Optimization and approximation on systems of geometric objects

van Leeuwen, E.J.

Chapter 8

Domination on Geometric Intersection Graphs

This chapter only treats the minimum dominating set problem on geometric intersection graphs. Although on general graphs the approximability of Minimum Dominating Set has been settled [156, 197, 66, 108], the problem is still open on numerous graph classes, including several classes of geometric intersection graphs.

In studying approximation algorithms for fundamental graph optimization problems on geometric intersection graphs, we demonstrated the power of the geometric shifting technique to approximate these problems. In particular, we were able to obtain better polynomial-time approximation schemes for Maximum Independent Set and Minimum Vertex Cover on unit disk graphs (Chapter 6) and on general disk graphs (Chapter 7). Moreover, we found a better ptas for Minimum (Connected) Dominating Set on unit disk graphs (Chapter 6), again using the shifting technique. These algorithms extend to intersection graphs of (unit) fat objects in any constant dimension and (at least partially) to the weighted case (see Section 6.3.5 and 7.4).

Interestingly, as pointed out by Erlebach, Jansen, and Seidel [103], these techniques do not seem sufficient for handling Minimum Dominating Set on intersection graphs of objects of different sizes. As far as we know, there are no results on intersection graphs of arbitrary disks, squares, etc., beyond the $(1 + \ln n)$-approximation ratio of the greedy algorithm [156, 197, 66]. In particular, we know of no constant-factor approximation algorithm or approximation hardness results. In this chapter, we address this open problem by studying Minimum Dominating Set on intersection graphs of different types of fat objects and providing new insights into its approximability.

In Section 8.2, we present a new general approach to deriving approximation algorithms for Minimum Dominating Set on geometric intersection graphs. We apply it to obtain the first constant-factor approximation algorithms for Minimum Dominating Set on intersection graphs of pairwise homothetic polygons with a constant number of corners and on intersection graphs of rectangles of bounded aspect-ratio.

We also obtain a constant-factor approximation algorithm for Minimum Dominating Set on disk graphs of constant ply (see Section 8.4). A surprising corollary of this is a constant integrality gap of the standard linear program (LP) for Minimum Dominating Set on planar graphs. For disk graphs of bounded ply, we can improve this result to a $(3 + \epsilon)$-approximation algorithm by using a new variant of the shifting technique. This algorithm extends to intersection graphs of fat objects of bounded ply and constant dimension.

The type of fat objects one considers has a strong impact on the approximability of Minimum Dominating Set, as shown in Section 8.5. We prove that on intersection graphs of n convex fat objects, approximation ratio $(1 - \epsilon) \ln n$ is not achievable in polynomial time for any $\epsilon > 0$, unless $\mathrm{NP} \subset \mathrm{DTIME}(n^{\mathrm{O}(\log \log n)})$. This also holds on intersection graphs of pairwise homothetic objects. Finally, we solve an open problem of Chlebík and Chlebíková [65], who asked whether their APX-hardness results for Minimum Dominating Set on intersection graphs of d-dimensional axis-parallel boxes if $d \geq 3$ extend to the case where $d = 2$. We affirm this by showing that Minimum Dominating Set is APX-hard on rectangle intersection graphs.

8.1 Small ϵ-Nets

The core of the algorithmic results of Section 8.2 relies on the availability of small ϵ-nets. Given a universe $\mathbb{U}$, a family $\mathcal{S}$ of subsets of $\mathbb{U}$ (called *objects*), and a (positive) weight function w over $\mathcal{S}$, we say that $\mathcal{R} \subseteq \mathcal{S}$ is an *ϵ-net* for $\mathcal{S}$ if any element $u \in \mathbb{U}$ for which $\sum_{s \in \mathcal{S} : u \in s} w(s) > \epsilon W$ is covered by $\mathcal{R}$ (i.e. $u \in \bigcup \mathcal{R}$), where $W = \sum_{s \in \mathcal{S}} w(s)$. In the classic definition of an ϵ-net, it assumed that all weights are equal to 1. That is, $\mathcal{R} \subseteq \mathcal{S}$ is a *binary ϵ-net* for $\mathcal{S}$ if any element $u \in \mathbb{U}$ covered by more than $\epsilon |\mathcal{S}|$ sets in $\mathcal{S}$ is also covered by $\mathcal{R}$. The *size* of a (binary) ϵ-net is the cardinality of $\mathcal{R}$.

We should note that in a way there are two definitions of an ϵ-net, that are essentially dual to each other [143, 68]. In the covering version of ϵ-nets, described above, we aim to select objects to cover elements that are covered by a lot of objects. In the dual definition, the hitting version, we need to select elements to hit all objects containing a large number of elements. Here we only need the covering variant and thus disregard the hitting version.

There have been several results on ϵ-nets in the past (e.g. [143, 35, 170, 205, 62, 44, 68, 188, 226]). The most general result is the following. Given a (finite) universe $\mathbb{U}$ and a family $\mathcal{S}$ of subsets of $\mathbb{U}$, let $S(u) = \{s \in S \mid u \in s\}$ for any $S \subseteq \mathcal{S}$. Then the *dual Vapnik-Chervonenkis dimension* or *dual VC-dimension* of $(\mathbb{U}, \mathcal{S})$ is equal to the cardinality of a largest set $S \subseteq \mathcal{S}$ for which $\{S(u) \mid u \in \mathbb{U}\}$ equals the power set of S [143].

Theorem 8.1.1 ([170]) *Suppose that $(\mathbb{U}, \mathcal{S})$ has dual VC-dimension d. Then for any $\epsilon > 0$ that is sufficiently small with respect to d there is a binary ϵ-net for $\mathcal{S}$ of size at most $(d/\epsilon) \cdot (\log(1/\epsilon) + 2 \log \log(1/\epsilon) + 3)$.*

There are many examples of set systems with constant dual VC-dimension. For instance, recall from Chapter 3 the representation of an arbitrary graph as an intersection graph. Given a graph G, let $\mathbb{U} = E(G)$ and $\mathcal{S} = \{S_v \mid v \in V(G)\}$, where $S_v = \{(u,v) \in E(G) \mid u \in V(G)\}$ for any $v \in V(G)$. This set system can easily be shown to have dual VC-dimension at most 2. Hence, by Theorem 8.1.1, it has an ϵ-net of size $\mathrm{O}(\frac{1}{\epsilon}\log\frac{1}{\epsilon})$. One can however improve on this bound.

Theorem 8.1.2 *Let* $(\mathbb{U}, \mathcal{S})$ *be induced by a graph* G *(as described above) and let* w *be a positive weight function over* $\mathcal{S}$*. Then one can find an* ϵ*-net of* $\mathcal{S}$ *of size at most* $2/\epsilon$ *in linear time.*

Proof: We need to cover all elements of $\mathbb{U}$ covered by sets of $\mathcal{S}$ with total weight exceeding ϵW. Any $u \in \mathbb{U}$ is in at most 2 sets of $\mathcal{S}$, say s_u^1 and s_u^2. If $w(s_u^1) + w(s_u^2) > \epsilon W$, then $\max\{w(s_u^1), w(s_u^2)\} > \epsilon W/2$. Hence $\mathcal{R} = \{s \in \mathcal{S} \mid w(s) > \epsilon W/2\}$ is an ϵ-net. Moreover, $|\mathcal{R}| < 2/\epsilon$. □

The bound of Theorem 8.1.2 is essentially tight. For $m > 0$, let $G = K_{2m}$ and let $(\mathbb{U}, \mathcal{S})$ be the set system induced by G. Set $w(s) = 1$ for each $s \in \mathcal{S}$ and set $\epsilon = 1/(m + \frac{1}{4})$. An ϵ-net for $(\mathbb{U}, \mathcal{S})$ is equal to a vertex cover of all edges $(u,v) \in E(G)$ for which $w(u) + w(v) > \epsilon W$. Clearly, $w(u) + w(v) = 2 > \epsilon \cdot 2m$ for each $(u,v) \in E(G)$. But each vertex cover of G needs at least $2m - 1$ vertices, while $2/\epsilon < 2m + 1$. As m tends to infinity, this is tight.

For geometric intersection graphs one can prove similar bounds. A family $\mathcal{S}$ of subsets of $\mathbb{U} = \mathbb{R}^2$ is a family of *pseudo-disks* if the sets in $\mathcal{S}$ are bounded by simple closed Jordan curves, such that each pair of curves intersects at most twice. Examples are families of disks, squares, or homothetic polygons. Given such $\mathbb{U}$ and $\mathcal{S}$, the next theorem follows from results of Chazelle and Friedman [62], Clarkson and Varadarajan [68], and Kedem et al. [161].

Theorem 8.1.3 *For any* $\epsilon > 0$*, there is a binary* ϵ*-net for* $\mathcal{S}$ *of size* $\mathrm{O}(1/\epsilon)$*.*

Such a net can be found by a randomized algorithm with polynomial expected running time [62, 68]. By derandomizing the algorithm using the method of conditional expectations, we can prove that a binary ϵ-net as in Theorem 8.1.3 can be found in time polynomial in $|\mathcal{S}|$ and $1/\epsilon$ [62, 256].

The above results are actually corollaries of more general theorems that relate the size of the ϵ-net to the union complexity of the set $\mathcal{S}$. An extensive treatment may be found in [62, 68, 256].

Pyrga and Ray [226] recently improved on Theorem 8.1.3 and the associated algorithms. The ϵ-nets following from their results also have size $\mathrm{O}(1/\epsilon)$, but with a much better hidden constant. Moreover, both the analysis and the algorithm needed to compute the net are easier.

Theorem 8.1.4 *For any* $\epsilon > 0$*, one can obtain a binary* ϵ*-net for* $\mathcal{S}$ *of size* $\mathrm{O}(1/\epsilon)$ *in time polynomial in* $|\mathcal{S}|$ *and* $1/\epsilon$*.*

Linear-sized ϵ-nets also exist for three-dimensional objects. Clarkson and Varadarajan [68] showed that an ϵ-net exists for unit cubes. This result was subsequently generalized by Laue [188].

Theorem 8.1.5 ([188]) *For any $\epsilon > 0$, one can obtain a binary ϵ-net of size $O(1/\epsilon)$ for a set $\mathcal{S}$ of translates of a fixed three-dimensional polytope in time polynomial in $|\mathcal{S}|$ and $1/\epsilon$.*

Note that the above algorithms find binary ϵ-nets. One can transform them into algorithms to find a (weighted) ϵ-net at relatively small cost.

Definition 8.1.6 *Algorithm A is a* net finder with size-function g *for $(\mathbb{U}, \mathcal{S})$ if for any $\epsilon > 0$ and any (positive) weight function w over $\mathcal{S}$, A gives an ϵ-net for $(\mathbb{U}, \mathcal{S})$ of size at most $g(1/\epsilon)$ in time polynomial in $|\mathcal{S}|$, $1/\epsilon$, and the size of a representation of w.*

The definition of a *binary net finder with size-function g* is similar. We will always assume the size-function g to be nondecreasing.

Proposition 8.1.7 ([45]) *If A is a binary net finder with size-function g for some $(\mathbb{U}, \mathcal{S})$, then there is a net finder A' with size-function $g'(1/\epsilon) = g(2/\epsilon)$ for $(\mathbb{U}, \mathcal{S})$.*

Proof: Let some $\epsilon > 0$ and some (positive) weight function w over $\mathcal{S}$ be given. Scale the weights to w' such that $W' = \sum_{s \in \mathcal{S}} w'(s) = |\mathcal{S}|$. Take $\lceil w'(s) \rceil$ copies of each $s \in \mathcal{S}$ and denote the resulting set of objects by $\mathcal{S}'$. Then

$$|\mathcal{S}'| = \sum_{s \in \mathcal{S}} \lceil w'(s) \rceil < \sum_{s \in \mathcal{S}} (1 + w'(s)) = W' + |\mathcal{S}| = 2|\mathcal{S}| = 2W'.$$

Choose $\epsilon' = \epsilon/2$ and apply A to $\mathcal{S}'$ and ϵ'. This gives an ϵ'-net for $\mathcal{S}'$ of size $g(2/\epsilon)$. Since $\epsilon'|\mathcal{S}'| < \epsilon W'$, it induces an ϵ-net of $\mathcal{S}$ with respect to w', and hence with respect to w as well. Observe that the above algorithm takes time polynomial in $|\mathcal{S}|$, $1/\epsilon$, and the size of the representation of w. □

8.2 Generic Domination

We give a generic approach to approximating Minimum Dominating Set, particularly on geometric intersection graphs. To this end, we introduce the novel notion of $\preceq$-dominating sets, which we then use in combination with ϵ-nets to approximate Minimum Dominating Set.

Let $\preceq$ be a binary reflexive relation on the vertices of a graph G. For example, if G is some geometric intersection graph with representation $\mathcal{S}$, $u \preceq v$ if the size of $\mathcal{S}(u)$ is at most the size of $\mathcal{S}(v)$. We say that $v \in V(G)$ is *$\preceq$-larger* than $u \in V(G)$ if $u \preceq v$. Denote by $N_{\preceq}(u) = \{v \in V(G) \mid (u,v) \in E(G), u \preceq v\}$ the set of $\preceq$-larger neighbors of some $u \in V(G)$ and let $N_{\preceq}[u] = N_{\preceq}(u) \cup \{u\}$ denote u's closed $\preceq$-larger neighborhood. Similarly, we define $N_{\succeq}(u) = \{v \in V(G) \mid (u,v) \in E(G), v \preceq u\}$ and $N_{\succeq}[u] = N_{\succeq}(u) \cup \{u\}$.

Definition 8.2.1 *Given a graph G and a binary reflexive relation $\preceq$ on the vertices of G, $C \subseteq V(G)$ is a* $\preceq$-dominating set *of G if for any $u \in V(G)$, $u \in C$ or there is a $\preceq$-larger neighbor of u in C.*

Alternatively, $C \subseteq V(G)$ is a *$\preceq$-dominating set* of G if $C \cap N_{\preceq}[u] \neq \emptyset$ for all $u \in V(G)$. Observe that $\preceq$-dominating sets are a proper generalization of ordinary dominating sets. Simply take $\preceq$ to be the complete relation, i.e. $u \preceq v$ for all $u, v \in V(G)$. Moreover, the definition of $\preceq$-dominating set is sound, as $V(G)$ is a $\preceq$-dominating set of G, regardless of the definition of $\preceq$.

For a given relation $\preceq$, one can try to find a relation between the cardinality of a smallest dominating and of a smallest $\preceq$-dominating set.

Definition 8.2.2 *Given a graph G and a binary reflexive relation $\preceq$ on $V(G)$, the* $\preceq$-factor *is the cardinality of a minimum $\preceq$-dominating set divided by the cardinality of a minimum dominating set.*

Clearly, the $\preceq$-factor is at least 1 for any relation $\preceq$. Knowing an upper bound on the $\preceq$-factor is more interesting however, as this leads to one of the main theorems of this chapter.

Theorem 8.2.3 *Let $(\mathbb{U}, \mathcal{S})$ be a set system for which a net finder with size-function g exists and let $\preceq$ be a binary reflexive relation on the vertices of $G = G[\mathcal{S}]$ with $\preceq$-factor at most c_1 such that for any $u \in V(G)$ there exist at most c_2 elements of $\mathbb{U}$ in $\mathcal{S}(u)$ jointly hitting all $\mathcal{S}(v)$ with $v \in N_{\preceq}(u)$. If the cardinality of a minimum dominating set of G is k, then one can find a dominating set of G of cardinality at most $g(c_1 c_2 k)$ in time polynomial in $|\mathcal{S}|$.*

Proof: Consider the standard integer LP of the minimum $\preceq$-dominating set problem:

$$\begin{aligned} z_I^* \quad = \quad \min \quad & \sum_{u \in V(G)} x_u \\ \text{s.t.} \quad & \sum_{v \in N_{\preceq}[u]} x_v \geq 1 \quad \forall u \in V(G) \\ & x_u \in \{0,1\} \qquad \forall u \in V(G) \end{aligned}$$

Observe that $z_I^* \leq c_1 k$. Relax the above integer LP by replacing its last constraint by $x_u \geq 0 \ \forall u \in V(G)$. Let x^* be a vector attaining the optimum fractional value z^*. Because for any $u \in V(G)$, all $\mathcal{S}(v)$ with $v \in N_{\preceq}(u)$ can be jointly hit by c_2 elements in $\mathcal{S}(u)$, each $\mathcal{S}(u)$ contains an element p such that $\sum_{v: p \in \mathcal{S}(v)} x_v^* \geq 1/c_2$. Call such an element *heavily covered.*

Now define a weight function w by $w(\mathcal{S}(u)) := x_u^* |\mathcal{S}|/z^*$. Let $W = \sum_{u \in V(G)} w(\mathcal{S}(u))$ and $\epsilon = 1/(c_2 z^*)$. Following the previous observation, this implies that any object $s \in \mathcal{S}$ contains an element p such that

$$\sum_{v: p \in \mathcal{S}(v)} w(\mathcal{S}(v)) = \sum_{v: p \in \mathcal{S}(v)} x_v^* |\mathcal{S}|/z^* = (|\mathcal{S}|/z^*) \cdot \sum_{v: p \in \mathcal{S}(v)} x_v^* \geq |\mathcal{S}|/(c_2 z^*) = \epsilon W.$$

Hence an ϵ-net $\mathcal{R} \subseteq \mathcal{S}$ for this choice of ϵ will cover all heavily covered elements. But then $\mathcal{R}$ induces a dominating set of G. Moreover,

$$|\mathcal{R}| \leq g(c_2 z^*) \leq g(c_2 z_I^*) \leq g(c_1 c_2 k).$$

Finally note that $\mathcal{R}$ can be found in time polynomial in $|\mathcal{S}|$. The optimum solution to the linear program can be found in polynomial time [163, 159]. Hence the weights of the weight function can be represented using a polynomial number of bits and therefore the ϵ-net can be found in polynomial time. $\square$

Observe that if instead of a (weighted) net finder we only have a binary net finder with size-function g, then the above algorithm yields a dominating set of cardinality at most $g(2c_1c_2k)$ by Proposition 8.1.7.

The running time of the algorithm described in Theorem 8.2.3 is determined by the time it takes to find the ϵ-net and to solve the linear program. The latter takes $\mathrm{O}(n^{3.5}\log^2 n)$ time [159], where we ignore some sublogarithmic terms. Young [275] showed that a $(1+\delta)$-approximate solution to the linear program can be found much quicker, in $\mathrm{O}(n^2 \log n/\delta^2)$ time. If we use such a solution in Theorem 8.2.3, the dominating set has cardinality $g((1+\delta)c_1c_2k)$.

The proof of Theorem 8.2.3 solves a linear program and finds an ϵ-net once, following a technique of Even, Rawitz, and Shahar [107]. Alternatively, one could use the iterative reweighting technique proposed by Brönnimann and Goodrich [45], where an ϵ-net is constructed in every iteration. In this chapter, finding the ϵ-net is usually quite expensive and hence we prefer the technique of Even, Rawitz, and Shahar. Moreover, it makes for an easier proof.

Another consequence of the proof of Theorem 8.2.3 is a bound on the integrality gap of the standard LP of Minimum Dominating Set. The *integrality gap* of an LP is the ratio of its optimum integral value and its optimum fractional value. For this bound, we need a fractional equivalent of the $\preceq$-factor.

Definition 8.2.4 *Given a graph G and a binary reflexive relation $\preceq$ on $V(G)$, the* fractional $\preceq$-factor *is the ratio of the optimum fractional value of the standard LP for Minimum $\preceq$-Dominating Set and the optimum fractional value of the standard LP for Minimum Dominating Set.*

For all relations $\preceq$ described in this chapter, we can find the same bound on the $\preceq$-factor as on the fractional $\preceq$-factor. It is not clear whether this is a coincidence.

We can now prove a fractional equivalent of Theorem 8.2.3.

