Accreting millisecond X-ray pulsars, from accretion disk to magnetic poles
Patruno, A.

Link to publication

Citation for published version (APA):
Patruno, A. (2009). Accreting millisecond X-ray pulsars, from accretion disk to magnetic poles

General rights
It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

Disclaimer/Complaints regulations
If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: http://uba.uva.nl/en/contact, or a letter to: Library of the University of Amsterdam, Secretariat, Singel 425, 1012 WP Amsterdam, The Netherlands. You will be contacted as soon as possible.
The long-term evolution of the spin, pulse shape, and orbit of the accretion-powered millisecond pulsar SAX J1808.4−3658


Abstract

We present a 7 yr timing study of the 2.5 ms X-ray pulsar SAX J1748, an X-ray transient with a recurrence time of ≈2 yr, using data from the *Rossi X-ray Timing Explorer* covering 4 transient outbursts (1998–2005). We verify that the 401 Hz pulsation traces the spin frequency fundamental and not a harmonic. Substantial pulse shape variability, both stochastic and systematic, was observed during each outburst. Analysis of the systematic pulse shape changes suggests that, as an outburst dims, the X-ray “hot spot” on the pulsar surface drifts longitudinally and a second hot spot may appear. The overall pulse shape variability limits the ability to measure spin frequency evolution within a given X-ray outburst (and calls previous $\dot{\nu}$ measurements of this source into question), with typical upper limits of $|\dot{\nu}| \lesssim 2.5 \times 10^{-14}$ Hz s$^{-1}$ ($2\sigma$). However, combining data from all the outbursts shows with high ($6\sigma$) significance that the pulsar is undergoing long-term spin down at a rate $\dot{\nu} = (-5.6 \pm 2.0) \times 10^{-16}$ Hz s$^{-1}$, with most of the spin evolution occurring during X-ray quiescence. We discuss the possible contributions of magnetic pro-
peller torques, magnetic dipole radiation, and gravitational radiation to the measured spin down, setting an upper limit of $B < 1.5 \times 10^8$ G for the pulsar’s surface dipole magnetic field and $Q/I < 5 \times 10^{-9}$ for the fractional mass quadrupole moment. We also measured an orbital period derivative of $\dot{P}_{\text{orb}} = (3.5 \pm 0.2) \times 10^{-12}$ s s$^{-1}$. This surprising large $\dot{P}_{\text{orb}}$ is reminiscent of the large and quasi-cyclic orbital period variation observed in the so-called “black widow” millisecond radio pulsars. This further strengthens previous speculation that SAX J1748 may turn on as a radio pulsar during quiescence. In an appendix we derive an improved (0.15 arcsec) source position from optical data.
4.1 Introduction

The growing class of accretion-powered millisecond X-ray pulsars discovered by the *Rossi X-Ray Timing Explorer* (*RXTE*) has verified the hypothesis that old millisecond pulsars obtained their rapid spins through sustained accretion in X-ray binaries. These objects provide a versatile laboratory. The X-ray pulse shapes arising from the magnetically channeled accretion flow can constrain the compactness (and hence the equation of state) of the neutron star. Tracking the arrival times of these X-ray pulses allows us to measure the pulsar spin evolution, which directly probes magnetic disk accretion torque theory in a particularly interesting regime (Psaltis & Chakrabarty 1999) and also allows exploration of torques arising from other mechanisms such as gravitational wave emission (Bildsten & Cumming 1998). There have been several reports of significant spin evolution in accreting millisecond pulsars, some with implied torques that are difficult to reconcile with standard accretion torque theory (Markwardt et al. 2003a; Morgan et al. 2003; Burderi et al. 2006, 2007). However, a variety of effects (including limited data spans, pulse shape variability, and non-Gaussian noise sources) can complicate the interpretation of these measurements. In this paper, we address these difficulties using a comprehensive analysis of the most extensive data set.

Of the eight accretion-powered millisecond pulsars currently known, the first one remains the best-studied example. The X-ray transient SAX J1808 was discovered during an outburst in 1996 by the *BeppoSAX* Wide Field Cameras (in ’t Zand et al. 1998). Timing analysis of *RXTE* data from a second outburst in 1998 revealed the presence of a 401 Hz (2.5 ms) accreting pulsar in a 2 hr binary (Wijnands & van der Klis 1998; Chakrabarty & Morgan 1998). The source is a recurrent X-ray transient, with subsequent ≈1 month long X-ray outbursts detected in 2000, 2002, and 2005; it is the only known accreting millisecond pulsar for which pulsations have been detected during multiple outbursts. Faint quiescent X-ray emission has also been observed between outbursts, although no pulsations were detected (Stella et al. 2000; Campana et al. 2002; Heinke et al. 2007). A source distance of 3.4–3.6 kpc is estimated from X-ray burst properties (in ’t Zand et al. 2001; Galloway et al. 2006). The pulsar is a weakly magnetized neutron star ([1–10] × 10^8 G at the surface; Psaltis & Chakrabarty 1999) while the mass donor is likely an extremely low-mass (≈0.05 M_⊙) brown dwarf (Bildsten & Chakrabarty 2001). SAX J1808 is the only source known to exhibit all three of the rapid X-ray variability phenomena associated with neutron stars in LMXBs: accretion-powered millisecond pulsations (Wijnands & van der Klis 1998), millisecond oscillations during thermonuclear X-ray bursts (Chakrabarty et al. 2003b), and kilohertz
4. The long-term evolution of the spin, pulse shape, and orbit of the accretion-powered millisecond pulsar SAX J1808.4–3658

quasi-periodic oscillations (Wijnands et al. 2003). After submitting this paper, we learned of an independent analysis by di Salvo et al. (2008) reporting an increasing orbital period (see §4.3.7).

An optical counterpart has been detected both during outburst (Roche et al. 1998; Giles et al. 1999) and quiescence (Homer et al. 2001). The relatively high optical luminosity during X-ray quiescence has led to speculation that the neutron star may be an active radio pulsar during these intervals (Burderi et al. 2003; Campana et al. 2004), although radio pulsations have not been detected (Burgay et al. 2003). Transient unpulsed radio emission (Gaensler et al. 1999; Rupen et al. 2005) and an infrared excess (Wang et al. 2001; Greenhill et al. 2006), both attributed to synchrotron radiation in an outflow, have been reported during X-ray outbursts.

In this paper, we describe our application of phase-connected timing solutions for each outburst to study the spin history and pulse profile variability of SAX J1808, providing the first look at the evolution of an accretion-powered X-ray pulsar. In section 4.2, we outline our analysis methods, noting the difficulties raised by pulse profile noise and describing a new technique to obtain a minimum-variance estimate of the spin phase in the presence of such noise. In section 4.3, we present the results of this analysis. In particular, we observe that the source is spinning down between outbursts, the binary orbital period is increasing, and the pulse profiles change in a characteristic manner as the outbursts progress. Finally, in section 4.4, we discuss the implications of these results to the properties of the neutron star and accretion geometry.

4.2 X-ray Observations and Data Analysis

4.2.1 RXTE data reduction

The RXTE Proportional Counter Array (PCA; Jahoda et al. 1996) has repeatedly observed SAX J1808, primarily during outburst. These observations total 307 separate pointings and an exposure time of 1,371 ks from 1998 through 2005. The PCA comprises five identical gas-filled proportional counter units (PCUs) sensitive to X-rays between 2.5 and 60 keV. Each PCU has an effective area of 1200 cm$^2$. It is uncommon for all five PCUs to be active: some are periodically disabled to decrease their rates of electrical breakdown (Jahoda et al. 2006). The average number of active PCUs has declined as the RXTE ages, and most observations during the 2002 and 2005 outbursts of SAX J1808 only include two or three.

All but three of the observations of SAX J1808 were taken with the E_125US_64M_0_1S mode, which records the arrival of each photon with a
Table 4.1: Observations analyzed for each outburst

<table>
<thead>
<tr>
<th>Date</th>
<th>Data range (MJD)</th>
<th>Time (ks)</th>
<th>Avg. No. of PCUs</th>
<th>Observation IDs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1998 Apr</td>
<td>50914.8 – 50939.6</td>
<td>178.1</td>
<td>4.67</td>
<td>30411-01-**</td>
</tr>
<tr>
<td>2000 Feb</td>
<td>51564.1 – 51601.9</td>
<td>126.8</td>
<td>3.74</td>
<td>40035-01-01-00 – 40035-01-04-01</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>40035-05-02-00 – 40035-05-18-00</td>
</tr>
<tr>
<td>2002 Oct</td>
<td>52562.1 – 52602.8</td>
<td>714.5</td>
<td>3.25</td>
<td>70080-01-**</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>70080-03-05-00 – 70080-03-24-00</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>70080-03-25-01, 70518-01-**</td>
</tr>
<tr>
<td>2005 Jun</td>
<td>53523.0 – 53581.4</td>
<td>284.3</td>
<td>2.84</td>
<td>91056-01-01-01 – 91056-01-04-01</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>91418-01-01-00 – 91418-01-07-00</td>
</tr>
</tbody>
</table>

Note. — The ranges of observation IDs given here are for numerically sorted IDs, which do not always reflect temporal order.

We consistently used an energy cut of roughly 2–15 keV for our timing analysis. While the source is readily detectable in the PCA at higher energies, the background dominates above 15 keV, especially in the dimmer tails of

---

1 We shifted the photons to the geocenter rather than the solar system barycenter since the TEMPO pulse timing program, which we used to fit phase models to the arrival times, was designed for radio timing and thus expects photon arrival times at some point on the Earth. TEMPO itself performed the barycentric corrections using the quoted position.
the outbursts. Excluding these high-energy counts optimized the detection of pulsations when the source was dim, providing a longer baseline for our timing analysis.

### 4.2.2 Pulse timing analysis with tempo

The core of this analysis resembles the work long-done for radio pulsars and slowly rotating X-ray pulsars. We first folded intervals of data according to a phase timing model to obtain pulse profiles. We next compared the profile from each interval to a template profile in order to calculate the offset between the observed and the predicted pulse times of arrival (TOAs). We then improved the initial phase model by fitting it to these TOA residuals.

We used the TEMPO pulsar timing program\(^2\), version 11.005, to calculate pulse arrival times from a phase model and to improve a phase model by fitting it to arrival time residuals. TEMPO reads in a list of TOAs and a set of parameters describing the pulsar timing model. It then adjusts the model to minimize the timing residuals between the predicted and observed arrival times. The output files include a revised timing model, a covariance matrix for the fit parameters, and a list of the timing residuals. TEMPO also includes a predictive mode, which takes a timing model and generates a series of polynomial expansions that give the model’s pulse arrival times during a specified time interval. TEMPO has been a standard tool of the radio pulsar community for decades and is well-tested at the microsecond-level accuracies with which we are measuring TOAs.

Our timing models fit for the following parameters: the pulsar spin frequency and (if necessary) the first-order frequency derivative; the times and magnitudes of any instantaneous changes in the frequency; and the orbital parameters. Our models supplied, but did not fit, the position of the source from Appendix 4.5. Because we fit the outbursts separately, the \(\approx 1\) month of data that each provides was not sufficient to improve the source position: the position of SAX J1808 (in particular, its right ascension) is degenerate with the frequency and frequency derivative on such timescales.

To parametrize the orbit of SAX J1808, we used TEMPO’s ELL1 binary model, which employs the Laplace parameters \(e \sin \omega\) and \(e \cos \omega\), where \(e\) is the eccentricity and \(\omega\) the longitude of periastron passage. This parametrization avoids the degeneracy of \(\omega\) in low-eccentricity systems (Deeter et al. 1981). For most of the fits, we held \(e = 0\) and solely fit the projected semimajor axis \(a_x \sin i\), the orbital period \(P_{\text{orb}}\), and the time of ascending node\(^3\) \(T_{\text{asc}}\). As a

---

\(^2\)http://www.atnf.csiro.au/research/pulsar/tempo/

\(^3\)Past pulsar timing of SAX J1808 uses the \(T_{90}\) fiducial, marking a time at which the
test of this assumption, we also repeated the fits allowing $e$ to vary. It was always consistent with zero.

This analysis depends critically on the accurate calculation and processing of the TOAs. To verify our results, they were independently calculated using two entirely separate data pathways. One corrected the RXTE count data to the geocenter and used TEMPO to barycenter the TOAs, as described in the previous section; the other used the FTOOL\textsuperscript{4} faxbary to barycenter the count data. Independent codes were then used to divide the count data into 512 s intervals, fold it according to a phase model, and measure the pulse times of arrival. Finally, we used both TEMPO and its replacement, TEMPO2 (Hobbs et al. 2006), to process the TOAs and refine the timing models. In all cases, the agreement between the final timing models was good.

