Front interactions in a three-component system

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Chapter 1

Introduction

Pattern formation is a very lively field of research within the nonlinear sciences, where the traditional disciplines of mathematics, physics, chemistry, and biology merge, interact, and exchange ideas. Reaction-diffusion equations (RDEs) serve as relevant, often simplified models within several branches of these fields. For instance, pulses traveling through nerve cells, as well as vegetation patterns [44] and stripes on zebras [37] are modeled by RDEs, see Figure 1.1. From a mathematical perspective, RDEs are arguably the simplest nonlinear partial differential equations (PDEs) that exhibit complex patterns observed in many natural systems such as spiral waves and spatio-temporal chaos which are observed in many natural systems, see Figure 1.1. Therefore, RDEs can be considered as the key prototype models in which one can begin to develop a fundamental understanding of complex patterns.

Localized structures form a special class of solutions to these RDEs related to the aforementioned patterns. These structures are solutions to the PDE remaining close to a trivial background state except in one or more localized spatial regions, see Figure 1.2. Fronts and pulses are the most well-known and well-studied localized structures in one spatial dimension; spots, spirals, and stripes are examples of localized structures in two dimensions. Localized patterns can be seen as the foundation for the mathematical analysis of more complex patterns.

In recent years, significant progress has been made in our mathematical understanding of the simplest localized structures. These being fronts and pulses that are stationary or move with a constant speed through a 1-dimensional domain [59]. In general, the behavior of localized structures is less well-ordered: those structures interact with each other and thus also move with different velocities, see Figure 1.2. At present, there is a well-developed theory that describes the interaction of fronts and pulses in the weak interaction regime [19, 20, 55]. In this regime these fronts or pulses are ‘far away’ from each other, meaning, all
Figure 1.1: In the left frame, one sees pattern formation on the coats of two zebras [73]. In the right frame, one sees pattern formation at the edge of the Negev Dessert. The dark stripes correspond to vegetation, whereas the lighter stripes correspond to sand. Note that the scale of this frame is in decimeters. Both phenomena are modeled by RDEs.

Figure 1.2: In the left frame, we plotted a localized stationary 2-pulse or 4-front solution of the three-component system under investigation in this thesis, see (1.1.3). Note that two of the components ($V$ and $W$) interact strongly, while the $U$-component interacts only via their background states. In the right frame, we plotted an interacting 3-front solution of the same system; only the $U$-component is plotted. In the context of Figure 1.1: black/dark corresponds to $U = -1$, and white/light to $U = 1$. 
components of the structure interact only through their trivial background states mentioned above. However, there is no mathematical theory that explains the interaction of fronts and pulses in the strong interaction regime where the fronts and pulses are close to each other, and all solution components are far from equilibrium in the regions between the fronts and pulses. In that regime, interesting behavior such as collision, repulsion, annihilation, and self-replication of fronts and pulses can be observed; in a daily life setting, one could, for instance, think of the collision of two stern waves. In between the weak and strong interaction regimes lies a third regime, the semi-strong interaction regime, where certain components of the fronts or pulses interact via the background state, while the remaining components interact strongly with each other, see Figure 1.2. Understanding this regime is a fundamental next step in furthering our understanding of how localized structures interact. In this thesis, we take a first step in that direction for a specific RDE.

1.1 Model equations

1.1.1 Physical background

In the mid-nineties, the physicist H. G. Purwins studied pattern formation in gas-discharges, the effect that creates light in fluorescent tubes, see Figures 1.3 and 1.4. He mainly considered the interactions of localized states with each other and first modeled his observations by a two-component RDE

\[
\begin{align*}
U_t &= D_U \Delta U + f(U) - \kappa_3 V + \kappa_1, \\
\tau V_t &= D_V \Delta V + U - V,
\end{align*}
\]  

(1.1.1)
where $f(U)$ was taken to be a cubic polynomial. Numerical simulations revealed that certain 2-dimensional structures – such as traveling spots – were unstable; they were, nevertheless, observed in experiments – an apparent contradiction. To rectify this situation, Purwins introduced a third component to his model [54, 60],

$$
\begin{align*}
U_t &= D_U \Delta U + f(U) - \kappa_3 V - \kappa_4 W + \kappa_1 , \\
\tau V_t &= D_V \Delta V + U - V , \\
\theta W_t &= D_W \Delta W + U - W .
\end{align*}
$$

(1.1.2)

It turned out that this extended model supports stable traveling spots and a variety of other interesting localized solutions, as well. In subsequent years, many variations of this model were studied extensively by several research groups of both physicists ([5, 32, 54, 60, 65, 70]) and mathematicians ([50–53, 71]). These studies went beyond the original gas-discharge context and established the Purwins system as a paradigm model within the field of pattern formation that can be used to investigate the interaction of fronts, pulses, and spots.

### 1.1.2 Mathematical background

Besides its physical background and the richness of dynamics it exhibits, exemplified by such patterns as breathing pulses, scattering pulses, and bouncing pulses,
the model (1.1.2) is also particularly interesting from a mathematical point of view. Even within the class of RDEs, the extended Purwins model has a remarkably transparent structure, especially in the scaling we consider in this thesis. In the next chapter, we will show that (1.1.2) can be rescaled, under certain circumstances, into a particularly simple form. In one spatial dimension, this rescaled version is given by

\[
\begin{align*}
U_t &= U_{\xi\xi} + U - U^3 - \varepsilon(\alpha V + \beta W + \gamma), \\
\tau V_t &= \frac{1}{\varepsilon^2} V_{\xi\xi} + U - V, \\
\theta W_t &= \frac{D^2}{\varepsilon^2} W_{\xi\xi} + U - W,
\end{align*}
\]

(1.1.3)

with

\[
0 < \varepsilon \ll 1; \quad D > 1; \quad 0 < \tau, \theta \ll \varepsilon^{-3}; \quad (\xi, t) \in \mathbb{R} \times \mathbb{R}^+; \quad \alpha, \beta, \gamma \in \mathbb{R} \quad \text{and} \quad \mathcal{O}(1) \quad \text{with respect to} \quad \varepsilon.
\]

This equation is the central object of study of this thesis. Note that we consider it in a single spatial dimension, since we use the dynamical systems approach of spatial dynamics to study it. The model is singularly perturbed (see Section 1.2.1), bistable (see Section 1.2.2), and its \( U \)-component is only weakly coupled to its \( V, W \)-components which, in turn, are linearly coupled back to \( U \). Moreover, explicit nonlinear effects occur directly only through the \( U^3 \) term in the first PDE. As a consequence, localized states typically interact in a semi-strong fashion, see Figure 1.2. The combination of these properties makes this model amenable to a rigorous mathematical analysis, something which has not yet been accomplished before for three-component systems.

In this thesis, we study (1.1.3) to gain insight in the interactions of fronts and pulses in the semi-strong interaction regime.

