A game for the Borel functions
Semmes, B.T.

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Chapter 1

Introduction

This thesis is divided into two parts. In the first part, we present a game-theoretic characterization of the Borel functions. We define a Wadge-style game, $G(f)$, and prove the following theorem:

1.0.1. Theorem. A function $f : \omega^\omega \to \omega^\omega$ is Borel $\iff$ Player II has a winning strategy in $G(f)$.

In the second part of the thesis, we turn our attention to the analysis of low-level Borel functions, summarized by the following diagram:

$$
\Lambda_{1,3} \quad \Lambda_{2,3} \\
\Lambda_{1,2} \quad \Lambda_{3,3} \\
\Lambda_{1,1} \quad \Lambda_{2,2}
$$

The notation $\Lambda_{m,n}$ denotes the class of functions $f : A \to \omega^\omega$ such that $A \subseteq \omega^\omega$ and $f^{-1}[Y]$ is $\Sigma_n^0$ in the relative topology of $A$ for any $\Sigma_m^0$ set $Y$. The two main results of the second part of the thesis are decomposition theorems for the $\Lambda_{2,3}$ and $\Lambda_{3,3}$ functions.

1.0.2. Theorem. A function $f : \omega^\omega \to \omega^\omega$ is $\Lambda_{2,3}$ $\iff$ there is a $\Pi_2^0$ partition $\langle A_n : n \in \omega \rangle$ of $\omega^\omega$ such that $f \upharpoonright A_n$ is Baire class 1.

1.0.3. Theorem. A function $f : \omega^\omega \to \omega^\omega$ is $\Lambda_{3,3}$ $\iff$ there is a $\Pi_2^0$ partition $\langle A_n : n \in \omega \rangle$ of $\omega^\omega$ such that $f \upharpoonright A_n$ is continuous.
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These results extend the decomposition theorem of John E. Jayne and C. Ambrose Rogers for the \( \Lambda_{2,2} \) functions.

1.0.4. **Theorem (Jayne, Rogers).** A function \( f : \omega^\omega \to \omega^\omega \) is \( \Lambda_{2,2} \) \( \iff \) there is a closed partition \( \langle A_n : n \in \omega \rangle \) of \( \omega^\omega \) such that \( f \upharpoonright A_n \) is continuous.

It should be noted that Jayne and Rogers proved a more general version of Theorem 1.0.4 [6]. In this thesis, however, we only prove decomposition theorems for total functions on the Baire space.

The author was motivated by two questions of Alessandro Andretta:

1. Is there a Wadge-style game for the (total) \( \Lambda_{3,3} \) functions?
2. Is Theorem 1.0.3 true?

In the second part of the thesis, we answer both questions affirmatively. The result for the Borel functions was obtained accidentally, while the author was investigating questions (1) and (2).

A brief summary follows. In Chapter 2, we define the tree game and show that it characterizes the Borel functions. In Chapter 3, we begin our analysis of low-level Borel functions with the three simplest classes.

\[
\Lambda_{1,2} \subset \Lambda_{2,2} \subset \Lambda_{1,1}
\]

In preparation for Chapters 4 and 5, we prove the Jayne-Rogers theorem and prove that the above containments are proper. In Chapter 4, we extend the analysis to the \( \Lambda_{1,3} \) and \( \Lambda_{2,3} \) functions.

\[
\Lambda_{1,3} \subset \Lambda_{2,3} \subset \Lambda_{1,2} \subset \Lambda_{2,2} \subset \Lambda_{1,1}
\]
We prove the decomposition theorem for $\Lambda_{2,3}$ and prove that the additional containments are proper. In Chapter 5, we complete the picture with an analysis of the $\Lambda_{3,3}$ functions.

1.1 Background

Unless otherwise indicated, we use notation that is standard in descriptive set theory. For all undefined terms, we refer the reader to [8].