### 4.2.3 TOA calculation in the presence of profile noise

Special care must be taken when measuring the pulse TOAs for rapidly rotating accretion-powered pulsars. In these systems, the pulse profiles exhibit variability on timescales of $\sim 10$ hr and longer that is well in excess of the Poisson noise expected from counting statistics. In this section, we develop a procedure to obtain a minimum-variance estimate of the timing residuals in the presence of such noise.

We use the term “noise” with respect to spin timing analysis simply to mean phase variability of one or more harmonics that does not seem to be due to underlying spin frequency changes. While some of this profile variability may in fact be quite ordered — distinctive pulse shape changes that occur in every outburst, for instance — we cannot model all of them and thus consider the unmodeled profile variability as “noise” from the phase-timing perspective. In this section, we attempt to minimize the impact of such variability on the accuracy of our pulse arrival times by favorably weighting data from less-noisy harmonics; in §4.2.4 we describe a Monte Carlo technique to estimate its impact on the timing model parameters.

To calculate the TOAs, we divided the timing data into 512 s intervals and determined one pulse arrival time per interval. We chose this length because it provides sufficient counts to make accurate measurements in the dim tails of the outbursts, while it still is short enough to sample within the 7249 s orbital period. This is necessary to improve the binary model and resolve any additional short-timescale variability.

For each outburst, we used TEMPO’s predictive mode to generate a series of

mean longitude is $90^\circ$. Since its orbit is circular, $T_{\text{90}} = T_{\text{asc}} + P_{\text{orb}}/4$.

\textsuperscript{4}http://heasarc.gsfc.nasa.gov/ftools/
4. The long-term evolution of the spin, pulse shape, and orbit of the accretion-powered millisecond pulsar SAX J1808.4−3658

...polynomial expansions predicting the times of pulse arrivals at the geocenter. These ephemerides are based on the revised optical position, our best-known orbital parameters, and a simple, constant-frequency spin model. Using the expansions, we calculated the expected phase for each photon arrival time. For each 512 s interval, we then divided the phases into \( n \) phase bins in order to create folded pulse profiles.

We then decomposed the profiles into their Fourier components. For a given folded profile, let \( x_j \) designate the number of photons in the \( j \)th phase bin, and \( N_{ph} = \sum_{j=1}^{n} x_j \) is the total number of photons. The complex amplitude of the \( k \)th harmonic is then

\[
a_k = \sum_{j=1}^{n} x_j \exp\left(2\pi ijk/n\right). \tag{4.1}\]

Throughout this paper, we number the harmonics such that the \( k \)th harmonic is \( k \) times the frequency of the 401 Hz fundamental. Since our analysis for the most part handles the phases and amplitudes of harmonics separately, we define these quantities explicitly as follows:

\[
A_k \exp\left[2\pi ik(\phi_k + \Delta\phi_k)\right] = 2a_k. \tag{4.2}\]

Here we are interested in the amplitude\(^5\) \( A_k \) and the phase residual \( \Delta\phi_k \), which we measure relative to a fixed phase offset \( \phi_k \). We include these offsets because we are principally interested in measuring the phase deviations from a fixed template profile, which we obtain by transforming the overall folded pulse profile from an outburst:\(^6\)

\[
A'_k \exp\left(2\pi ik\phi_k\right) = 2 \sum_{j=1}^{n} x'_j \exp\left(2\pi ijk/n\right). \tag{4.3}\]

Here \( x'_j \) and \( N'_{ph} \) give the phase bin counts and total counts for the template pulse profile.

Note that we define the phases such that shifting a fixed pulse profile by some phase \( \Delta\phi \) produces the same shift in the phase of each of its harmonic: \( \Delta\phi_k = \Delta\phi \). Hence the unique phases for each \( \Delta\phi_k \) range from 0 to 1/k.

\(^5\)We define \( A_k \) such that it is the actual amplitude, in photons, of the observed pulsations. We must therefore include both the positive and negative frequency components (which are equal for real signals), introducing the factor of 2 on the right-hand side of eqs. (4.2) and (4.3).

\(^6\)In the presence of sudden pulse profile changes, we may use multiple templates during a single outburst, thus using different values of \( \phi_k \) on either side of the change. Further description is at the end of this section.
Positive phase residuals corresponding to time lags: $\Delta \phi_k > 0$ indicates that the $k$th harmonic arrived later than predicted by the model.

The uncertainty in the phase residuals $\Delta \phi_k$ due to Poisson noise (derived in Appendix 4.6) are

$$\sigma_k = \frac{\sqrt{2N_{\text{ph}}}}{2\pi k A_k}. \quad (4.4)$$

For our analysis, we rejected phase measurements with uncertainties greater than 0.1 ms (i.e., 0.04 cycles). Generally, this cut only removed points in the tails of the outbursts, where the flux was low.

The measured fractional rms amplitudes are

$$r_k = \frac{A_k}{\sqrt{2} (N_{\text{ph}} - B)}, \quad (4.5)$$

where $B$ is the approximate number of background events within our energy range and time interval, estimated using the FTOOL pcabackest.\(^7\) The $r_k$ add in quadrature: the total rms fractional amplitude for a pulse profile described with $m$ harmonics is $r = \left( \sum_{k=1}^{m} r_k^2 \right)^{1/2}$. Uncertainties on the fractional amplitudes are computed using the method described by Groth (1975b) and Vaughan et al. (1994), which accounts for the addition of noise to the complex amplitude of the signal. The probability that the detection of a harmonic is due solely to Poisson noise is $\exp(-P_k)$, where $P_k = \frac{1}{4} \frac{A_k^2}{N_{\text{ph}}}$ is the unit-normalized power for the $k$th harmonic. For a fuller review of Fourier techniques in X-ray timing, we defer to van der Klis (1989).

Since each harmonic provides an independent measurement of the phase residual, we can combine them to provide the overall phase residual for the sample pulse. We obtain the optimal estimator by weighting each measurement according to its variance:

$$\Delta \phi = \sum_{k=1}^{m} w_k \Delta \phi_k / \sum_{k=1}^{m} w_k; \quad (4.6)$$

$$w_k = \frac{1}{\sigma_k^2} = \frac{k^2 A_k^2}{N_{\text{ph}}}. \quad (4.7)$$

Thus far, this analytical method closely parallels the work long done on spin-powered pulsars (e.g., Taylor 1993, Appendix A).

However, there are some essential differences that must be taken into account when dealing with accretion-powered pulsars. Unlike spin-powered pulsars, which usually show one or more sharp, asymmetric pulses per cycle, there

\(^7\)http://heasarc.gsfc.nasa.gov/docs/xte/recipes/pcabackest.html
is little harmonic content in the pulsations of SAX J1808 beyond $k = 2$, so we truncate the series there. Additionally, while the individual pulses of radio pulsars show appreciable variability from one period to the next, their integrated profiles are very stable (Manchester & Taylor 1977). In such cases, one expects that the pulse fractions of sample pulses are similar to the template ($A_k/N_{ph} \approx A'_k/N'_{ph}$) and that the harmonics reflect a common phase residual ($\Delta \phi_k \approx \Delta \phi$) that traces the rotational phase of the star. Indeed, the standard template-matching analysis is predicated on these assumptions. Furthermore, any variability is assumed to be due to Poisson noise, which is of equal magnitude at all timescales (i.e., it is white noise). In contrast, the accretion-powered pulsars show substantial pulse profile variability. Beyond the usual Poisson noise ($\sigma_k$ from eq. [4.4]), three additional issues complicate the usual approach of template matching: long-timescale correlations (i.e., red noise) in the observed pulse fractions, with each harmonic’s $A_k$ varying independently; red noise in the phase offsets $\Delta \phi_k$; and sudden pulse profile changes, in which the phase offset between the two measured harmonics changes drastically on the timescale of the observations.

In their timing analysis of 283 s accretion-powered pulsar Vela X-1, Boynton et al. (1984) partially address the issue of intrinsic pulse profile noise in the harmonics. In the most general case, the amplitude of the variability in the phase residuals $\Delta \phi_k$ is different for each harmonic $k$. This is the case for both Vela X-1 and SAX J1808. They correct for the harmonic dependence of these fluctuations by scaling the phase residuals of each harmonic by constants chosen such that the phase residuals all have the same amplitude of variability (Boynton & Deeter 1985b). Thus the influence of particularly noisy harmonics was diminished, and they were able to measure with much greater accuracy the underlying spin of Vela X-1.

Our approach was similar. For each outburst, we measured the total rms amplitude of the phase residuals for each harmonic with respect to a best-fit constant-frequency model. These residuals will represent the combined effect of the Poisson noise and any intrinsic profile noise:

$$\sigma_{k,\text{rms}}^2 = \langle \sigma_k^2 \rangle + \sigma_{k,\text{int}}^2.$$  

We calculated the Poisson contribution $\langle \sigma_k^2 \rangle$ as a weighted mean$^8$ of the results from equation (4.4), giving us a value for $\sigma_{k,\text{int}}^2$. We then incorporate this

$^8$To calculate the mean of the variances $\sigma_k^2$, we weight each according to equation (4.7). Labeling each uncertainty as $\sigma_{kl}$ for intervals $l = 1, \ldots, N_{\text{int}}$, we have $\langle \sigma_k^2 \rangle = \left( \sum_{l=1}^{N_{\text{int}}} \sigma_{kl}^{-2} / N_{\text{int}} \right)^{-1}$. This weighting scheme prevents large variances during the tails of the outbursts from skewing the results.
additional uncertainty into our weighting to determine $\Delta \phi$:

$$w_k = \frac{1}{\sigma_k^2 + \sigma_{k,\text{int}}^2}.$$  \hfill (4.9)

$\sigma_k^2$ changes from one TOA measurement to the next due to the variability of the pulse fraction and count rate; there is no assumption that these are constant, as there is in the case of standard template fitting. $\sigma_{k,\text{int}}^2$ is a constant measured independently for each harmonic of each outburst. The result is a minimum-variance estimator for $\Delta \phi_k$. For instance, if the 802 Hz second harmonic has smaller intrinsic fluctuations than the fundamental, then our method presumes that it better reflects the spin of the NS and will weight it more strongly.

Sudden pulse profile changes are somewhat simpler to deal with. We use a different template pulse profile (and hence different measurements on either side of the change of $\phi_k$, defined in eq. [4.3]). We only modeled one such sudden profile change in this way: at the end of the main body of the 2002 outburst (around MJD 52576), the fundamental phase $\phi_1$ experienced a shift while the second harmonic, $\phi_2$, remained constant. The stability of $\phi_2$ allowed us to phase connect across the feature, as Burderi et al. (2006) also noted.

Our distinction between sudden profile changes and pulse profile noise is admittedly somewhat arbitrary. We make it solely in the interest of best estimating the rotational phase of the star — we are not claiming to model some underlying difference in physical processes. In 2002, the phase residuals on either side of the modeled pulse profile change were quite stable, albeit with different values of $\phi_1$. This stability in both harmonics makes it a good candidate for such treatment. In contrast, the phase residuals of both harmonics during the 2005 outburst show greater amplitude fluctuations at nearly all timescales. While its residuals and the 2002 residuals follow a similar pattern at the end of the main body of the outburst (the phases of the second harmonic remain roughly constant, while the phases of the fundamental drop appreciably), the phase of the fundamental continues to fluctuate wildly after this event rather than settling down on a “new” template profile. Therefore we elect to attribute these profile changes to intrinsic noise and weight the relatively stable second harmonic more strongly.

However, when the phases of both harmonics are continuously changing, the ability to define pulse arrival times breaks down, and the data are of little use for determining the spin of the star. For SAX J1808, the pulse profile is changing throughout the rises and peaks of the outbursts, so we excluded these data from our measurements of the spin frequency. We did include these data when calculating the orbital parameters. Since the timescale for these pulse profile changes ($\gtrsim 10$ hr) is many times the orbital period, they tend to average out and have little impact on these measurements.
The resulting phase residuals give the best estimator for the offset between the measured and predicted pulse arrival times. By adding these offsets to the phases predicted by the TEMPO ephemerides, we arrived at more accurate pulse arrival times for each interval.

4.2.4 Parameter fitting and uncertainty estimation

After measuring the pulse times of arrival, we input them into TEMPO to re-fit the timing solution. In order to interpret the resulting models, we must understand the nature of the noise in the TOAs and how it affects the model parameters. The harmonic weighting system described above makes the optimal choice to mitigate the phase variability due to a particularly noisy harmonic, but the TOA residuals are still of substantially greater magnitude than would be expected from Poisson noise alone. This leaves us with the task of estimating the fit uncertainties in the presence of such noise. These uncertainties are crucial to our construction of timing models, as they are needed to estimate the significance of fit components, such as frequency derivatives and instantaneous frequency changes.