1.2 Basic concepts

In this section, we lay down certain basic features of (1.1.3) in relatively simple terminology. We have chosen to do so by means of a number of specific examples. This way, we can solidify the introduction of these concepts by means of concrete calculations. First, we expound on the notion of a singular perturbation by looking into simple yet instructive perturbations of cubic polynomials and ordinary differential equations (ODEs). Next, we introduce the concept of bistability. In Section 1.3, we proceed in a similar fashion: we exemplify the three main mathematical tools used in this thesis by three relatively simple examples. These examples are chosen for convenience in introducing the mathematical techniques we use, and wherever possible we specify if there is a connection between the example and some part of the full model (1.1.3).
1.2.1 Singular perturbations

In simple terms, a singularly perturbed problem is a problem containing a small parameter, usually denoted by $\varepsilon$, with the property that one cannot obtain a valid approximation to all of its solutions by setting this small parameter to zero. This feature of singularly perturbed problems is in stark contrast with the behavior of regularly perturbed problems, where one can directly obtain such an approximation by setting the small parameter to zero. To obtain more insight what differentiates a singularly and a regularly perturbed setting, we now consider several simple examples.

A cubic algebraic equation

In the first example, we contrast the behavior of the roots of the regularly perturbed cubic equation

$$x^3 - x^2 + \varepsilon = 0, \quad 0 < \varepsilon \ll 1,$$

with that of the roots of the singularly perturbed equation

$$\varepsilon x^3 - x^2 + 1 = 0, \quad 0 < \varepsilon \ll 1. \quad (1.2.1)$$

In both cases, we consider the behavior of these roots as we let $\varepsilon \to 0$. A priori, one may think that approximations of the zeroes of either equation can be (formally) obtained from the reduced equation which is obtained by setting $\varepsilon \to 0$ in the original equation. However, this is only true for the regularly perturbed equation; its reduced equation has the zeroes $x_r^0 = 0$, with multiplicity two, and $x_r^0 = 1$. It is simple to show that these values are the limiting values of the roots for $\varepsilon \neq 0$ as we let $\varepsilon \to 0$. The reduced equation corresponding to the singularly perturbed equation has, on the contrary, only two zeroes $x_s^0 = \pm 1$, since it is of degree two instead of three. The third zero of the singularly perturbed equation has ‘disappeared’ in the limit $\varepsilon \to 0$; in fact, it is $O(\varepsilon^{-1})$ and thus becomes arbitrarily large for $\varepsilon \to 0$. More explicitly, for $\varepsilon = 0.01$, the zeroes of the singularly perturbed equation are approximately $x_s^* = 1.005, x_s^* = -0.9951$, and $x_s^* = 99.99$, while the zeroes of the regularly perturbed equation approximately read $x_r^* = 0.1057, x_r^* = -0.09554$, and $x_r^* = 0.9898$.

So, by naively setting $\varepsilon \to 0$, one loses information on one of the zeroes of the singularly perturbed equation, while this is not the case for the regularly perturbed one.

The asymptotic behavior of the third zero of the singularly perturbed equation can, naturally, also be determined. Indeed, by rescaling the variable $y := \varepsilon x$, (1.2.1) transforms into

$$y^3 - y^2 + \varepsilon^2 = 0.$$
The corresponding reduced equation is now of degree three, and it also yields the root \( y_0 = 1 \) apart from the the double zero solution. In the original variable, this first root becomes \( x_0 = \varepsilon^{-1} \) and it corresponds to the leading order approximation of the third root (which is not captured by the reduced equation of (1.2.1)); the double zero solution, in turn, corresponds to the bounded roots \( x_0^* = \pm 1 \).

**Ordinary differential equations**

The terminology introduced above can be transplanted easily to the field of differential equations. An example of a singularly perturbed ODE is the equation for a rescaled damped oscillator,

\[
\varepsilon \ddot{x} + \dot{x} + x = 0, \quad 0 < \varepsilon \ll 1,
\]

\[x(0) = A_1, \quad \dot{x}(0) = A_2, \tag{1.2.2}\]

where \( \dot{\cdot} = \frac{d}{dt} \), and \( A_1, A_2 \) are arbitrary real numbers. Setting \( \varepsilon \to 0 \) in this equation yields the reduced equation

\[\dot{x} + x = 0.\]

The solution to this first order ODE is readily found to be \( x_0(t) = Ce^{-t} \). This formula, in turn, yields that \( x_0(0) = C = A_1 \) and \( \dot{x}_0(0) = -C = A_2 \). Therefore, the reduced problem cannot satisfy both initial conditions, unless \( A_1 = -A_2 \).

In this particular example, the general solution to the full equation (1.2.2) can be explicitly found to be

\[x(t) = C_1 e^{-(\frac{1 + \sqrt{1-4\varepsilon}}{2\varepsilon})t} + C_2 e^{-(\frac{1 - \sqrt{1-4\varepsilon}}{2\varepsilon})t} = C_1 e^{-(\frac{1}{\varepsilon} + \mathcal{O}(1))t} + C_2 e^{-(1 + \mathcal{O}(\varepsilon))t}, \]

where \( C_1 \) and \( C_2 \) are determined by the initial conditions. We see that, here also, one of the exponents is of \( \mathcal{O}(\varepsilon^{-1}) \) and thus becomes arbitrarily large for \( \varepsilon \to 0 \).

So, here again, setting \( \varepsilon \to 0 \) leads to the loss of crucial information. This fact is most readily evident through our observation above that the reduced ODE cannot accommodate both initial conditions. However, the reduced dynamics actually describes the long time behavior of the full system, see Figure 1.5 and Section 1.3.1 (where the reduced dynamics are termed slow dynamics). On the other hand, this is a mere coincidence: if one changes the \( +\dot{x} \) into a \( -\dot{x} \) in (1.2.2) the new reduced dynamics does no longer describe the asymptotic behavior of the full system, since the stability type of the fixed point at \( x = 0 \) changes from attractive to repulsive.

A prototype example of a regularly perturbed ODE is the first order ODE

\[\dot{x} = -(1 - \varepsilon)x, \quad 0 < \varepsilon \ll 1,\]

\[x(0) = A_1. \tag{1.2.3}\]
Figure 1.5: In the left frame, we plotted the phase portrait of the singularly perturbed rescaled damped oscillator (1.2.2) (solid line) and of its reduced problem (dotted line). Note that all initial conditions eventually approach 0. Observe that, to leading order, all initial conditions align with the linear subspace associated with the reduced problem. In the right frame, we plotted the time series of the specific initial condition $A_1 = 1$ and $A_2 = 14.36$. Note that we set $\varepsilon = 0.1$ in both frames.

The solution to this equation is given by

$$x(t) = A_1 e^{-(1-\varepsilon)t} = A_1 e^{-t} + O(\varepsilon),$$

while the solution to the corresponding reduced equation reads $x_0(t) = A_1 e^{-t}$. In this case, then, the dynamics of the reduced equation offers a good approximation of the full dynamics for all time $t$, see Figure 1.6. A slightly more complicated version of this simple first order ODE will be analyzed in more detail in Section 1.3.3 in order to introduce the renormalization group (RG) method.

