p}$. Due to symmetry, it is sufficient to consider angles β with $\frac{1}{4}\pi \leq \beta \leq \frac{1}{2}\pi$ (see Figure 7.4 and 7.5).

In Figure 7.4 and 7.5, i is the point in the middle between c and c^p. Then

$$\|i\| = \tfrac{1}{2}\left(\|c^p\| - \tfrac{1}{2}\right) + \tfrac{1}{2} = \tfrac{1}{2}\left(\|c^p\| + \tfrac{1}{2}\right)$$

and

$$\|s\| = \sqrt{1 - \|i - c\|^2} = \sqrt{1 - \left(\tfrac{1}{2}\left(\|c^p\| - \tfrac{1}{2}\right)\right)^2}.$$

This implies that

$$\begin{aligned} \|l\| &= \sqrt{\|i\|^2 + \|s\|^2} \\ &= \sqrt{\tfrac{1}{4}\|c^p\|^2 + \tfrac{1}{4}\|c^p\| + \tfrac{1}{16} + 1 - \tfrac{1}{4}\|c^p\|^2 + \tfrac{1}{4}\|c^p\| - \tfrac{1}{16}} \\ &= \sqrt{1 + \tfrac{1}{2}\|c^p\|}. \end{aligned}$$

It remains to determine $\|c^p\|$. We distinguish two cases.

If $\frac{1}{2}\pi - \tan^{-1}(\frac{1}{2}) \leq \beta \leq \frac{1}{2}\pi$, then α is between 0 and $\tan^{-1}(\frac{1}{2})$. But then we can easily see that $\|c^p\| = \frac{1}{\cos\alpha}$, and thus

$$\|l\| = \sqrt{1 + \frac{1}{2\cos\alpha}}$$

We next compute the derivative

$$\frac{\mathrm{d}\,\|l\|}{\mathrm{d}\alpha} = \frac{1}{\sqrt{1 + \frac{1}{2\cos\alpha}}} \cdot \frac{1}{4\cos^2\alpha} \cdot \sin\alpha$$

For $0 \leq \alpha \leq \tan^{-1}(\frac{1}{2})$, $\frac{\mathrm{d}\|l\|}{\mathrm{d}\alpha}$ is strictly positive. Hence for $\frac{1}{2}\pi - \tan^{-1}(\frac{1}{2}) \leq \beta \leq \frac{1}{2}\pi$, $\|l\|$ is at least $\sqrt{1 + \frac{1}{2\cos 0}} = \sqrt{3/2} \approx 1.225$.

If $\frac{1}{4}\pi \leq \beta \leq \frac{1}{2}\pi - \tan^{-1}(\frac{1}{2})$, then $0 \leq \alpha' \leq \frac{1}{4}\pi$. Using the Cosine Law,

$$\|c^p\| = \sqrt{\tfrac{1}{2} + \tfrac{1}{4} - 2\tfrac{1}{\sqrt{2}}\tfrac{1}{\sqrt{4}} \cdot \cos\left(\alpha' + \tfrac{3}{4}\pi\right)} = \sqrt{\tfrac{3}{4} - \tfrac{1}{2}\sqrt{2} \cdot \cos\left(\alpha' + \tfrac{3}{4}\pi\right)}.$$

Then

$$\|l\| = \sqrt{1 + \tfrac{1}{2}\sqrt{\tfrac{3}{4} - \tfrac{1}{2}\sqrt{2} \cdot \cos\left(\alpha' + \tfrac{3}{4}\pi\right)}}.$$

We again look at the derivative

$$\tfrac{\mathrm{d}\|l\|}{\mathrm{d}\alpha'} = \tfrac{1}{16}\sqrt{2} \cdot \tfrac{1}{\|l\|}\tfrac{1}{\|c^p\|} \cdot \sin\left(\alpha' + \tfrac{3}{4}\pi\right).$$

If $0 \leq \alpha' \leq \frac{1}{4}\pi$, then $\frac{\mathrm{d}\|l\|}{\mathrm{d}\alpha'}$ is nonnegative. Hence for $\frac{1}{4}\pi \leq \beta \leq \frac{1}{2}\pi - \tan^{-1}(\frac{1}{2})$, $\|l\|$ is at least $\sqrt{1 + \frac{1}{2}\sqrt{5/4}} \approx 1.249$.