We noted in the previous section that we treat the TOA residuals as noise, despite some of their variability arising from pulse shape changes that recur in every outburst. This is not a bad approximation: because we fit our models separately for each outburst, correlations in the pulse profile variability between outbursts are not relevant. Furthermore, the power spectra of the TOA residuals (see Fig. 4.4 in §4.3.3) resemble the power-law noise spectra typically observed in actual red noise processes, so treating it as such is reasonable.

We initially used the simplest possible timing model when fitting the TOAs of each outburst in TEMPO: a circular orbit and a constant frequency. (Note that we fit independent models for each outburst. The uncertainties were too large to phase connect between outbursts.) When this simple model proved insufficient to account for the phase residuals, we introduced a nonzero $\dot{\nu}$ and instantaneous frequency changes, as needed. However, there is a danger of overfitting the data. It is important to recognize that some of the features in the residuals are probably pulse profile variability rather than spin evolution. We took great care in our attempts to distinguish between the $\dot{\nu}$ measurements and the artifacts of intrinsic timing noise.

The colored nature of the timing noise in both harmonics is the primary difficulty in the interpretation of the parameter fits. TEMPO assumes that the TOA uncertainties one gives it are white and approximately Gaussian, as is the case of pure Poisson noise. As a result, it systematically underestimates the uncertainties in the fitted parameters in the presence of timing noise. Red timing noise is particularly problematic, because it dominates on the long
timescales on which $\nu$ and $\dot{\nu}$ measurements depend.

Instead of adopting this white noise assumption of TEMPO, we estimated confidence intervals for $\nu$ and $\dot{\nu}$ using Monte Carlo simulations of the timing residuals of each outburst. After using TEMPO to obtain the best fit for a timing model, we calculated the power spectrum $P(f)$ of the timing residuals that TEMPO output. This spectrum is a convolution of the true noise spectrum and the sampling function; most notably, there is excess power around 1 d, an artifact of RXTE observations often being scheduled approximately a day apart, and at the RXTE orbital period of 96 min due to Earth occultations. We applied a low-pass filter to remove these peaks in an attempt to approximate the underlying noise spectrum, $P'(f)$:

$$P'(f) = P(f) \times \left[ (1 - A) \exp\left(-f^2 \tau_c^2\right) + A \right]. \quad (4.10)$$

$\tau_c$ gives the time scale for the low-pass cutoff. $A$ gives the fraction of high-frequency noise to let through, reproducing the short-timescale scatter (principally but not entirely Poisson) that we observed within each observation. Typical values were 3 d and 10–20%.

We then created thousands of sets of artificial phase residuals with the noise properties of the filtered spectrum, $P'(f)$. To reproduce the sampling irregularities, we removed all points at times absent in the original data. The parameters of the low-pass filter were tuned such that the mean power spectrum of the resulting Monte Carlo residuals was as close as possible to the original power spectrum, $P(f)$. For each set of residuals, we measured the frequency of the best linear fit (or, if our TEMPO model fit for $\dot{\nu}$, the frequency derivative of the best quadratic fit). The standard deviations of these measurements provided uncertainty estimates for the respective parameter, this time more accurately accounting for the noise spectrum.

We relied solely on TEMPO to calculate the uncertainties in the binary orbit parameters. While the intrinsic pulse profile noise spectrum is colored on the timescale of days to weeks, the phase residuals are approximately white on timescales equal to and shorter than the 2 hr orbital period. The amplitude of the short-timescale variability is roughly 1.5 times what one would expect from counting noise alone, so we scaled our Poisson-derived phase uncertainties accordingly when estimating the uncertainties of the orbital parameter fits. This rescaling makes TEMPO’s uncertainty estimates for the orbital parameters reasonably accurate. One important consistency check of this simple approach worked nicely: the 1998, 2002, and 2005 measurements of $P_{\text{orb}}$ and $a_\times \sin i$, two parameters that should be the same for each outburst at our level of accuracy, were indeed found to be constant, with reduced $\chi^2$ statistics close to unity.

In deriving new binary and spin parameters, the new values sometimes differed considerably from the parameters with which we initially folded the
Figure 4.1: The light curves, phase residuals, and fractional amplitudes for all four outbursts. The top panels show the fundamental and harmonic background-subtracted light curves for each outburst. The strips along the top of the graphs indicate the times of observations; stars indicate the times of thermonuclear X-ray bursts. The second panels show the phase residuals relative to a constant frequency of 140.972072 Hz, with black points giving the phases of the fundamental and grey points indicating the second harmonic. Positive phases indicate pulse arrivals later than predicted by the constant frequency model. The error bars reflect the statistical errors only, as outlined in equation (4.4). These data have been divided so that there is only one point per observation. The error bars represent the standard deviation of one of the four outbursts. The bottom plot shows the fractional amplitudes of the fundamental and harmonic.
data for TOA calculations. If our orbital model improved substantially, we
iterated the above procedure, calculating new times of arrival for each 512 s
interval and refitting. Because the orbit introduces a periodic frequency mod-
ulation with amplitude \( \Delta \nu = \nu_0 \cdot 2\pi a \sin i/cP_{\text{orb}} > 1/512 \text{s} \), an inaccurate
orbital ephemeris can significantly reduce detection strength. In contrast, the
spin frequency is remarkably stable through all the observations, so there was
no need to recalculate TOAs upon the relatively minor revisions to the spin
model.

4.3 Results

The results of our pulse timing solutions are shown in Figure 4.1, which com-
pares the light curves, phase residuals, and fractional amplitudes for each
outburst. Inspecting the best-fit frequency lines in the phase residual plots,
it is clear that a constant pulse profile attached to a constant-frequency ro-
tator does not adequately describe the observed residuals. We consider five
sources of phase residuals relative to a best-fit constant-frequency model: Pois-
son timing noise, intrinsic pulse profile noise, sudden and well-defined pulse
profile changes, additional spin frequency derivatives, and instantaneous fre-
quency changes in the underlying rotation of the star. In this section, we will
consider all these possible contributions to the residuals and their relationships
with each other and the other properties of each outburst.

4.3.1 Light curves of the outbursts

The light curves of each outburst are quite similar in shape. We divide them
into four stages: the rise, which was only definitively captured in 2005 and took
\( \approx 5 \text{d} \); the short-lived peak at a 2–25 keV flux of \((1.9–2.6) \times 10^{-9} \text{ erg cm}^{-2} \text{s}^{-1}\),
equal to a luminosity of \((4.7–6.4) \times 10^{35} \text{ erg s}^{-1}\) using the distance of 3.5 kpc
and bolometric correction of \(L_{\text{bol}}/L_{2–25 \text{ keV}} = 2.12\) derived by Galloway et al.
(2006); a slow decay\(^9\) in luminosity, lasting 10–15 d, until the source reaches
approximately \(8 \times 10^{-10} \text{ erg cm}^{-2} \text{s}^{-1}(= 2.0 \times 10^{35} \text{ erg s}^{-1})\); and a sudden
drop followed by low-luminosity flaring as the outburst flickers out, with the
timescale between flares on the order of 5 d. Figure 4.2 shows a cartoon of a
typical outburst from SAX J1808, with each of these stages labeled.

\(^9\)Some authors (e.g., Cui et al. 1998; Burderi et al. 2006) refer to this part of the outburst
as the “exponential decay” stage, based on the approximately exponential dimming of the
1998 and 2002 outbursts. However, the fall off of the 2005 outburst during this stage is
closer to linear, so we simply refer to it as the “slow decay” stage to contrast it with the
more rapid luminosity drop at its conclusion.
4. The long-term evolution of the spin, pulse shape, and orbit of the accretion-powered millisecond pulsar SAX J1808.4−3658

Figure 4.2: The anatomy of a typical outburst from SAX J1808. The features of the light curve and their fluxes and timescales are similar to those observed during the 1998, 2002, and 2005 outbursts. Bolometric luminosities assume a distance of 3.5 kpc and a correction of $L_{\text{bol}}/L_{2-25\text{ keV}} = 2.12$ (Galloway et al. 2006).

The RXTE first collected high-resolution timing data from SAX J1808 during the 1998 April outburst. Data from the RXTE All Sky Monitor (ASM) show that the peak luminosity occurred approximately three days before the first PCA observation. (See Fig. 2 of Galloway et al. 2006 for a comparison of the ASM and PCA light curves; note that its 1998 plot does not include two raster scans analyzed here.) Unfortunately, PCA observations stopped shortly after the body of the outburst and do not sample the tail.

SAX J1808 was discovered to be again in outburst when it emerged from behind the Sun in 2000 January. Coverage of the outburst was limited and included only the outburst tail. Wijnands et al. (2001) comment on the erratic nature of the flaring during the tail. ASM data indicate that the peak occurred 2 weeks prior to the first PCA observation and that we are observing the later, dimmer stage of the flaring tail. Comparison of the PCA data with the 2002 and 2005 outbursts suggests likewise.

The 2002 outburst, detected in mid-October and observed for the next two months, was the brightest, had the best PCA coverage, and included the
4.3 Results

detection of four extremely bright thermonuclear X-ray bursts during its peak. Its light curve was very similar in shape to the 1998 outburst.

In 2005 June, SAX J1808 was again in outburst. This time, the detection preceded the peak by a few days, providing a full sampling of the light curve. This outburst was somewhat dimmer, with a peak luminosity of only 70% of the 2002 peak and a correspondingly shorter slow-decay stage. The subsequent rapid decay and flaring tail look quite similar to the other outbursts.

4.3.2 Characteristic pulse profile changes

Just as the light curves of each outburst were quite similar, the evolution of the pulse profile during each outburst was remarkably consistent. Figure 4.3 illustrates the full range of pulse profiles that we observed from SAX J1808. In many instances, the similarity of the pulse profiles between outbursts is quite striking. In this section, we describe how these profiles change throughout the outbursts.

We observed the outburst rise, labeled as profile 1 in Figure 4.3, exclusively during the 2005 outburst. The profiles are smooth and asymmetric, with a slow rise followed by a more rapid drop-off after the peak. There is no sign of a second peak.

We observed the outburst maxima during 2002 and 2005. The similarity of the pulse profile evolution between the two outbursts is remarkable. During the first half of the maxima (labeled as profile set 2), the profiles show a secondary bump lagging the main pulse. Compared to the burst rise, the fractional amplitude has decreased somewhat. During the second half of the maximum, the pulse becomes broader, subsuming the lagging secondary bump. (See profile set 3.) This change appears to be gradual: in both outbursts, a mid-peak observation exhibited an intermediate pulse profile.

Profile set 4 shows the pulse profiles during the slow decay stages of the outbursts. The 1998 and 2002 profiles are quite similar: the pulses are somewhat asymmetric, rising more steeply than they fall. In both outbursts, this pulse profile is very stable during the approximately 10 d of the decay in luminosity. During the 2005 outburst, this asymmetry is more pronounced, and the profile varies between observations. Initially, the pulse exhibited a small lagging bump (profile 4A), quite similar to the pulse profile during the first half of the outburst maximum. The relative size of that bump varied substantially, in some observations appearing as a small secondary peak (profile 4B). Over the course of the decline, the source switched back and forth between a double-peaked and single-peaked profile as indicated in the figure. A given state would typically be seen for two or three observations (1–2 d) before switching to the other. Profile set 5 covers the rapid drop in flux at
4. The long-term evolution of the spin, pulse shape, and orbit of the
accretion-powered millisecond pulsar SAX J1808.4−3658

the end of the outbursts. During 1998, the pulse profile was quite stable and
did not appreciably change during this drop, although its fractional amplitude
increased somewhat. In contrast, the 2002 and 2005 outbursts show a major
pulse profile shift concurrent with the drop in luminosity. Prior to the drop,
the pulses in set 4 show a quick rise and a slower fall. After the drop, the
asymmetry of the 2002 and 2005 profiles reverses: profiles 5B show a slow rise
and a quick drop. In terms of harmonic components, these changes represent
a shift in the phase of the fundamental by approximately 0.15 cycles as it
went from leading the second harmonic to lagging behind it. The phase of the
harmonic did not change. In both outbursts, observations during the \( \approx 2 \) d
of rapid luminosity decline reveal an intermediate stage in which the main
pulse is momentarily symmetric (profiles 5A). During this transition, small
but significant secondary pulses are present.

During the flaring tail of the outburst (profile set 6), the pulse profile again
showed substantial variability. In 2002, the profile repeatedly switched be-
tween an asymmetric pulse (profile 6A, identical to the pulse profile at the
end of the rapid dimming stage) and a double-peaked profile (profile 6B). The
double peaked pulse profile occurs principally (but not exclusively) at the end
of the flares, as their luminosity declines. These pulse profile changes are
almost entirely the result of changing fractional amplitudes of the harmonic
components; the phase offset between the fundamental and second harmonic
remains for the most part constant. A notable exception occurs during the
decay of the first flare at around MJD 52582. At this time the phase of the
fundamental jumped by \( \approx 0.2 \) cycles, indicating a sudden lag of this amount
behind its previous arrival time. By the next observation, less than two hours
later, the phase residual of the fundamental returned to its previous value.