We use the symbol $\subseteq$ for containment and $\subset$ for proper containment. For sets $A$ and $B$, we let $^BA$ denote the set of functions that map $B$ to $A$. The notation $^{<B}A$ denotes

$$\bigcup_{b \in B} ^bA$$

and we define $^{\leq B}A := ^{<B}A \cup ^BA$. In particular, $^{<\omega}A$ is the set of finite sequences of natural numbers and $^{\leq\omega}A$ is $^{<\omega}A \cup ^{\omega}A$.

For a finite sequence $s \in ^{<\omega}A$, we define $[s]_A := \{x \in ^{\omega}A : s \subseteq x\}$. If the $A$ is understood from the context, we may simply write $[s]$. We use the symbol $\cdot$ for concatenation of sequences. For $n \in \omega$, let $s^n$ denote the sequence $s \cdot s \cdot \ldots \cdot s$, with $s$ appearing $n$ times, and let $s^*$ denote the infinite sequence $s \cdot s \cdot s \cdot \ldots$ in $^{\omega}A$. If $s$ is a singleton sequence, $\langle a \rangle$, then when concatenating we may write $a$ instead of $\langle a \rangle$ without danger of confusion. Thus, we may write $a^n$ instead of $\langle a \rangle^n$, and the reader will realize that we mean concatenation of sequences and not exponentiation. The notation $\text{lh}(s)$ is used for the length of $s$, so $\text{lh}(s) := \text{dom}(s)$. If $s$ is non-empty, we define $\text{pred}(s) := s \upharpoonright \text{lh}(s) - 1$ to be the immediate predecessor of $s$. The set of immediate successors of $s$ is denoted by $\text{succ}_A(s) := \{s^a : a \in A\}$. If the $A$ is understood from the context, we may write $\text{succ}(s)$.

We say that a set $T \subseteq ^{<\omega}A$ is a tree if $s \subseteq t \in T \Rightarrow s \in T$. For a set $T \subseteq ^{<\omega}A$, we define $\text{tree}(T) := \{s : \exists t \in T(s \subseteq t)\}$. For a tree $T \subseteq ^{<\omega}A$ and $s \in ^{<\omega}A$, we define $T[s] := \{t \in T : t \subseteq s \text{ or } s \subseteq t\}$. The notation $\text{tn}(T)$ is used to denote the terminal nodes of $T$, so $\text{tn}(T) := \{s \in T : t \supset s \Rightarrow t \notin T\}$. The notation $[T]$ is used to denote the set of infinite branches of $T$, so $[T] := \{x \in ^{\omega}A : \forall n \in \omega(x \upharpoonright n \in T)\}$. The tree $T$ is linear if $s \subseteq t$ or $t \subseteq s$ for all $s, t \in T$. The tree $T$ is finitely branching if $s \in T \Rightarrow \text{succ}(s) \cap T$ is finite. A function $\phi : T \rightarrow ^{<\omega}B$ is monotone if $s \subseteq t \in T \Rightarrow \phi(s) \subseteq \phi(t)$ and length-preserving if $\text{lh}(\phi(s)) = \text{lh}(s)$. A function $\phi : ^{<\omega}A \rightarrow ^{<\omega}B$ is infinitary if

$$\bigcup_{s \subseteq x} \phi(s)$$

is infinite for every $x \in ^{\omega}A$. 
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There is a minor ambiguity regarding the [ ] notation: if $\emptyset$ is considered to be a sequence in $\langle A \rangle$, then $[\emptyset] = \omega A$. If, however, we view $\emptyset$ as a tree, then $[\emptyset] = \emptyset$. From the context, it will be clear which meaning is intended.

We work in the theory $\text{ZF} + \text{DC}(\mathbb{R})$: that is to say, $\text{ZF}$ with dependent choice over the reals. In terms of topological spaces, we will be working exclusively with the Cantor space, the Baire space, and subspaces of the Baire space. If we are considering a subspace $A \subseteq \omega \omega$, we will always use the relative topology as the topology of $A$.