A second example of a regularly perturbed ODE is offered by the equation for a rescaled oscillator with small damping,

$$\ddot{x} + \varepsilon \dot{x} + x = 0, \quad x(0) = A_1, \quad \dot{x}(0) = A_2. \tag{1.2.4}$$

The solution to this full equation is given by

$$x(t) = C_1 e^{\frac{1}{2}(\varepsilon+\sqrt{\varepsilon^2-4})t} + C_2 e^{\frac{1}{2}(\varepsilon-\sqrt{\varepsilon^2-4})t} = A_2 \sin t + A_1 \cos t + O(\varepsilon t), \tag{1.2.5}$$

whereas the solution to the reduced equation reads $x_0(t) = A_2 \sin t + A_1 \cos t$. After an $O(\varepsilon^{-1})$ time scale the expansion of (1.2.5) is no longer well-ordered, and secular terms get a leading order influence. However, since the reduced equation misses the effect of the small damping, these secular terms do not appear in the solution to the reduced equation. Therefore, trajectories of the full problem do
1.2. Basic concepts

Figure 1.6: In the left frame, we plotted the solutions to the regularly perturbed equation (1.2.4) (solid line) and to the reduced equation (dotted line). The initial conditions here were $A_1 = 3$, $A_2 = 2$, and $\varepsilon = 0.1$. The trajectory corresponding to the reduced equation mimics the dynamics of the full equation, to leading order and over a bounded time interval. After some time, however, the dynamics of the reduced equation is completely different from the actual full dynamics. In the right frame, we plotted the solutions to the full and reduced equations corresponding to (1.2.3). The initial condition here was $A_1 = 3$, whereas $\varepsilon = 0.1$. Note that the trajectory of the reduced equation describes the dynamics of the full equation to leading order for all time.

Indeed follow the trajectories of the reduced problem to leading order but only up to $O(\varepsilon^{-1})$ time. This observation serves to highlight the important fact that the reduced equation of a regularly perturbed problem can also yield insufficient approximations, see Figure 1.6.

In conclusion, whereas one observes a radical change in the dimensionality of the dynamics of the singularly perturbed system (made explicit foremost by a reduction in the dimensionality of the system) when one sets the small parameter to zero, this is not the case for regularly perturbed problems.

1.2.2 Stability and bistability

The concept of bistability is best illustrated by Figure 1.7. Consider the double-well (two minima) potential landscape plotted in the figure. Because of friction, a marble dropped in this landscape eventually ends up in one of the two wells. For a given, sufficiently large but not too large, friction, the initial position of the marble fully determines the well in which it ends up resting. For instance, marble 1 in Figure 1.7 ends up in the first well, while marble 2 ends up in the second well. However, there is also the possibility that the marble lands exactly on the peak (maximum) in between the two wells (marble 3 in Figure 1.7). In this case, the marble does not move at all. This situation is highly unlikely of
course, since an arbitrarily small perturbation would cause the marble to move into one of the two wells. Mathematically, we call this peak location unstable, as opposed to the two wells which are termed stable. Accordingly, we say that this double-well landscape is of bistable nature.

Mathematically, a simple example of a bistable ODE is

\[ \dot{x} = x - x^3, \quad x(0) = A_1. \tag{1.2.6} \]

This first order ODE has two attracting (stable) fixed points at \( x = \pm 1 \) and an unstable fixed point at \( x = 0 \). This unstable fixed point acts as a separatrix between the two attractors at \( x = \pm 1 \), that is, negative initial conditions tend to \(-1\) and positive initial conditions go to \(+1\) without trajectories originating from one region being able to cross into the other, see Figure 1.8. Moreover, note the connection between (1.2.6) and the (graph of the) potential in Figure 1.7: \( x - x^3 > 0 \) in the regions where \( \dot{x} > 0 \) and \( x - x^3 < 0 \) in the regions where \( \dot{x} < 0 \). Mathematically speaking, \( x - x^3 \) corresponds to minus the derivative of \(-\frac{1}{2}x^2 + \frac{1}{4}x^4\), the graph of which corresponds to the potential of Figure 1.8.

Perhaps the simplest bistable PDE [26, 45] reads,

\[ U_t = U_{xx} + U - U^3, \]

which is identical to the \( U \)-component of (1.1.3) with \( \varepsilon = 0 \). Besides the trivial fixed points at \( U(x, t) \equiv \pm 1 \) and at \( U(x, t) \equiv 0 \), this PDE also has the spatially inhomogeneous stationary solutions \( U(x, t) = \pm \tanh (\frac{1}{2} \sqrt{2}x) \) that connect \( x = \mp 1 \) to \( x = \pm 1 \). In Section 1.3.2, we analyze the stability of one of these inhomogeneous
Figure 1.8: In the left frame, we plotted the phase portrait of the bistable ODE (1.2.6). The arrows indicate the direction in which a trajectory evolves. For instance, trajectories starting in $x = -0.5$ end up in $x = -1$, while trajectories starting in $x = 0.5$ end up in $x = 1$. In the right frame, we plotted the time series of several initial conditions. Positive initial conditions tend to $+1$ and negative initial conditions go to $-1$. Therefore, the unstable fixed point at $x = 0$ acts as a separatrix.

solutions to introduce the concept of the Evans function, see also Sections 2.2.1 and 3.4.1.

1.3 More advanced concepts

The three main mathematical tools which we use in this thesis are geometric singular perturbation theory (Chapter 2), the Evans function (Chapter 3), and the RG method (Chapter 4). Proceeding along the lines of the previous section, we introduce and explain the basic ideas underlying these concepts by means of three simple examples.

1.3.1 Geometric singular perturbation theory

Consider the reaction kinetics of the two-component limit model of (1.1.3) (with \( \tau = \varepsilon^{-1}, \alpha = -1, \beta = 0, \) and \( \gamma = 0 \)),

\[
\begin{align*}
\dot{u} &= u - u^3 + \varepsilon v, \\
\dot{v} &= \varepsilon (u - v),
\end{align*}
\]