The tail of the 2005 outburst is more chaotic. The fractional amplitudes
and phases both exhibit strong red noise, producing a pulse profile that is
sometimes asymmetric with a slow rise and quick fall (6A); at other times
asymmetric with a quick rise and slow fall (6C; not shown, but basically just
the reverse of profile 6A); and in one instance clearly double-peaked (6B). The
observations were sparse and generally short, so it was impossible to better
c characterize the evolution of these pulse profile fluctuations. The flaring tail
of the 2000 outburst was quite similar, with a highly variable pulse profile
that included double-peaked profiles and asymmetric single pulses of both
orientations. We did not include it because the observations were few and
sparse.
4.3 Results

Figure 4.3: A comprehensive view of the 2–15 keV pulse profiles observed from SAX J1808. Each pulse profile was calculated by folding the observations within the indicated time intervals using the best-fit constant-frequency model of each outburst, so any movement of the peaks reflects the phase offsets from the constant frequency. The profiles are background-subtracted, normalized such that the phase bins have a mean value of unity, and plotted on 0.80–1.20. Thus the plotted profiles accurately show the change in fractional amplitude during the outburst. The profiles are numbered according their position within the outburst: 1 indicates the burst rise; 2, the beginning of the outburst maximum; 3, the end of the maximum; 4, the slow decay stage; 5, the steep luminosity drop marking the end of the main outburst; and 6, the flaring tail. During some parts of the burst, two pulse profiles are present, with the source switching between them. In these cases, we show both profiles and label the regions of the light curve in which they occurred accordingly. The solid black line shows the fluxes from the PCA observations; the grey boxes show the fluxes from the ASM daily averages.
4. The long-term evolution of the spin, pulse shape, and orbit of the accretion-powered millisecond pulsar SAX J1808.4−3658

Table 4.2: Noise properties of the outbursts

<table>
<thead>
<tr>
<th>Date</th>
<th>Fundamental</th>
<th>Second Harmonic</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(\langle \sigma_1^2 \rangle^{1/2} )</td>
<td>(\sigma_{1,\text{int}} )</td>
</tr>
<tr>
<td>1998 Apr</td>
<td>0.007</td>
<td>0.014</td>
</tr>
<tr>
<td>2000 Feb</td>
<td>0.023</td>
<td>0.052</td>
</tr>
<tr>
<td>2002 Oct</td>
<td>0.012</td>
<td>0.016</td>
</tr>
<tr>
<td>2005 Jun</td>
<td>0.013</td>
<td>0.061</td>
</tr>
</tbody>
</table>

Note. — All phases are in cycles (i.e., fractions of the 2.5 ms spin period). \(\langle \sigma_1^2 \rangle^{1/2} \) gives the mean contribution of Poisson noise; \(\sigma_{1,\text{int}} \) is the amplitude of pulse profile variability in excess of the Poisson noise; and \(\gamma_{\text{PLN}} \) is the slope of the power law best fit to the spectrum of \(\sigma_{1,\text{int}}^2 \). We did not attempt to estimate power law noise slopes for the 2000 outburst because of its low-quality data.

4.3.3 Noise properties of the timing residuals

To measure the spin phase of SAX J1808 using the formalism developed in §4.2.3, we must characterize the variability of the harmonic components. This variability encompasses both the pulse profile changes discussed in the previous section as well as any noise in the spin phase of the star.

In our analysis of the phase residuals, we took into account the rms amplitude of the intrinsic pulse profile noise in each harmonic, \(\sigma_{k,\text{int}}^2 \) defined in equation (4.8). A casual glance at the phase residuals of Figure 4.1 reveals that the magnitudes of these fluctuations vary substantially between outbursts. Table 4.2 summarizes these amplitudes for each outburst and compares them to the mean amplitudes of their Poisson noise, \(\langle \sigma_k^2 \rangle^{1/2} \). These values are then used in equation (4.9). For instance, in the 2002 outburst the fundamental is more heavily weighted in measuring the spin phase than the second harmonic, while in 2005 the opposite is true.

The scatter of the phase residuals between observations is generally greater than the scatter within an observation, suggesting that the pulse profile noise is red. Power spectra of the phase residuals, shown in Figure 4.4, confirm this. We estimated these power spectra using Fourier transforms of the residuals from equally spaced 512 s bins. Here we have not attempted to deconvolve the uneven sampling periodicities at 1 d and 96 min due to the RXTE observation schedule and orbit. There are no peaks at the 2 hr binary orbital period, indicating that the pulse profile is independent of orbital phase.

The resulting noise powers are around 2 decades higher at long periods.
4.3 Results

Figure 4.4: Power spectra of the phase residuals for the fundamental (black squares) and the second harmonic (grey circles) relative to the best-fit constant-frequency models. (The 2002 model also includes a phase shift in the fundamental to account for the profile change at MJD 52577.) The dashed lines show the power level due to counting statistics, a white-noise contribution proportional to $\langle \sigma_k^2 \rangle$. The data points show the powers $P_k(f)$, from which we have subtracted the contribution of counting statistics. These powers are normalized such that $\int_{10^{-3}}^{10^{-7}} Hz P_k(f) df = \langle \sigma_k^2 \rangle_{int}$, as defined in equation (4.8). The vertical dotted lines show the relevant time scales for the spectra: the 96 min and $\approx 1$ d periodicities of the RXTE observations, and the 121 min SAX J1808 orbital period.

($\approx 3$ d or longer) than at short periods for 1998, and even more for 2005. The 2002 outburst spectra exhibit less profile noise at long timescales, but still are somewhat red. Poisson statistics produce an uncolored lower limit
on noise. This white noise dominates at timescales shorter than the orbital period, except in the case of the particularly noisy fundamental of 2005. The spectra of the intrinsic profile noise (i.e., the spectra after subtracting off the Poisson contribution) roughly followed a power law noise spectrum, which we parametrized as $P_k(f) \propto f^{-\gamma_{PLN}}$. The best-fit values of $\gamma_{PLN}$, listed in Table 4.2, varied from roughly 0.4 to 1.

### 4.3.4 Fractional amplitudes of the harmonics

In our time-domain discussion of the pulse profiles, an apparent trend is the tendency of the pulses to become narrower, more asymmetric, or doubly peaked — generally speaking, to become less sinusoidal — as the outburst’s flux decreases. In the frequency domain, the relation is striking: the fractional amplitude of the 802 Hz second harmonic, $r_2$, strongly anticorrelates with the
background-subtracted 2–25 keV flux, $f_x$, as shown in Figure 4.5. This power-law dependency has a slope of $-0.50 \pm 0.01$. The agreement with the data is excellent for such a simple model, giving a reduced $\chi^2$ statistic of $\chi^2_{\nu} = 1.15$ with 1816 degrees of freedom. It spans two and a half decades in luminosity and includes every detected harmonic amplitude from all four outbursts.

In terms of the pulse profile, the second harmonic contributes in two ways. If its peak is $45^\circ$ out of phase with the peak of the fundamental, it will produce an asymmetric pulse profile (e.g., profile 6A in Fig. 4.3). If it is in phase, a narrower primary pulse with a small second peak will result (as in profile 6B). If the components are $90^\circ$ out of phase, the profile will be profile 6B flipped, but we never observed such a configuration.

To further understand the influence of flux on the pulse profile, we decomposed the second harmonic’s fractional amplitude into its asymmetric and double-peaked components,

$$r_{2,\text{asym}} = r_2 |\sin 4\pi \psi|$$
$$r_{2,\text{dp}} = r_2 |\cos 4\pi \psi|,$$

where $\psi$ is the phase offset between the peaks of the two harmonics: $\psi = (\phi_2 + \Delta \phi_2) - (\phi_1 + \Delta \phi_1)$. The resulting plots have substantially more scatter than Figure 4.3 due to the uncertainty of $\psi$, which is considerable, particularly at low fluxes. However, they both roughly conform to the $r_2 \propto f_x^{-1/2}$ power law. We conclude that the decrease in flux increases the asymmetry of the pulses and the presence of secondary pulses in approximately equal measure.

In contrast, the fractional amplitude of the fundamental behaves unpredictably. During the slow-decay stage of 1998, it is unvarying and strong, at a constant 5.5% rms. During this stage of 2002, it is weaker (4%) and somewhat variable; during 2005, it is weaker still and erratically changing by up to a full percent between observations. Its behavior is more consistent in the tail. In all outbursts, the fractional amplitude of the fundamental varies widely, usually (but not always) having its maxima around the peaks of the flares and its minima during the fading portion of the flares.

For the most part, a pulse profile model only including the fundamental and second harmonic adequately describes the folded profiles. However, folding long stretches of data does sometimes result in the detection of a third harmonic with fractional amplitudes ranging up to $\approx 0.25\%$ rms. We do not reliably detect any higher harmonics.

### 4.3.5 Upper limits on the subharmonics

With some assumptions, we can strongly constrain the presence of subharmonics and half-integral harmonics. The most straightforward approach is to
Table 4.3. Upper limits on subharmonics and half-integral harmonics

<table>
<thead>
<tr>
<th>Harmonic Factor</th>
<th>Upper limit (%)</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>100.2</td>
<td>0.017</td>
<td>0.019</td>
<td>0.52</td>
</tr>
<tr>
<td>1/2</td>
<td>200.5</td>
<td>0.022</td>
<td>0.024</td>
<td>0.45</td>
</tr>
<tr>
<td>3/2</td>
<td>601.5</td>
<td>0.018</td>
<td>0.021</td>
<td>0.43</td>
</tr>
<tr>
<td>5/2</td>
<td>1002.4</td>
<td>0.026</td>
<td>0.024</td>
<td>0.42</td>
</tr>
</tbody>
</table>

These background-corrected upper limits are quoted at the 95% confidence level. These limits result from combining all the observations (column A), combining only bright observations (B), and not combining any observations (C). See the text for more details.

*Frequencies listed here are approximate. The upper limits were obtained using exact multiples of the best-fit constant-ν models.*

Fold all the observations using multiples of the best-fit frequency models from each outburst. The amplitude of the resulting profile will give an upper limit. The resulting 95% confidence upper limits are listed in column A of Table 4.3. However, this approach is only statistically valid if the uncorrected fractional amplitude (i.e., the fractional amplitude relative to the source counts and the background) is constant. Clearly this assumption is false. Aside from the varying proportion of source photons, the background-subtracted fractional amplitudes of the fundamental and the second harmonic fluctuate throughout the outburst, spanning nearly an order of magnitude in the tails of the burst. There is no reason to believe that a subharmonic would not fluctuate similarly.

Column B of Table 4.3 takes the more moderate approach of only folding together observations during which the 2–25 keV flux exceeds the value of $5 \times 10^{-10}$ erg cm$^{-2}$ s$^{-1}$, thereby only including the main body of the outbursts. Background photons are thus a much smaller contribution, and the fractional amplitudes of the observed two harmonics were relatively stable during these times. Nevertheless, we still are folding enough photons to obtain very stringent upper limits: in the case of the 200 Hz subharmonic, we get a 95% confidence upper limit of 0.024% rms. We feel that these numbers are our most reliable, not making unreasonable assumptions about the fractional
amplitude fluctuations.

For completeness, we also include the most conservative upper limits, which make no assumptions whatsoever about the fractional amplitudes of the subharmonics. For instance, it would be possible in principle for the subharmonic to be present only during a single observation and zero-amplitude everywhere else. To constrain the resulting upper limits at least somewhat, we again only used observations during which the source was brighter than $5 \times 10^{-10}$ erg cm$^{-2}$ s$^{-1}$ and that had at least $10^6$ counts. These single-observation limits are tabulated in column C.

The stringent upper limits of column B provide the best evidence yet that the spin frequency of the star is indeed 401 Hz. If the star was spinning at 200.5 Hz, with two antipodal hot spots each emitting pulses to produce the observed frequency, a 200.5 Hz subharmonic would almost certainly be present.

### 4.3.6 Spin frequency measurements and constraints

We initially performed the simplest possible fits to the phase residuals of each outburst: constant-frequency models. We did not include the data at the very beginning of the 2002 and 2005 outbursts, where pulse profile changes during the rise and peak obscure any variations in the phase. We also excluded the residuals during 2002’s mid-outburst pulse profile change, but included the residuals of the fundamental on both sides of the shift by using different profile templates before and after it.