Theorem 8.2.5 *Let $(\mathbb{U}, \mathcal{S})$ be a set system for which a net finder with size-function g exists and let $\preceq$ be a binary reflexive relation on the vertices of $G = G[\mathcal{S}]$ with fractional $\preceq$-factor at most c_3 such that for any $u \in V(G)$ there exist at most c_2 elements of $\mathbb{U}$ in $\mathcal{S}(u)$ jointly hitting all $\mathcal{S}(v)$ with $v \in N_{\preceq}(u)$. If the optimum fractional value of the standard LP for Minimum Dominating Set is z^*, then one can find a dominating set of G of cardinality at most $g(c_2c_3z^*)$ in time polynomial in $|\mathcal{S}|$.*

Proof: Let $z^*_{\preceq}$ denote the optimum fractional value of the standard LP for Minimum $\preceq$-Dominating Set on G. Following the proof of Theorem 8.2.3, one can find a dominating set of cardinality at most $g(c_2z^*_{\preceq})$ in polynomial time. As $z^*_{\preceq} \leq c_3z^*$, this dominating set has cardinality at most $g(c_2c_3z^*)$. $\square$

In other words, the integrality gap is at most $g(c_2c_3z^*)/z^*$.

Again if only a binary net finder with size-function g exists, then one can find a dominating set of G of cardinality at most $g(2c_2c_3z^*)$ in polynomial time. This implies that the integrality gap is at most $g(2c_2c_3z^*)/z^*$.

As an example and to demonstrate the generality of Theorem 8.2.3 and Theorem 8.2.5, we apply them to general graphs. For a graph G, let $\Delta(G)$ denote the maximum degree of a vertex of $V(G)$.

Theorem 8.2.6 *Minimum Dominating Set has a polynomial-time* $2\Delta(G)$*-approximation algorithm on any graph* G*. Moreover, the integrality gap is at most* $2\Delta(G)$*.*

Proof: Let G be any graph and $(\mathbb{U}, \mathcal{S})$ a representation of G, i.e. $\mathbb{U} = E(G)$ and $\mathcal{S} = \{S_v \mid v \in V(G)\}$, where $S_v = \{(u,v) \in E(G) \mid u \in V(G)\}$ for any $v \in V(G)$. Define a binary relation $\preceq$ such that $u \preceq v$ for all $u, v \in V(G)$. Observe that the (fractional) $\preceq$-factor is 1. For any $u \in V(G)$, $N_{\preceq}(u) = N(u)$, and thus there exist (at most) $\Delta(G)$ elements of $\mathbb{U}$ in $\mathcal{S}(u)$ that jointly hit all $\mathcal{S}(v)$ with $v \in N_{\preceq}(u)$. Simply take all edges incident to u. Theorem 8.1.2 showed that any graph G with representation $(\mathbb{U}, \mathcal{S})$ has an ϵ-net of size $2/\epsilon$, which can be found in polynomial time. The theorem statement follows from Theorem 8.2.3 and Theorem 8.2.5. $\square$

Note that the above algorithm can only guarantee an approximation ratio of $2\Delta(G)$, whereas a greedy algorithm giving ratio $1 + \ln \Delta(G)$ exists [156, 197, 66, 149]. Theorem 8.2.6 merely serves as an indication of the power of Theorem 8.2.3 and Theorem 8.2.5. The real challenges and offered improvements lie with geometric intersection graphs.

8.3 Dominating Set on Geometric Intersection Graphs

The main result of this section is a constant-factor approximation algorithm for Minimum Dominating Set on intersection graphs of homothetic convex polygons. The constant depends on the number of corners (i.e. the complexity) of the base polygon. We also show that on intersection graphs of regular polygons the dependence on the complexity of the base polygon can be reduced. Although homotheticity is crucial in the analysis of these results, we show that on intersection graphs of axis-parallel rectangles that are not necessarily homothetic, but have constant aspect-ratio, one can obtain a constant-factor approximation algorithm as well. A discussion of disk graphs is deferred to Section 8.4.

8.3.1 Homothetic Convex Polygons

We show that Minimum Dominating Set on intersection graphs of homothetic convex polygons with r corners has a polynomial-time $\mathrm{O}(r^4)$-approximation algorithm. We require two auxiliary results before we are ready to prove this.

First we need a way to bound the (fractional) $\preceq$-factor of a relation $\preceq$. The next two lemmas hold for arbitrary graphs.

Lemma 8.3.1 *Let $\preceq$ be a binary reflexive relation on the vertices of a graph G such that for any $u \in V(G)$ a minimum $\preceq$-dominating set for $U_u = \{v \mid v \not\preceq u, v \in N(u)\}$ has cardinality at most c. Then the $\preceq$-factor is at most $c+1$.*

Proof: Consider some dominating set C of G and for each $u \in C$, let C_u be a minimum $\preceq$-dominating set of U_u. We claim that $C' = C \cup \bigcup_{u \in C} C_u$ is a $\preceq$-dominating set of G. For suppose that there is some $v \in V(G) - C'$ that is not $\preceq$-dominated by a vertex in C'. Because C is a dominating set of G and $C \subseteq C'$, $v \in U_u$ for some $u \in C$. But then v is $\preceq$-dominated by C_u and hence by C', a contradiction. Finally, note that $|C'| \leq (c+1) \cdot |C|$. Therefore the $\preceq$-factor is at most $c+1$. □

Observe that one only needs an upper bound on $|C_u|$ for vertices u appearing in the dominating set C.

Lemma 8.3.2 *Let $\preceq$ be a binary reflexive relation on the vertices of some graph G such that for any $u \in V(G)$ a minimum $\preceq$-dominating set for $U_u = \{v \mid v \not\preceq u, v \in N(u)\}$ has cardinality at most c. Then the fractional $\preceq$-factor is at most $c+1$.*

Proof: Let x^* be an optimal fractional solution to the standard LP for Minimum Dominating Set, with value z^*. For any $u \in V(G)$, let C_u be a minimum $\preceq$-dominating for U_u. Set x'_v to x^*_v for each $v \in V(G)$ and then add x^*_u to x'_v for each $u \in V(G)$ with $v \in C_u$. Then for any $u \in V(G)$,

$$\sum_{v \in N_{\preceq}[u]} x'_v \geq \sum_{v \in N_{\preceq}[u]} x^*_v + \sum_{v \in N[u] - N_{\preceq}[u]} x^*_v = \sum_{v \in N[u]} x^*_v \geq 1.$$

Hence x' is a solution to the standard LP for Minimum $\preceq$-Dominating Set. It has value

$$\sum_{u \in V(G)} x'_u \leq \sum_{u \in V(G)} (c+1) x^*_u = (c+1) \cdot z^*.$$

Therefore the fractional $\preceq$-factor is at most $c+1$. □

We are now ready to present the relation used in the approximation algorithm. Call the straight line segment between two corners of a convex polygon a *chord*. Observe that some chords correspond to sides of the polygon and that each chord is contained in the polygon. Let $G = G[\mathcal{S}]$ be the intersection graph of a set $\mathcal{S}$ of homothetic convex polygons. Define a relation $\preceq_{1/3}$ as follows. For any two vertices $u, v \in V(G)$, let $v \preceq_{1/3} u$ if $\mathcal{S}(u)$ contains a corner of $\mathcal{S}(v)$ or $\mathcal{S}(u)$ covers at least one third of a chord of $\mathcal{S}(v)$.

The next lemma is crucial to the analysis of the approximation algorithm. For its proof, recall the following definitions of points and lines of a triangle.

An *altitude* of a corner is the straight line through this corner, perpendicular to the side opposite the corner. A *median* of a corner is the straight line through this corner and the midpoint of the opposite side. The intersection point of the medians of a triangle is its *centroid* or *barycenter*.

Lemma 8.3.3 *Let $G = G[\mathcal{S}]$ be the intersection graph of a collection $\mathcal{S}$ of homothetic convex polygons with r corners for some $r \geq 3$. Then the $\preceq_{1/3}$-factor is at most $2r(r-2)+1$.*

Proof: Consider a dominating set C of G such that for each $u \in C$ there is no $v \in V(G)$ for which $\mathcal{S}(v)$ strictly contains $\mathcal{S}(u)$. Clearly, G always has a dominating set with this property that is also a minimum dominating set. Let $u \in C$ and consider the set $U = \{v \mid v \not\preceq_{1/3} u, v \in N(u)\}$. Following Lemma 8.3.1, it suffices to bound the cardinality of a minimum $\preceq_{1/3}$-dominating set of U by $2r(r-2)$ to prove the lemma.

Observe that for any $v \in U$, $\mathcal{S}(u)$ does not contain a corner of $\mathcal{S}(v)$. As the polygons are convex and homothetic, each $\mathcal{S}(v)$ with $v \in U$ must contain a corner of $\mathcal{S}(u)$. Consider a corner p of $\mathcal{S}(u)$ and let $U_p = \{v \mid v \in U, p \in \mathcal{S}(v)\}$ be the set of vertices $v \in U$ for which $p \in \mathcal{S}(v)$. Because $\mathcal{S}(v)$ has no corner in $\mathcal{S}(u)$ for each $v \in U_p$, there must be precisely one side of $\mathcal{S}(v)$ that intersects $\mathcal{S}(u)$. This side is not incident with the corner of $\mathcal{S}(v)$ corresponding to p. Let $U_{p,s}$ be the set of vertices $v \in U_p$ for which side s of $\mathcal{S}(v)$ intersects $\mathcal{S}(u)$.

For any p, s, reduce $\mathcal{S}(u)$ and each $\mathcal{S}(v)$ with $v \in U_{p,s}$ to the triangle induced by the corner corresponding to p and the side corresponding to s. This gives a collection $\mathcal{S}'$ of homothetic triangles all containing p, but no triangle $\mathcal{S}'(v)$ with $v \in U_{p,s}$ contains $\mathcal{S}'(u)$ or has a corner in $\mathcal{S}'(u)$. Moreover, the sides of the triangles correspond to chords of the original polygons.

Assume without loss of generality that one side of the triangles of $\mathcal{S}'$ is parallel to the x-axis and that p corresponds to the left corner of $\mathcal{S}'(u)$. Now let $v_t \in U_{p,s}$ be a vertex such that the top corner of $\mathcal{S}'(v_t)$ has the largest distance to the altitude of the left corner of $\mathcal{S}'(u)$ among all top corners of triangles in $U_{p,s}$. Similarly, let v_r be a vertex such that the right corner $\mathcal{S}'(v_r)$ has the largest distance to the altitude of the left corner of $\mathcal{S}'(u)$. We claim that v_t and v_r form a $\preceq_{1/3}$-dominating set of $U_{p,s}$.

Let w be a vertex in $U_{p,s}$ (see Figure 8.1). We may assume that $\mathcal{S}'(w)$ has no corner in $\mathcal{S}'(v_t)$, $\mathcal{S}'(v_r)$, or $\mathcal{S}'(u)$. Then $\mathcal{S}'(w)$ contains a corner of $\mathcal{S}'(v_t)$, $\mathcal{S}'(v_r)$, and $\mathcal{S}'(u)$. Furthermore, by the choice of v_t and v_r, $\mathcal{S}'(w)$ cannot strictly contain either $\mathcal{S}'(v_t)$ or $\mathcal{S}'(v_r)$, as the top or right corner of $\mathcal{S}'(w)$ would be further from the altitude than the top or right corner of $\mathcal{S}'(v_t)$ or $\mathcal{S}'(v_r)$ respectively.

Observe that there must be a side of $\mathcal{S}'(w)$ such that p is at least as far from this side as the centroid of $\mathcal{S}'(w)$. Suppose w.l.o.g. that $\mathcal{S}'(v_r)$ protrudes this side of $\mathcal{S}'(w)$. Then the corner of $\mathcal{S}'(v_r)$ in $\mathcal{S}'(w)$ is at least as far from this side as p, and thus at least as far from the side as the centroid of $\mathcal{S}'(w)$. An easy calculation shows that $\mathcal{S}'(v_r)$ covers at least one third of the side of $\mathcal{S}'(w)$. But

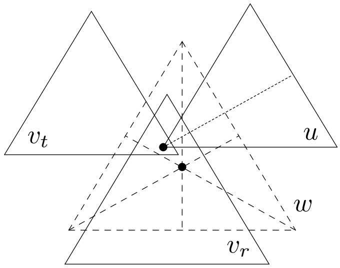

Figure 8.1: Triangles $\mathcal{S}'(u)$, $\mathcal{S}'(v_t)$, $\mathcal{S}'(v_r)$, and $\mathcal{S}'(w)$ of the proof of Lemma 8.3.3. The two dots represent p and the centroid of w. The dotted line inside $\mathcal{S}'(u)$ is the altitude of p.

then $\mathcal{S}(v_r)$ covers at least one third of a chord of $\mathcal{S}(w)$ and hence $w \preceq_{1/3} v_r$. Therefore v_t and v_r are a $\preceq_{1/3}$-dominating set of $U_{p,s}$.

As each of the r corners of the base polygon has $r-2$ sides not incident with it, U has a $\preceq_{1/3}$-dominating set of cardinality at most $2r(r-2)$. Following Lemma 8.3.1, the $\preceq_{1/3}$-factor is at most $2r(r-2)+1$. □

Combining Lemma 8.3.3 with Theorem 8.2.3, we obtain the following result.

Theorem 8.3.4 *Let $r \geq 3$ be an integer. There is a polynomial-time* $\mathrm{O}(r^4)$-*approximation algorithm for Minimum Dominating Set on intersection graphs of homothetic convex r-polygons.*

Proof: Let $G = G[\mathcal{S}]$ be the intersection graph of a collection $\mathcal{S}$ of homothetic convex r-polygons for some $r \geq 3$. Use the relation $\preceq_{1/3}$. Lemma 8.3.3 showed that the $\preceq_{1/3}$-factor is at most $2r(r-2)+1$. To hit all $\preceq_{1/3}$-larger neighbors of a vertex, place a point on each corner of the corresponding polygon and two on all chords, such that each chord is divided into three equal parts. This gives a total of $r + 2\binom{r}{2} = r^2$ points. Observe that homothetic convex polygons form a set of pseudo-disks. The theorem statement now follows from Theorem 8.1.4 and Theorem 8.2.3. □

This also implies an $\mathrm{O}(r^4)$-approximation algorithm for Minimum Connected Dominating Set on intersection graphs of homothetic convex r-polygons for $r \geq 3$ by using Proposition 6.3.24.

Another consequence of Theorem 8.3.4 is a constant-factor approximation algorithm for Minimum Dominating Set on max-tolerance (interval) graphs, because Kaufmann et al. [160] proved that max-tolerance graphs are intersection graphs of isosceles right triangles.

Using a similar proof as for Lemma 8.3.3, we can show that the fractional $\preceq_{1/3}$-factor is at most $2r(r-2)+1$. Then the following may be easily proved from Theorem 8.2.5.

Theorem 8.3.5 *Let $r \geq 3$ be an integer. The integrality gap of the standard LP for Minimum Dominating Set on intersection graphs of homothetic convex r-polygons is* $\mathrm{O}(r^4)$.

8.3.2 Regular Polygons

If the given polygons are homothetic regular polygons, then we can improve on the analysis of the previous section. We distinguish between regular polygons with an odd and with an even number of corners. Let $G = G[\mathcal{S}]$ be the intersection graph of a set $\mathcal{S}$ of homothetic odd regular polygons. Define a relation $\preceq_{1/2}$ such that for any $u, v \in V(G)$, $u \preceq_{1/2} v$ if $\mathcal{S}(v)$ contains a corner of $\mathcal{S}(u)$ or $\mathcal{S}(v)$ covers at least half of a side of $\mathcal{S}(u)$.

Lemma 8.3.6 *Let $G = G[\mathcal{S}]$ be the intersection graph of a set $\mathcal{S}$ of homothetic odd regular polygons with r corners for some odd integer $r \geq 5$. Then the $\preceq_{1/2}$-factor is at most $2r+1$.*

Proof: Let C be a dominating set such that for each $u \in C$ there is no $v \in V(G)$ for which $\mathcal{S}(u) \subset \mathcal{S}(v)$. Consider the set $U = \{v \mid v \not\preceq_{1/2} u, v \in N(u)\}$ for some $u \in V(G)$. For each corner p of $\mathcal{S}(u)$, let $U_p = \{v \mid v \in U, p \in \mathcal{S}(v)\}$ be the set of vertices in U for which the corresponding polygon contains p. Because $\mathcal{S}(u)$ does not contain a corner of $\mathcal{S}(v)$ for any $v \in U_p$ and the polygons are odd and regular, $\mathcal{S}(u)$ protrudes the same side of each $\mathcal{S}(v)$ with $v \in U_p$.

Similar to Lemma 8.3.3, let v_t and v_b be two vertices for which this side of the corresponding polygons extends furthest in either direction. Then any $\mathcal{S}(w)$ with $w \in U$ is at most twice as large as $\mathcal{S}(v_t)$ or $\mathcal{S}(v_b)$, or this would contradict the choice of v_t or v_b. We may assume that $\mathcal{S}(v_t)$ and $\mathcal{S}(v_b)$ contain no corner of $\mathcal{S}(w)$, otherwise $w \preceq_{1/2} v_t$ or $w \preceq_{1/2} v_b$. Since $\mathcal{S}(w)$ intersects $\mathcal{S}(v_t)$ and $\mathcal{S}(v_b)$, the largest of $\mathcal{S}(v_t)$ and $\mathcal{S}(v_b)$ covers at least half of a side of $\mathcal{S}(w)$. Hence $w \preceq_{1/2} v_t$ or $w \preceq_{1/2} v_b$. But then v_t and v_b form a $\preceq_{1/2}$-dominating set of U. It follows immediately from Lemma 8.3.1 that the $\preceq_{1/2}$-factor is at most $2r+1$. $\square$

Theorem 8.3.7 *Let $r \geq 3$ be an odd integer. There is a polynomial-time $\mathrm{O}(r^2)$-approximation algorithm for Minimum Dominating Set on intersection graphs of homothetic regular r-polygons.*

Proof: The case when $r = 3$ follows from Theorem 8.3.4. So let $G = G[\mathcal{S}]$ be the intersection graph of a set $\mathcal{S}$ of homothetic regular r-polygons for some odd integer $r \geq 5$. Observe that all $\preceq_{1/2}$-larger neighbors of a $u \in V(G)$ can be hit by the corners of $\mathcal{S}(u)$ and the midpoint of each side. Then Theorem 8.1.4 and Theorem 8.2.3 immediately give the theorem. $\square$

Furthermore, we can adapt Lemma 8.3.6 to bound the fractional $\preceq_{1/2}$-factor. Therefore the integrality gap of the standard LP of Minimum Dominating Set on intersection graphs of homothetic regular r-polygons for odd integers $r \geq 3$ is $\mathrm{O}(r^2)$ as well by Theorem 8.2.5.

For homothetic even regular polygons, we use a completely different relation to improve on the approximation ratio attained by the algorithm of Theorem 8.3.4. We require the following consequence of Lemma 8.3.1. A binary relation $\preceq$ is a *preorder* if it is both reflexive and transitive. It is *total* if $u \preceq v$ or $v \preceq u$ for any pair u, v.

Lemma 8.3.8 *Let $\preceq$ be a total preorder on the vertices of a graph G such that for any $u \in V(G)$ the cardinality of any independent set of $N_{\preceq}(u)$ is bounded by c. Then the $\preceq$-factor is at most $c + 1$.*

Proof: Find a $\preceq$-dominating set of $N_{\preceq}(u)$ as follows. Since $\preceq$ is a total preorder, there is a $v \in N_{\preceq}(u)$ that is maximum, i.e. $w \preceq v$ for each $w \in N_{\preceq}(u)$. Observe that v $\preceq$-dominates $N(v) \cap N_{\preceq}(u)$. Now remove $N[v]$ from $N_{\preceq}(u)$ and iterate. This yields a $\preceq$-dominating set of $N_{\preceq}(u)$ that is also an independent set. Hence it has cardinality at most c. It follows from Lemma 8.3.1 that the $\preceq$-factor is at most $c + 1$. □

A similar lemma can be proved for the fractional $\preceq$-factor.