where \( \dot{} = \frac{d}{dt}, u, v \in \mathbb{R}^1, \) and \( 0 < \varepsilon \ll 1. \) This small parameter \( \varepsilon \) gives the system its singular character. With a change of time scale, \( \hat{t} := \varepsilon t, \) this system is
transformed into
\[
\begin{align*}
\varepsilon u' &= u - u^3 + \varepsilon v, \\
v' &= u - v,
\end{align*}
\]
with \( ' = \frac{d}{dt} \). The time scale corresponding to \( t \) is termed the fast time scale, whereas the time scale corresponding to \( \hat{t} \) is called slow. Accordingly, the former system is called the fast system, while the latter one is called the slow system. These two systems are equivalent for \( \varepsilon \neq 0 \) but have very different reduced equations, i.e., limit systems as \( \varepsilon \to 0 \). The fast reduced system (FRS) reads
\[
\dot{u} = u - u^3 \quad \text{and} \quad v \equiv v_0 \in \mathbb{R}. \tag{1.3.2}
\]
This is a 1-parameter family of 1-dimensional systems parametrized by \( v \equiv v_0 \), a constant in \( \mathbb{R} \). The slow reduced system (SRS), on the other hand, is given by
\[
\begin{align*}
0 &= u - u^3, \\
v' &= u - v, \tag{1.3.3}
\end{align*}
\]
which is a 1-dimensional differential-algebraic system. In fact, the SRS is only defined exactly on the fixed points of the FRS by virtue of these points coinciding with the solutions to that constraint. Both reduced systems are simpler and of lower dimension compared to the full systems and therefore easier to analyze; see Figure 1.9 for their dynamics.
The idea underlying geometric singular perturbation theory is to analyze the full system (with \( \varepsilon \neq 0 \) but sufficiently small) by suitably combining the dynamics of the two limiting systems, see [33, 39, 43, 64] for instance. The foundations of the theory were laid out by Fenichel [27, 28], and the theory is accordingly also called Fenichel theory. We first need to recall certain definitions before we can state Fenichel’s persistence theorems.

Consider the following slow-fast system (in fast formulation)

\[
\begin{align*}
\dot{u} &= f(u, v; \varepsilon), \\
\dot{v} &= \varepsilon g(u, v; \varepsilon).
\end{align*}
\]

A manifold \( \mathcal{M}_\varepsilon \) is said to be *locally invariant* under the flow generated by (1.3.4) if there exists a neighborhood \( V \supset \mathcal{M} \) such that no orbit can leave \( \mathcal{M} \) without leaving \( V \).

A locally invariant manifold \( \mathcal{M}_0 \) is called *normally hyperbolic* if the eigenvalues of the linearization \( D_u f(u, v; 0) \) restricted to \( \mathcal{M}_0 \) are bounded away from the imaginary axis.

The *stable manifold* \( W^s(\mathcal{M}) \) and *unstable manifold* \( W^u(\mathcal{M}) \) of a manifold \( \mathcal{M} \) which is locally invariant under a flow \( \phi(t; \cdot) \) are defined as follows:

\[
\begin{align*}
W^s(\mathcal{M}) &= \{ y \mid d(\phi(t; y), \mathcal{M}) \to 0 \text{ as } t \to +\infty \}, \\
W^u(\mathcal{M}) &= \{ y \mid d(\phi(t; y), \mathcal{M}) \to 0 \text{ as } t \to -\infty \},
\end{align*}
\]

with \( \phi(0; y) = y \) and \( d(\cdot, \cdot) \) is the usual Euclidean distance.

Fenichel’s first persistence theorem states that, if the FRS has a normally hyperbolic invariant manifold \( \mathcal{M}_0 \), then, for \( \varepsilon \) small enough, the full system possesses a locally invariant slow manifold \( \mathcal{M}_\varepsilon \) that is \( \mathcal{O}(\varepsilon) \) close to \( \mathcal{M}_0 \) in the \( C^1 \) topology. Moreover, Fenichel’s second persistence theorem states that the full system also possesses locally invariant stable and unstable manifolds \( W^{s,u}(\mathcal{M}_\varepsilon) \) which are \( \mathcal{O}(\varepsilon) \) close to the stable and unstable manifolds \( W^{s,u}(\mathcal{M}_0) \) of the FRS.

We are ready to apply these the two persistence theorems to the problem at hand. The three manifolds \( \mathcal{M}_0^{\pm 1,0} := \{(u, v) \mid u = \pm 1, u = 0\} \) are normally hyperbolic, since each \( \mathcal{M}_0^{\pm 1,0} \) is invariant with respect to the FRS (1.3.2) and

\[
\left. \frac{d}{du}(u - u^3) \right|_{\mathcal{M}_0^{\pm 1,0}} = 1 - 3u^2 \big|_{\mathcal{M}_0^{\pm 1,0}} = \{-2, 1\}.
\]

Moreover,

\[
\begin{align*}
W^s(\mathcal{M}_0^{\pm 1}) &= \mathbb{R}_\pm \setminus \{0\} \times \mathbb{R}, \\
W^u(\mathcal{M}_0^{\pm 1}) &= \mathcal{M}_0^{\pm 1}, \\
W^s(\mathcal{M}_0^0) &= \mathcal{M}_0^0, \\
W^u(\mathcal{M}_0^0) &= [-1, 1] \setminus \{0\} \times \mathbb{R}.
\end{align*}
\]
By Fenichel’s first persistence theorem, the full system (1.3.1) possesses locally invariant slow manifolds $M_{\varepsilon}^{\pm 1,0}$ that are $O(\varepsilon)$ close to $M_{0}^{\pm 1,0}$, respectively. Note that $M_{0}^{\pm 1,0}$ themselves are no longer (locally) invariant manifolds as $\varepsilon \neq 0$. We now (formally) determine $M_{\varepsilon}^{\pm 1,0}$. By Fenichel theory, we know that $M_{\varepsilon}^{\pm 1,0}$ can be expressed as a graph $u = h_{\varepsilon}^{\pm 1,0}(v)$; local invariance then yields

$$u = h_{\varepsilon}^{\pm 1,0}(v) \Rightarrow \dot{u} = \frac{dh_{\varepsilon}^{\pm 1,0}}{dv} \Rightarrow \dot{v} = \varepsilon \frac{dh_{\varepsilon}^{\pm 1,0}}{dv}(u - v).$$

On the other hand, $\dot{v} = u - u^3 + \varepsilon v$. Combining these two expressions for $\dot{u}$, we obtain an equation for $h_{\varepsilon}^{\pm 1,0}(v)$,

$$\varepsilon \frac{dh_{\varepsilon}^{\pm 1,0}}{dv}(u - v) = u - u^3 + \varepsilon v.$$

Expanding $h_{\varepsilon}^{\pm 1,0}$ asymptotically in $\varepsilon$, $h_{\varepsilon}^{\pm 1,0}(v) = h_{0}^{\pm 1,0}(v) + \varepsilon h_{1}^{\pm 1,0}(v) + O(\varepsilon^2)$, and equating $O(1)$ terms of course yields the normally hyperbolic manifolds $M_{0}^{\pm 1,0}$; $h_{0}^{0} = 0$ and $h_{0}^{\pm 1} = \pm 1$. Substituting these expressions in the $O(\varepsilon)$ terms of the expansion yields,

$$h_{1}^{0} + v = 0 \quad \Rightarrow \quad M_{0}^{0} = \{(u,v) \mid u = h_{\varepsilon}^{0}(v) = -\varepsilon v + O(\varepsilon^2)\},$$

$$-2h_{1}^{\pm 1} + v = 0 \quad \Rightarrow \quad M_{\varepsilon}^{\pm 1} = \{(u,v) \mid u = h_{\varepsilon}^{\pm 1}(v) = \pm 1 + \frac{\varepsilon v}{2} + O(\varepsilon^2)\}.$$