For a metrizable space $X$, the Borel hierarchy $\Sigma^0_\alpha(X)$, $\Pi^0_\alpha(X)$, and $\Delta^0_\alpha(X) := \Sigma^0_\alpha(X) \cap \Pi^0_\alpha(X)$ is defined as usual for $\alpha < \omega_1$. If the space $X$ is understood, then we may write $\Sigma^0_\alpha$, $\Pi^0_\alpha$, and $\Delta^0_\alpha$. Above the Borel sets lies the projective hierarchy $\Sigma^1_\alpha(X)$, $\Pi^1_\alpha(X)$, and $\Delta^1_\alpha(X) := \Sigma^1_1(X) \cap \Pi^1_1(X)$. In terms of the projective hierarchy, we will only need the classical fact that the Borel sets are equal to $\Delta^1_1$ for Polish spaces. If $X$ and $Y$ are metrizable spaces, then $f : X \to Y$ is continuous if $f^{-1}[U]$ is open for every open set $U$ of $Y$, and a function $f : X \to Y$ is Baire class 1 if $f^{-1}[U]$ is $\Sigma^0_\alpha$ for every open set $U$ of $Y$. Recursively, for $1 < \xi < \omega_1$, $f : X \to Y$ is Baire class $\xi$ if it is the pointwise limit of functions $f_n : X \to Y$, where each $f_n$ is Baire class $\xi_n$ with $\xi_n < \xi$. A function $f : X \to Y$ is Borel if $f^{-1}[U]$ is Borel for every open (equivalently, Borel) set of $Y$.

By the classical work of Lebesgue, Hausdorff, and Banach, if $Y$ is also separable, then a function $f : X \to Y$ is Baire class $\xi$ if $f^{-1}[U]$ is $\Sigma^0_\xi$ in $X$ for every open set $U$ of $Y$. So, in this case, the Borel functions are equal to the union of the Baire class $\xi$ functions. If, in addition, $X$ is separable and zero-dimensional, then $f$ is Baire class 1 iff $f$ is the pointwise limit of continuous functions. We will be working with functions $f : A \to \omega \omega$ with $A \subseteq \omega \omega$, so the above facts will hold.

We define $\Lambda_{m,n}$ to be the set of functions $f : A \to \omega \omega$ such that $A \subseteq \omega \omega$ and $f^{-1}[Y]$ is $\Sigma^0_\alpha$ for any $\Sigma^0_\alpha$ set $Y$. Thus, for example, “$\Lambda_{1,1}$” is the same as continuous, “$\Lambda_{1,2}$” is the same as Baire class 1, and “$\Lambda_{1,3}$” is the same as Baire class 2.

The $\subseteq$ containments for the $\Lambda_{m,n}$ classes are trivial.

1.1.1. Proposition. For $m, n \geq 1$, $\Lambda_{m+1,n} \subseteq \Lambda_{m,n}$ and $\Lambda_{m,n} \subseteq \Lambda_{m+1,n+1}$.

1.1.2. Proposition. For $m, n \geq 1$ and $k \geq 0$, $\Lambda_{m,n} \subseteq \Lambda_{m+k,n+k}$.

1.1.3. Proposition. Let $A \subseteq \omega \omega$, $f : A \to \omega \omega$, and $m, n \geq 1$. Then $f \in \Lambda_{m,n} \iff f^{-1}[Y]$ is $\Pi^0_n$ in the relative topology of $A$ for any $Y \in \Pi^0_m$.

1.1.4. Lemma. Let $n \geq m \geq 2$, $A \subseteq \omega \omega$, $f : A \to \omega \omega$, and suppose that there is a partition $\langle A_i : i \in \omega \rangle$ of $A$ such that $A_i$ is $\Pi^0_{n-1}$ in the relative topology of $A$ and $f \upharpoonright A_i$ is $\Lambda_{1,n-m+1}$. Then $f$ is $\Lambda_{m,n}$.
1.1. Background

Let \( Y \in \Sigma^0_m \) and \( Y_j \in \Pi^0_{m-1} \) such that \( Y = \bigcup_j Y_j \). It follows that

\[
f^{-1}[Y] = \bigcup_i (f \upharpoonright A_i)^{-1}[Y]
= \bigcup_i \bigcup_j (f \upharpoonright A_i)^{-1}[Y_j]
= \bigcup_i \bigcup_j A \cap X_{i,j}, \text{ where } X_{i,j} \in \Pi^0_{n-1}
= A \cap X, \text{ where } X \in \Sigma^0_n.
\]

For the second to last equality, note that \( f \upharpoonright A_i \in \Lambda_{m-1,n-1} \) by Proposition 1.1.2 (take \( k = m - 2 \)).