Now let $G = G[\mathcal{S}]$ be the intersection graph of a collection $\mathcal{S}$ of homothetic regular r-polygons for some even integer $r \geq 2$. Define a total preorder $\preceq_{\mathsf{Leb}}$ on $V(G)$ such that $u \preceq_{\mathsf{Leb}} v$ for $u, v \in V(G)$ if the Lebesgue measure of $\mathcal{S}(u)$ is at most the Lebesgue measure of $\mathcal{S}(v)$.

Lemma 8.3.9 *Let $G = G[\mathcal{S}]$ be the intersection graph of a collection $\mathcal{S}$ of homothetic convex compact sets in $\mathbb{R}^2$. Then the $\preceq_{\mathsf{Leb}}$-factor is at most 5 if $\mathcal{S}$ is a collection of homothetic parallelograms and at most 6 otherwise.*

Proof: Following Lemma 8.3.8, it suffices to bound the cardinality of any independent set of $N_{\preceq_{\mathsf{Leb}}}(u)$ for each $u \in V(G)$ by 4 and 5 respectively. So for some $u \in V(G)$, define a set $\mathcal{S}' = \{\mathcal{S}'(v) \mid v \in N_{\preceq_{\mathsf{Leb}}}[u]\}$ of translated copies of $\mathcal{S}(u)$ such that $\mathcal{S}'(v) \subseteq \mathcal{S}(v)$ and $\mathcal{S}'(v) \cap \mathcal{S}'(u) \neq \emptyset$ for each $v \in N_{\preceq_{\mathsf{Leb}}}[u]$. An independent set of $N_{\preceq_{\mathsf{Leb}}}(u)$ corresponds to one of $G[\mathcal{S}']$, and vice versa.

We now apply a result of Kim, Kostochka, and Nakprasit [167], who showed that if H is the intersection graph of a set of translated copies of a fixed convex compact set in the plane with $\omega(H) \geq 2$, then the maximum degree of H is at most $4\omega(H) - 4$ if this fixed set is a parallelogram and at most $6\omega(H) - 7$ otherwise, where $\omega(H)$ is the cardinality of a maximum clique of H. Let H' be the subgraph of $G[S']$ induced by u and any independent set of $G[\mathcal{S}']$ (i.e. of $N_{\preceq_{\mathsf{Leb}}}(u)$). Then $\omega(H') = 2$ and thus the degree of u in H' is bounded by 4 and 5 respectively. The lemma follows. □

Note that the bounds of this lemma are tight, as demonstrated by a suitable representation of $K_{1,5}$ and $K_{1,6}$ respectively.

Theorem 8.3.10 *Let $r \geq 2$ be an even integer. There is a polynomial-time $\mathrm{O}(r)$-approximation algorithm for Minimum Dominating Set on intersection graphs of homothetic regular r-polygons.*

Proof: Use the relation $\preceq_{\mathsf{Leb}}$. Lemma 8.3.9 proved that the $\preceq_{\mathsf{Leb}}$-factor is at most 6. All $\preceq_{\mathsf{Leb}}$-larger neighbors of a vertex can be hit by placing a point on each corner of the corresponding polygon. The theorem statement then follows from Theorem 8.1.4 and Theorem 8.2.3. $\square$

It follows from Theorem 8.2.5 that the integrality gap of the standard LP of Minimum Dominating Set is $\mathrm{O}(r)$ on intersection graphs of homothetic regular r-polygons for even integers $r \geq 2$.

Note that although the algorithm of Theorem 8.3.10 also applies to Minimum Dominating Set on intersection graphs of homothetic regular 2-polygons (i.e. interval graphs), a linear-time exact algorithm exists in this case [61] and the integrality gap of the standard LP is 1 [47].

8.3.3 More General Objects

Observe that the proof of Theorem 8.3.10 goes through for arbitrary homothetic parallelograms. In fact, we can extend Theorem 8.3.7 and Theorem 8.3.10 to the following theorem. An *affine regular polygon* is any polygon that can be obtained from a regular polygon by an invertible affine transformation.

Theorem 8.3.11 *For any integer $r \geq 2$, there is a polynomial-time approximation algorithm for Minimum Dominating Set on intersection graphs of homothetic affine regular r-polygons, attaining approximation ratio $\mathrm{O}(r)$ if r is even and $\mathrm{O}(r^2)$ otherwise.*

Proof: Let $\mathcal{S}$ be a collection of homothetic affine regular r-polygons for some $r \geq 2$. Apply the inverse affine transformation to transform $\mathcal{S}$ into a collection $\mathcal{S}'$ of homothetic regular r-polygons and note that $G[\mathcal{S}] = G[\mathcal{S}']$. The theorem statement is now immediate from Theorem 8.3.7 and Theorem 8.3.10. $\square$

A consequence of this result is a constant-factor approximation algorithm for intersection graphs of homothetic rectangles. By placing a mild restriction on the type of rectangles, we can drop the homotheticity constraint.

We consider intersection graphs of axis-parallel rectangles whose aspect-ratio is constant. The *aspect-ratio* of a rectangle is the length of its longest side divided by the length of its shortest side.

Lemma 8.3.12 *Let $\mathcal{S}$ be a collection of axis-parallel rectangles with aspect-ratio at most c for some $c \geq 1$. Then for any $\epsilon > 0$, one can obtain a binary ϵ-net of size $\mathrm{O}(c/\epsilon)$ in time polynomial in $|\mathcal{S}|$ and c/ϵ.*

Proof: Construct a set of homothetic squares $\mathcal{S}'$ by replacing each rectangle $s \in \mathcal{S}$ by at most c axis-parallel squares, the union of which is precisely s. Now use Theorem 8.1.4 to find an ϵ'-net for $\mathcal{S}'$, where $\epsilon' = \epsilon/c$. $\square$

Theorem 8.3.13 *For any integer $c \geq 1$, there is a polynomial-time $\mathrm{O}(c^3)$-approximation algorithm for Minimum Dominating Set on intersection graphs of axis-parallel rectangles with aspect-ratio at most c.*

Proof: Let $G = G[\mathcal{S}]$ be the intersection graph of a collection $\mathcal{S}$ of axis-parallel rectangles with aspect-ratio at most c, for some integer $c \geq 1$. Consider a $\preceq_{\mathsf{Leb}}$-larger neighbor v of some vertex u. Without loss of generality, $\mathcal{S}(u)$ is a $1 \times c$ rectangle. Then all sides of $\mathcal{S}(v)$ have length at least 1 and $\mathcal{S}(v)$ contains a corner of $\mathcal{S}(u)$ or covers at least a $1/c$-fraction of a long side of $\mathcal{S}(u)$. Hence $2c + 2$ points in $\mathcal{S}(u)$ suffice to hit all $\preceq_{\mathsf{Leb}}$-larger neighbors. But then any independent set of $N_{\preceq_{\mathsf{Leb}}}(u)$ has cardinality at most $2c+2$ and the $\preceq_{\mathsf{Leb}}$-factor is at most $2c+3$ by Lemma 8.3.8. The theorem now follows from Lemma 8.3.12 and Theorem 8.2.3. □

The result of Theorem 8.3.13 does not seem to extend to similarly defined variants of regular pentagons, regular hexagons, or other regular polygons.

To show that Theorem 8.2.3 may also be applied beyond two dimensions, we prove the following theorem about Minimum Dominating Set on intersection graphs of translated copies of an affine three-dimensional box. We should note that the results of Section 6.3.3 imply the existence of a ptas for this case.

Theorem 8.3.14 *There exists a constant-factor approximation algorithm for Minimum Dominating Set on intersection graphs of translated copies of an affine three-dimensional box.*

Proof: Using the idea of the proof of Theorem 8.3.11, we may assume that we are given the intersection graph $G = G[\mathcal{S}]$ of a set $\mathcal{S}$ of translated copies of a three-dimensional box. It is easy to see that the $\preceq_{\mathsf{Leb}}$-factor is at most 9 by Lemma 8.3.8 and that any $\preceq_{\mathsf{Leb}}$-larger neighborhood can be hit by 8 points. Hence, following a result by Laue [188] (see Theorem 8.1.5), we may apply Theorem 8.2.3 with a linear function g and the theorem follows. □

Since Theorem 8.1.5 applies to translated copies of any fixed three-dimensional polytope, it seems likely that the above theorem could be extended to more general or more complex three-dimensional objects.

8.4 Disk Graphs of Bounded Ply

The obvious class of intersection graphs missing in the above discussion is the class of disk graphs. We proved in Chapter 6 that Minimum Dominating Set has a ptas on unit disk graphs, but this scheme does not carry over to general disk graphs. The ideas developed in Chapter 6 also seem to be insufficient to handle this problem. Finally, even though the $\preceq_{\mathsf{Leb}}$-factor is at most 6 for disk graphs, we do not know how to apply Theorem 8.2.3. The problem (when using $\preceq_{\mathsf{Leb}}$) is that we cannot choose a constant number of points inside a disk to hit all $\preceq_{\mathsf{Leb}}$-larger neighbors. All $\preceq_{\mathsf{Leb}}$-larger neighbors of a disk can be hit by a constant number of points, but some would have to lie outside the disk. Unfortunately, Theorem 8.2.3 does not seem to extend to this case.

If we know however that the ply of the set of disks representing the disk graph is bounded, then the above techniques do work and we obtain a constant-factor approximation algorithm. We give these algorithms below, in order of descending approximation ratio. Recall that the *ply* of a set of objects is the maximum over all points p of the number of objects strictly containing p.

8.4.1 Ply-Dependent Approximation Ratio

The approximation ratio of the first approximation algorithms we present depend (linearly) on the ply of the set of disks representing the disk graph.

Lemma 8.4.1 *Given a set of disks of ply γ, the cardinality of the closed $\preceq_{\mathsf{Leb}}$-larger neighborhood of any disk is at most 9γ.*

The proof uses an area bound in a manner similar to Lemma 7.1.1 (see also Miller et al. [210]). We can now immediately prove the following result.

Theorem 8.4.2 *There is a polynomial-time $\mathrm{O}(\gamma)$-approximation algorithm for Minimum Dominating Set on disk graphs of ply γ.*

Proof: By Lemma 8.4.1, all $\preceq_{\mathsf{Leb}}$-larger neighbors of a disk can be hit by at most 9γ points. Lemma 8.3.9 shows that the $\preceq_{\mathsf{Leb}}$-factor is at most 6. The theorem now follows from Theorem 8.1.4 and Theorem 8.2.3. □

A different technique improves on Theorem 8.4.2. We essentially give a second general approach to approximate Minimum Dominating Set using $\preceq$-dominating sets, but this time without using ϵ-nets.

Theorem 8.4.3 *Let G be a graph and let $\preceq$ be a binary reflexive relation on the vertices of G with fractional $\preceq$-factor at most c_3. Suppose that the maximum cardinality of the $\preceq$-larger closed neighborhood of any $u \in V(G)$ is at most c_2. Then the integrality gap of the standard LP for Minimum Dominating Set on G is at most c_2c_3.*

Proof: From Definition 8.2.4, the integrality gap of (the standard LP for) Minimum $\preceq$-Dominating Set on G multiplied by the fractional $\preceq$-factor is an upper bound to the integrality gap of (the standard LP for) Minimum Dominating Set on G. By assumption, the fractional $\preceq$-factor is at most c_3. Hence it suffices to bound the integrality gap of Minimum $\preceq$-Dominating Set on G.

We transform the minimum $\preceq$-dominating set problem on G to an instance of Minimum Set Cover. Let $\mathbb{U} = V(G)$ and $\mathcal{S} = \{\mathcal{S}(v) \mid v \in V(G)\}$ where $\mathcal{S}(v) = \{u \mid v \in N_{\preceq}[u]\}$. Hochbaum [148] showed that the integrality gap of Minimum Set Cover is bounded by the element frequency. The element frequency of $(\mathbb{U}, \mathcal{S})$ is at most the maximum cardinality of any $\preceq$-larger closed neighborhood of G, which is at most c_2 by assumption.

Observe that a (fractional) $\preceq$-dominating set of G corresponds directly to a (fractional) set cover of $(\mathbb{U}, \mathcal{S})$ and vice versa. Hence the integrality gap of Minimum $\preceq$-Dominating Set on G is at most c_2. This gives a bound on the integrality gap of Minimum Dominating Set on G of $c_2 c_3$. $\square$

Theorem 8.4.4 *The integrality gap of the standard LP for Minimum Dominating Set on disk graphs of ply γ is at most 54γ. If $\gamma = 1$, then the gap is at most 42. Hence the gap on planar graphs is at most 42.*

Proof: By Lemma 8.3.9, the fractional $\preceq_{\mathsf{Leb}}$-factor is at most 6. The maximum cardinality of any $\preceq_{\mathsf{Leb}}$-larger closed neighborhood of G is at most 9γ by Lemma 8.4.1. Hence the gap is at most 54γ by Theorem 8.4.3. If $\gamma = 1$, then the maximum cardinality of any $\preceq_{\mathsf{Leb}}$-larger closed neighborhood of $\mathcal{S}$ is at most 7, yielding the bound of 42 on the gap. As planar graphs are disks graphs of ply 1 [169, 210], the gap on planar graphs is at most 42. $\square$

Although a ptas for Minimum Dominating Set on planar graphs is known [22], we are not aware of any previous results on the integrality gap of the standard LP for Minimum Dominating Set on this class of graphs.

The reduction from Minimum $\preceq$-Dominating Set to Minimum Set Cover given in the proof of Theorem 8.4.3 can be exploited algorithmically.

Theorem 8.4.5 *Let G be a graph and let $\preceq$ be a binary reflexive relation on the vertices of G with $\preceq$-factor at most c_1. Suppose that the maximum cardinality of the $\preceq$-larger closed neighborhood of any $u \in V(G)$ is at most c_2. Then there is a linear-time $c_1 c_2$-approximation algorithm for Minimum Dominating Set on G.*

Proof: Transform the minimum $\preceq$-dominating set instance on G to an instance of Minimum Set Cover, as in Theorem 8.4.3. Bar-Yehuda and Even [24] proved that Minimum Set Cover has a linear-time approximation algorithm with approximation ratio at most the maximum element frequency. Following the proof of Theorem 8.4.3, the maximum element frequency is at most c_2. As the $\preceq$-factor is at most c_1, the theorem follows. $\square$

Using the proof of Theorem 8.4.4, we can then show the following.

Theorem 8.4.6 *There exists a linear-time (54γ)-approximation algorithm for Minimum Dominating Set on disk graphs of ply γ.*

Note that the approximation ratio improves to 42 on disk graphs of ply 1, i.e. on planar graphs.

Theorem 8.4.3 and 8.4.5 also have implications for Minimum Dominating Set on general graphs. Following Lemma 8.3.1 and 8.3.2, the (fractional) $\preceq$-factor of any relation $\preceq$ is at most the maximum cardinality of any $\preceq$-larger closed neighborhood of G.

Corollary 8.4.7 *Let G be a graph and let $\preceq$ be a binary reflexive relation on the vertices of G. Suppose that the maximum cardinality of the $\preceq$-larger closed neighborhood of any $u \in V(G)$ is at most c. Then the integrality gap of the standard LP for Minimum Dominating Set on G is at most c^2. Moreover, there is a linear-time c^2-approximation algorithm for Minimum Dominating Set on G.*

Clearly, $c \leq \Delta(G)$ for any relation $\preceq$, yielding an integrality gap of $\Delta^2(G)$ and a linear-time $\Delta^2(G)$-approximation algorithm for Minimum Dominating Set on any graph G. This is far worse than the $(1+\ln \Delta(G))$-approximation algorithm for Minimum Dominating Set known in the literature [156, 197, 66, 149]. One could however imagine that a relation $\preceq$ for which c is minimum over all relations $\preceq$ beats this bound.

Theorem 8.4.8 *Let G be a graph. We can find in polynomial time a binary reflexive relation $\preceq$ such that the maximum cardinality of any $\preceq$-larger closed neighborhood of G is minimized.*

Proof: First note that there is an asymmetric binary reflexive relation $\preceq$ attaining the minimum. Now observe that an asymmetric binary reflexive relation $\preceq$ on G corresponds to an orientation $\vec{G}$ of G and vice versa. Simply let $u \preceq v$ if and only if there is a directed edge from u to v in $\vec{G}$. Hence it suffices to find an orientation $\vec{G}$ of G minimizing the maximum out-degree of any vertex. Using a result of Frank and Gyárfás [111], such an orientation can be found in polynomial time. $\square$

If an upper bound to the maximum cardinality of any $\preceq$-larger closed neighborhood of G is known for some relation $\preceq$, then we can bound the approximation ratio of the algorithm of Corollary 8.4.7.

Theorem 8.4.9 *There exists a linear-time $(9\gamma)^2$-approximation algorithm for Minimum Dominating Set on disk graphs of ply γ, even if no representation of the graph is given.*

Proof: By Lemma 8.4.1, a disk graph G of ply γ has a binary reflexive relation $\preceq$ for which the maximum cardinality of any $\preceq$-larger closed neighborhood of G is at most 9γ, namely $\preceq_{\mathsf{Leb}}$. The theorem follows from Theorem 8.4.8 and Corollary 8.4.7. $\square$

Note that to apply the approximation algorithm, one does not need to know the ply of the given disk graph. The fact that the graph has a disk representation of ply γ only turns up in the analysis of the approximation factor.

8.4.2 A Constant Approximation Ratio

We can improve the approximation ratio further by using the shifting technique. We show that Minimum $\preceq_{\mathsf{Leb}}$-Dominating Set on n-vertex disk graphs of bounded ply, i.e. of ply $\gamma = \gamma(n) = \mathrm{o}(\log n)$, has an eptas. Because the $\preceq_{\mathsf{Leb}}$-factor is at most 6, this implies the existence of a $(6+\epsilon)$-approximation algorithm for Minimum Dominating Set on such disk graphs.

We use the shifting technique in the way outlined in Chapter 7. Assume that we are given a set of disks $\mathcal{D}$ such that the smallest disk has radius 1/2. We aim to find a small $\preceq_{\mathsf{Leb}}$-dominating set of $G = G[\mathcal{D}]$.

Partition the disks into levels. A disk of radius r has level j ($j \in \mathbb{Z}_{\geq 0}$) if $2^{j-1} \leq r < 2^j$. The level of the largest disk is denoted by l. The set $\mathcal{D}_{=j}$ is defined as the set of disks in $\mathcal{D}$ having level j. Similarly, we can define $\mathcal{D}_{\geq j}$ as the set of disks in $\mathcal{D}$ having level at least j, and so on.

For each level j, define a grid induced by horizontal lines $y = hk2^j$ and vertical lines $x = vk2^j$ ($h, v \in \mathbb{Z}$) for some odd integer $k \geq 7$, whose value we determine later. The grid formed in this way partitions the plane into squares of size $k2^j \times k2^j$, called *j-squares*. Furthermore, any j-square is contained in precisely one $(j+1)$-square and each $(j+1)$-square contains exactly four j-squares, denoted by $S_1, \ldots, S_4$. These four squares are *siblings* of each other. The set of disks intersecting a j-square S is denoted by $\mathcal{D}^S$, while the set of disks intersecting the boundary of S is denoted by $\mathcal{D}^{\mathrm{b}(S)}$. Similarly, $\mathcal{D}^{\mathrm{i}(S)} = \mathcal{D}^S - \mathcal{D}^{\mathrm{b}(S)}$ is the set of disks fully contained in the interior of S, $\mathcal{D}^{\mathrm{c}(S)}$ denotes the set of disks whose center is contained in S, and $\mathcal{D}^{+(S)} = \bigcup_{i=1}^4 \mathcal{D}^{\mathrm{b}(S_i)} - \mathcal{D}^{\mathrm{b}(S)}$ is the set of disks intersecting the boundary of at least one of the four children of S, but not the boundary of S itself. The meaning of combinations such as $\mathcal{D}^{\mathrm{b}(S)}_{\leq j}$ should be self-explaining. The level of a square S is denoted by $j(S)$.