By Fenichel’s second persistence theorem, $M_{\varepsilon}^{\pm 1}$ have local 2-dimensional stable manifolds, $W^s(M_{\varepsilon}^{\pm 1})$, and $M_{\varepsilon}^{0}$ has a local 2-dimensional unstable manifold, $W^u(M_{\varepsilon}^{0})$. Moreover, since the full problem (1.3.1) is planar, all initial conditions in between $M_{\varepsilon}^{\pm 1}$ and $M_{\varepsilon}^{0}$ lie in both $W^u(M_{\varepsilon}^{0})$ and $W^s(M_{\varepsilon}^{-1})$, and hence are forward asymptotic to $M_{\varepsilon}^{-1}$. A similar statement can be made for all initial conditions between $M_{\varepsilon}^{0}$ and $M_{\varepsilon}^{\pm 1}$. Moreover, since $M_{\varepsilon}^{\pm 1}$ possess attracting fixed points $\pm u_f$, respectively, $M_{\varepsilon}^{0}$ acts as a separatrix between $W^s(-u_f)$ and $W^s(u_f)$. In Figure 1.10, we plotted the dynamics of (1.3.1).

### 1.3.2 The Evans function

In this section, which can be seen as background information to the first part of Section 3.4.1, we introduce the notion of the Evans function. In particular, we use an Evans function to determine explicitly the linear stability properties of a heteroclinic stationary solution to a bistable PDE, that is, to determine the spectrum of the linearized flow corresponding to a certain time independent solution joining two distinct trivial states of the PDE. Note that, under certain rather general conditions, which apply to the specific PDE under consideration in this section, linear stability of a solution implies that the solution is also nonlinear stable [59].

For concreteness of presentation, we work with the bistable PDE introduced in
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Figure 1.10: In the left frame, we plotted the dynamics of the FRS and the SRS of (1.3.1), see Figure 1.9. In the right frame, we plotted the dynamics of the full system with $\varepsilon = 0.1$. The fixed points of the system are $(0,0)$ and $\pm u_f = \pm (1.0488, 1.0488)$ (to four decimals places). The former is unstable, while the latter two are stable. Observe the locally invariant slow manifolds $M_{\varepsilon}^{\pm 1,0}$ close to $M_{0}^{\pm 1,0}$, respectively, whose existence are guaranteed by Fenichel’s first persistence theory. The asymptotic dynamics on $M_{\varepsilon}^{\pm 1,0}$ is given by the SRS to leading order. For the two attracting manifolds $M_{\varepsilon}^{\pm 1}$ this can be seen in the plot. Moreover, note that $M_{\varepsilon}^{0}$ acts as a separatrix between $W^s(-u_f)$ and $W^s(u_f)$.
Section 1.2.2,

\[ U_t = U_{xx} + U - U^3, \]  
(1.3.5)

which also forms the backbone of the three-component system (1.1.3). Equation (1.3.5) possesses the heteroclinic stationary solutions \( U(x, t) = u^\pm_h(x) := \pm \tanh \left( \frac{1}{2} \sqrt{2} x \right) \), see also Section 2.2.1. Here, we analyze the solution \( u^+_h(x) \) connecting \( U = -1 \) (at \( x = -\infty \)) to \( U = +1 \) (at \( x = \infty \)), see Figure 1.11. Then, by symmetry considerations, we replicate our results for the remaining heteroclinic solution \( u^-_h(x) \). For notational convenience, we drop the superscript ‘+’ in \( u^+_h(x) \).

To determine the linear stability properties of \( u_h(x) \), we consider the (standard) small perturbation of \( u_h(x) \)

\[ U(x, t) = u_h(x) + e^{\lambda t} u(x), \]  
(1.3.6)

where \( x \in \mathbb{R}, \lambda \in \mathbb{C} \), and \( u \) is an integrable function. Note that this is a realistic assumption: nonintegrable perturbations cannot be assumed to be small in any natural topology. Plugging this perturbation into (1.3.5), and linearizing we obtain the stability problem

\[ 0 = u_{xx} + (1 - 3(u_h)^2 - \lambda) u = u_{xx} + \left( 3 \text{sech}^2 \left( \frac{1}{2} \sqrt{2} x \right) - (\lambda + 2) \right) u. \]  
(1.3.7)

Thus, the linearized operator reads

\[ \mathcal{L} = \frac{d^2}{dx^2} + \left( 3 \text{sech}^2 \left( \frac{1}{2} \sqrt{2} x \right) - 2 \right); \]

its spectrum is equal to the union of its point spectrum and its essential or continuous spectrum. The part of the spectrum in the left half of the complex plane
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corresponds to stable eigendirections, whereas spectrum in the right half plane corresponds to unstable eigendirections, see (1.3.6).

The essential spectrum $\sigma_{\text{ess}}$ covers instabilities under perturbations ‘at infinity’. Usually, it is straightforward to determine this essential spectrum for a localized solution [59, 63]; here, $\sigma_{\text{ess}} = \{ \lambda \in \mathbb{R} \mid \lambda \leq -2 \}$. Roughly speaking, this can be concluded as follows: for $|x|$ large, (1.3.7) is to leading order given by

$$u_{xx} - (\lambda + 2)u = 0,$$

with the general solution reading

$$u(x) = A_1 e^{\sqrt{\lambda + 2} x} + A_2 e^{-\sqrt{\lambda + 2} x}.$$

This solution becomes oscillatory, also in the neighborhood of infinity, for $\lambda \in (-\infty, -2] = \sigma_{\text{ess}}$. So, although these solutions $u(x)$ are bounded, they fail to be integrable for $\lambda \in \sigma_{\text{ess}}$ and thus do not belong the function space we settled on in advance.

The point spectrum, that is, the set of isolated eigenvalues corresponding to localized integrable eigenfunctions, is usually much harder to determine. It is here where the Evans function, an analytic function whose zeroes correspond to these isolated eigenvalues [1, 21–24], comes into play. Generally, and for $\lambda \notin \sigma_{\text{ess}}$, it is possible to construct two sets $E_1$ and $E_2$ of solutions to the linearized system in such a way that $E_1$ forms a basis for the subspace of solutions that approach zero at positive spatial infinity, while $E_2$ forms a basis for the subspace of solutions that approach zero at negative spatial infinity. Together $E_1$ and $E_2$ span the solution space of the linear problem. Since the eigenfunctions corresponding to isolated eigenvalues necessarily converge to zero in both spatial limits, as they are integrable, any eigenfunction is an element of both $E_1$ and $E_2$. The Evans function $\mathcal{D}$ is defined as the determinant of the Wronskian of the fundamental matrix solution generated by $E_1$ and $E_2$. In general, this Evans function is nonzero, i.e., $E_1$ and $E_2$ together form a basis of the solution space. However, for any specific value of $\lambda$ to be in the point spectrum, $E_1$ and $E_2$ must be linearly dependent, and therefore the Wronskian must be zero. Thus, zeroes of the Evans function $\mathcal{D}$ correspond to isolated eigenvalues in the point spectrum of the linearized operator [1, 59, 63]. Note that, an Evans function is only determined up to a scaling constant by construction. Therefore, there exists a whole 1-dimensional subspace of Evans functions rather than a unique Evans function.