Observe that the radius of the smallest circle enclosing R is $\frac{1}{2} + \frac{1}{2}\sqrt{2} \approx 1.207$. Since $\|l\| \geq \sqrt{3/2} \approx 1.225$, $\|l\| > \frac{1}{2} + \frac{1}{2}\sqrt{2}$. Hence the two intersection points of the boundaries of $\mathsf{disk}(c, 1)$ and $\mathsf{disk}(c^p, 1)$ are not in A. $\square$

Given any set of nonintersecting disks intersecting the unit square, but not intersecting the origin, with centers $c_1, \ldots, c_k$, we can thus find an equivalent set of disks with centers $c_1^p, \ldots, c_k^p$, which are also nonintersecting and intersect the unit square, but not the origin. So we may assume that all disk centers of disks not intersecting the origin are on R.

Theorem 7.5.2 *The maximum number of nonintersecting unit disks, intersecting the unit square, but not intersecting the origin, is at most* 7.

Proof: Because the centers of any such a set of unit disks lie on R, the distance between any two centers on R must be at least 1 as well (follows from the triangle inequality). Because R has length $4+\pi \approx 7.142$, there can be at most 7 such centers on R. $\square$

Corollary 7.5.3 *The number of nonintersecting unit disks intersecting the unit square is at most* 8.

In the proof of the above theorem, we used that the distance on R between any two disk centers must be at least 1. By considering this distance more closely, we can improve the bound.

Theorem 7.5.4 *The number of nonintersecting unit disks, intersecting the unit square, but not intersecting the origin, is at most* 6.

Proof: For sake of contradiction, assume $c_1, \ldots, c_7$ are the centers of 7 such unit disks. Partition R into *corners* and *straights* as shown in Figure 7.3. A point on a separator is assumed to belong to the adjacent corner. Then the 4 corners and the 4 straights partition R. Furthermore, a corner or a straight can contain at most one disk center c_i ($1 \leq i \leq 7$). Then there is either a corner or a straight which does not contain a disk center c_i.

Suppose there is a corner which does not contain a disk center. Consider a corner which *does* contain a disk center (see Figure 7.6). We determine the length on R of edges $\overline{c_i c_{i+1}}$ and $\overline{c_i c_{i-1}}$, depending on α. So let $l = \|\overline{c_i c_{i+1}}\|_R + \|\overline{c_i c_{i-1}}\|_R$. We know that $\|\overline{c_i c_{i+1}}\| \geq 1$. Using that triangle T is an isosceles triangle, $\beta = \frac{\pi - \alpha}{2}$. Then basic trigonometry gives that the part of the straight covered by $\overline{c_i c_{i+1}}$ has length at least

$$\sqrt{1 - \left(\tfrac{1}{2} - \tfrac{1}{2}\cos\alpha\right)^2} - \tfrac{1}{2}\sin\alpha.$$

But then

$$\|\overline{c_i c_{i+1}}\|_R \geq \sqrt{1 - \left(\tfrac{1}{2} - \tfrac{1}{2}\cos\alpha\right)^2} - \tfrac{1}{2}\sin\alpha + \tfrac{1}{2}\alpha.$$

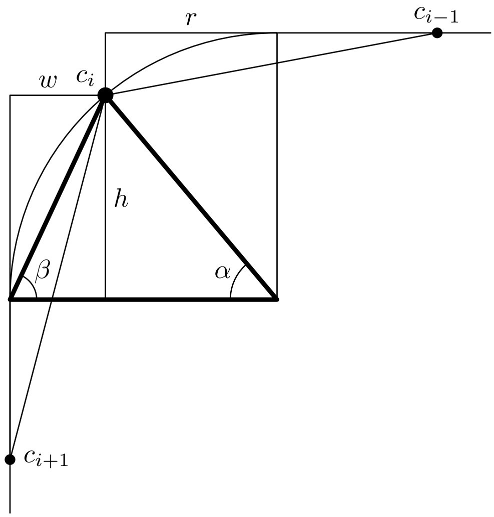

Figure 7.6: The triangle T is formed by the three thick lines. As T is an isosceles triangle, $\beta = \frac{\pi-\alpha}{2}$. From the figure, we derive that $\|h\| = \frac{1}{2}\sin\alpha$, $\|r\| = \frac{1}{2}\cos\alpha$, and $\|w\| = \frac{1}{2} - \frac{1}{2}\cos\alpha$.

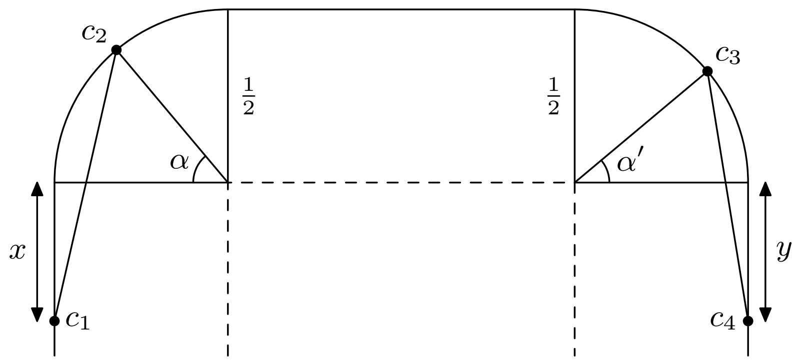

Figure 7.7: The top straight contains no disk center. Clearly, $\|x\|+\|y\|$ is minimal if $\alpha = \alpha' = \frac{1}{2}$.