Similarly, let $\mathcal{D}^{\mathrm{or}(S)}$ denote the set of disks having their center outside a j-square S and intersecting a band of width 2^j along the outer boundary of S. This band is called the *outer ring* of S. We also define several *inner rings*. Let $\mathcal{D}^{\mathrm{ir}_{j'}(S)} \subseteq \mathcal{D}^{\mathrm{c}(S)}$ denote the set of disks having their center inside S and intersecting a band of width $2^{j'}$ along the inner boundary of S. Observe that this implies that $\mathcal{D}^{\mathrm{ir}_{j(S)+\lceil \log k \rceil}(S)} = \mathcal{D}^{\mathrm{c}(S)}$. For convenience, we also define $\mathcal{D}^{\overline{\mathrm{ir}}(S)} = \bigcup_{j' \geq 0} \mathcal{D}^{\mathrm{ir}_{j'}(S)}_{\geq j'} = \bigcup_{j' \geq 0} \mathcal{D}^{\mathrm{ir}_{j'}(S)}_{=j'}$. Now define $\mathcal{D}^{\overline{+\mathrm{r}}(S)} = \bigcup_{i=1}^4 \mathcal{D}^{\overline{\mathrm{ir}}(S_i)} - \mathcal{D}^{\overline{\mathrm{ir}}(S)}$, extending the notion of $\mathcal{D}^{+(S)}$ we had before.

We now prove the following auxiliary theorem. Let $\mathcal{D}$ be a set of disks of ply γ and let OPT be a $\preceq_{\mathsf{Leb}}$-dominating set of $\mathcal{D}$ of minimum cardinality.

Theorem 8.4.10 *Let $\mathcal{D}$ be a set of disks of ply γ and $k \geq 7$ an odd positive integer. Then in $\mathrm{O}(k^2 n^2\, 2^{(80k-68)\gamma/\pi} 3^{(64k-60)\gamma/\pi})$ time, we can find a $\preceq_{\mathsf{Leb}}$-dominating set C of $\mathcal{D}$ such that $|C| \leq \sum_S \left(\left| OPT^{\mathrm{c}(S)}_{=j(S)} \right| + \left| OPT^{\mathrm{or}(S)}_{=j(S)} \right| \right)$, where the sum is over all squares S.*

We perform bottom-up dynamic programming on the j-squares. Observe that for each j-square S, disks in $\mathcal{D}^{\mathrm{c}(S)}_{\leq j}$ can be $\preceq_{\mathsf{Leb}}$-dominated by disks in $\mathcal{D}^{\mathrm{c}(S)}$ and $\mathcal{D}^{\mathrm{or}(S)}$. Following the approach developed in Chapter 7, we consider the status of disks in $\mathcal{D}^{\mathrm{or}(S)}_{>j}$. However, the outer ring of a j-square might partially overlap sibling j-squares, creating a problem when 'gluing' results together. Therefore we also consider the status of disks in the inner ring(s).

During the dynamic programming, we compute a $\preceq_{\mathsf{Leb}}$-dominating set of $\mathcal{D}^{\mathrm{c}(S)} - \mathcal{D}^{\overline{\mathrm{ir}}(S)}_{>j}$, given the status of disks in $\mathcal{D}^{\mathrm{or}(S)}_{>j} \cup \mathcal{D}^{\overline{\mathrm{ir}}(S)}_{>j}$ and using disks in $\mathcal{D}^{\mathrm{c}(S)} - \mathcal{D}^{\overline{\mathrm{ir}}(S)}_{>j}$ and $\mathcal{D}^{\mathrm{or}(S)}_{\leq j}$. A disk in $\mathcal{D}^{\mathrm{or}(S)}_{>j}$ is either in the dominating set, or it is not. A disk in $\mathcal{D}^{\overline{\mathrm{ir}}(S)}_{>j}$ has three possible statuses: either it is in the dominating set, or it is $\preceq_{\mathsf{Leb}}$-dominated by a disk in $\mathcal{D}^{\mathrm{or}(S)}_{>j} \cup \mathcal{D}^{\mathrm{c}(S)}_{>j}$, or it is $\preceq_{\mathsf{Leb}}$-dominated by a yet undetermined disk. We define for each j-square S and any two disjoint sets $W_1 \subseteq \mathcal{D}^{\mathrm{or}(S)}_{>j} \cup \mathcal{D}^{\overline{\mathrm{ir}}(S)}_{>j}$ and $W_2 \subseteq \mathcal{D}^{\overline{\mathrm{ir}}(S)}_{>j}$ the function $\mathsf{size}(S, W_1, W_2)$ as

$$\min\left\{ |T| \;\middle|\; T \subseteq \mathcal{D}^{\mathrm{or}(S)}_{=j} \cup \left(\mathcal{D}^{\mathrm{c}(S)} - \mathcal{D}^{\overline{\mathrm{ir}}(S)}_{>j}\right); \right.$$
$$\left. W_1 \cup T \preceq_{\mathsf{Leb}}\text{-dominates } W_2 \cup \left(\mathcal{D}^{\mathrm{c}(S)} - \mathcal{D}^{\overline{\mathrm{ir}}(S)}_{>j}\right)\right\}$$

if $j = 0$ and

$$\min\left\{ |U| + \sum_{i=1}^{4} \mathsf{size}\left(S_i, \left(W_1 \cup U\right) \cap \left(\mathcal{D}^{\mathrm{or}(S_i)}_{>j-1} \cup \mathcal{D}^{\overline{\mathrm{ir}}(S_i)}_{>j-1}\right), X_i\right) \;\middle|\; \right.$$
$$U \subseteq \mathcal{D}^{\overline{+\mathrm{r}}(S)}_{>j-1} \cup \mathcal{D}^{\mathrm{or}(S)}_{=j} \cup \mathcal{D}^{\mathrm{ir}_j(S)}_{=j}$$
$$\left. X_i = \left(\left(W_2 \cup \mathcal{D}^{\overline{+\mathrm{r}}(S)}_{>j-1} \cup \mathcal{D}^{\mathrm{ir}_j(S)}_{=j}\right) - N_{\succeq_{\mathsf{Leb}}}\left[W_1 \cup U\right]\right) \cap \mathcal{D}^{\overline{\mathrm{ir}}(S_i)}_{>j-1} \right\}$$

if $j > 0$. Here the minimum over an empty set is ∞. Let $\mathsf{sol}(S, W_1, W_2)$ be the subset of $\mathcal{D}$ attaining $\mathsf{size}(S, W_1, W_2)$, or $\emptyset$ if $\mathsf{size}(S, W_1, W_2) = \infty$. The meaning of W_1 and W_2 is as follows. The disks in W_1 are dominators, whereas the disks in W_2 need to be $\preceq_{\mathsf{Leb}}$-dominated by disks in W_1 or $\mathcal{D}^{\mathrm{c}(S)} - \mathcal{D}^{\overline{\mathrm{ir}}(S)}_{>j}$.

Properties of the size- and sol-Functions

Functions size and sol are reasonably easy to compute, as we will show later. First, we prove that the size and sol functions attain the properties set forth in Theorem 8.4.10.

Lemma 8.4.11 $\sum_{S;\, j(S)=l} \mathsf{size}(S, \emptyset, \emptyset) \leq \sum_S \left(\left|\mathcal{C}^{\mathrm{c}(S)}_{=j(S)}\right| + \left|\mathcal{C}^{\mathrm{or}(S)}_{=j(S)}\right|\right)$*, where* $\mathcal{C}$ *is any* $\preceq_{\mathsf{Leb}}$*-dominating set.*

Proof: We apply induction on the level j and show that the following invariant holds for any j-square S:

$$\mathsf{size}\left(S, \left(\mathcal{C}^{\mathrm{or}(S)}_{>j} \cup \mathcal{C}^{\overline{\mathrm{ir}}(S)}_{>j}\right), \mathcal{D}^{\overline{\mathrm{ir}}(S)}_{>j} - \mathcal{C}^{\overline{\mathrm{ir}}(S)}_{>j} - N_{\succeq_{\mathsf{Leb}}}\left[\mathcal{C} - \mathcal{C}^{\mathrm{c}(S)} - \mathcal{C}^{\mathrm{or}(S)}\right]\right)$$
$$\leq \quad \left|\mathcal{C}^{\mathrm{c}(S)}_{>j}\right| - \left|\mathcal{C}^{\overline{\mathrm{ir}}(S)}_{>j}\right| + \sum_{S' \subseteq S} \left(\left|\mathcal{C}^{\mathrm{c}(S')}_{=j(S')}\right| + \left|\mathcal{C}^{\mathrm{or}(S')}_{=j(S')}\right|\right).$$

For $j = 0$, the invariant holds from the definition of size, as

$$\begin{aligned} |T| &\leq \left|\mathcal{C}_{=j}^{\mathrm{or}(S)}\right| + \left|\mathcal{C}^{\mathrm{c}(S)}\right| - \left|\mathcal{C}_{>j}^{\overline{\mathrm{ir}}(S)}\right| \\ &= \left|\mathcal{C}_{=j}^{\mathrm{or}(S)}\right| + \left|\mathcal{C}_{=j}^{\mathrm{c}(S)}\right| + \left|\mathcal{C}_{>j}^{\mathrm{c}(S)}\right| - \left|\mathcal{C}_{>j}^{\overline{\mathrm{ir}}(S)}\right|. \end{aligned}$$

So assume that $j > 0$ and that the invariant holds for all j-squares with $j' < j$. Then from the description of size and by applying induction,

$$\begin{aligned} &\mathsf{size}\left(S, \left(\mathcal{C}_{>j}^{\mathrm{or}(S)} \cup \mathcal{C}_{>j}^{\overline{\mathrm{ir}}(S)}\right), \mathcal{D}_{>j}^{\overline{\mathrm{ir}}(S)} - \mathcal{C}_{>j}^{\overline{\mathrm{ir}}(S)} - N_{\succeq_{\mathsf{Leb}}}\left[\mathcal{C} - \mathcal{C}^{\mathrm{c}(S)} - \mathcal{C}^{\mathrm{or}(S)}\right]\right) \\ &\leq \left|\mathcal{C}_{>j-1}^{\overline{+\mathrm{r}}(S)}\right| + \left|\mathcal{C}_{=j}^{\mathrm{or}(S)}\right| + \left|\mathcal{C}_{=j}^{\mathrm{ir}_j(S)}\right| \\ &\quad + \sum_{i=1}^{4} \mathsf{size}\left(S_i, \left(\mathcal{C}_{>j-1}^{\mathrm{or}(S_i)} \cup \mathcal{C}_{>j-1}^{\overline{\mathrm{ir}}(S_i)}\right),\right. \\ &\qquad\qquad \left.\mathcal{D}_{>j-1}^{\overline{\mathrm{ir}}(S_i)} - \mathcal{C}_{>j-1}^{\overline{\mathrm{ir}}(S_i)} - N_{\succeq_{\mathsf{Leb}}}\left[\mathcal{C} - \mathcal{C}^{\mathrm{c}(S_i)} - \mathcal{C}^{\mathrm{or}(S_i)}\right]\right) \\ &\leq \left|\mathcal{C}_{>j-1}^{\overline{+\mathrm{r}}(S)}\right| + \left|\mathcal{C}_{=j}^{\mathrm{or}(S)}\right| + \left|\mathcal{C}_{=j}^{\mathrm{ir}_j(S)}\right| \\ &\quad + \sum_{i=1}^{4}\left(\left|\mathcal{C}_{>j-1}^{\mathrm{c}(S_i)}\right| - \left|\mathcal{C}_{>j-1}^{\overline{\mathrm{ir}}(S_i)}\right|\right) + \sum_{i=1}^{4}\sum_{S_i' \subseteq S_i}\left(\left|\mathcal{C}_{=j(S_i')}^{\mathrm{c}(S_i')}\right| + \left|\mathcal{C}_{=j(S_i')}^{\mathrm{or}(S_i')}\right|\right) \\ &= \left|\mathcal{C}_{>j-1}^{\overline{+\mathrm{r}}(S)}\right| + \left|\mathcal{C}_{=j}^{\mathrm{or}(S)}\right| + \left|\mathcal{C}_{=j}^{\mathrm{ir}_j(S)}\right| \\ &\quad + \left|\mathcal{C}_{>j-1}^{\mathrm{c}(S)}\right| - \left|\mathcal{C}_{>j-1}^{\overline{\mathrm{ir}}(S)}\right| - \left|\mathcal{C}_{>j-1}^{\overline{+\mathrm{r}}(S)}\right| + \sum_{i=1}^{4}\sum_{S_i' \subseteq S_i}\left(\left|\mathcal{C}_{=j(S_i')}^{\mathrm{c}(S_i')}\right| + \left|\mathcal{C}_{=j(S_i')}^{\mathrm{or}(S_i')}\right|\right) \\ &= \left|\mathcal{C}_{>j}^{\mathrm{c}(S)}\right| - \left|\mathcal{C}_{>j}^{\overline{\mathrm{ir}}(S)}\right| + \left|\mathcal{C}_{=j}^{\mathrm{c}(S)}\right| + \left|\mathcal{C}_{=j}^{\mathrm{or}(S)}\right| \\ &\quad + \sum_{i=1}^{4}\sum_{S_i' \subseteq S_i}\left(\left|\mathcal{C}_{=j(S_i')}^{\mathrm{c}(S_i')}\right| + \left|\mathcal{C}_{=j(S_i')}^{\mathrm{or}(S_i')}\right|\right) \\ &= \left|\mathcal{C}_{>j}^{\mathrm{c}(S)}\right| - \left|\mathcal{C}_{>j}^{\overline{\mathrm{ir}}(S)}\right| + \sum_{S' \subseteq S}\left(\left|\mathcal{C}_{=j(S')}^{\mathrm{c}(S')}\right| + \left|\mathcal{C}_{=j(S')}^{\mathrm{or}(S')}\right|\right). \end{aligned}$$

The first inequality above is the crucial one. We give an explicit proof. Let $W_1 = \mathcal{C}_{>j}^{\mathrm{or}(S)} \cup \mathcal{C}_{>j}^{\overline{\mathrm{ir}}(S)}$, $W_2 = \mathcal{D}_{>j}^{\overline{\mathrm{ir}}(S)} - \mathcal{C}_{>j}^{\overline{\mathrm{ir}}(S)} - N_{\succeq_{\mathsf{Leb}}}\left[\mathcal{C} - \mathcal{C}^{\mathrm{c}(S)} - \mathcal{C}^{\mathrm{or}(S)}\right]$, and $U = \mathcal{C}_{>j-1}^{\overline{+\mathrm{r}}(S)} \cup \mathcal{C}_{=j}^{\mathrm{or}(S)} \cup \mathcal{C}_{=j}^{\mathrm{ir}_j(S)}$. We claim that the inequality holds for this choice of U.

First we show that $(W_1 \cup U) \cap \left(\mathcal{D}_{>j-1}^{\mathrm{or}(S_i)} \cup \mathcal{D}_{>j-1}^{\overline{\mathrm{ir}}(S_i)}\right) = \mathcal{C}_{>j-1}^{\mathrm{or}(S_i)} \cup \mathcal{C}_{>j-1}^{\overline{\mathrm{ir}}(S_i)}$ for any $i = 1, \ldots, 4$. Note that

$$W_1 \cup U = \mathcal{C}_{>j}^{\mathrm{or}(S)} \cup \mathcal{C}_{>j}^{\overline{\mathrm{ir}}(S)} \cup \mathcal{C}_{>j-1}^{\overline{+\mathrm{r}}(S)} \cup \mathcal{C}_{=j}^{\mathrm{or}(S)} \cup \mathcal{C}_{=j}^{\mathrm{ir}_j(S)}$$

$$
\begin{aligned}
&= \mathcal{C}^{\mathrm{or}(S)}_{>j-1} \cup \mathcal{C}^{\overline{\mathrm{ir}}(S)}_{>j-1} \cup \mathcal{C}^{\overline{+\mathrm{r}}(S)}_{>j-1} \\
&= \mathcal{C}^{\mathrm{or}(S)}_{>j-1} \cup \bigcup_{i=1}^{4} \mathcal{C}^{\overline{\mathrm{ir}}(S_i)}_{>j-1}.
\end{aligned}
$$

But then $(W_1 \cup U) \cap \left(\mathcal{D}^{\mathrm{or}(S_i)}_{>j-1} \cup \mathcal{D}^{\overline{\mathrm{ir}}(S_i)}_{>j-1}\right) = \mathcal{C}^{\mathrm{or}(S_i)}_{>j-1} \cup \mathcal{C}^{\overline{\mathrm{ir}}(S_i)}_{>j-1}$ for any $i = 1, \ldots, 4$.

For the third parameter, we observe that for any $i = 1, \ldots, 4$

$$
\begin{aligned}
X_i &= \left(\left(W_2 \cup \mathcal{D}^{\overline{+\mathrm{r}}(S)}_{>j-1} \cup \mathcal{D}^{\mathrm{ir}_j(S)}_{=j}\right) - N_{\succeq_{\mathsf{Leb}}}[W_1 \cup U]\right) \cap \mathcal{D}^{\overline{\mathrm{ir}}(S_i)}_{>j-1} \\
&= \Bigg(\left(\mathcal{D}^{\overline{\mathrm{ir}}(S)}_{>j} \cup \mathcal{D}^{\overline{+\mathrm{r}}(S)}_{>j-1} \cup \mathcal{D}^{\mathrm{ir}_j(S)}_{=j}\right) - N_{\succeq_{\mathsf{Leb}}}\left[W_1 \cup U\right] \\
&\qquad - N_{\succeq_{\mathsf{Leb}}}\left[\mathcal{C} - \mathcal{C}^{\mathrm{c}(S)} - \mathcal{C}^{\mathrm{or}(S)}\right]\Bigg) \cap \mathcal{D}^{\overline{\mathrm{ir}}(S_i)}_{>j-1} \\
&= \mathcal{D}^{\overline{\mathrm{ir}}(S_i)}_{>j-1} - N_{\succeq_{\mathsf{Leb}}}\left[W_1 \cup U\right] - N_{\succeq_{\mathsf{Leb}}}\left[\mathcal{C} - \mathcal{C}^{\mathrm{c}(S)} - \mathcal{C}^{\mathrm{or}(S)}\right] \\
&\subseteq \mathcal{D}^{\overline{\mathrm{ir}}(S_i)}_{>j-1} - \mathcal{C}^{\overline{\mathrm{ir}}(S_i)}_{>j-1} - N_{\succeq_{\mathsf{Leb}}}\left[\mathcal{C} - \mathcal{C}^{\mathrm{c}(S_i)} - \mathcal{C}^{\mathrm{or}(S_i)}\right].
\end{aligned}
$$

Because for any W and any $X_i \subseteq X_i'$, $\mathsf{size}(S_i, W, X_i) \leq \mathsf{size}(S_i, W, X_i')$, the first inequality is correct.