We now proceed to determine the point spectrum of (1.3.7) and the corresponding eigenfunctions with the help of an Evans function. Upon rescaling $y$ via $y := \frac{1}{2}\sqrt{2}x$, the stability problem (1.3.7) reads

$$0 = u_{yy} + (6\text{sech}^2 y - P^2) u,$$

(1.3.8)
where \( P^2 := 2\lambda + 4 \). To avoid that \( \lambda \in \sigma_{\text{ess}} \), \( P^2 \) should satisfy \( P^2 \in \mathbb{C}\{(-\infty,0]\). It follows that we can select the square root branch in such a way that \( P \in \mathbb{C}^+ = \{ P \in \mathbb{C} \mid \Re(P) > 0 \} \). It is hard to solve this problem straightforwardly, but this can be done via a reduction to (so-called) hypergeometric functions. To do so, we introduce the function \( F(y) := u(y) \cosh^P y \) and the independent variable \( z := \frac{1}{2} (1 - \tanh y) \). Since \( y \in (\infty, \infty) \), it follows that \( z \in (0,1) \) and

\[
\frac{d}{dy} = \frac{d}{dz} \frac{dz}{dy} = -\frac{\operatorname{sech}^2 y}{2} \frac{d}{dz} = -2z(1-z) \frac{d}{dz}.
\]

Therefore, (1.3.8) transforms to

\[
0 = F_{yy} - 2P \tanh(y) F_y - (P^2 + P - 6) \operatorname{sech}^2(y) F
= z(1-z) F_{zz} + (P+1)(1-2z) F_z - (P+3)(P-2) F,
\]

where in the two steps the respective factors \( \operatorname{sech}^P y \) and \( 4z(1-z) \) are divided out. This is a hypergeometric differential equation. The solution space of the general hypergeometric differential equation,

\[
z(1-z) F_{zz} + (c - (a + b + 1) z) F_z - ab F = 0,
\]

is spanned by the hypergeometric series

\[
F(a, b|c|z) := 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\ldots(a+n-1)b(b+1)\ldots(b+n-1)}{n!c(c+1)\ldots(c+n-1)} z^n
\]

and \( z^{1-c} F(a-c+1, b-c+1|2-c|z) \), see [48] for example. Here, \( a = P+3, b = P-2, \) and \( c = P+1, \) and therefore \( F(P+3, P-2|P+1|z) \) and \( z^{-P} F(3, -2|1-P|z) \) are two independent solutions of (1.3.9). Note that \( F(3, -2|1-P|z) \) is a quadratic polynomial in \( z \),

\[
F(3, -2|1-P|z) = 1 - \frac{6}{1-P} z + \frac{12}{(1-P)(2-P)} z^2.
\]

Next, we define a scaled version of \( z^{-P} F(3, -2|1-P|z) \) which is also a solution to (1.3.9)

\[
J^-(z; P) := (2z)^{-P}(1-P)(2-P) F(3, -2|1-P|z).
\]

Note that the scaling factor \( 2^{-P} \) is incorporated only for our convenience. Now, \( J^-(z; P) \) solves (1.3.9), has no singularities at \( P = 1, 2, \) and \( \lim_{z \to 1} J^-(z; P) \) is bounded. Working backwards, we find then that

\[
u^-(y; P) := \operatorname{sech}^P(y) J^-(z; P), \quad \text{with} \quad z = \frac{1}{2} (1 - \tanh y),
\]

solves (1.3.8) and \( \lim_{y \to -\infty} u^-(y; P) = 0 \). By the reversibility symmetry of equation (1.3.8) (\( \text{i.e.}, \) its invariance under the change \( y \to -y \))

\[
u^+(y; P) := u^-(y; P) = \operatorname{sech}^P(y) J^-(1-z; P)
\]
is also a solution of (1.3.8) and \( \lim_{y \to -\infty} u^+(y; P) = 0 \). Hence, we have explicitly constructed two independent solutions of (1.3.8). Plainly, \( u^-(y; P) \) corresponds to the set \( E_2 \), whereas \( u^+(y; P) \) corresponds to \( E_1 \).

Using these independent solutions, we now define the Evans function

\[
\tilde{\mathcal{D}}(P) := \begin{vmatrix}
- \frac{d}{dy} u^-(y; P) & u^+(y; P) \\
\frac{d}{dy} u^-(y; P) & \frac{d}{dy} u^+(y; P)
\end{vmatrix}.
\]

Note that the function \( \tilde{\mathcal{D}} \) is independent of \( y \) by Abel’s identity, see also [1]. Note also that, since \( u^+ \) has been obtained from \( u^- \) by an application of the reversibility symmetry, \( u^+ \) and \( u^- \) can only be dependent for values of \( P \) if \( u^\pm = \pm u^- \). This implies that the eigenfunctions are either odd or even. In general, \( \mathcal{D}(P) \) cannot be computed explicitly. However, in the context of this example, a tedious yet direct computation shows that

\[
\tilde{\mathcal{D}}(P) = u^-(y; P) \frac{d}{dy} u^+(y; P) - u^+(y; P) \frac{d}{dy} u^-(y; P) = \operatorname{sech}^P(y) J^- (z; P) \left( J^- (1 - z; P) \frac{d}{dy} \left( \operatorname{sech}^P(y) \right) \right)
+ \operatorname{sech}^P(y) \frac{d^2}{dy^2} J^- (1 - z; P)) - \operatorname{sech}^P(y) J^- (1 - z; P)
\left( J^- (z; P) \frac{d}{dy} \left( \operatorname{sech}^P(y) \right) + \operatorname{sech}^P(y) \frac{d^2}{dy^2} J^- (z; P)) \right)
\]

\[
= \frac{\operatorname{sech}^2 y - \operatorname{sech}^2 (1 - z)}{2^{2p - 1} z^2 (1 - z)^p} \left( (1 - z)(Q(z; P)Q_z (1 - z; P) - Q(1 - z; P)Q_z (z; P)) - PQ(z; P)Q(1 - z; P) \right)
= 2P \left( 12z(1 - z) (-12z^2 + 12z + P^2 - 4) - (144z^4 - 288z^3 - 12P^2z^2 + 192z^2 + 12P^2z - 48z + P^4 - 5P^2 + 4P) \right)
= -2P \left( P^2 - 1 \right) \left( P^2 - 4 \right).
\]