1.1.5. Lemma. Let \( n \in \omega \) with \( n > 0 \). Let \( A \subseteq \omega^\omega \), \( h : A \rightarrow \omega^\omega \), and suppose that \( A = B_0 \cup B_1 \) such that \( B_0 \) and \( B_1 \) are \( \Sigma^0_{n+1} \) in \( A \) and \( B_0 \cap B_1 = \emptyset \). If there is a \( \Pi^0_n \) partition \( \langle B_{b,m} : m \in \omega \rangle \) of \( B_0 \) and a \( \Pi^0_n \) partition \( \langle B_{1,m} : m \in \omega \rangle \) of \( B_1 \), then there is a \( \Pi^0_n \) partition \( \langle A_m : m \in \omega \rangle \) of \( A \) that refines the partitions \( B_{b,m} \) and \( B_{1,m} \): for every \( i \in \omega \), there is a \( b < 2 \) and a \( j \in \omega \) such that \( A_i \subseteq b_{h,j} \).

Proof. We begin by noting that we cannot simply take the sets \( B_{b,m} \) to be the partition, since \( B_{b,m} \) is not necessarily \( \Pi^0_n \) in \( A \). For \( b < 2 \) and \( m \in \omega \), let \( B'_{b,m} \) be \( \Pi^0_n \) in \( A \) such that \( B_{b,m} = B'_{b,m} \cap B_b \). Let \( C_{b,m} \) be \( \Pi^0_n \) in \( A \) and pairwise disjoint such that \( B_b = \bigcup C_{b,m} \). Note that for any \( i \) and \( j \), \( C_{b,i} \cap B'_{b,j} = C_{b,i} \cap B_{b,j} \) is \( \Pi^0_n \) in \( A \). The sets \( C_{b,i} \cap B_{b,j} \) form the desired partition of \( A \).

We end this section with a brief note about \( \Gamma \)-completeness, following the discussion in [8] on page 169. Suppose \( \Gamma \) is a class of sets in Polish spaces. In other words, for any Polish space \( X \), \( \Gamma(X) \subseteq \mathcal{P}(X) \). If \( Y \) is a Polish space, then \( A \subseteq Y \) is \( \Gamma \)-complete if \( A \in \Gamma(Y) \) and \( B \leq_W A \) for any \( B \in \Gamma(X) \), where \( X \) is a zero-dimensional Polish space. Note that if \( A \) is \( \Gamma \)-complete, \( B \in \Gamma \), and \( A \leq_W B \), then \( B \) is \( \Gamma \)-complete.

1.1.6. Theorem (Wadge). Let \( X \) be a zero-dimensional Polish space. Then \( A \subseteq X \) is \( \Sigma^0_\xi \)-complete iff \( A \in \Sigma^0_\xi \setminus \Pi^0_\xi \).

1.1.7. Fact. The set \( \{ x \in \omega^2 : \exists i \forall j \geq i (x(j) = 0) \} \) is \( \Sigma^0_3 \)-complete.

Let \( \gamma, \cdot, \cdot \) be the bijection \( \omega \times \omega \rightarrow \omega \):

\[
\gamma 0, 0 = 0,
\gamma 0, j + 1 = \gamma j, 0 + 1,
\gamma i + 1, j - 1 = \gamma i, j + 1.
\]

1.1.8. Fact. The set \( \{ x \in \omega^2 : \exists i \exists j (\gamma i, j = 1) \} \) is \( \Sigma^0_3 \)-complete.