Since l is the level of the largest disk, for any j-square S with $j \geq l$, $\mathcal{C}^{\mathrm{or}(S)}_{>j} \cup \mathcal{C}^{\overline{\mathrm{ir}}(S)}_{>j} = \emptyset$, $\mathcal{D}^{\mathrm{c}(S)}_{>j} = \emptyset$, and $\mathcal{D}^{\overline{\mathrm{ir}}(S)}_{>j} = \emptyset$. Hence

$$
\begin{aligned}
\sum_{S;\, j(S)=l} \mathsf{size}(S, \emptyset, \emptyset) &\leq \sum_{S;\, j(S)=l} \sum_{S' \subseteq S} \left(\left|\mathcal{C}^{\mathrm{c}(S')}_{=j(S')}\right| + \left|\mathcal{C}^{\mathrm{or}(S')}_{=j(S')}\right|\right) \\
&= \sum_{S} \left(\left|\mathcal{C}^{\mathrm{c}(S)}_{=j(S)}\right| + \left|\mathcal{C}^{\mathrm{or}(S)}_{=j(S)}\right|\right).
\end{aligned}
$$

This proves the lemma. □

It follows that if OPT is a minimum $\preceq_{\mathsf{Leb}}$-dominating set, then

$$
\sum_{S;\, j(S)=l} \mathsf{size}(S, \emptyset, \emptyset) \leq \sum_{S} \left(\left|OPT^{\mathrm{c}(S)}_{=j(S)}\right| + \left|OPT^{\mathrm{or}(S)}_{=j(S)}\right|\right).
$$

Lemma 8.4.12 $\bigcup_{S;\, j(S)=l} \mathsf{sol}(S, \emptyset, \emptyset)$ *is a* $\preceq_{\mathsf{Leb}}$*-dominating set.*

Proof: For any j-square S and any two disjoint sets $W_1 \subseteq \mathcal{D}^{\mathrm{or}(S)}_{>j} \cup \mathcal{D}^{\overline{\mathrm{ir}}(S)}_{>j}$, $W_2 \subseteq \mathcal{D}^{\overline{\mathrm{ir}}(S)}_{>j}$, we claim that $W_1 \cup \mathsf{sol}(S, W_1, W_2)$ is a $\preceq_{\mathsf{Leb}}$-dominating set of $W_2 \cup (\mathcal{D}^{\mathrm{c}(S)} - \mathcal{D}^{\overline{\mathrm{ir}}(S)}_{>j})$ if $\mathsf{size}(S, W_1, W_2) \neq \infty$.

Apply induction on j. If $j = 0$, this follows trivially from the definition of size and sol. So assume that $j > 0$ and that the claim holds for all j'-squares with $j' < j$.

Suppose that $\mathsf{size}(S, W_1, W_2) \neq \infty$ for two disjoint sets $W_1 \subseteq \mathcal{D}^{\mathrm{or}(S)}_{>j} \cup \mathcal{D}^{\overline{\mathrm{ir}}(S)}_{>j}$, $W_2 \subseteq \mathcal{D}^{\overline{\mathrm{ir}}(S)}_{>j}$. Let $U \subseteq \mathcal{D}^{\overline{+\mathrm{r}}(S)}_{>j-1} \cup \mathcal{D}^{\mathrm{or}(S)}_{=j} \cup \mathcal{D}^{\mathrm{ir}_j(S)}_{=j}$ attain the minimum in the definition of size for W_1 and W_2. Because $\mathsf{size}(S, W_1, W_2) \neq \infty$, it must be that $\mathsf{size}(S_i, W^i, X_i) \neq \infty$ for $i = 1, \ldots, 4$ as well, where

$$W^i = (W_1 \cup U) \cap \left(\mathcal{D}^{\mathrm{or}(S_i)}_{>j-1} \cup \mathcal{D}^{\overline{\mathrm{ir}}(S_i)}_{>j-1}\right)$$

and

$$X_i = \left(\left(W_2 \cup \mathcal{D}^{\overline{+\mathrm{r}}(S)}_{>j-1} \cup \mathcal{D}^{\mathrm{ir}_j(S)}_{=j}\right) - N_{\succeq_{\mathsf{Leb}}}\left[W_1 \cup U\right]\right) \cap \mathcal{D}^{\overline{\mathrm{ir}}(S_i)}_{>j-1}.$$

Then by induction, $W^i \cup \mathsf{sol}(S_i, W^i, X_i)$ is a $\preceq_{\mathsf{Leb}}$-dominating set of $X_i \cup \left(\mathcal{D}^{\mathrm{c}(S_i)} - \mathcal{D}^{\overline{\mathrm{ir}}(S_i)}_{>j-1}\right)$. Observe that

$$\begin{aligned}
&\bigcup_{i=1}^{4} W^i \cup \bigcup_{i=1}^{4} \mathsf{sol}(S_i, W^i, X_i) \\
&\quad= \bigcup_{i=1}^{4} \left((W_1 \cup U) \cap \left(\mathcal{D}^{\mathrm{or}(S_i)}_{>j-1} \cup \mathcal{D}^{\overline{\mathrm{ir}}(S_i)}_{>j-1}\right)\right) \cup \bigcup_{i=1}^{4} \mathsf{sol}(S_i, W^i, X_i) \\
&\quad\subseteq W_1 \cup U \cup \bigcup_{i=1}^{4} \mathsf{sol}(S_i, W^i, X_i) \\
&\quad= W_1 \cup \mathsf{sol}(S, W_1, W_2)
\end{aligned}$$

and

$$\bigcup_{i=1}^{4} \left(X_i \cup \left(\mathcal{D}^{\mathrm{c}(S_i)} - \mathcal{D}^{\overline{\mathrm{ir}}(S_i)}_{>j-1}\right)\right) = \bigcup_{i=1}^{4} X_i \cup \left(\mathcal{D}^{\mathrm{c}(S)} - \mathcal{D}^{\overline{\mathrm{ir}}(S)}_{>j-1} - \mathcal{D}^{\overline{+\mathrm{r}}(S)}_{>j-1}\right).$$

Since $W_1 \cup \mathsf{sol}(S, W_1, W_2)$ also $\preceq_{\mathsf{Leb}}$-dominates $N_{\succeq_{\mathsf{Leb}}}[W_1 \cup U]$, we can derive that $W_1 \cup \mathsf{sol}(S, W_1, W_2)$ $\preceq_{\mathsf{Leb}}$-dominates

$$\begin{aligned}
&\bigcup_{i=1}^{4} X_i \cup \left(\mathcal{D}^{\mathrm{c}(S)} - \mathcal{D}^{\overline{\mathrm{ir}}(S)}_{>j-1} - \mathcal{D}^{\overline{+\mathrm{r}}(S)}_{>j-1}\right) \cup N_{\succeq_{\mathsf{Leb}}}[W_1 \cup U] \\
&\qquad\supseteq \bigcup_{i=1}^{4} \left(\left(W_2 \cup \mathcal{D}^{\overline{+\mathrm{r}}(S)}_{>j-1} \cup \mathcal{D}^{\mathrm{ir}_j(S)}_{=j}\right) \cap \mathcal{D}^{\overline{\mathrm{ir}}(S_i)}_{>j-1}\right) \\
&\qquad\quad \cup \left(\mathcal{D}^{\mathrm{c}(S)} - \mathcal{D}^{\overline{\mathrm{ir}}(S)}_{>j-1} - \mathcal{D}^{\overline{+\mathrm{r}}(S)}_{>j-1}\right) \\
&\qquad= W_2 \cup \left(\mathcal{D}^{\mathrm{c}(S)} - \mathcal{D}^{\overline{\mathrm{ir}}(S)}_{>j}\right).
\end{aligned}$$

From the previous lemma, we know that $\sum_{S;\, j(S)=l} \mathsf{size}(S, \emptyset, \emptyset) \neq \infty$. Hence $\bigcup_{S;\, j(S)=l} \mathsf{sol}(S, \emptyset, \emptyset)$ is a $\preceq_{\mathsf{Leb}}$-dominating set of $\bigcup_{S;\, j(S)=l} \mathcal{D}^{\mathrm{c}(S)}$. Because each disk is in $\mathcal{D}^{\mathrm{c}(S)}$ for some l-square S, this is a $\preceq_{\mathsf{Leb}}$-dominating set of $\mathcal{D}$. □

Computing the size- and sol-Functions

We apply the methods outlined in Chapter 7. We show again that it is sufficient to size and sol for a limited number of j-squares.

The definition of nonempty and empty is slightly different than usual. We say that that a j-square S is *nonempty* if S or the outer ring of S is intersected by a level j disk and *empty* otherwise.

The definition of relevant remains the same, modulo the new definition of nonempty. A j-square S is *relevant* if one of its three siblings is nonempty or there is a nonempty square S' containing S, such that S' has level at most $j + \lceil \log k \rceil$ (so each nonempty j-square is relevant). Note that this definition induces $\mathrm{O}(k^2 n)$ relevant squares. A relevant square S is said to be a *relevant child* of another relevant square S' if $S \subset S'$ and there is no third relevant square S'', such that $S \subset S'' \subset S'$. Conversely, if S is a relevant child of S', S' is a *relevant parent* of S.

Lemma 8.4.13 *For each relevant* 0*-square* S*, all* size- *and* sol*-values for* S *can be computed in* $\mathrm{O}\big(nk\gamma\, 2^{(16k+32)\gamma/\pi} 3^{(40k-12)\gamma/\pi}\big)$ *time.*

Proof: We use area bounds to bound the cardinality of sets we are interested in. By Lemma 7.1.1, $\left|\mathcal{D}^{\mathrm{or}(S)}_{>j}\right| \leq 16(k+2)\gamma/\pi$. To bound $\left|\mathcal{D}^{\overline{\mathrm{ir}}(S)}_{>j}\right|$, note that $\left|\mathcal{D}^{\mathrm{ir}_{j+1}(S)}_{\geq j+1}\right| \leq (20k-60)\gamma/\pi$ and that for any $j' > j$, $\left|\mathcal{D}^{\mathrm{ir}_{j'}(S)}_{\geq j'} - \mathcal{D}^{\mathrm{ir}_{j'-1}(S)}_{\geq j'-1}\right| \leq (3 \cdot 2^{j-j'+3}k - 60)\gamma/\pi$. Hence

$$\left|\mathcal{D}^{\overline{\mathrm{ir}}(S)}_{>j}\right| \leq \left(\left(20 + 3 \sum_{j'=j+2}^{\infty} 2^{j-j'+3}\right)k - 60\right)\gamma/\pi \leq (32k-60)\gamma/\pi.$$

Therefore we can enumerate all disjoint sets $W_1 \subseteq \mathcal{D}^{\mathrm{or}(S)}_{>j} \cup \mathcal{D}^{\overline{\mathrm{ir}}(S)}_{>j}$, $W_2 \subseteq \mathcal{D}^{\overline{\mathrm{ir}}(S)}_{>j}$ in $\mathrm{O}\big(2^{(16k+32)\gamma/\pi} 3^{(32k-60)\gamma/\pi}\big)$ time.

Using Lemma 7.1.2, the pathwidth of $\mathcal{D}^{\mathrm{or}(S)}_{=j} \cup \big(\mathcal{D}^{\mathrm{c}(S)} - \mathcal{D}^{\overline{\mathrm{ir}}(S)}_{>j}\big)$ can be bounded by $8(k+6)\gamma/\pi$. By adapting the algorithm of Corollary 5.3.9, we can find the set T required by the definition of size and sol in $\mathrm{O}(nk\gamma\, 3^{(8k+48)\gamma/\pi})$ time. The lemma follows. □

Now assume that the size- and sol-values of all relevant children of a j-square S are known.

Lemma 8.4.14 *For each relevant* j*-square* S $(j > 0)$ *with relevant* $(j-1)$*-square children, in* $\mathrm{O}(2^{(80k-68)\gamma/\pi} 3^{(64k-60)\gamma/\pi})$ *time all* size- *and* sol*-values for* S *can be computed.*

Proof: Using similar ideas as in Lemma 8.4.13 and Lemma 7.1.1, we can show that $\left|\mathcal{D}^{\mathrm{or}(S)}_{\geq j}\right| \leq (40k+60)\gamma/\pi$ and

$$\left|\mathcal{D}^{\overline{\mathrm{ir}}(S)}_{\geq j}\right| \leq \left(\left(40 + 3 \sum_{j'=j+1}^{\infty} 2^{j-j'+3}\right)k - 60\right)\gamma/\pi \leq (64k-60)\gamma/\pi.$$

Now bound $\left|\mathcal{D}_{\geq j}^{\overline{+\mathrm{r}}(S)}\right|$. Note that $\left|\bigcup_{i=1}^{4}\mathcal{D}_{\geq j}^{\mathrm{ir}_j(S_i)} - \mathcal{D}_{\geq j}^{\mathrm{ir}_j(S)}\right| \leq (32k-128)\gamma/\pi$ and for any $j' > j$,

$$\begin{aligned}\left|\left(\bigcup_{i=1}^{4}\mathcal{D}_{\geq j'}^{\mathrm{ir}_{j'}(S_i)} - \mathcal{D}_{\geq j'}^{\mathrm{ir}_{j'}(S)}\right) - \left(\bigcup_{i=1}^{4}\mathcal{D}_{\geq j'-1}^{\mathrm{ir}_{j'-1}(S_i)} - \mathcal{D}_{\geq j'-1}^{\mathrm{ir}_{j'-1}(S)}\right)\right| \\ \leq \quad (2^{j-j'+3}k - 12)\cdot\gamma/\pi.\end{aligned}$$

Then

$$\begin{aligned}\left|\mathcal{D}_{\geq j}^{\overline{+\mathrm{r}}(S)}\right| &\leq \left(32k - 128 + \sum_{j'=j+1}^{\infty} 2^{j-j'+3}k\right)\cdot\gamma/\pi \\ &= (40k-128)\gamma/\pi.\end{aligned}$$

The lemma now follows from the definition of size and sol. □

Lemma 8.4.15 *For each relevant j-square S ($j > 0$) with no relevant children of level $j-1$, all* size- *and* sol-*values for S can be computed in* $\mathrm{O}(n\,2^{44\gamma/\pi}3^{16\gamma/\pi})$ *time.*

Proof: Following the proof of Lemma 7.2.6, $\mathcal{D}_{\geq j}^{\overline{\mathrm{ir}}(S)} = \mathcal{D}_{\geq j+\lceil\log k\rceil}^{\overline{\mathrm{ir}}(S)}$. Then Lemma 7.1.1 shows that $\left|\mathcal{D}_{\geq j}^{\overline{\mathrm{ir}}(S)}\right| \leq \left|\mathcal{D}_{\geq j+\lceil\log k\rceil}^{\mathrm{c}(S)}\right| \leq 16\gamma/\pi$. Lemma 7.2.6 implies that $\left|\mathcal{D}_{\geq j}^{\overline{+\mathrm{r}}(S)}\right| = 0$ and $\mathcal{D}_{\geq j}^{\mathrm{or}(S)} = \mathcal{D}_{\geq j+\lceil\log k\rceil}^{\mathrm{or}(S)}$ and thus $\left|\mathcal{D}_{\geq j}^{\mathrm{or}(S)}\right| \leq 44\gamma/\pi$. Then from the proof of Lemma 7.2.6 and the definition of size and sol, we can compute all size- and sol-values in $\mathrm{O}(n\,2^{44\gamma/\pi}3^{16\gamma/\pi})$ time. □

Lemma 8.4.16 *The value of* $\sum_{S;\,j(S)=l}\mathsf{size}(S,\emptyset,\emptyset)$ *can be computed in time* $\mathrm{O}(k^2n^2\,2^{(80k-68)\gamma/\pi}3^{(64k-60)\gamma/\pi})$.

Proof: Follows from Lemma 8.4.13, Lemma 8.4.14, Lemma 8.4.15, and the proof of Lemma 7.2.7. The number of relevant squares is $\mathrm{O}(k^2n)$. □

Proof of Theorem 8.4.10: Follows directly from Lemmas 8.4.11, 8.4.12, and 8.4.16. □

The Approximation Algorithm

The shifting technique can now be applied as follows. For an integer a ($0 \leq a \leq k-1$), call a line of level j *active* if it has the form $y = (hk + a2^{l-j})2^j$ or $x = (vk + a2^{l-j})2^j$ ($h, v \in \mathbb{Z}$). The active lines partition the plane into j-squares as before, although shifted with respect to the shifting parameter a. However, we can still apply the algorithm of Theorem 8.4.10.

Let C_a denote the set returned by the algorithm for the j-squares induced by shifting parameter a ($0 \leq a \leq k-1$) and let $C_{\min}$ be a smallest such set.

Lemma 8.4.17 *$|C_{\min}| \leq (1+24/k)\cdot|OPT|$, where OPT is a minimum $\preceq_{\mathsf{Leb}}$-dominating set.*

Proof: Define $\mathcal{D}_a^{\mathrm{or}}$ as the set of disks intersecting the outer ring of a j-square S at their level, i.e. $\mathcal{D}_a^{\mathrm{or}} = \bigcup_S \mathcal{D}_{=j(S)}^{\mathrm{or}(S)}$. Clearly a disk of level j can be in $\mathcal{D}_a^{\mathrm{or}}$ for at most 8 values of a. Therefore $\sum_{a=0}^{k-1} |OPT \cap \mathcal{D}_a^{\mathrm{or}}| \leq 8 \cdot |OPT|$. Furthermore, for any fixed value of a, any level j disk can intersect the outer ring of at most 3 j-squares. It follows from Lemma 8.4.11 that

$$|C_a| \leq \sum_S \left(\left|\mathcal{C}_{=j(S)}^{\mathrm{c}(S)}\right| + \left|\mathcal{C}_{=j(S)}^{\mathrm{or}(S)}\right|\right) \leq |OPT| + 3|OPT \cap \mathcal{D}_a^{\mathrm{or}}|.$$

Then

$$k \cdot |C_{\min}| \leq \sum_{a=0}^{k-1} |C_a| \leq \sum_{a=0}^{k-1} (|OPT| + 3|OPT \cap \mathcal{D}_a^{\mathrm{or}}|) \leq (k+24)\cdot|OPT|.$$

Hence $|C_{\min}| \leq (1+24/k)\cdot|OPT|$. □

Combining Theorem 8.4.10 and Lemma 8.4.17, we obtain the following approximation scheme.

Theorem 8.4.18 *There is an eptas for Minimum $\preceq_{\mathsf{Leb}}$-Dominating Set on n-vertex disk graphs of bounded ply, i.e. of ply $\gamma = \gamma(n) = \mathrm{o}(\log n)$.*

Proof: Consider any $\epsilon > 0$. Choose k as the largest odd integer such that $(64k-60)\gamma/\pi \leq \log_3 n$. If $k < 7$, output $V(G)$. Otherwise, apply the algorithm of Lemma 8.4.10 and compute $C_{\min}$ in $\mathrm{O}(n^4 \log^3 n)$ time. Furthermore, if $\gamma = \gamma(n) = \mathrm{o}(\log n)$, there is a c_ϵ such that $k \geq 24/\epsilon$ and $k \geq 7$ for all $n \geq c_\epsilon$. Therefore, if $n \geq c_\epsilon$, it follows from Lemma 8.4.17 and the choice of k that $C_{\min}$ is a $(1+\epsilon)$-approximation to the optimum. Hence there is a fiptas$^\omega$ for Minimum $\preceq_{\mathsf{Leb}}$-Dominating Set on n-vertex disk graphs of bounded ply, i.e. of ply $\gamma = \gamma(n) = \mathrm{o}(\log n)$. The theorem follows from Theorem 2.2.4. □

Observe that the above theorem extends to intersection graphs of fat objects of any constant dimension and the weighted case. Because the $\preceq_{\mathsf{Leb}}$-factor is at most 6 for disk graphs, we also obtain the following result.

Theorem 8.4.19 *There is an algorithm that gives for any $\epsilon > 0$ a $(6+\epsilon)$-approximation for Minimum Dominating Set on disk graphs with n vertices and of bounded ply, i.e. of ply $\gamma = \gamma(n) = \mathrm{o}(\log n)$, in time $f(1/\epsilon)\cdot n^{\mathrm{O}(1)}$ for some computable function f of $1/\epsilon$.*

Similar constant-factor approximation algorithms exist for Minimum Dominating Set on intersection graphs of other fat objects of bounded ply, constant dimension, and constant $\preceq_{\mathsf{Leb}}$-factor. For example, a $(5+\epsilon)$-approximation algorithm on square graphs or a $(13+\epsilon)$-approximation algorithm on 3-dimensional ball graphs follow from Theorem 8.4.18.

8.4.3 A Better Constant

Although the above approach yields a constant-factor approximation algorithm for Minimum Dominating Set on disk graphs of bounded ply, we can also approximate it directly, i.e. without using $\preceq_{\mathsf{Leb}}$-dominating sets. This gives an easier algorithm with a better approximation ratio. To this end, we apply the shifting technique in a novel fashion.

Let $k \geq 9$ be an odd multiple of 3, let $\mathcal{D}$ be partitioned into levels and the plane into j-squares. We prove the following auxiliary theorem.