Similarly,

$$\|\overline{c_i c_{i-1}}\|_R \geq \sqrt{1 - \left(\tfrac{1}{2} - \tfrac{1}{2}\sin\alpha\right)^2} - \tfrac{1}{2}\cos\alpha + \tfrac{1}{4}\pi - \tfrac{1}{2}\alpha.$$

Hence

$$l \geq \sqrt{1 - \left(\tfrac{1}{2} - \tfrac{1}{2}\cos\alpha\right)^2} + \sqrt{1 - \left(\tfrac{1}{2} - \tfrac{1}{2}\sin\alpha\right)^2} - \tfrac{1}{2}\sin\alpha - \tfrac{1}{2}\cos\alpha + \tfrac{1}{4}\pi.$$

Then the derivative is

$$\begin{aligned}\frac{\mathrm{d}l}{\mathrm{d}\alpha} \quad = \quad & -\tfrac{1}{2}\frac{1}{\sqrt{1-\left(\frac{1}{2}-\frac{1}{2}\cos\alpha\right)^2}}\left(\tfrac{1}{2} - \tfrac{1}{2}\cos\alpha\right)\sin\alpha \\ & +\tfrac{1}{2}\frac{1}{\sqrt{1-\left(\frac{1}{2}-\frac{1}{2}\sin\alpha\right)^2}}\left(\tfrac{1}{2} - \tfrac{1}{2}\sin\alpha\right)\cos\alpha \\ & -\tfrac{1}{2}\cos\alpha + \tfrac{1}{2}\sin\alpha.\end{aligned}$$

If $0 \leq \alpha < \frac{1}{4}\pi$, then $\frac{\mathrm{d}l}{\mathrm{d}\alpha} < 0$. If $\frac{1}{4}\pi < \alpha \leq \frac{1}{2}\pi$, $\frac{\mathrm{d}l}{\mathrm{d}\alpha} > 0$. If $\alpha = \frac{1}{4}\pi$, $\frac{\mathrm{d}l}{\mathrm{d}\alpha} = 0$. So $l = \|\overline{c_i c_{i+1}}\|_R + \|\overline{c_i c_{i-1}}\|_R$ is minimal if $\alpha = \frac{1}{4}\pi$ and has value $l_{\min} \approx 2.057$. Because the disk centers can be numbered arbitrarily, we may assume that c_2, c_4, and c_6 are on corners. Using symmetry, we may also assume that they are on counter-clockwise consecutive corners, as shown in Figure 7.3. Then

$$\begin{gathered}(\|\overline{c_1 c_2}\|_R + \|\overline{c_2 c_3}\|_R) + (\|\overline{c_3 c_4}\|_R + \|\overline{c_4 c_5}\|_R) + (\|\overline{c_5 c_6}\|_R + \|\overline{c_6 c_7}\|_R) + \|\overline{c_7 c_1}\|_R \\ \geq 3 l_{\min} + 1 \approx 7.17\end{gathered}$$

This is larger than the length of R, which is a contradiction.

So there must be a straight which does not contain a disk center. Using symmetry, we may assume that this is the top straight (see Figure 7.7). Note that $0 \leq \alpha, \alpha' \leq \frac{1}{2}\pi$. We minimize $\|x\| + \|y\|$. Trivially, this minimum is attained if $\alpha = \alpha' = \frac{1}{2}\pi$. In this case,

$$\|x\| + \|y\| = 2\left(\sqrt{1 - \left(\tfrac{1}{2}\right)^2} - \tfrac{1}{2}\right) = \sqrt{3} - 1.$$

Hence

$$\begin{gathered}(\|\overline{c_1 c_2}\|_R + \|\overline{c_2 c_3}\|_R + \|\overline{c_3 c_4}\|_R) + \|\overline{c_4 c_5}\|_R + \|\overline{c_5 c_6}\|_R + \|\overline{c_6 c_7}\|_R + \|\overline{c_7 c_1}\|_R \\ \geq (\sqrt{3} - 1 + \tfrac{1}{2}\pi + 1) + 4 \approx 7.30\end{gathered}$$

This is larger than the length of R, which is a contradiction. Therefore there can be at most 6 nonintersecting unit disks intersecting the unit square, but not intersecting the origin. □

Using this upper bound and the lower bound given before, we have proved Theorem 7.3.4.