Recalling the definition of \( P \), we find that the Evans function associated with the linear stability of the front solution \( u_b(x) \) of (1.3.5) reads

\[
\mathcal{D}(\lambda) = \tilde{\mathcal{D}}(P(\lambda)) = -8\sqrt{2} \lambda \left( \lambda + \frac{3}{2} \right) \sqrt{-\lambda + \frac{3}{2}}.
\]

The zeroes of this Evans function, and thus the isolated eigenvalues of (1.3.7), are \( \lambda = 0 \) and \( \lambda = -\frac{3}{2} \). The zero eigenvalue is due to the translation invariance of the system (see also the next paragraph), and therefore the heteroclinic solution \( u_b(x) \) is linearly stable as \( \lambda < 0 \). Note also that the essential spectrum corresponds to \( \{ \lambda \in \mathbb{R} \mid \lambda \leq -2 \} \), so that \( \mathcal{D}(\lambda) \) is purely imaginary for \( \lambda \in \sigma_{\text{ess}} \). Thus for this system also the essential spectrum can be obtained from the Evans function calculation.
Since the original problem (1.3.5) is translation invariant – that is, every translated version (in $x$) of a solution to (1.3.5) is also a solution to (1.3.5) – $\lambda = 0$ was expected to be an eigenvalue with corresponding eigenfunction equal to (a scaled version of) the derivative of the heteroclinic solution $u_h(x)$. A short computation shows that

$$\lambda = 0 \implies P = 2 \implies J^-(z; 2) = 3 \implies u^-(y; 2) = 3 \text{sech}^2 y = u^+(x; 2).$$

Thus, $\lambda = 0$ indeed yields an even eigenfunction that is a scaled version $u'_h(x)$ – this is given by $\frac{1}{2} \sqrt{2} \text{sech}^2 \left( \frac{1}{2} \sqrt{2} x \right)$ – namely $3 \text{sech}^2 \left( \frac{1}{2} \sqrt{2} x \right)$. The other eigenvalue, $\lambda = -\frac{3}{2}$, yields

$$\lambda = -\frac{3}{2} \implies P = 1 \implies J^-(z; 1) = -3(1 - 2z) \implies u^-(y; 1) = -3 \text{sech} y \tanh y = -u^+(y; 1).$$

Thus, the second (odd) eigenfunction is given by $-3 \tanh \left( \frac{1}{2} \sqrt{2} x \right) \text{sech} \left( \frac{1}{2} \sqrt{2} x \right)$, see Figure 1.11.

1.3.3 The renormalization group method

There are a variety of methods which entail a form or another of renormalization, and all these methods are called RG methods. Originally, renormalization was developed in field theory and in the theory of phase transitions [42, 61, 66, 67] to cope with irregularities in critical exponents. More recently, the RG method has been adapted to deal with various perturbation and asymptotic problems arising both in ODEs and PDEs [7]. These new methods use either a continuous invariance condition [7, 10] or a discrete one [68, 69]; see also [49] for a discussion and examples of this RG method. Another type of an RG method using scaling invariance to prove rigorously that the solutions to a nonlinear parabolic PDEs have a particular asymptotic form, was developed in [6] and has originated in statistical physics. This method has also been used to prove stability of solutions, see [18].

The aforementioned RG methods also differ from the RG method we use in this thesis. Our method was developed in [55] and uses renormalization to establish the existence and nonlinear stability properties of certain special solutions to PDEs and ODEs. Moreover, it was also used in [16] to validate a formally derived system of ODEs describing the dynamics of semi-strongly interacting fronts. It is for this purpose that we employ the RG method, see Chapter 4.

We remark that in many physical problems, the dynamics of the positions of fronts are referred to as collective coordinates, see for example [3]. In [19, 20, 55], it is proved that for a certain type of weakly interacting structures, a reduction to such collective coordinates always possible is. However, in this thesis we are
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Dealing with semi-strongly interacting structures. For such a structures, the author is not aware of methods other than that developed in [55] (and used in this thesis) to prove rigorously the collective coordinates reduction.

To explain the basic ideas behind the RG method we use in Chapter 4 [16, 31, 47, 55], we consider the first order nonautonomous ODE,

\[ \dot{x} = -(1 - \varepsilon(t))x, \quad 0 < |\varepsilon(t)| \leq \varepsilon_0 < 1, \]

(1.3.10)

for \( \varepsilon_0 \) small enough (see below). Note that for \( \varepsilon(t) \equiv \varepsilon \), (1.3.10) can be solved explicitly: \( x(t) = A_0 e^{-\varepsilon(1-\varepsilon)t} \), see (1.2.3). It follows that there is a globally attracting fixed point at \( x = 0 \). This is, of course, also the case for any general \( \varepsilon(t) \) satisfying \( 0 < |\varepsilon(t)| \leq \varepsilon_0 \ll 1 \). However, for the sake of demonstration, we pretend that both the exact solution of (1.3.10) and the existence of the attracting fixed point at the origin are unavailable. Using the RG method, we identify the attracting fixed point at \( x = 0 \) and prove that it is, indeed, globally attracting.

The variation of constants formula applied to (1.3.10) yields

\[ x(t) = e^{-t}A_0 + \int_0^t e^{-(t-s)}\varepsilon(s)x(s)ds. \]

(1.3.11)

Defining

\[ y(t) := \sup_{0 < t' < t} e^{t'}|x(t')| \]

and multiplying both sides of (1.3.11) by \( e^t \), we obtain the estimate

\[ e^t x(t) \leq |A_0| + \varepsilon_0 \int_0^t y(s)ds \leq |A_0| + \varepsilon_0 ty(t). \]

Note that the last term is secular: it becomes arbitrarily large for increasing \( t \). Taking the supremum over all time up to time \( \tau \) yields

\[ y(\tau) \leq |A_0| + \varepsilon_0 \tau y(\tau) \implies y(\tau) \leq \frac{|A_0|}{1 - \varepsilon_0 \tau}. \]

It seems that, according to this inequality, the secular term yields blow up for too large \( \tau \), see also Figure 1.12. A prototypical area of applications of the RG method concerns problems on which the estimates exhibit similar secular behavior. Problems solved by using the RG method often have this property of an uncontrollable secular term. Using these naive estimates, we have an \textit{a priori} finite time control over the solution,

\[ |x(t)| \leq 2e^{-t}|A_0|, \quad t \in \left(0, \frac{1}{2\varepsilon_0}\right). \]