Theorem 8.4.20 *Let $\mathcal{D}$ be a set of disks of ply γ, $k \geq 9$ an odd multiple of 3, and OPT a minimum dominating set. Then we can obtain in time $\mathrm{O}(k^2 n^2\, 3^{32k\gamma/\pi} 2^{16k\gamma/\pi} 4^{16(k+1)\gamma/\pi})$ a set $C \subseteq \mathcal{D}$ dominating $\bigcup_S \mathcal{D}^{\mathrm{i}(S)}_{=j(S)}$ such that $|C| \leq \sum_S \left| OPT^S_{=j(S)} \right|$, where the union and sum are over all squares S.*

The set C is computed by performing bottom-up dynamic programming on the j-squares. For each j-square S, we consider each possible dominating set for $\mathcal{D}^{\mathrm{i}(S)}$, given the status of disks in $\mathcal{D}^{\mathrm{b}(S)}_{>j}$. A disk in $\mathcal{D}^{\mathrm{b}(S)}_{>j}$ can have one of three statuses: either it is a dominator, or it is dominated by a disk in $\mathcal{D}^S$, or it is dominated by a yet undetermined disk. Now define for each j-square S and any two disjoint sets $W_1, W_2 \subseteq \mathcal{D}^{\mathrm{b}(S)}_{>j}$ the function $\mathsf{size}(S, W_1, W_2)$ as

$$\begin{cases} \min\left\{ |T| \,\middle|\, T \subseteq \mathcal{D}^{\mathrm{b}(S)}_{=j} \cup \mathcal{D}^{\mathrm{i}(S)}_{\geq j};\ W_1 \cup T \text{ dominates } \mathcal{D}^{\mathrm{i}(S)}_{\geq j} \cup W_2 \right\} & \text{if } j = 0; \\ \min\Big\{ |U| + \sum_{i=1}^{4} \mathsf{size}\big(S_i, (W_1 \cup U)^{\mathrm{b}(S_i)}, X_i\big) \,\Big|\, & \\ \qquad U \subseteq \mathcal{D}^{+(S)}_{>j-1} \cup \mathcal{D}^{\mathrm{b}(S)}_{=j},\ X_i \subseteq \mathcal{D}^{\mathrm{b}(S_i)}_{>j-1} & \\ \qquad \{X_1, \ldots, X_4\} \text{ decomposes } W_2 \cup (\mathcal{D}^{+(S)}_{>j-1} - U) \Big\} & \text{if } j > 0. \end{cases}$$

Here we define the minimum over an empty set to be ∞ and we say a family of pairwise disjoint sets $\{A_1, \ldots, A_m\}$ *decomposes* (or is a *decomposition* of) some set $\mathcal{A}$ if $A_i \subseteq \mathcal{A}$ for each i and $\bigcup_i A_i = \mathcal{A}$. Note that this definition explicitly allows empty sets.

Let $\mathsf{sol}(S, W_1, W_2)$ be the subfamily of $\mathcal{D}$ attaining $\mathsf{size}(S, W_1, W_2)$, or $\emptyset$ if $\mathsf{size}(S, W_1, W_2)$ is ∞. In the function parameters, the set W_1 is used for disks of $\mathcal{D}^{\mathrm{b}(S)}_{>j}$ that will be in the dominating set, while W_2 is used to denote the subset of $\mathcal{D}^{\mathrm{b}(S)}_{>j}$ that should be dominated by a disk in D^S. Note that one actually only needs to consider sets $W_2 \subseteq \mathcal{D}^{\mathrm{b}(S)}_{>j} - N[W_1]$, but doing so would not improve the theoretical performance of the algorithm and might complicate its analysis.

Properties of the size- and sol-Functions

We start again by showing that size and sol are the functions that we need.

Lemma 8.4.21 $\sum_{S;\, j(S)=l} \mathsf{size}(S,\emptyset,\emptyset) \leq \sum_S \left|\mathcal{C}^S_{=j(S)}\right|$, *where* $\mathcal{C}$ *is any dominating set.*

Proof: We prove using induction that the following inequality holds for all j-squares S:

$$\mathsf{size}\left(S, \mathcal{C}^{\mathrm{b}(S)}_{>j}, N^{\mathrm{ii}(S)}(\mathcal{C}^S)^{\mathrm{b}(S)}_{>j}\right) \leq \left|\mathcal{C}^{\mathrm{i}(S)}_{>j}\right| + \sum_{S' \subseteq S} \left|\mathcal{C}^{S'}_{=j(S')}\right|.$$

Here $N^{\mathrm{ii}(S)}(X)$ is the set of disks $d \notin X$ such that d intersects some $d' \in X$ (i.e. $d \in N(X)$) and $d \cap d'$ intersects S.

The base case is trivial, since $\mathcal{C}^{\mathrm{b}(S)}_{>0} \cup \mathcal{C}^{\mathrm{i}(S)}_{>0} \cup \mathcal{C}^S_{=0} = \mathcal{C}^S$ clearly dominates $\mathcal{D}^{\mathrm{i}(S)}_{\geq j} \cup N^{\mathrm{ii}(S)}(\mathcal{C}^S)^{\mathrm{b}(S)}_{>j}$. For the inductive step, we can show that

$$\begin{aligned}
&\mathsf{size}\left(S, \mathcal{C}^{\mathrm{b}(S)}_{>j}, N^{\mathrm{ii}(S)}(\mathcal{C}^S)^{\mathrm{b}(S)}_{>j}\right)\\
&\qquad\leq \left|\mathcal{C}^{+(S)}_{>j-1}\right| + \left|\mathcal{C}^{\mathrm{b}(S)}_{=j}\right| + \sum_{i=1}^{4} \mathsf{size}\left(S_i, \mathcal{C}^{\mathrm{b}(S_i)}_{>j-1}, N^{\mathrm{ii}(S_i)}(\mathcal{C}^{S_i})^{\mathrm{b}(S_i)}_{>j-1}\right)\\
&\qquad\leq \left|\mathcal{C}^{+(S)}_{>j-1}\right| + \left|\mathcal{C}^{\mathrm{b}(S)}_{=j}\right| + \sum_{i=1}^{4} \left|\mathcal{C}^{\mathrm{i}(S_i)}_{>j-1}\right| + \sum_{i=1}^{4} \sum_{S'_i \subseteq S_i} \left|\mathcal{C}^{S'_i}_{=j(S'_i)}\right|\\
&\qquad= \left|\mathcal{C}^{+(S)}_{>j-1}\right| + \left|\mathcal{C}^{\mathrm{b}(S)}_{=j}\right| + \left|\mathcal{C}^{\mathrm{i}(S)}_{>j-1}\right| - \left|\mathcal{C}^{+(S)}_{>j-1}\right| + \sum_{i=1}^{4} \sum_{S'_i \subseteq S_i} \left|\mathcal{C}^{S'_i}_{=j(S'_i)}\right|\\
&\qquad= \left|\mathcal{C}^{\mathrm{i}(S)}_{>j}\right| + \left|\mathcal{C}^S_{=j}\right| + \sum_{i=1}^{4} \sum_{S'_i \subseteq S_i} \left|\mathcal{C}^{S'_i}_{=j(S'_i)}\right|\\
&\qquad= \left|\mathcal{C}^{\mathrm{i}(S)}_{>j}\right| + \sum_{S' \subseteq S} \left|\mathcal{C}^{S'}_{=j(S')}\right|.
\end{aligned}$$

The first inequality above is the crucial one and that it should hold is not obvious. We give an explicit proof.

Suppose that to obtain

$$\mathsf{size}\left(S, \mathcal{C}^{\mathrm{b}(S)}_{>j}, N^{\mathrm{ii}(S)}(\mathcal{C}^S)^{\mathrm{b}(S)}_{>j}\right)$$

using the definition of size, we consider $U = \mathcal{C}^{+(S)}_{>j-1} \cup \mathcal{C}^{\mathrm{b}(S)}_{=j}$. As for $i = 1, \ldots, 4$,

$$\begin{aligned}
(W_1 \cup U)^{\mathrm{b}(S_i)} &= \left(\mathcal{C}^{\mathrm{b}(S)}_{>j} \cup \mathcal{C}^{+(S)}_{>j-1} \cup \mathcal{C}^{\mathrm{b}(S)}_{=j}\right)^{\mathrm{b}(S_i)}\\
&= \left(\mathcal{C}^{\mathrm{b}(S)}_{>j-1} \cup \mathcal{C}^{+(S)}_{>j-1}\right)^{\mathrm{b}(S_i)}\\
&\overset{\text{by def.}}{=} \left(\bigcup_{m=1}^{4} \mathcal{C}^{\mathrm{b}(S_m)}_{>j-1}\right)^{\mathrm{b}(S_i)}\\
&= \mathcal{C}^{\mathrm{b}(S_i)}_{>j-1},
\end{aligned}$$

the second parameter of the inductive call is correct.

So what about the third parameter? We claim that

$$W_2 \cup (\mathcal{D}^{+(S)}_{>j-1} - U) \;\subseteq\; \bigcup_{i=1}^{4} N^{\mathrm{ii}(S)}(\mathcal{C}^S)^{\mathrm{b}(S_i)}_{>j-1} \;\subseteq\; \bigcup_{i=1}^{4} N^{\mathrm{ii}(S_i)}(\mathcal{C}^{S_i})^{\mathrm{b}(S_i)}_{>j-1}.$$

Because $\mathcal{C}$ is a dominating set and every disk in $\mathcal{D}^{+(S)}_{>j-1}$ must be dominated by a disk in $\mathcal{D}^S$,

$$\begin{aligned} N^{\mathrm{ii}(S)}(\mathcal{C}^S)^{+(S)}_{>j-1} &= N(\mathcal{C}^S)^{+(S)}_{>j-1} \\ &= \mathcal{D}^{+(S)}_{>j-1} - \mathcal{C}^{+(S)}_{>j-1} \\ &= \mathcal{D}^{+(S)}_{>j-1} - U. \end{aligned}$$

Then

$$\begin{aligned} W_2 \cup (\mathcal{D}^{+(S)}_{>j-1} - U) &= N^{ii(S)}(\mathcal{C}^S)^{\mathrm{b}(S)}_{>j} \cup N^{\mathrm{ii}(S)}(\mathcal{C}^S)^{+(S)}_{>j-1} \\ &\subseteq N^{\mathrm{ii}(S)}(\mathcal{C}^S)^{\mathrm{b}(S)}_{>j-1} \cup N^{\mathrm{ii}(S)}(\mathcal{C}^S)^{+(S)}_{>j-1} \\ &\overset{\text{by def.}}{=} \bigcup_{i=1}^{4} N^{\mathrm{ii}(S)}(\mathcal{C}^S)^{\mathrm{b}(S_i)}_{>j-1}, \end{aligned}$$

thus proving the first part of the claim.

To prove the second part, consider any disk d in $N^{\mathrm{ii}(S)}(\mathcal{C}^S)^{\mathrm{b}(S_i)}_{>j-1}$ for all $i \in I \subseteq \{1, \ldots, 4\}$. As $d \in N^{\mathrm{ii}(S)}(\mathcal{C}^S)$, there must be some $h \in \{1, \ldots, 4\}$ such that $d \in N^{\mathrm{ii}(S_h)}(\mathcal{C}^S)$, i.e. $d \in N^{\mathrm{ii}(S_h)}(\mathcal{C}^{S_h})$. Furthermore, it is clear that $h \in I$. But then $d \in N^{\mathrm{ii}(S_h)}(\mathcal{C}^{S_h})^{\mathrm{b}(S_h)}_{>j-1}$. This proves the claim.

Following the claim, there exists a decomposition $\{X_1, \ldots, X_4\}$ of $W_2 \cup (\mathcal{D}^{+(S)}_{>j-1} - U)$ such that $X_i \subseteq N^{\mathrm{ii}(S_i)}(\mathcal{C}^{S_i})^{\mathrm{b}(S_i)}_{>j-1}$ and hence $X_i \subseteq \mathcal{D}^{\mathrm{b}(S_i)}_{>j-1}$. Therefore for such X_i and the chosen set U,

$$\begin{aligned} &\mathsf{size}\left(S, \mathcal{C}^{\mathrm{b}(S)}_{>j}, N^{\mathrm{ii}(S)}(\mathcal{C}^S)^{\mathrm{b}(S)}_{>j}\right) \\ &\quad\le |U| + \sum_{i=1}^{4} \mathsf{size}\Big(S_i, (W_1 \cup U)^{\mathrm{b}(S_i)}, X_i\Big) \\ &\quad= \left|\mathcal{C}^{+(S)}_{>j-1}\right| + \left|\mathcal{C}^{\mathrm{b}(S)}_{=j}\right| + \sum_{i=1}^{4} \mathsf{size}\left(S_i, \mathcal{C}^{\mathrm{b}(S_i)}_{>j-1}, X_i\right) \\ &\quad\le \left|\mathcal{C}^{+(S)}_{>j-1}\right| + \left|\mathcal{C}^{\mathrm{b}(S)}_{=j}\right| + \sum_{i=1}^{4} \mathsf{size}\left(S_i, \mathcal{C}^{\mathrm{b}(S_i)}_{>j-1}, N^{\mathrm{ii}(S_i)}(\mathcal{C}^{S_i})^{\mathrm{b}(S_i)}_{>j-1}\right), \end{aligned}$$

where the last inequality follows from $X_i \subseteq N^{\mathrm{ii}(S_i)}(\mathcal{C}^{S_i})^{\mathrm{b}(S_i)}_{>j-1}$. This proves the inequality of the previous page.

We now know that

$$\mathsf{size}\left(S, \mathcal{C}^{\mathrm{b}(S)}_{>j}, N^{\mathrm{ii}(S)}(\mathcal{C}^S)^{\mathrm{b}(S)}_{>j}\right) \le \left|\mathcal{C}^{\mathrm{i}(S)}_{>j}\right| + \sum_{S' \subseteq S} \left|\mathcal{C}^{S'}_{=j(S')}\right|.$$

Since l is the level of the largest disk, $\mathcal{C}^{\mathrm{i}(S)}_{>j} = \emptyset$ and $\mathcal{C}^{\mathrm{b}(S)}_{>j} = \emptyset$, and thus $N^{\mathrm{ii}(S)}(\mathcal{C}^S)^{\mathrm{b}(S)}_{>j} = \emptyset$ for all j-squares S with $j \geq l$. Hence

$$\sum_{S;\, j(S)=l} \mathsf{size}(S, \emptyset, \emptyset) \leq \sum_{S;\, j(S)=l} \sum_{S' \subseteq S} \left|\mathcal{C}^{S'}_{=j(S')}\right| = \sum_S \left|\mathcal{C}^S_{=j(S)}\right|.$$

This proves the lemma. $\square$

It follows that if OPT is a minimum dominating set, then

$$\sum_{S;\, j(S)=l} \mathsf{size}(S, \emptyset, \emptyset) \leq \sum_S \left|OPT^S_{=j(S)}\right|.$$

Lemma 8.4.22 $\bigcup_{S;\, j(S)=l} \mathsf{sol}(S, \emptyset, \emptyset)$ *is a dominating set for* $\bigcup_S \mathcal{D}^{\mathrm{i}(S)}_{=j(S)}$.

Proof: For any j-square S and any two disjoint $W_1, W_2 \subseteq \mathcal{D}^{\mathrm{b}(S)}_{>j}$, we claim that $W_1 \cup \mathsf{sol}(S, W_1, W_2)$ dominates $W_2 \cup \mathcal{D}^{\mathrm{i}(S)}_{>j} \cup \bigcup_{S' \subseteq S} \mathcal{D}^{\mathrm{i}(S')}_{=j(S')}$ if $\mathsf{size}(S, W_1, W_2) \neq \infty$. Apply induction on j. The case $j = 0$ is trivial, so assume that $j > 0$ and that the claim holds for all $j' < j$.

Suppose that $\mathsf{size}(S, W_1, W_2) \neq \infty$ for some j-square S and for disjoint $W_1, W_2 \subseteq \mathcal{D}^{\mathrm{b}(S)}_{>j}$. Let $U, X_1, \ldots, X_4$ attain the minimum in the definition of size. Then $\mathsf{size}(S_i, (W_1 \cup U)^{\mathrm{b}(S_i)}, X_i) \neq \infty$ for $i = 1, \ldots, 4$. By induction, $(W_1 \cup U)^{\mathrm{b}(S_i)} \cup \mathsf{sol}(S_i, (W_1 \cup U)^{\mathrm{b}(S_i)}, X_i)$ dominates $X_i \cup \mathcal{D}^{\mathrm{i}(S_i)}_{>j-1} \cup \bigcup_{S'_i \subseteq S_i} \mathcal{D}^{\mathrm{i}(S'_i)}_{=j(S'_i)}$ for $i = 1, \ldots, 4$. Observe that

$$\begin{aligned}
&\bigcup_{i=1}^{4} \left((W_1 \cup U)^{\mathrm{b}(S_i)} \cup \mathsf{sol}\left(S_i, (W_1 \cup U)^{\mathrm{b}(S_i)}, X_i\right)\right) \\
&\qquad\qquad = W_1 \cup U \cup \bigcup_{i=1}^{4} \mathsf{sol}\left(S_i, (W_1 \cup U)^{\mathrm{b}(S_i)}, X_i\right) \\
&\qquad\qquad = W_1 \cup \mathsf{sol}(S, W_1, W_2)
\end{aligned}$$

and that

$$\begin{aligned}
&\bigcup_{i=1}^{4} \left(X_i \cup \mathcal{D}^{\mathrm{i}(S_i)}_{>j-1} \cup \bigcup_{S'_i \subseteq S_i} \mathcal{D}^{\mathrm{i}(S'_i)}_{=j(S'_i)} \right) \\
&\qquad = W_2 \cup \left(\mathcal{D}^{+(S)}_{>j-1} - U\right) \cup \bigcup_{i=1}^{4} \left(\mathcal{D}^{\mathrm{i}(S_i)}_{>j-1} \cup \bigcup_{S'_i \subseteq S_i} \mathcal{D}^{\mathrm{i}(S'_i)}_{=j(S'_i)} \right).
\end{aligned}$$

Because $W_1 \cup \mathsf{sol}(S, W_1, W_2)$ dominates U, $W_1 \cup \mathsf{sol}(S, W_1, W_2)$ dominates

$$W_2 \cup \mathcal{D}^{+(S)}_{>j-1} \cup \bigcup_{i=1}^{4} \left(\mathcal{D}^{\mathrm{i}(S_i)}_{>j-1} \cup \bigcup_{S'_i \subseteq S_i} \mathcal{D}^{\mathrm{i}(S'_i)}_{=j(S'_i)} \right) = W_2 \cup \mathcal{D}^{\mathrm{i}(S)}_{>j} \cup \bigcup_{S' \subseteq S} \mathcal{D}^{\mathrm{i}(S')}_{=j(S')}.$$

This proves the claim.

We know that $\sum_{S;\, j(S)=l} \mathsf{size}(S, \emptyset, \emptyset) \neq \infty$ from Lemma 8.4.21. Since each disk has level at most l, $\bigcup_{S;\, j(S)=l} \mathsf{sol}(S, \emptyset, \emptyset)$ dominates $\bigcup_S \mathcal{D}^{\mathrm{i}(S)}_{=j(S)}$. □

Computing the size- and sol-Functions

We use the same definitions of (non)empty and relevant (child) as in Chapter 7. That is, a j-square is *nonempty* if it is intersected by a level j disk and *empty* otherwise. A j-square S is *relevant* if one of its three siblings is nonempty or there is a nonempty square S' containing S, such that S' has level at most $j + \lceil \log k \rceil$ (so each nonempty j-square is relevant). Note that this definition induces $\mathrm{O}(k^2 n)$ relevant squares. A relevant square S is said to be a *relevant child* of another relevant square S' if $S \subset S'$ and there is no third relevant square S'', such that $S \subset S'' \subset S'$. Conversely, if S is a relevant child of S', then S' is a *relevant parent* of S.