The resulting frequency measurements are shown in Figure 4.6 and summarized in Table 4.4. These data clearly indicate that the source is spinning down. The probability that the actual spin frequency is constant or increasing is less than $10^{-9}$ given the uncertainty estimates. These uncertainties do assume that our optical position is exact, but the position error is excluded because its effects are highly correlated; for instance, the 1998 and 2002 outbursts are six months apart on the calendar, so a position offset would produce equal and opposite frequency displacements for the measurements from these outbursts. There is no position that would provide a statistically feasible constant or increasing frequency.

The linear fit through the measured frequencies is not particularly good: its $\chi^2$ statistic is 9.7 with 2 degrees of freedom, yielding a probability of about 1% that the frequencies are drawn from a linear progression. Once again, changing the source position does not significantly change the result or improve the fit, and changes in the position by more than the $1 \sigma$ uncertainty along the ecliptic substantially worsen the linear fit. To estimate the uncertainties of the linear slope in light of this poor fit, we rescaled the measurement errors
4. The long-term evolution of the spin, pulse shape, and orbit of the accretion-powered millisecond pulsar SAX J1808.4—3658

![Graph showing constant-frequency measurements of the SAX J1808 outbursts, showing the spin down of the star. The frequencies are relative to $\nu_0 = 400.97521000$ Hz. The error bars are estimated using Monte Carlo simulations of phase residuals with the same noise properties as the actual outburst; they do not account for the uncertainty in the source position. The $\times$’s mark what the frequencies would be if the fit source position differed from the actual position by $2\sigma$ along the ecliptic plane in the direction of increasing RA. The same position error in the decreasing RA direction would move the frequency points by an equal amount in the opposite sense.]

Figure 4.6: Constant-frequency measurements of the SAX J1808 outbursts, showing the spin down of the star. The frequencies are relative to $\nu_0 = 400.97521000$ Hz. The error bars are estimated using Monte Carlo simulations of phase residuals with the same noise properties as the actual outburst; they do not account for the uncertainty in the source position. The $\times$’s mark what the frequencies would be if the fit source position differed from the actual position by $2\sigma$ along the ecliptic plane in the direction of increasing RA. The same position error in the decreasing RA direction would move the frequency points by an equal amount in the opposite sense.

such that reduced $\chi^2$ statistic would be unity. The resulting first-order spin derivative is $\dot{\nu} = (-5.6 \pm 2.0) \times 10^{-16}$ Hz s$^{-1}$. The large 1\,\sigma uncertainty reflects the uncertainty in the slope of the frequency change, not in the observation that the source is spinning down. The probability that the frequency is not decreasing is less than $10^{-9}$, as mentioned above, a confidence of better than 6\,\sigma.

Fitting second-order frequency models established that $\dot{\nu}$ is consistent with zero during all the outbursts. These measurements are particularly sensitive to pulse profile variations, so care must be taken to not overfit such features. We again exclude the initial observations of the 1998, 2002, and 2005 outbursts, because the pulse profile changes would induce large non-zero $\dot{\nu}$ measurements that most likely do not reflect the spin of the underlying neutron star. Using
4.3 Results

Table 4.4. Best-fit constant frequencies, and their $\dot{\nu}$ upper limits

<table>
<thead>
<tr>
<th>Data included (MJD)</th>
<th>$\nu - \nu_0$ $^a$ ((\mu)Hz)</th>
<th>$\dot{\nu}$ $^b$ ($10^{-14}$ Hz s$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1998 Apr 50914.8 – 50936.9</td>
<td>0.371 ± 0.018</td>
<td>(−7.5, 7.3)</td>
</tr>
<tr>
<td>2000 Feb 51564.0 – 51601.9</td>
<td>0.254 ± 0.012</td>
<td>(−1.1, 1.4)</td>
</tr>
<tr>
<td>2002 Oct 52565.0 – 52602.8$^c$</td>
<td>0.221 ± 0.006</td>
<td>(−1.3, 2.5)</td>
</tr>
<tr>
<td>2005 Jun 53529.6 – 53581.5</td>
<td>0.195 ± 0.016</td>
<td>(−0.5, 2.4)</td>
</tr>
</tbody>
</table>

$^a$The frequencies are relative to $\nu_0 = 400.975210$ Hz.

$^b$95% confidence intervals from the Monte Carlo simulations.

$^c$Excluding MJD 52575.7–52577.7.

TEMPO to find the best-fit $\dot{\nu}$’s and applying Monte Carlos to estimate their uncertainties, we arrived at the 95% confidence intervals of Table 4.4. Excluding the 1998 outburst, which had the shortest span of timing data and thus the most poorly constrained $\dot{\nu}$, these 95% confidence upper limits were all of order $|\dot{\nu}| \lesssim 2.5 \times 10^{-14}$ Hz s$^{-1}$.

The uncertainties in the measurement of the frequency preclude phase connection between outbursts. During the 920 d gap between the 2002 and 2005 outbursts, the 6 nHz frequency uncertainty from 2002 would accumulate to a phase uncertainty of 0.5 cycles; the $2 \times 10^{-16}$ Hz s$^{-1}$ uncertainty in the long-term spin down would contribute 0.6 cycles. Worse, these estimates are best-case scenarios, since they assume that the spin down is constant.

During the 1998 and 2002 outbursts, we observed an abrupt change in the slope of the phase residuals at the end of the main outburst. We modeled these apparent instantaneous changes of frequency by including frequency glitches in our TEMPO fits. (While the TEMPO glitch models are useful in describing the data, we do not believe that we observed actual sudden changes in the spin frequency of the star, a point discussed in detail in §4.4.3.) Figure 4.7 shows the phase residuals of these outbursts and their best-fit glitch models. These models only employ an instantaneous change in frequency; including a phase jump or introducing a $\dot{\nu}$ after the events did not significantly improve the fits.$^{10}$

In both outbursts, these apparent frequency changes coincide with the sud-

$^{10}$During the 2002 outburst, the absence of a phase jump refers only to the second harmonic, which we believe is a better tracer of the neutron star spin during this period of time (in agreement with the conclusions of Burderi et al. 2006).
The long-term evolution of the spin, pulse shape, and orbit of the accretion-powered millisecond pulsar SAX J1808.4−3658

Figure 4.7: Comparison of the 1998 and 2002 glitch-like events. The phase plots show the phase residuals relative to a constant-frequency model for the fundamental (black points) and the harmonic (grey points), binned such that there is one point per observation. The black lines indicate the best timing models fit by TEMPO. The middle plot shows the 1998 and 2002 light curves for comparison. The 1998 light curve has been vertically offset by 1 erg cm$^{-2}$ s$^{-1}$ for clarity. The data are displayed such that the apparent changes in frequency are aligned. Notice that this alignment also has the effect of closely matching up the light curves.

den drop in flux that marks the transition from the slow-decay stage to the flaring tail stage. At the same time, the fractional amplitudes of the fundamental and harmonic increase, and, in the case of 2002, the pulse profile change occurs. (This pulse profile change, discussed earlier in §4.3.2, is apparent in Fig. 4.7 as the rapid advance of the fundamental phase.) If we view the phase residuals with respect to the pre-transition frequencies, as is the case in Figure 4.7, the residuals following the transition skew upward, indicating
progressively increasing lags. This effect is more pronounced in 1998, but its coverage is far better in 2002. If we were to interpret these changes in slope as abrupt spin frequency changes, they would represent drops of 0.21 \( \mu \text{Hz} \) and 0.03 \( \mu \text{Hz} \) for 1998 and 2002, respectively. (Again, we consider this scenario unlikely; see §4.4.3.) If we instead interpret them as the motion of a radiating spot, the drift rates would be 6.5° d\(^{-1}\) and 1.0° d\(^{-1}\), retrograde. The total observed shifts between the start of the flaring tail and the loss of the signal are substantial: 0.15 cycles (54°) in 1998 and 0.06 cycles (22°) in 2002. The data are not good enough to distinguish whether these drifts are continuous. For instance, it is possible that the hot spot made a retrograde jump every time there was a flare.

We did not observe the main body of the 2000 outburst, so we cannot measure whether the apparent frequency decreased when it entered the flaring tail stage. But if it did, and if the decrease in the apparent frequency was of similar magnitude to that observed in 1998 and 2002, then including the main body of the 2000 outburst would raise the overall frequency of the outburst somewhat. This correction might put it in line with the other frequency measurements in Figure 4.6, reducing the large \( \chi^2 \) statistic of the constant-\( \dot{\nu} \) fit. Therefore we cannot conclude that the change in the observed frequency from one outburst to the next is incompatible with a linear progression.

During the 2005 outburst, the substantial pulse profile noise during the tail prevented us from measuring a change in apparent frequency. The uncertainty in the measurement of the frequency during the tail was 0.03 \( \mu \text{Hz} \), as estimated using Monte Carlo simulations of the profile noise, and the phase residuals jumped by as much as 0.1 cycles from one observation to the next. If there was a smaller drift, as seen during the 2002 outburst, we would not necessarily detect it.

### 4.3.7 Evolution of the binary orbit

We fit the orbital parameters separately for each outburst. Table 4.5 lists the results. As expected, the values of \( a \sin i \) and \( P_{\text{orb}} \) were consistent among the outbursts. The fit parameters \( e \sin \omega \) and \( e \cos \omega \) were consistent with zero. We used them to improve significantly on previous upper limits on the eccentricity.

The measured time of ascending node advanced with each outburst, relative to the times expected if the period was constant. Figure 4.8 shows these \( T_{\text{asc}} \) residuals. A quadratic provides a good fit (\( \chi^2 = 1.01 \) with a single degree of freedom), yielding a constant orbital period derivative of \( \dot{P}_{\text{orb}} = (3.5 \pm 0.2) \times 10^{-12} \text{ s}^{-1} \) and a significance of 15.6\( \sigma \). In an independent analysis of the same data, di Salvo et al. (2008) report a consistent value for \( \dot{P}_{\text{orb}} \). They derive a smaller uncertainty and larger \( \chi^2 \), most likely reflecting an underestimate of
4. The long-term evolution of the spin, pulse shape, and orbit of the accretion-powered millisecond pulsar SAX J1808.4−3658

Figure 4.8: Measurement of an orbital period derivative. The points show the observed times of ascending node, relative to the expected times for a constant period. The \( T_{\text{asc}} \) of each outburst comes progressively later, indicating a period derivative of \((3.5 \pm 0.2) \times 10^{-12} \, \text{s} \, \text{s}^{-1}\).

The orbital phase measurement errors.

Table 4.6 summarizes all the parameters for the pulse timing of SAX J1808.

4.4 Discussion

Our analysis of multiple outbursts from SAX J1808 allows us to greatly improve our understanding of the behavior of this low-mass X-ray binary. By comparing the observed frequency from each outburst, we can see the long-term spin down, which is too small to be detectable from a single outburst. Comparison of the pulse profiles from each outburst lead us to conclude that we are seeing characteristic, repeated profile changes as the outbursts progress, rather than a purely random noise process. Finally, fitting of the orbital parameters over the seven years of observation provides a greatly improved orbital ephemeris.
4.4 Discussion

Table 4.5: Binary parameter measurements from each outburst

<table>
<thead>
<tr>
<th>Outburst</th>
<th>( P_{\text{orb}} ) (s)</th>
<th>( a \sin i ) (light-ms)</th>
<th>( T_{\text{asc}} ) (MJD, TDB)</th>
<th>( e \sin \omega ) (10(^{-6}))</th>
<th>( e \cos \omega ) (10(^{-6}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1998 Apr</td>
<td>7249.1563(18)</td>
<td>62.8080(46)</td>
<td>50921.7584194(12)</td>
<td>-60 ± 64</td>
<td>-86 ± 64</td>
</tr>
<tr>
<td>2000 Feb</td>
<td>---</td>
<td>---</td>
<td>51591.8019861(40)</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>2002 Oct</td>
<td>7249.1565(6)</td>
<td>62.8147(31)</td>
<td>52570.0186514(9)</td>
<td>8 ± 57</td>
<td>41 ± 57</td>
</tr>
<tr>
<td>2005 Jun</td>
<td>7249.1547(24)</td>
<td>62.8282(109)</td>
<td>53524.9944192(32)</td>
<td>-173 ± 83</td>
<td>53 ± 83</td>
</tr>
</tbody>
</table>

Note. — We excluded the 2000 outburst when calculating everything but \( T_{\text{asc}} \) because its data were noisy and sparse.