(1.3.12)
The idea behind the RG method is to choose a specific new initial condition \( A_1 \) at time \( t = \frac{1}{2\varepsilon_0} \) – the moment when (1.3.12) loses its validity – and to repeat the above procedure using this new initial condition \( A_1 \). (Note that we can also renormalize at any other moment smaller than \( \frac{1}{\varepsilon_0} \). This specific choice of \( A_1 \) is

\[
A_1 := 2e^{-\frac{1}{2\varepsilon_0}}|A_0|,
\]

namely, the upperbound of (1.3.12) at \( t = \frac{1}{2\varepsilon_0} \). Using this new initial condition, we now consider the initial value problem

\[
\dot{x} = -(1 - \varepsilon(t))x, \quad 0 < |\varepsilon(t)| \leq \varepsilon_0 \ll 1,
\]

\[
x \left( \frac{1}{2\varepsilon_0} \right) = A_1.
\]

The variation of constants formula combined with the above estimates yields, then,

\[
|x(t)| \leq 2e^{-\left(t - \frac{1}{2\varepsilon_0}\right)}|A_1| = 4e^{-t}|A_0|, \quad t \in \left( \frac{1}{2\varepsilon_0}, \frac{1}{\varepsilon_0} \right).
\]

Repeating this renormalization procedure, we obtain

\[
|A_{n+1}| := 2e^{-\frac{1}{2\varepsilon_0}}|A_n| = 2^{(n+1)}e^{-\frac{n+1}{2\varepsilon_0}}|A_0|.
\]

An easy induction shows, then, that

\[
|x(t)| \leq 2e^{-\left(t - \frac{n}{2\varepsilon_0}\right)}|A_N| = 2^{(n+1)}e^{-t}|A_0|, \quad t \in \left( \frac{n}{2\varepsilon_0}, \frac{n+1}{2\varepsilon_0} \right).
\]

Thus,

\[
|x(t)| \leq 2|A_0|e^{-t(1 - \varepsilon_0 \log 2)} \quad \text{for all } t.
\]

Since \( \varepsilon_0 \) is a small enough, which in this particular case means \( \varepsilon_0 < \frac{1}{2 \log 2} \), we obtain a net contraction of the initial condition \( A_0 \). Therefore, there is a globally attracting fixed point of (1.3.10) at \( x(t) = 0 \). See Figure 1.12.

In Chapter 4, we use this same method to establish the attractivity of a set spanned by \( N \)-front solutions. There, however, we do not choose a different initial condition when we renormalize; instead, we choose a different position of the fronts we linearize about.

### 1.4 Outline

As we already remarked, system (1.1.3) is the central object of study in this thesis. Our ultimate goal is to understand the semi-strong front and pulse interactions...
Figure 1.12: Schematic depiction of the main idea underlying the RG method. After a finite time, the secular term becomes too large (diverges) and one has to renormalize. For (1.3.10), this entails to choosing a new initial condition.

present in this system. Before we can thoroughly study those interactions, we need to first study the existence and stability properties of the ‘simple’ asymptotic states. That is, we need to analyze the existence and stability of several stationary or uniformly traveling 1- or 2-pulse solutions (2- or 4-front solutions). In all of these cases, and for both the existence and stability problems, we find that we can reduce the system of PDEs to a 6-dimensional system of first order ODEs in the spatial variable $\xi$, so that the analysis reduces to investigating these ODEs. Note that these ODEs turn out to be autonomous and nonlinear for the existence problem and nonautonomous but linear for the stability problem.

In Chapter 2, we consider the existence problem. This chapter was published in 2009 under the title *Pulse dynamics in a three-component system: existence analysis* in the *Journal of Dynamics and Differential Equations*, and it is joint work with A. Doelman and T.J. Kaper. The main mathematical tool we use in that chapter is geometric singular perturbation theory (see Section 1.3.1).

In Chapter 3, we consider the nonlinear stability problem. This chapter was published in 2008 under the title *Pulse dynamics in a three-component system: Stability and bifurcations* in the journal *Physica D*, and it is also joint work with A. Doelman and T.J. Kaper. The main mathematical tool used here is the Evans function (see Section 1.3.2).
Chapter 1. Introduction

It should be noted that these first two chapters are more than merely preliminary work towards the interaction analysis of the final chapter, as they deal with several other issues. For example, we analyze the way in which the second inhibitor, the $W$-component, alters the dynamics of the two-component limit model (1.1.1). More specifically, we map out the type of dynamics exhibited by the three-component model which is absent from the two-component limit model (1.1.1). As in the case of the original Purwins system (1.1.2), several solutions only exist or are only stable in the extended model (1.1.3). For instance, stationary 2-pulse solutions do not exist in the two-component limit model (Section 2.6) but do exist in the extended three-component model (Section 2.5.2). In a similar vein, stationary 1-pulse solutions can only emerge from a saddle-node bifurcation in the extended system; this is not the case in the limit system, see Corollary 3.4.2 and Figure 3.3. We also study the possible bifurcations of stable stationary 1-pulse solutions. Besides the aforementioned saddle-node bifurcation, a 1-pulse solution can bifurcate into a breathing 1-pulse solution (Hopf bifurcation) or into a traveling 1-pulse solution as the bifurcation parameter $\tau$ keeps increasing, see Sections 2.4 and 3.5.2. Note that the latter bifurcation can be either supercritical or subcritical, see Lemma 2.4.1 and Figure 3.11. Moreover, observe that these bifurcations only occur for large $\tau = O(\varepsilon^{-2})$. This makes the bifurcation analysis much more involved compared to the existence and stability analysis of stationary 1-pulse solutions.

In the final chapter, we analyze the semi-strong front and pulse interactions. Using the RG method (see Section 1.3.3), we derive systems of ODEs describing the motion of the various fronts of an $N$-front solution. This derivation contains a fully nonlinear PDE analysis, in contrast to the previous chapters. Eventually, we analyze the derived ODEs to understand the semi-strong front and pulse dynamics. This chapter has been recently submitted for publication under the title *Front interactions in a three-component system*, and it is joint work with A. Doelman, T.J. Kaper, and K. Promislow.

Finally, we would like to clarify that the material in this thesis only offers a the first few glimpses into the dynamics generated by (1.1.3). One of the main ‘gaps’ in our current understanding is related to the fact that we only study interactions in the parameter regime where $\tau = O(1)$, see the last chapter. In particular, we do not study the much more interesting regime $\tau = O(\varepsilon^{-2})$, where uniformly traveling, breathing, and stationary 1-pulse solutions co-exist and interact, see Chapters 2 and 3. The reason behind that is quite fundamental: in that regime, the essential spectrum associated with the stability of an $N$-front solution lies asymptotically close to the imaginary axis. As a consequence, some of the central estimates on which the RG method is built no longer hold. Our analysis is also limited in the sense that we only consider patterns in one spatial dimension; a natural and challenging future project concerns the analysis of the planar variant of (1.1.3). This is a natural next step, for instance to make a bet-
ter comparison with the (numerical) observations obtained for the original model (1.1.2). Two *a priori* bottlenecks for this analysis concern the extensions of, first, the dynamical system-informed *spatial dynamics* approach and of the RG method to a 2-dimensional setting. To the former bottleneck there exist solutions in the literature, while the latter bottleneck is still largely unsolved.