Lemma 8.4.23 *For each relevant* 0*-square* S*, all* size- *and* sol*-values for* S *can be computed in* $\mathrm{O}\big(nk\gamma\, 3^{(24k+32)\gamma/\pi}\big)$ *time.*

Proof: We use the bounds of Lemma 7.1.1 and Lemma 7.1.2. Then $\left|\mathcal{D}^{\mathrm{b}(S)}_{>0}\right| \leq 16k\gamma/\pi$ and all disjoint $W_1, W_2 \subseteq \mathcal{D}^{\mathrm{b}(S)}_{>0}$ can be enumerated in $\mathrm{O}(3^{16k\gamma/\pi})$ time. Furthermore, as the pathwidth of $\mathcal{D}^{\mathrm{b}(S)}_{=j} \cup \mathcal{D}^{\mathrm{i}(S)}_{>j}$ can be bounded by $8(k+4)\gamma/\pi$, we can adapt the algorithm of Corollary 5.3.9 to find the appropriate minimum dominating set in $\mathrm{O}(nk\gamma\, 3^{(8k+32)\gamma/\pi})$ time. The lemma follows. □

Now assume that the size- and sol-values of all relevant children of a j-square S are known.

Lemma 8.4.24 *For each relevant* j*-square* S $(j > 0)$ *with relevant* $(j-1)$*-square children, in* $\mathrm{O}(3^{32k\gamma/\pi} 2^{16k\gamma/\pi} 4^{16(k+1)\gamma/\pi})$ *time all* size- *and* sol*-values for* S *can be computed.*

Proof: Using Lemma 7.1.1, we can show that $\left|\mathcal{D}^{\mathrm{b}(S)}_{\geq j}\right| \leq 32k\gamma/\pi$ and $\left|\mathcal{D}^{+(S)}_{\geq j}\right| \leq 16k\gamma/\pi$. Hence all disjoint $W_1, W_2 \subseteq \mathcal{D}^{\mathrm{b}(S)}_{>j}$ and all $U \subseteq \mathcal{D}^{+(S)}_{>j-1} \cup \mathcal{D}^{\mathrm{b}(S)}_{=j}$ can be enumerated in $\mathrm{O}(3^{32k\gamma/\pi} 2^{16k\gamma/\pi})$ time. To enumerate all decompositions $\{X_1, \ldots, X_4\}$ of $W_2 \cup (\mathcal{D}^{+(S)}_{>j-1} - U)$ for fixed W_2 and U such that $X_i \subseteq \mathcal{D}^{\mathrm{b}(S_i)}_{>j-1}$, it suffices to consider decompositions of disks in an 'extended cross'. Following Lemma 7.1.1, the number of disks intersecting it is at most $16(k+1)\gamma/\pi$. The lemma follows. □

Lemma 8.4.25 *For each relevant* j*-square* S $(j > 0)$ *with no relevant children of level* $j-1$ *all* size- *and* sol*-values for* S *can be computed in* $\mathrm{O}(n\, 2^{64\gamma/\pi} 3^{32\gamma/\pi})$ *time.*

Proof: From the proof of Lemma 7.2.6, we know that $\mathcal{D}^{\mathrm{b}(S)}_{>j} = \mathcal{D}^{\mathrm{b}(S)}_{\geq j+\lceil \log k \rceil}$, $\mathcal{D}^{+(S)}_{>j-1} = \emptyset$, and $\mathcal{D}^{\mathrm{b}(S)}_{=j} = \emptyset$. Then for any disjoint $W_1, W_2 \subseteq \mathcal{D}^{\mathrm{b}(S)}_{>j}$, we can show that $\mathsf{size}(S, W_1, W_2)$ equals

$$\begin{cases} 0 & \text{if } S \text{ has no relevant children and} \\ & \quad W_1 \text{ dominates } W_2; \\ \infty & \text{if } S \text{ has no relevant children and} \\ & \quad W_1 \text{ doesn't dominate } W_2; \\ \min\limits_{\{X_{S''}\}} \sum\limits_{S''} \mathsf{size}\left(S'', W_1^{\mathrm{b}(S'')}, X_{S''}\right) & \text{otherwise.} \end{cases}$$

The sum is over all relevant children S'' of S and $\{X_{S''}\}$ decomposes W_2.

Since $\mathcal{D}^{\mathrm{b}(S)}_{>j} = \mathcal{D}^{\mathrm{b}(S)}_{\geq j+\lceil \log k \rceil}$, $\left|\mathcal{D}^{\mathrm{b}(S)}_{>j}\right| \leq 32\gamma/\pi$. Then all disjoint $W_1, W_2 \subseteq \mathcal{D}^{\mathrm{b}(S)}_{>j}$ can be enumerated in $\mathrm{O}(3^{32\gamma/\pi})$ time.

For fixed W_1 and W_2, we can compute $\mathsf{size}(S, W_1, W_2)$ in $\mathrm{O}(n)$ time if S has no relevant children. Otherwise, number the relevant children of S arbitrarily, $S''_1, \ldots, S''_m$. Now compute $\mathsf{size}(S, W_1, W_2)$ using the following function s. For any $X \subseteq W_2$,

$$\begin{aligned} s^1(X) &= \mathsf{size}(S''_1, W_1^{\mathrm{b}(S'')}, X) \\ s^i(X) &= \min\nolimits_{X_{S''_i} \subseteq X} \{\mathsf{size}(S''_1, W_1^{\mathrm{b}(S'')}, X) + s^{i-1}(X - X_{S''_i})\} \end{aligned}$$

Then $\mathsf{size}(S, W_1, W_2) = s^m(W_2)$. One can thus compute $\mathsf{size}(S, W_1, W_2)$ in $\mathrm{O}(n\, 2^{64\gamma/\pi})$ time, as $m = \mathrm{O}(n)$. The lemma follows. $\square$

Lemma 8.4.26 *The value of* $\sum_{S;\, j(S)=l} \mathsf{size}(S, \emptyset, \emptyset)$ *can be computed in time* $\mathrm{O}(k^2 n^2 \gamma\, 3^{32k\gamma/\pi} 2^{16k\gamma/\pi} 4^{16(k+1)\gamma/\pi})$.

Proof: Follows from Lemma 8.4.23, Lemma 8.4.24, Lemma 8.4.25, and the proof of Lemma 7.2.7. $\square$

Proof of Theorem 8.4.20: Follows directly from Lemmas 8.4.21, 8.4.22, and 8.4.26. $\square$

The Approximation Algorithm

The shifting technique can now be applied as follows. For an integer a ($0 \leq a \leq k-1$), call a line of level j *active* if it has the form $y = (hk + a2^{l-j})2^j$ or $x = (vk + a2^{l-j})2^j$ ($h, v \in \mathbb{Z}$). The active lines partition the plane into j-squares as before, although shifted with respect to the shifting parameter a. However, we can still apply the algorithm of Theorem 8.4.20.

Let C_a denote the set returned by the algorithm for the j-squares induced by shifting parameter a ($0 \leq a \leq k-1$). Instead of considering each set C_a individually, we join three such sets to ensure that we have a dominating set.

So let $C_i^3 = C_i \cup C_{i+k/3} \cup C_{i+2k/3}$ for each $i = 0, \ldots, k/3-1$. This is properly defined, as k is a multiple of 3. Denote the smallest such set by $C_{\min}^3$. We claim that $C_{\min}^3$ is a dominating set of cardinality at most $(3+36/k) \cdot |OPT|$, where OPT is a minimum dominating set.

To prove this claim, let $\mathcal{D}_a^b$ be the set of disks intersecting the boundary of a j-square S at their level, i.e. $\mathcal{D}_a^b = \bigcup_S \mathcal{D}_{=j(S)}^{\mathrm{b}(S)}$.

Lemma 8.4.27 *C_i^3 is a dominating set of $\mathcal{D}$ for any $i = 0, \ldots, k/3-1$.*

Proof: We claim that any disk is in at most two of the sets $\mathcal{D}_i^b, \mathcal{D}_{i+k/3}^b, \mathcal{D}_{i+2k/3}^b$. A level j disk is in $\mathcal{D}_a^b$ if and only if it intersects an active line of level j for a. We showed in Lemma 7.2.8 that any disk intersects an active horizontal line for at most two values of a and an active vertical line for at most two values of a. It is easy to see from the proof of this lemma that the intersections with an active horizontal line and similarly the intersections with an active vertical line must occur for consecutive values of a (modulo k). Since $k \geq 9$ is an odd multiple of 3, $k/3 > 1$, and thus $i, i+k/3, i+2k/3$ are nonconsecutive integers (modulo k). It follows that any disk is in at most two of the sets $\mathcal{D}_i^b, \mathcal{D}_{i+k/3}^b, \mathcal{D}_{i+2k/3}^b$, as claimed.

Lemma 8.4.22 shows that C_a is a dominating set of $\bigcup_S \mathcal{D}_{=j(S)}^{\mathrm{i}(S)} = \mathcal{D} - \mathcal{D}_a^b$. Given the previous argument, $(\mathcal{D} - \mathcal{D}_i^b) \cup (\mathcal{D} - \mathcal{D}_{i+k/3}^b) \cup (\mathcal{D} - \mathcal{D}_{i+2k/3}^b) = \mathcal{D}$. Hence $C_i^3 = C_i \cup C_{i+k/3} \cup C_{i+2k/3}$ is a dominating set of $\mathcal{D}$. □

Lemma 8.4.28 *$|C_{\min}^3| \leq (3+36/k) \cdot |OPT|$, where OPT is a minimum dominating set.*

Proof: Following the proof of Lemma 7.2.8, a level j disk is in $\mathcal{D}_a^b$ for at most 4 different values of a. Therefore $\sum_{a=0}^{k-1} |OPT \cap \mathcal{D}_a^b| \leq 4 \cdot |OPT|$. Furthermore, for any fixed value of a, any level j disk can intersect at most 4 j-squares. It follows from Theorem 8.4.20 that

$$\begin{aligned}
|C_a| &\leq \sum_S \left| OPT_{=j(S)}^S \right| \\
&\leq \sum_S \left(\left| OPT_{=j(S)}^{\mathrm{i}(S)} \right| + \left| OPT_{=j(S)}^{\mathrm{b}(S)} \right| \right) \\
&\leq |OPT| + 3|OPT \cap \mathcal{D}_a^b|.
\end{aligned}$$

Then

$$\begin{aligned}
\tfrac{1}{3}k \cdot |\mathcal{C}_{\min}^3| &\leq \sum_{i=0}^{k/3-1} |C_i^3| \\
&\leq \sum_{a=0}^{k-1} |C_a|
\end{aligned}$$

$$\begin{aligned}&\leq \sum_{a=0}^{k-1}\left(|OPT| + 3|OPT \cap \mathcal{D}_a^b|\right)\\&\leq (k+12)\cdot |OPT|.\end{aligned}$$

Hence $|C^3_{\min}| \leq (3 + 36/k) \cdot |OPT|$. □

Combining Theorem 8.4.20 and Lemma 8.4.28, we obtain the following approximation algorithm.

Theorem 8.4.29 *There is an algorithm that gives for any $\epsilon > 0$ a $(3+\epsilon)$-approximation for Minimum Dominating Set on disk graphs with n vertices and of bounded ply, i.e. of ply $\gamma = \gamma(n) = \mathrm{o}(\log n)$, in time $f(1/\epsilon)\cdot n^{\mathrm{O}(1)}$ for some computable function f of $1/\epsilon$.*

Proof: Consider any $\epsilon > 0$. Choose k as the largest odd multiple of 3 such that $32k\gamma/\pi \leq \log_3 n$. If $k < 9$, output $V(G)$. Otherwise, apply the algorithm of Theorem 8.4.20 and compute $C^3_{\min}$ in $\mathrm{O}(n^5 \log^3 n)$ time. Furthermore, if $\gamma = \gamma(n) = \mathrm{o}(\log n)$, there is a c_ϵ such that $k \geq 36/\epsilon$ and $k \geq 9$ for all $n \geq c_\epsilon$. Therefore, if $n \geq c_\epsilon$, it follows from Lemma 8.4.28 and the choice of k that $C^3_{\min}$ is a $(3+\epsilon)$-approximation to the optimum. The theorem now follows from the proof of Theorem 2.2.4. □

We can obtain analogous approximation algorithms on intersection graphs of fat objects of bounded ply and of any constant dimension.

8.5 Hardness of Approximation

We have seen that although Minimum Dominating Set is a challenging problem on intersection graphs of arbitrary-sized geometric objects, it still is approximable on a variety of classes of geometric intersection graphs. We show however that there are also classes of geometric intersection graphs for which no constant-factor approximation algorithm or approximation scheme can exist, under certain complexity assumptions.

We prove that Minimum Dominating Set on intersection graphs of convex polygons or of homothetic polygons is as hard as on general graphs. That is, it is not approximable within $(1-\epsilon)\ln n$ for any $\epsilon > 0$, unless $\mathrm{NP} \subset \mathrm{DTIME}(n^{\mathrm{O}(\log\log n)})$. This nicely complements Theorem 8.3.4, where we gave a constant-factor approximation algorithm on intersection graphs of homothetic convex polygons. Hence it seems that both convexity and homotheticity are essential properties of the objects when designing a constant-factor approximation algorithm.

We also gain further insight into Minimum Dominating Set on disk graphs. We show that on a collection of fat almost-disks, Minimum Dominating Set is as hard as on general graphs. Even if the fat objects have constant description complexity, the problem is still APX-hard.

Finally, we solve an open problem of Chlebík and Chlebíková [65] by proving that Minimum Dominating Set is APX-hard on rectangle intersection graphs. This result extends to intersection graphs of ellipses. We should note that all hardness results given here extend to Minimum Connected Dominating Set.

8.5.1 Intersection Graphs of Polygons

We consider the approximability of Minimum Dominating Set on intersection graphs of polygons. First we show that convexity of the objects is no guarantee for the existence of a constant-factor approximation algorithm. Instead of just looking at arbitrary convex polygons, we prove a stronger result.

Recall from Chapter 3 that a *polygon-circle graph* is the intersection graph of a set of polygons for which all corners lie on a fixed circle. Note that in a polygon-circle graph, all polygons are convex. This graph class is a generalization of *circle graphs*, which are intersection graphs of chords of a fixed circle. On circle graphs, we know that Minimum Dominating Set has a $(2+\epsilon)$-approximation algorithm [76], but no ptas unless P=NP [75]. The slight generalization to polygon-circle graphs however makes Minimum Dominating Set much more difficult.

Theorem 8.5.1 *Minimum Dominating Set on polygon-circle graphs is not approximable within* $(1-\epsilon)\ln n$ *for any* $\epsilon > 0$*, unless NP* $\subset$ *DTIME(*$n^{\mathrm{O}(\log\log n)}$*).*

Proof: We give a gap-preserving reduction from Minimum Set Cover. Consider an instance $(\mathbb{U}, \mathcal{S})$ of Minimum Set Cover and assume that $\mathbb{U} = \bigcup \mathcal{S}$ and $\mathbb{U} = \{1, \ldots, n\}$. Fix a circle and $n+1$ points on this circle, numbered $p_1, \ldots, p_{n+1}$ in order of appearance on the circle. Construct a polygon P_j for each set $\mathcal{S}_j$ as the convex hull of the set $\{p_i \mid i \in \mathcal{S}_j\} \cup \{p_{n+1}\}$. Furthermore, place a tiny polygon around each point p_i such that these tiny polygons are pairwise disjoint.

Observe that any polygon dominated by a tiny polygon is also dominated by some polygon P_j. It is now easy to see that the optima of the minimum set cover instance and the constructed instance of Minimum Dominating Set on polygon-circle graphs are the same. As the construction can be computed in polynomial time, this gives a gap-preserving reduction. The theorem then follows from Feige's inapproximability result for Minimum Set Cover [108]. □

A direct consequence of Theorem 8.5.1 is an inapproximability result on intersection graphs of convex polygons.

Corollary 8.5.2 *Unless NP* $\subset$ *DTIME(*$n^{\mathrm{O}(\log\log n)}$*), Minimum Dominating Set on intersection graphs of convex polygons cannot be approximated within* $(1-\epsilon)\ln n$ *for any* $\epsilon > 0$.

We give a similar result for intersection graphs of fat convex polygons later.

If the polygons are not convex, but translated copies of a fixed polygon, the approximability of Minimum Dominating Set does not change.

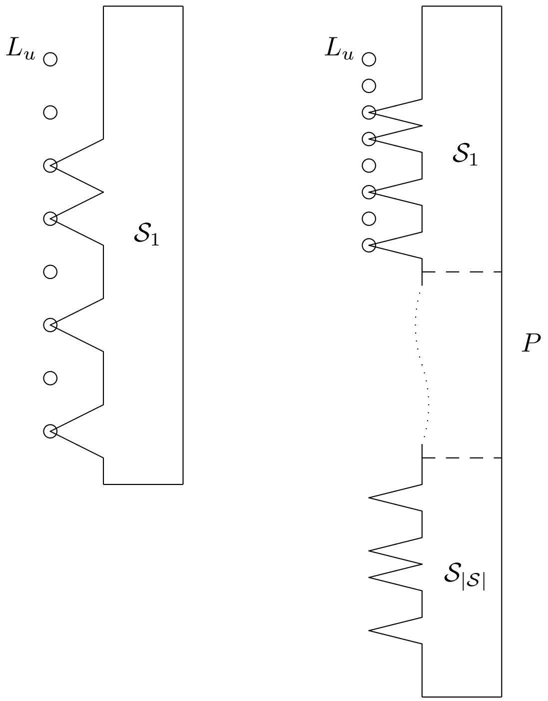

Figure 8.2: The left figure shows the rectangle constructed for $\mathcal{S}_1$. The right figure shows the combination of the rectangles for $\mathcal{S}_1, \ldots, \mathcal{S}_{|\mathcal{S}|}$. This is the base polygon P. The small circles represent the L_u, which are homothetic copies of P.

Theorem 8.5.3 *Minimum Dominating Set on intersection graphs of homothetic polygons is not approximable within* $(1-\epsilon)\ln n$ *for any* $\epsilon > 0$*, unless NP* $\subset$ *DTIME(*$n^{O(\log\log n)}$*).*

Proof: We give a similar reduction as in the proof of Theorem 8.5.1. Given an instance $(\mathbb{U}, \mathcal{S})$ of Minimum Set Cover, place for each $u \in \mathbb{U}$ a polygon L_u in the plane, such that these polygons are aligned in a column (see Figure 8.2). Now construct a rectangle next to this column. Deform the long side by placing small bulges on it such that the deformed rectangle intersects an L_u if and only if $u \in \mathcal{S}_1$ (see Figure 8.2). Do this for each set in $\mathcal{S}$ and stack these rectangles. This is the base polygon P. By taking a translated copy of P for each set in $\mathcal{S}$ and ensuring that the L_u are homothetic copies of P, we can build the same graph as in Theorem 8.5.1. The theorem follows. □

The hardness results of Corollary 8.5.2 and Theorem 8.5.3 complement Theorem 8.3.4, where we gave an $O(r^4)$-approximation algorithm on intersection graphs of homothetic convex polygons with r corners.

The approximability of Minimum Dominating Set on intersection graphs of convex polygons or of homothetic polygons with r corners has yet to be determined. The APX-hardness on circle graphs [75] implies APX-hardness on intersection graphs of convex polygons with two (or more) corners. Hence no ptas exists, unless P=NP [16]. Using the gadget of Theorem 8.5.1, we can give a slightly weaker result, but by an easier proof. We use that Minimum k-Set Cover is APX-hard for any $k \geq 3$ (by reduction from Minimum Vertex Cover on graphs of degree at most 3 [9]). *Minimum k-Set Cover* is the variation of Minimum Set Cover where each set has cardinality at most k.