Table 4.6: Combined timing parameters for SAX J1808

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orbital period, ( P_{\text{orb}} ) (s)</td>
<td>7249.156961(14)</td>
</tr>
<tr>
<td>Orbital period derivative, ( \dot{P}_{\text{orb}} ) (10(^{-12}) s s(^{-1}))</td>
<td>3.48(23)</td>
</tr>
<tr>
<td>Projected semimajor axis, ( a \sin i ) (light-ms)</td>
<td>62.8132(24)</td>
</tr>
<tr>
<td>Time of ascending node, ( T_{\text{asc}} ) (MJD, TDB)</td>
<td>52499.9602477(10)</td>
</tr>
<tr>
<td>Eccentricity, ( e ) (95% confidence upper limit)</td>
<td>&lt; 1.2 \times 10^{-4}</td>
</tr>
<tr>
<td>Spin frequency, ( \nu ) (Hz)</td>
<td>400.975210240(11)</td>
</tr>
<tr>
<td>Spin frequency derivative, ( \dot{\nu} ) (10(^{-16}) Hz s(^{-1}))</td>
<td>-5.6(2.0)</td>
</tr>
</tbody>
</table>

\( aP_{\text{orb}} \) and \( \nu \) are specified for the time \( T_{\text{asc}} \).
4.4.1 Long-term spin down

By observing the mean spin frequency of each outburst, we found that SAX J1808 is spinning down at a rate of \( \dot{\nu} = (-5.6 \pm 2.0) \times 10^{-16} \) Hz s\(^{-1}\). This spin down results in a loss of rotational energy at a rate of \( \dot{E} = 4\pi^2 I \dot{\nu} = 9 \times 10^{33} \) erg s\(^{-1}\), assuming a canonical value of \( I = 10^{45} \) g cm\(^2\) for the neutron star (NS) moment of inertia.

Most of this spin down occurs during X-ray quiescence; accretion torques during the outbursts play a minimal role. Over the seven years of our observations, the mean outburst frequency decreases by \( \nu_{2005} - \nu_{1998} = -0.18 \pm 0.02 \) \( \mu \)Hz. Let us suppose that this frequency change happens only during the X-ray outbursts (which have a duty cycle of \( \lesssim 5\% \)). Since the outburst light curves are quite similar, it is reasonable to presume that each would contribute roughly the same frequency shift, thus splitting this frequency change into three equal steps. If the spin down is due to a constant \( \dot{\nu}_{\text{outburst}} \) that acts during the \( \approx 20 \) d of each outburst,\(^{11}\) then

\[
\dot{\nu}_{\text{outburst}} \approx \frac{-0.18 \ \mu\text{Hz}}{3 \times 20 \ \text{d}} = -3.5 \times 10^{-14} \ \text{Hz s}^{-1}.
\]

By contrast, we were able to set stringent (95% confidence) upper limits of \( |\dot{\nu}| \lesssim 2.5 \times 10^{-14} \) Hz s\(^{-1}\) during the outbursts (Table 4.4). We conclude that the spin down is dominated by torques exerted during X-ray quiescence.

We will thus consider three possible sources of torque during quiescence: magnetic dipole radiation, the expulsion of matter by the magnetic field (i.e., the propeller effect), and gravitational radiation. In general, we assume that all three mechanisms contribute additively to the observed spin down of SAX J1808,

\[
N_{\text{obs}} = N_{\text{dipole}} + N_{\text{prop}} + N_{\text{gr}}.
\]

We discuss each below.

**Magnetic dipole torque**

A spinning dipolar magnetic field will produce a significant spin down during quiescence for the \( 10^8 \) G field strengths expected for a millisecond pulsar. Relativistic force-free MHD models of pulsar magnetospheres by Spitkovsky (2006) give a torque of \( N_{\text{dipole}} = -\mu^2(2\pi \nu/c)^3(1 + \sin^2 \alpha) \), where \( \mu \) is the magnetic dipole moment and \( \alpha \) is the angle between the magnetic and rotational

\(^{11}\) In reality, \( \dot{\nu} \) would almost certainly not be constant during as the accretion rate varies, but for argument’s sake we make the most conservative assumptions possible. A varying \( \dot{\nu} \) would require that it be sometimes greater than the value from equation (4.12), making it even less plausible that it would escape detection.
poles. Pulse profile modeling of the 1998 outburst by Poutanen & Gierliński (2003) suggests that the magnetic hot spot is not far from the rotational pole, separated by an angle of 5–20°. While other effects might also contribute to the spin down, the rotating magnetic field will always be present and provides an upper limit on the dipole moment:

$$\mu < 0.77 \times 10^{26} \left(1 + \sin^2 \alpha\right)^{-1/2} \times \left(\frac{I}{10^{45} \text{ g cm}^2}\right)^{1/2} \left(\frac{\nu}{401 \text{ Hz}}\right)^{-3/2} \times \left(\frac{-\dot{\nu}}{5.6 \times 10^{-16} \text{ Hz s}^{-1}}\right)^{1/2} \text{ G cm}^3. \quad (4.14)$$

For $\alpha = 15^\circ$, this upper limit on the dipole is $0.75 \times 10^{26} \text{ G cm}^3$, yielding a field strength of roughly $B = 1.5 \times 10^8 \text{ G}$ at the magnetic poles.$^{12}$ We emphasize that this upper limit on the magnetic field is for a purely dipolar field. The presence of higher-order multipoles would require a stronger field at the NS surface to produce the observed $\dot{\nu}$. This field estimate is consistent with the limits implied by accretion physics (see §4.4.5).

If magnetic dipole torque is a significant contributor to the spin down of SAX J1808, then the source may behave like a rotation-powered pulsar during quiescence, producing radio pulsations and a particle wind. The heating of the companion by a particle wind has been invoked as an explanation of why the companion is significantly brighter than expected in the optical. Burderi et al. (2003) predicted a dipole moment of $\mu = 5 \times 10^{26} \text{ G cm}^3$ based on the optical observations, somewhat higher than our approximate upper limit on $\mu$, but most likely within the uncertainties of the model. A similar analysis by Campana et al. (2004) found the needed irradiation luminosity to be $L_x = (4 \pm 1) \times 10^{33} \text{ erg s}^{-1}$, compatible with the observed $\dot{E} = 9 \times 10^{33} \text{ erg s}^{-1}$ loss of rotational energy. No radio emission has been detected during quiescence. The upper limits of 0.5 mJy (Gaensler et al. 1999; Burgay et al. 2003) are not particularly constraining.

The X-ray luminosities of isolated millisecond pulsars, for which magnetic dipole radiation is the primary spin-down mechanism, shows a strong correlation with their rates of rotational energy loss. From the tables compiled in Zavlin (2006) and Cameron et al. (2007), the 5–10 keV X-ray luminosity goes as $L_x \propto \dot{E}^{1.13}$ with less than a quarter decade of scatter. Based on this empirical relation, we would expect a quiescent luminosity for SAX J1808 of

$^{12}$The Spitkovsky (2006) formula for $N_{\text{dipole}}$ differs substantially from the classically derived torque due to a rotating dipole in a vacuum, $N_{\text{vac}} = \frac{4}{5} \mu^2 (2\pi\nu/c)^3 \sin^2 \alpha$, especially for small $\alpha$: for $\alpha = 15^\circ$, the derived limit is approximately one fifth of the vacuum value.
5 × 10^{30} \text{ erg s}^{-1}. However, this prediction is a factor of ten lower than the observed quiescent fluxes of 8 × 10^{31} \text{ erg s}^{-1} and 5 × 10^{31} \text{ erg s}^{-1} (Campana et al. 2002; Heinke et al. 2007), suggesting other mechanisms for quiescent emission are at work.

**Magnetic propeller torque**

The propeller effect offers another possible explanation for the observed spin down during quiescence. If the Keplerian corotation radius (defined as $r_{\text{co}} = (GM/4\pi^2\nu^2)^{1/3} \approx 31 \text{ km}$) is less than the magnetospheric radius $r_0$, at which point the infalling matter couples to the magnetic field, then the magnetic field will accelerate the matter, possibly ejecting it from the system (Illarionov & Sunyaev 1975). The torque exerted on the neutron star by propeller ejection of matter at a rate $\dot{M}_{\text{ej}}$ depends on the details of the interaction between the pulsar magnetosphere and the accretion disk. However, we can parametrize this torque as

$$N_{\text{prop}} = -n\dot{M}_{\text{ej}}(GMr_0)^{1/2} = -n(r_0/r_{\text{co}})^{1/2}\dot{M}_{\text{ej}}(GMr_{\text{co}})^{1/2}, \quad (4.15)$$

where the detailed physics determines the dimensionless torque $n$, which is zero for $r_0 = r_{\text{co}}$ and of order unity for $r_0 \gtrsim 1.1r_{\text{co}}$ (Eksi et al. 2005).

We can then roughly estimate the rate at which matter would need to be ejected from the system during quiescence to account for the observed spin down:

$$\dot{M}_{\text{ej}} < -2.3 \times 10^{-12} n^{-1}(r_0/r_{\text{co}})^{-1/2} \times \left(\frac{I}{10^{45} \text{ g cm}^2}\right)\left(\frac{M}{1.4 M_\odot}\right)^{-2/3}\left(\frac{\nu}{401 \text{ Hz}}\right)^{1/3} \times \left(\frac{-\dot{\nu}}{5.6 \times 10^{-16} \text{ Hz s}^{-1}}\right) M_\odot \text{ yr}^{-1}. \quad (4.16)$$

As a consistency check, we note that this upper limit does not exceed the predicted long-term mass transfer rate for the binary, $1 \times 10^{-11} M_\odot \text{ yr}^{-1}$, which is driven by gravitational radiation emission due to the binary orbit (Bildsten & Chakrabarty 2001). Indeed, not all the mass lost by the donor star will necessarily reach the pulsar magnetosphere during quiescence and be propelled outward; most of it would queue up in the accretion disk and later reach the NS during an outburst. Galloway et al. (2006) found that the mass transfer is roughly conservative, albeit with enough uncertainty that propeller mass loss as large as the above $\dot{M}_{\text{ej}}$ limit is not ruled out.
Even if propeller spin down provides the dominant quiescent torque, the resulting ejection of matter from the system would not greatly affect the binary orbit. The timescale for propeller spin down is proportional to the timescale for the ejection of mass: $\dot{P}_{\text{orb}}/P_{\text{orb}} \propto \dot{M}_{\text{ej}}/M_c$, where $M_c \approx 0.05 M_\odot$ is the mass of the companion. Applying the above $\dot{M}_{\text{ej}}$ gives $M_c/\dot{M}_{\text{ej}} = 20$ Gyr, far longer than the observed orbital evolution timescale of $P_{\text{orb}}/\dot{P}_{\text{orb}} = 66$ Myr. More refined calculations using the arguments of Tauris & van den Heuvel (2006) yield a propeller timescale of 6 Gyr, still far too large. Clearly there are other contributions to the orbital evolution; we discuss some in §4.4.7.

**Gravitational radiation torque**

A variety of mechanisms have been proposed in which rapidly rotating neutron stars can develop mass quadrupoles that give rise to gravitational radiation from the neutron star itself. These mechanisms include $r$-mode instabilities (Wagoner 1984; Andersson et al. 1999), accretion-induced variations in the density of the NS crust (Bildsten & Cumming 1998; Ushomirsky et al. 2000), distortion of the NS due to toroidal magnetic fields (Cutler 2002), and magnetically confined mountains at the magnetic poles (Melatos & Payne 2005). The loss of angular momentum due to gravitational radiation has been suggested as a mechanism to explain the absence of observed pulsars with spin frequencies faster than $\approx 730$ Hz (Chakrabarty et al. 2003b; Chakrabarty 2005) and makes millisecond pulsars a target for interferometric gravitational wave detectors.

The mass quadrupole moment of the star, $Q$, determines the torque produced by gravitational radiation: $N_{\text{gr}} = -\frac{32}{5} G Q^2 (2\pi \nu/c)^5$. For our measured $\dot{\nu}$, this sets an upper limit of

$$Q < 4.4 \times 10^{36} \left( \frac{I}{10^{45} \text{ g cm}^2} \right)^{1/2} \left( \frac{\nu}{401 \text{ Hz}} \right)^{-5/2} \times \left( \frac{-\dot{\nu}}{5.6 \times 10^{-16} \text{ Hz s}^{-1}} \right)^{1/2} \text{ g cm}^2,$$

or $Q \lesssim 10^{-8} I$. The strain amplitude of the resulting gravitational waves, averaged over all NS orientations, is $h_c = 115 G \nu^2 Q/dc^4$ (Brady et al. 1998), giving a characteristic strain at Earth of $h_c = 6 \times 10^{-28}$. This strain is undetectable by current or planned gravitational wave experiments. For Advanced LIGO, with a strain sensitivity of $\sim 3 \times 10^{-24}$ Hz$^{-1/2}$ in the 100–400 Hz range (Fritschel 2003), even a search using an accurate phase model would require years of integration time. Note that the dependence of $N_{\text{gr}}$ on the $\nu$ is very
strong, so it is quite possible that gravitational wave emission produces larger spin downs in faster (≈700 Hz) rotators.

4.4.2 Pulse profile variability

The evolution of the pulse profile is clearly not purely stochastic. With multiple outbursts, we are able to note for the first time that the pulse profile seems to take on similar shapes at similar times in the outbursts, as illustrated in Figure 4.3. These characteristic changes in the pulse profiles suggest that the emitting regions of the NS are changing shape and position as the outbursts progress. The consistency of these changes, along with the consistency of the outburst light curves, suggests that as the accretion disk empties onto the star, the geometry of the disk, the accretion funnels, and the resulting hot spots evolve in a similar manner for each outburst.

The most striking example of ordered pulse-profile evolution is the strong relationship between the harmonic content and luminosity: \( r_2 \propto L^{-1/2} \). Given the complexity of the system, its abidance by such a simple model is quite surprising. SAX J1808 is not alone in this behavior. At least two other millisecond pulsars, IGR J00291+5934 and XTE J1807–294, exhibit similar inverse correlations between the amplitude of their second harmonics and luminosity (Hartman et al. 2007, in prep.).

One possible explanation is recession of the accretion disk as the accretion rate drops, revealing the star’s previously occulted second hot spot. For rapidly rotating pulsars, partial occultation of the star by the accretion disk will be common. Assuming a mass of 1.4 \( M_\odot \), the co-rotation radius of SAX J1808 is \( r_{\text{co}} = 31 \text{ km} \approx 3R \), where \( R \) is the NS radius. Following the standard pulsar accretion model (e.g., Ghosh & Lamb 1979b), the inner edge of the accretion disk will be at roughly the Alfvén radius: \( r_0 \approx r_A \equiv (2GM)^{-1/7}M^{-2/7}\mu^{1/7} \). This truncation radius must be at \( r_0 < r_{\text{co}} \) for infalling matter to reach the NS surface. There are clear problems with the application of this model, which was developed for higher-field pulsars with \( r_0 \gg R \): the width of the transition region in which the magnetic field becomes dominant is on the same order as its distance to the star, muddling the definition of a truncation radius. Nevertheless, this simple model is still qualitatively instructive.

Since \( r_{\text{co}} \approx 3R \), neutron stars in systems with inclinations \( i \gtrsim 70^\circ \) will always be partially occulted during outburst. During the outbursts of SAX J1808, the peak fluxes at which the pulses are most sinusoidal are roughly a factor of 10 greater than the low fluxes at which the harmonics are more prevalent (cf. Fig. 4.5). As a result, the Alfvén radius will increase by a factor of \( r_{A,\text{tail}}/r_{A,\text{peak}} \approx 10^{2/7} \approx 2 \) as the source dims. Because the maximum Alfvén
radius is $\approx 3R$ during accretion, the radius during the peak of the outbursts must be $r_{A,\text{peak}} \lesssim \frac{9}{2}R$. At this separation, the star will be partially occulted if $i \gtrsim 45^\circ$. Thus the degree of occultation will depend on $\dot{M}$ for a wide range of inclinations. For $45^\circ \lesssim i \lesssim 70^\circ$, the disk will partially occult the NS above some critical $\dot{M}$. For $i \gtrsim 70^\circ$, the NS will always be partially occulted, with the degree of occultation increasing as $\dot{M}$ increases. Pulse profile modeling by Poutanen & Gierliński (2003) suggests that the system is at an inclination of $i > 65^\circ$.

The observations that show clearly double-peaked pulse profiles happen exclusively in the final, flaring tail stage of the outbursts, typically during the fading portion of a flare. In view of this model, one could imagine that the accretion disk is most recessed as the flares fade. One difficulty with this model is that the increased $r_{\text{2}}$ observed at low luminosities is not solely due to the appearance of doubly peaked pulse profiles; many profiles in this regime show single pulses, but with substantially greater asymmetry than typically seen at higher luminosities.

Another possible cause is the expansion of the hot spots during high accretion due to diffusive effects. Simulations of accretion flows by Romanova et al. (2004) demonstrate that as the fluence increases, the cross-sections of the accretion funnels grow. Modeling by Muno et al. (2002c) establishes that the harmonic content of the pulsations decreases as the size of the hot spot increases.

### 4.4.3 Motion of the hot spot

During the 1998, 2002, and 2005 outbursts, we observed clear trends in the phase residuals that suggest that the emitting regions do not remain at a fixed longitude. During 2002 and 2005, an abrupt phase change in the fundamental at the end of the main body of the outburst produces an advance of the pulse peak that corresponds to a shift of the hot spot by $\approx 50^\circ$ eastward. These shifts are simultaneous with and occur on the same 3–4 d timescale as the sudden drops in luminosity at the end of the main outbursts. During 1998 and 2002, the phase residuals of both harmonics begin gradually increasing during the flaring tails of the outburst, corresponding to a westward drift of the hot spots. Motion of the hot spot has also been suggested to explain phase residuals in GX 1+4 and RX J0812.4−3114 (Galloway et al. 2001) and XTE J1814−338 (Papitto et al. 2007).

These trends in the phase residuals almost certainly represent motion of the

---

13For a more natural description, we adopt the Earth-based convention of longitude: earlier pulse arrivals $\equiv$ prograde hot spot motion $\equiv$ eastward shift, and vice versa.
observed hot spot rather than frequency glitches. Glitches are rapid changes in the spin frequency of the NS due to imperfect coupling between the crust and more rapidly rotating, superfluidic lower layers (e.g., Anderson & Itoh 1975). This interaction occurs well below the accretion layer, and it would not be expected to coincide with or have the same timescale as rapid changes in the accretion rate.

When discussing the motion of the hot spots, the longitudes of the magnetic poles provide natural meridians from which to measure phase. Since their movement would require the realignment of currents in the core and crust, the magnetic poles remain at fixed positions for timescales far longer than the outbursts. The suppression of regions of the field due to accretion also occurs on long timescales (Cumming et al. 2001).

For high-field pulsars, the magnetospheric radius is far from the star, and the accretion column follows field lines that reach the NS surface near the magnetic pole. This is not necessarily the case for low-field pulsars. A closer accretion disk will intersect more curved field lines, which terminate farther from the poles. In the previous section, we described how the Alfvén radius can move outward from roughly $1.5R$ to $3R$ as the accretion rate drops. As the disk recesses, it will intersect increasingly curved field lines that are rooted closer to the poles, causing the hot spots at the bases of the accretion columns to also approach the poles.

This simple picture can explain the observed phase shift as the luminosity rapidly drops during the end of the 2002 and 2005 outbursts. In both cases, the luminosity decreases by about a factor of 4. A change in $\dot{M}$ by this magnitude would cause the Alfvén radius to move outward by a factor of 1.5 and the inner edge of the accretion disk to move outward by a similar amount. This change will almost certainly cause material removed from the inner edge of the disk to attach to a different set of field lines, with the larger radius favoring lines that attach closer to the pole. If the hot spot tends to be to the west of the pole, as seen in Romanova et al. (2004) for a magnetic pole an angle of $\alpha = 30^\circ$ from the rotational pole, then the attachment to different field lines would produce an eastward drift as observed. That said, these MHD simulations appear to have strong, chaotic dependencies on their parameters. (For $\alpha = 15^\circ$, the hot spot is south of the magnetic pole; for $30^\circ$, west; and for $45^\circ$, north!) More work is needed to better model these observations.

This scenario does not explain why the shift of the pulse peak would solely be expressed by a change in the fundamental; during these episodes in 2002 and 2005 the phase of the harmonic remains relatively constant. However, a movement of the hotspot toward the magnetic pole would most likely change the shape of the hotspot, possibly in a way that would preserve the phase of
the harmonic.

The slow drifts seen during the tails of the 1998 and 2002 outbursts are also difficult to explain. The flares during the tail cause the luminosity to change in a periodic manner, so we cannot expect a monotonic motion of the accretion disk’s inner edge. The net drift during the tail of the 2002 outburst is of the same magnitude as the rapid phase shift that happens right before the tail begins, suggesting the drift may be a relaxation of the accretion column back to its original location.

4.4.4 Comparison with previous spin frequency measurements

There have been a number of previous reports of short-term $\dot{\nu}$ measurements made during outbursts of several accreting millisecond pulsars including XTE J0929–314 (Galloway et al. 2002), SAX J1808 (Morgan et al. 2003; Burderi et al. 2006), XTE J1751–305 (Markwardt et al. 2003a), IGR J00291+5934 (Falanga et al. 2005b; Burderi et al. 2007), and XTE J1814–334 (Papitto et al. 2007). Some of the reported $\dot{\nu}$ values have been surprisingly large given the estimates of $\dot{M}$ during the outbursts, possibly violating a basic prediction of magnetic disk accretion theory: that accretion torques cannot exceed the characteristic torque $N_{\text{char}} = \dot{M}(GM_{\text{cor}})^{1/2}$ exerted by accreting Keplerian material at the corotation radius (e.g., Ghosh & Lamb 1979b).

In the particular case of SAX J1808, spin derivatives as large as a few times $10^{-13}$ Hz s$^{-1}$ near the outburst peak were reported (Morgan et al. 2003; Burderi et al. 2006), corresponding to accretion torques exceeding $N_{\text{char}}$ for this source. However, these studies calculated pulse phase residuals using only a single harmonic; Morgan et al. (2003) reported $\dot{\nu}$ detections using only the fundamental, while Burderi et al. (2006) measuring the phase from the second-harmonic alone after noting the sudden phase shift of the fundamental in the middle of the 2002 outburst. Our results in §3.6 indicate that both of these approaches are likely to be contaminated by pulse shape changes, at least in the case of SAX J1808.

Figure 4.9 illustrates this point. Taking the phases of the harmonic components as direct spin measurements can produce large values of $\dot{\nu}$ during the peak of the 2002 outburst. Fitting only the fundamental’s phase residuals during the first 10 d of the outburst, we find $\dot{\nu} = (−1.2 ± 0.4) \times 10^{-13}$ Hz s$^{-1}$. On the other hand, using only the second harmonic for the same interval, we find $\dot{\nu} = (5.3 ± 0.1) \times 10^{-13}$ Hz s$^{-1}$, in good agreement with the Burderi et al. (2006) measurement. Because the pulse shape is changing rapidly during this part of the outburst, the pulse arrival times cannot be accurately determined. We therefore cannot reliably use this part of the outburst to measure the spin
4. The long-term evolution of the spin, pulse shape, and orbit of the accretion-powered millisecond pulsar SAX J1808.4−3658

Figure 4.9: Fitting a frequency model using only the fundamental (black) or the harmonic (grey) produces non-zero ˙\(\nu\) measurements during the peak of the 2002 outburst. The data points are the 512 s phase residuals relative to the best constant-frequency model for the 2002 outburst. The solid lines give the best constant- ˙\(\nu\) models, fit solely to the fundamental or the second harmonic. The dashed line shows the constant-frequency model derived using both, combined via equation (4.9); this fit did not use the points prior to MJD 52565.

We thus conclude that the past measurements of short-term ˙\(\nu\) in SAX J1808 are unreliable. The analysis technique we described in §2.3 can mitigate the effects of pulse shape variability to some extent, but attempts to measure ˙\(\nu\) in accreting pulsars must properly account for these variability effects, and in some instances these effects may prevent such measurements. The ˙\(\nu\) measurements reported in other accreting millisecond pulsars must all be reevaluated in this light; all the apparent violations of the \(N \leq N_{\text{char}}\) limit predicted by theory may be owing to spurious measurements caused by pulse shape vari-
ability. However, at least some accreting millisecond pulsars are observed to have relatively stable pulse shapes, indicating that accurate short-term $\dot{\nu}$ measurements are possible and that previous measurement of these sources should be reliable.

### 4.4.5 Constraints on the magnetic field

We showed in §4.1.1 that the condition $N_{\text{dipole}} \leq N_{\text{obs}}$ implies that the magnetic dipole moment $\mu \lesssim 0.8 \times 10^{26} \text{ G cm}^3$. This limit is consistent with the range for $\mu$ implied by the observation of accretion-powered pulsations throughout the outbursts (Psaltis & Chakrabarty 1999). At low accretion rates, the field cannot be so strong that it centrifugally inhibits matter from reaching the NS; during times of high accretion, it must be strong enough to truncate the disk above the stellar surface in order for there to be pulsations.