Theorem 8.5.4 *Minimum Dominating Set on polygon-circle graphs of convex polygons with* r *corners is APX-hard for any* $r \geq 4$*. Hence it has no ptas, unless P=NP.*

Proof: We use the same gadget as in the proof of Theorem 8.5.1 and reduce from Minimum k-Set Cover, which is APX-hard for any $k \geq 3$. The gadget constructs polygons with at most $k+1$ corners. The theorem follows. □

Corollary 8.5.5 *Minimum Dominating Set on intersection graphs of convex polygons with* r *corners is APX-hard for any* $r \geq 4$*. Hence it has no ptas, unless P=NP.*

8.5.2 Intersection Graphs of Fat Objects

The approximation schemes for Maximum Independent Set and Minimum Vertex Cover on disk graphs (see Chapter 7) extend easily to intersection graphs of fat objects. It is unlikely that an approximation algorithm for Minimum Dominating Set extends this way, as on intersection graphs of fat objects that are almost-disks, Minimum Dominating Set becomes hard to approximate.

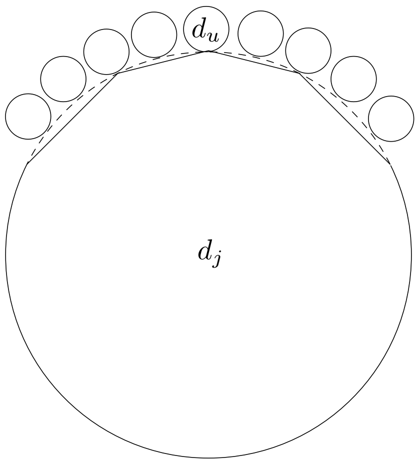

Figure 8.3: A cut-off disk d_j and the disks d_u for elements $u \in \mathbb{U}$ of Theorem 8.5.6.

Recall that a convex subset s of $\mathbb{R}^2$ is *α-fat* for some $\alpha \geq 1$ if the ratio between the radii of the smallest disk enclosing s and the largest disk inscribed in s is at most α [97].

Theorem 8.5.6 *For any $\alpha > 1$, Minimum Dominating Set on intersection graphs of α-fat objects is not approximable within $(1 - \epsilon) \ln n$ for any $\epsilon > 0$, unless NP $\subset$ DTIME($n^{O(\log \log n)}$).*

Proof: We reduce from Minimum Set Cover in a manner similar as in the proof of Theorem 8.5.1. For an instance $(\mathbb{U}, \mathcal{S})$ of Minimum Set Cover, construct an instance of Minimum Dominating Set as follows. Each $u \in \mathbb{U}$ corresponds to a 'small' disk d_u. Each $\mathcal{S}_j$ corresponds to a disk d_j with the top replaced by a polygonal structure such that d_j intersects d_u if and only if $u \in \mathcal{S}_j$ (see Figure 8.3). Packing the d_u close together makes the fatness of the construction arbitrarily close to 1. As any object dominated by a d_u is also dominated by a d_j for which $u \in S_j$, the optima of the two instances are equal. Moreover, the construction can be computed in polynomial time. The theorem then follows from Feige's result [108]. $\square$

An object has *constant description complexity* if it is a semialgebraic set defined by a constant number of polynomial (in)equalities of constant maximum degree [97]. The objects in the proof of Theorem 8.5.6 that model the $\mathcal{S}_j$ are the intersection of a disk with a polygon and thus we can describe each such d_j by one quadratic inequality and $|\mathcal{S}_j| + 1$ linear inequalities. Hence the objects in the construction of Theorem 8.5.6 might not have constant description complexity. So for constant description complexity objects, better approximation ratios than $\ln n$ could be attained. However, we can still prove APX-hardness by reducing from Minimum k-Set Cover with the same gadget.

Theorem 8.5.7 *For any $\alpha > 1$, Minimum Dominating Set on intersection graphs of α-fat objects of constant description complexity is APX-hard. Hence it has no ptas, unless P=NP.*

These results say something about intersection graphs of fat objects in general, and of fat almost-disks in particular. But we can easily prove similar results for almost-squares, bounded aspect-ratio almost-rectangles, almost-triangles, etc. Basically, if we slightly relax the shape constraints for a given object, Minimum Dominating Set on the intersection graphs of such relaxed objects is hard to approximate. Moreover, the above results can be used to derive hardness of approximation results for Minimum Connected Dominating Set.

8.5.3 Intersection Graphs of Rectangles

Chlebík and Chlebíková [65] proved that Minimum Dominating Set is APX-hard on intersection graphs of three-dimensional axis-parallel boxes and asked whether this result can be extended to only two dimensions. We prove that this is indeed the case.

Theorem 8.5.8 *Minimum Dominating Set on rectangle intersection graphs is APX-hard. Hence it has no ptas, unless P=NP.*

Proof: We give an L-reduction [220] from Minimum Vertex Cover on graphs of degree three, which is known to be APX-hard [9]. Consider an arbitrary instance x of Minimum Vertex Cover on graphs of degree three. Let $G = (V, E)$ be the graph of x and denote the cardinality of the smallest vertex cover in G by k. Number the vertices of V arbitrarily $v_1, \ldots, v_n$, where $n = |V|$. Now construct for each vertex v_i a horizontal and a vertical rectangle R_i^h and R_i^v and connect them as shown in Figure 8.4. Call the big rectangle used in the connection of R_i^h and R_i^v the *big plate* P_i of i and the two small rectangles the *small plates* of i. This models the vertices. Next we model the edges. If $(v_i, v_j) \in E$ for certain $i < j$, then add a small rectangle $S_{i,j}$ in the intersection of rectangles R_i^v and R_j^h (see Figure 8.4). This gives the instance $f(x)$ of Minimum Dominating Set on rectangle intersection graphs. Observe that this is indeed a polynomial-time computable function (even if not only the graph, but also the rectangles are part of the output).

Let C be a vertex cover of G of cardinality k. Let $R^h[C] = \{R_i^h \mid v_i \in C\}$ be the set of horizontal rectangles induced by C and similarly let $R^v[C]$ be the set of vertical rectangles induced by C. Furthermore, let $\overline{P[C]} = \{P_i \mid v_i \notin C\}$ be the big plates for which the corresponding vertex is not in C.

We claim that $D = R^h[C] \cup R^v[C] \cup \overline{P[C]}$ is a dominating set of G'. Let r be an arbitrary rectangle. Suppose that r is an $S_{i,j}$ for a certain i, j. Since C is a vertex cover, $v_i \in C$ or $v_j \in C$. Assume w.l.o.g. that $v_i \in C$. Then

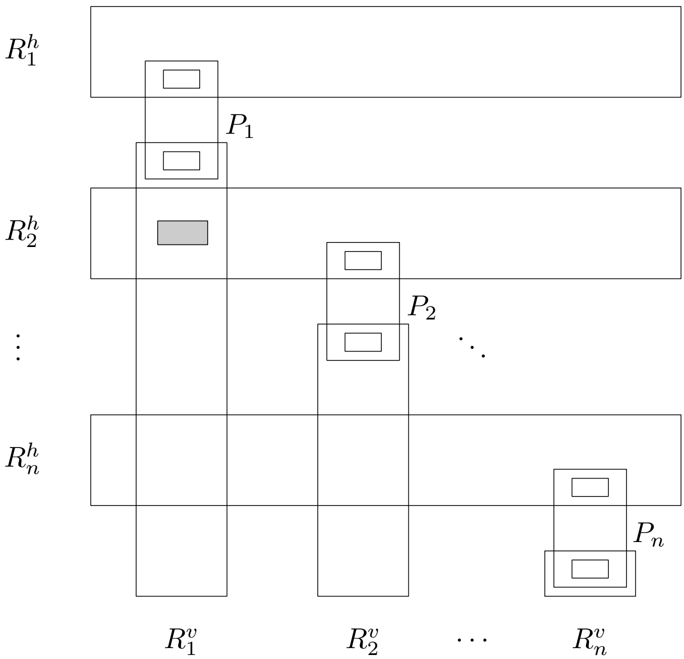

Figure 8.4: The intersection graph used in the proof of Theorem 8.5.8. If edge (v_1, v_2) is in E, then the shaded rectangle $S_{1,2}$ is in G'.

by construction, $R_i^h, R_i^v \in D$, and thus by construction of G', $S_{i,j}$ must be dominated. Suppose that r is a (big or small) plate of i. If $v_i \in C$, then $R_i^h, R_i^v \in D$, and thus the plate must be dominated. If $v_i \notin C$, then $P_i \in \overline{P[C]} \subseteq D$, and the plate is dominated. Using a similar argument, we can show that if r is R_i^h or R_i^v for certain i, it must be dominated. Hence D is a dominating set of G'.

Note that $|R^h[C]| = |R^v[C]| = |C| = k$. Furthermore, $|\overline{P[C]}| = n - k$. Since G has degree three, $k \geq n/4$. Hence

$$m^*(f(x)) \leq n - k + k + k \leq 4k + k = 5 \cdot m^*(x). \tag{8.1}$$

We now take a closer look at the cardinality of dominating sets of G'. Let D be an arbitrary dominating set of G'. Observe that the rectangles dominated by small plates and the $S_{i,j}$ are also dominated by the appropriate big plate or R_i^h respectively R_i^v. Hence we can replace these small plates and $S_{i,j}$'s and obtain a dominating set D' with $|D'| \leq |D|$, where all small plates and $S_{i,j}$ are dominated by big plates and rectangles of type R_i^h and R_i^v. Let $R^2[D'] = \{R_i^h, R_i^v \mid R_i^h, R_i^v \in D'\}$ be the rectangles for v_i for which both the horizontal and the vertical version occur in D', $R^1[D']$ the remaining rectangles of type R_i^h and R_i^v (i.e. rectangles for v_i for which only one version occurs in D'), and let $P[D']$ denote the big plates in D'. Furthermore, let $R[D'] = R^2[D'] \cup R^1[D']$. Note that $R^2[D'] \cap R^1[D'] = \emptyset$.

Consider $C = \{v_i \mid R_i^h \in D' \text{ or } R_i^v \in D'\}$. Since all $S_{i,j}$ are dominated by $R[D']$, C is a vertex cover. Observe that to dominate all plates of i, $P_i \in D'$, or $R_i^h, R_i^v \in D'$. This holds for all i. Thus $|P[D']| + |R^2[D']|/2 \geq n$. Also, as C is a vertex cover of G, $|R^1[D']| + |R^2[D']|/2 = |C| \geq k$.
Hence

$$\begin{aligned} |D'| &\geq |R^1[D']| + |R^2[D']| + |P[D']| \\ &\geq |R^1[D']| + |R^2[D']|/2 + n \\ &\geq k + n. \end{aligned} \tag{8.2}$$

Together with Equation 8.1, this implies that $m^*(f(x)) = n + k$.

Now suppose that $|D| = m^*(f(x)) + c$, for a certain $c \geq 0$. Then $|D'| \leq |D| = n + k + c$. Using Equation 8.2,

$$\begin{aligned} |R^1[D']| + |R^2[D']|/2 + n &\leq n + k + c \\ |R^1[D']| + |R^2[D']|/2 &\leq k + c \\ |C| &\leq m^*(x) + c \\ |C| - m^*(x) &\leq c. \end{aligned}$$

This gives an L-reduction from Minimum Vertex Cover on graphs of degree three to Minimum Dominating Set on rectangle intersection graphs with $\alpha = 5$ and $\beta = 1$. □

Note that this theorem holds even if the rectangles have to be axis-parallel or if no rectangle can be fully contained in another rectangle (by slightly changing the construction of Figure 8.4). Furthermore, the construction in the proof of Theorem 8.5.8 can be replicated using ellipses instead of rectangles. This gives the following theorem.

Theorem 8.5.9 *Minimum Dominating Set on ellipse intersection graphs is APX-hard. Hence it has no ptas, unless P = NP.*

The proof of Theorem 8.5.9 requires ellipses of relatively high eccentricity (of the order $\sqrt{1-n^{-2}}$) as 'R_i^h' and 'R_i^v'. Hence the proof does not immediately carry over to disk graphs.

The construction to prove the APX-hardness of Minimum Dominating Set in rectangle intersection graphs can be extended to prove the APX-hardness of Minimum Connected Dominating Set. In fact, the new construction generalizes the previous construction, as it can also be used to prove the APX-hardness of Minimum Dominating Set. Below, we give this generalized proof.

Theorem 8.5.10 *Minimum Connected Dominating Set on rectangle intersection graphs is APX-hard. Hence it has no ptas, unless P=NP.*

Proof: Consider again an arbitrary instance x of Minimum Vertex Cover on graphs of degree three. Let $G = (V = \{v_1, \ldots, v_n\}, E)$ be the graph of x and denote the cardinality of the smallest vertex cover of x by k. We keep the construction of Theorem 8.5.8 (see Figure 8.4) and extend it as follows (see Figure 8.5). For any big plate P_i, we add a horizontal plate P_i^h intersecting P_i and containing a single small rectangle, ensuring that P_i^h is in any connected dominating set. We also add three surrounding rectangles S_1, S_2, and S_3, each containing a single small rectangle, enforcing the presence of S_1, S_2, and S_3 in any connected dominating set. These rectangles are aligned such that S_1 intersects all horizontal rectangles R_i^h, S_2 intersects S_1 and all vertical rectangles R_i^v, and S_3 intersects S_2 and all horizontal plates P_i^h. The intersection graph G' of these rectangles is the function $f(x)$ for the L-reduction. It can be quickly verified that this is indeed a polynomial-time computable function (even if the rectangles are part of the output).

Let C be a vertex cover of G of cardinality k. Recall that $R^h[C] = \{R_i^h \mid v_i \in C\}$, $R^v[C] = \{R_i^v \mid v_i \in C\}$, and $\overline{P[C]} = \{P_i \mid v_i \notin C\}$. Let $P^h = \{P_i^h \mid i = 1, \ldots, n\}$. We claim that $D = R^h[C] \cup R^v[C] \cup \overline{P[C]} \cup P^h \cup \{S_1, S_2, S_3\}$ is a connected dominating set of G'. From the proof of Theorem 8.5.8 and the construction of G', it should be clear that D is a dominating set for G'.

To prove that D induces a connected subgraph of G', let $d, d' \in D$ be any two distinct rectangles in G'. We show there exists a path in $G'[D]$ between d and d'. If d or d' is in $R^h[C]$ ($R^v[C]$), it takes one step to reach S_1 (S_2). Similarly, if d or d' is in $\overline{P[C]}$, the appropriate horizontal plate can be used to reach S_3 in two steps. Thus from d or d', we can reach S_1, S_2, or S_3 in

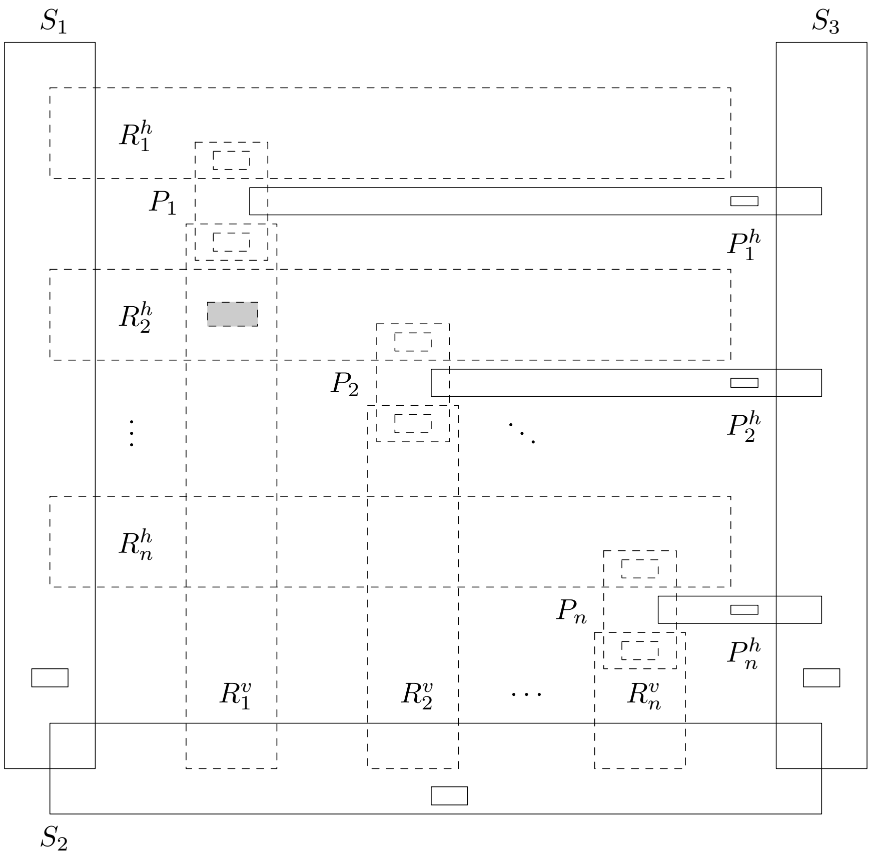

Figure 8.5: The intersection graph used in the APX-hardness proof of Minimum Connected Dominating Set. The rectangles that also appeared in Figure 8.4 have dashed boundaries.

the subgraph of G' induced by D in at most two steps. But since $\{S_1, S_2, S_3\}$ form a connected induced subgraph in G', this implies that D is a connected subgraph of G'. Hence D is a connected dominating set.

We now give an upper bound to $|D|$. Since the degree of G is three, $k \geq n/4$, and thus

$$\begin{aligned} m^*(f(x)) &\leq |D| \\ &= |R^h[C]| + |R^v[C]| + |\overline{P[C]}| + |P^h| + 3 \\ &= k + k + (n-k) + n + 3 \\ &= 2n + k + 3 \\ &\leq 9k + 3 \\ &\leq 12k \\ &= 12 \cdot m^*(x). \end{aligned} \tag{8.3}$$

Now let D be an arbitrary connected dominating set of G'. We may assume that D contains all P_i^h and S_1, S_2, S_3 (if not, the small rectangle contained in these rectangles is in D, which can be easily replaced by the bigger rectangle). Similarly, as already noted in the proof of Theorem 8.5.8, we may assume that all small plates and $S_{i,j}$ are dominated by big plates and rectangles R_i^h and R_i^v. Let $R^2[D'] = \{R_i^h, R_i^v \mid R_i^h, R_i^v \in D\}$ be the set of the rectangles for i for which both the horizontal and the vertical version occur in D, $R^1[D]$ the set of remaining rectangles of type R_i^h and R_i^v (i.e. rectangles for i for which only one version occurs in D), and let $P[D]$ denote the set of big plates in D. Furthermore, let $R[D] = R^2[D] \cup R^1[D]$. Note that $R^2[D] \cap R^1[D] = \emptyset$.

Consider $C = \{v_i \mid R_i^h \in D \text{ or } R_i^v \in D\}$. Since all $S_{i,j}$ are dominated by $R[D]$, C is a vertex cover. Observe that to dominate the small plates of i, $P_i \in D$, or both $R_i^h, R_i^v \in D$. This holds for all i. Therefore $|P[D]| + |R^2[D]|/2 \geq n$. Also, as C is a vertex cover for G, $|R^1[D]| + |R^2[D]|/2 = |C| \geq k$. Hence

$$\begin{aligned} |D| &\geq |R^1[D]| + |R^2[D]| + |P[D]| + |P^h| + 3 \\ &\geq |R^1[D]| + |R^2[D]|/2 + n + n + 3 \\ &\geq k + 2n + 3. \end{aligned} \tag{8.4}$$

Together with Equation 8.3, this implies that $m^*(f(x)) = 2n + k + 3$.

Now suppose that $|D| = m^*(f(x)) + c$, for a certain $c \geq 0$. Then $|D| = 2n + k + 3 + c$. Using Equation 8.4,

$$\begin{aligned} |R^1[D]| + |R^2[D]|/2 + 2n + 3 &\leq |D| \;=\; 2n + k + 3 + c \\ |R^1[D]| + |R^2[D]|/2 &\leq k + c \\ |C| &\leq m^*(x) + c \\ |C| - m^*(x) &\leq c. \end{aligned}$$

This gives an L-reduction with $\alpha = 12$ and $\beta = 1$. □

This reduction can also be extended to ellipse intersection graphs (where the ellipses have high eccentricity).