The dimmest observation in which we observed pulsations was in 1998, with a flux in the 2–25 keV band of $1.5 \times 10^{-11} \text{ erg cm}^{-2} \text{ s}^{-1}$; the brightest was at the peak of the 2002 outburst, $2.62 \times 10^{-9} \text{ erg cm}^{-2} \text{ s}^{-1}$. These fluxes, along with an improved estimate of the Eddington luminosity from observations of photospheric radius expansion bursts (Galloway et al. 2006), give us new limits on the range of accretion rates at which pulsations have been detected, relative to the Eddington rate $\dot{M}_E$: $\dot{M}_{\text{min}} = 1.8 \times 10^{-4} \dot{M}_E$ and $\dot{M}_{\text{max}} = 0.03 \dot{M}_E$. (We have made the usual assumption that $L \propto \dot{M}$.) These limits allow us to update the range for $\mu$ derived in Psaltis & Chakrabarty (1999), equations (11) and (12):

$$0.2 \times 10^{26} \text{ G cm}^3 \lesssim \mu \lesssim 6 \times 10^{26} \text{ G cm}^3.$$  

(4.18)

Taken together with the $N_{\text{dipole}}$ limit, we obtain a fairly narrow allowed range for the magnetic dipole moment,

$$0.2 \times 10^{26} \text{ G cm}^3 \lesssim \mu \lesssim 0.8 \times 10^{26} \text{ G cm}^3,$$

(4.19)

which corresponds to a surface dipole magnetic field strength of $(0.4–1.5) \times 10^8$ G. This field is relatively weak: the magnetic fields implied by the Australia Telescope National Facility Pulsar Catalog\(^\text{15}\) (Manchester et al. 2005) for millisecond pulsars range from $1.1 \times 10^8$ G to $14 \times 10^8$ G.

---

\(^{14}\)In deriving this range for $\mu$, we make the same conservative assumptions as Psaltis & Chakrabarty (1999): the Ghosh & Lamb (1991) boundary layer parameter ranges on $0.1 < \gamma_B(\dot{M}) < 1$; the NS mass is $1.4 M_\odot < M < 2.3 M_\odot$; and the NS radius is $10 \text{ km} < R < 15 \text{ km}$.

\(^{15}\)http://www.atnf.csiro.au/research/pulsar/psrcat/

Pulsars associated with clusters were excluded to minimize the impact of line-of-site accelerations. Field strengths were approximated using equation (4.14).
4. The long-term evolution of the spin, pulse shape, and orbit of the accretion-powered millisecond pulsar SAX J1808.4–3658

4.4.6 Constraints on accretion torques

Even though we did not detect an accretion-induced $\dot{\nu}$ during the outbursts, our new upper limits on $|\dot{\nu}|$ provide far stronger constraints on the accretion physics of low-$B$ systems such as SAX J1808 than previous measurements. Following the earlier analysis of the 1998 outburst by Psaltis & Chakrabarty (1999), the lower limit on the spin frequency derivative predicted by accretion torque theory during an outburst with an average accretion rate of $\dot{M}_{\text{avg}} \approx \frac{1}{3} \dot{M}_{\text{max}} \approx 0.01 \dot{M}_E$ is

$$\dot{\nu} \gtrsim 2 \times 10^{-14} \eta \left( \frac{I}{10^{45} \text{g cm}^2} \right)^{-1} \left( \frac{R}{10 \text{ km}} \right)^{3/2} \times \left( \frac{M}{1.4 M_\odot} \right)^{1/2} \left( \frac{\dot{M}_{\text{avg}}}{0.01 \dot{M}_E} \right) \text{Hz s}^{-1}, \quad (4.20)$$

where $\eta$ is a dimensionless parameter encapsulating the disk-magnetosphere interaction. (Refer to Ghosh & Lamb 1979b for a discussion of the physics that goes into this parameter.) $\eta$ is strongly dependent on the magnetospheric radius. For $r_0 \approx r_{\text{co}}$, the NS will be in spin equilibrium with the accreted matter and $\eta$ will be small. From the $\dot{\nu}$ confidence intervals in Table 4.4, the probability that we would have missed detecting the resulting $2 \times 10^{-14}$ Hz s$^{-1}$ spin up is 0.15%, suggesting that $\eta < 1$ and the source is near spin equilibrium during the outbursts.

4.4.7 Discussion of the increasing $P_{\text{orb}}$

Our seven year baseline for timing analysis provides the most precise measurements yet of the orbital period of SAX J1808. We find that the orbital period is increasing at a rate $\dot{P}_{\text{orb}} = 3.5(2) \times 10^{-12}$ s s$^{-1}$. This $\dot{P}_{\text{orb}}$ lies somewhat outside the 90% confidence upper limit set by Papitto et al. (2005) using the 1998–2002 outbursts, most likely owing to the more limited baseline available in that analysis.

It is interesting to compare our measurement with theoretical expectations. For orbital periods $\lesssim 3$ hr, mass transfer is LMXBs is driven by angular momentum losses due to gravitational radiation from the binary (Kraft et al. 1962), since magnetic braking torques are thought to be ineffective in this regime (Rappaport et al. 1983; Spruit & Ritter 1983). For SAX J1808, the $\dot{M}$ predicted by this mechanism is consistent with observationally inferred long-term average value of $\dot{M} = 1 \times 10^{-11} M_\odot \text{yr}^{-1}$ (Bildsten & Chakrabarty 2001). For conservative mass transfer from a degenerate (brown dwarf) donor, this predicts orbital expansion on a time scale $P_{\text{orb}}/\dot{P}_{\text{orb}} = 3$ Gyr (see, e.g., Tauris &
4.5 APPENDIX: Improved Optical Position for SAX J1808

An accurate source position is essential for high-precision pulsar timing. An incorrect position results in errors during the barycentering of X-ray arrival times, producing frequency offsets due to improperly corrected Doppler shifts (see, e.g., Manchester & Peters 1972). SAX J1808 lies only $\beta = -13.6^\circ$ below the ecliptic plane, so any errors during barycentering will be particularly pronounced. For example, a position error of $\epsilon = 0^\circ 2$ parallel to the plane
The long-term evolution of the spin, pulse shape, and orbit of the accretion-powered millisecond pulsar SAX J1808.4−3658 of the ecliptic produces frequency and frequency derivative offsets relative to \( \nu_0 \approx 401 \text{ Hz} \) of

\[
\Delta \nu = \nu_0 \epsilon (a_\oplus \cos \beta/c) (2\pi/P_\oplus) \cos \tau = 40 \cos \tau \text{ nHz} \tag{4.21}
\]
\[
\Delta \dot{\nu} = -\nu_0 \epsilon (a_\oplus \cos \beta/c) (2\pi/P_\oplus)^2 \sin \tau = -8 \times 10^{-15} \sin \tau \text{ Hz s}^{-1} \tag{4.22}
\]

Here \( \tau = 2\pi t/P_\oplus \) parametrizes the Earth’s orbit, with time \( t \) equal to zero when the Earth is closest to the source. These offsets are comparable with the expected timing uncertainties. Each outburst gives a baseline of about \( 2 \times 10^6 \text{ s} \) over which we can typically measure pulse arrival times with an accuracy of better than \( 25 \mu s \), or \( 1 \times 10^{-2} \) cycles, producing \( \sim 5 \text{ nHz} \) frequency uncertainties. By similar logic, we should be sensitive to \( \dot{\nu} \)'s as small as \( \sim 3 \times 10^{-15} \text{ Hz s}^{-1} \). In practice, the pulse shape noise observed in SAX J1808 makes the actual uncertainties somewhat greater than these back-of-the-envelope values, increasing the \( \nu \) uncertainty by a factor of \( \sim 2 \) and the \( \dot{\nu} \) uncertainty by a factor of \( \sim 10 \), but the frequency uncertainty is still substantially less than the offsets due to a \( 0''2 \) position error.

We observed the field of SAX J1808 with the Raymond and Beverly Sackler Magellan Instant Camera (MagIC) on the 6.5-m Baade (Magellan I) telescope on the night of 2001 June 13, using the \( r' \) filter. The seeing was \( 0''5 \). Figure 4.10 shows the results. After standard reduction, involving bias-subtraction and flatfielding, we attempted to register the field to the International Coordinate Reference System (ICRS). We examined three astrometric catalogs for this purpose: the Hubble Space Telescope Guide Star Catalog (GSC, which was used by Giles et al. 1999; Lasker et al. 1990), the USNO-B1.0 survey (Monet et al. 2003), and the Two-Micron All-Sky Survey (2MASS; Skrutskie et al. 2006). We selected stars from all three catalogs that were not saturated or blended on our image, and fit using the IRAF task \texttt{ccmap} for the position offset, rotation, and plate-scale. We found that we could obtain the best astrometry with 2MASS: with USNO and GSC, many stars that had consistent positions between 2MASS and our image had deviations of more than \( 0''2 \), the overall scatter was larger, and there were fewer stars. With 2MASS we fit using 70 stars across the \( 2' \) MagIC frame. With position residuals of \( 0''08 \) in each coordinate, we obtained a combined uncertainty of \( 0''08/\sqrt{70} = 0''01 \). Therefore, our astrometric uncertainty is dominated by the \( \approx 0''15 \) position uncertainty of 2MASS.

To verify our position, we checked for stars on the MagIC image from the Second US Naval Observatory CCD Astrograph Catalog (UCAC2; Zacharias et al. 2004). These are highly accurate positions (individual uncertainties of 20–40 mas) for relatively bright (\( \approx 15 \text{ mag} \)) stars taken with a CCD at a current epoch (1996–1998) and with proper motions. We found three unsaturated
4.5 APPENDIX: Improved Optical Position for SAX J1808

Figure 4.10: A 30′′ portion of our r′-band Magellan image. The counterpart of SAX J1808 is indicated by the tick marks: it is the north-west object of the close pair at the center. We also indicate three 2MASS stars that we used for astrometry with the circles. The changing grayscale levels across the image reflects poor correction for the four-amplifier readout of MagIC but does not affect our astrometry.

UCAC2 stars on our image: 16259696, 16259777, and 16259680. We measured their positions on our image and compared the positions derived from the 2MASS solution to those from UCAC2, updated to epoch 2001.45. We found no net shift, and the offsets are less than 0″16 in all cases. (We note that the stars are toward the edge of the image, where residual image distortions may be present, in contrast to SAX J1808 which is at the center of the image). Therefore we believe that our solution using 2MASS is indeed accurate to our stated uncertainty of 0″15.

We then measured the position of SAX J1808 on the image and transformed the position to the ICRS. The position that we find is: R.A. = 18$h^0m^27.s^62$, Decl. = $-36^\circ58^\prime43^\prime\prime3$, equinox J2000.0, with uncertainty 0″15. This is 1″5 from the Giles et al. (1999) position, twice their quoted 0″8 uncertainty. But with many more reference stars of higher quality over a smaller field (Giles et al. 1999 used 5 GSC stars over a 4′ field), and CCD data taken at a more recent epoch (1998 for 2MASS, vs. 1987–1996 for GSC and 1981 for USNO), this new position should be more accurate.

\[\text{The USNO does not recommend GSC for current use: see http://ad.usno.navy.mil/star/star\_cats\_rec.shtml#gsc2.2.}\]
4.6 Derivation of Phase Uncertainties

Derivation of the uncertainties of the phase residuals, as given in equation (4.4), follows from our definition of the phases,

\[ A_k \exp(2\pi i k \phi_k) = 2 \sum_{j=1}^{n} x_j \exp(2\pi i j k / n), \]

(4.23)

where we have divided our phases into \( n \) bins, each containing \( x_j \) photons. Inverting to solve for \( \phi_k \),

\[ \phi_k = \frac{1}{2\pi i k} \left( \ln \sum_{j=1}^{n} x_j E_{jk} - \ln \frac{A_k}{2} \right), \]

(4.24)

where we define the constants \( E_{jk} \equiv \exp(2\pi i j k / n) \) for the sake of brevity.

For relatively low fractional amplitudes (certainly the case throughout this paper), each phase bin will contain approximately the same number of photons:

\[ x_j \approx \frac{N_{\text{ph}}}{n}, \]

with variances \( (\sigma x_j)^2 \approx \frac{N_{\text{ph}}}{n} \) due to Poisson counting statistics. These add in quadrature to give the variance in \( \phi_k \):

\[ \sigma_k^2 = \left| \sum_{j=1}^{n} \left( \frac{\partial \phi_k}{\partial x_j} \right)^2 (\sigma x_j)^2 \right| = \left| \sum_{j=1}^{n} \left( \frac{1}{2\pi k} \sum_{j'=1}^{n} x_{j'} E_{jk'} \right)^2 \left( \frac{N_{\text{ph}}}{n} \right) \right|. \]

(4.25)

Summing the exponentials, we have \[ \left| \sum_{j=1}^{n} E_{jk}^2 \right| = \frac{1}{2} n. \] From the definition of \( A_k \) in equation (4.23), \[ \left| \sum_{j=1}^{n} x_j E_{jk} \right| = \frac{1}{2} A_k. \] Substituting these in, we reach our estimate of the phase uncertainty:

\[ \sigma_k = \sqrt{\frac{2N_{\text{ph}}}{2\pi k A_k}}. \]