‘Knowable’ as ‘known after an announcement’

Balbiani, P.; Baltag, A.; van Ditmarsch, H.; Herzig, A.; Hoshi, T.; de Lima, T.

Published in:
Review of Symbolic Logic

DOI:
10.1017/S1755020308080210

THE REVIEW OF SYMBOLIC LOGIC
Volume 1, Number 3, October 2008

‘KNOWABLE’ AS ‘KNOWN AFTER AN ANNOUNCEMENT’*

PHILIPPE BALBIANI
IRIT, Université de Toulouse

ALEXANDRU BALTAG
Computer Science Laboratory, Oxford University

HANS VAN DITMARSCH
Department of Computer Science, University of Otago
and IRIT, Université de Toulouse

ANDREAS HERZIG
IRIT, Université de Toulouse

TOMOHIRO HOSHI
Philosophy Department, Stanford University

and

TIAGO DE LIMA
Department of Technology Management, Eindhoven University of Technology

Abstract. Public announcement logic is an extension of multiagent epistemic logic with dynamic operators to model the informational consequences of announcements to the entire group of agents. We propose an extension of public announcement logic with a dynamic modal operator that expresses what is true after *any* announcement: $\Diamond\varphi$ expresses that there is a truthful announcement ψ after which φ is true. This logic gives a perspective on Fitch’s knowability issues: For which formulas φ, does it hold that $\varphi \to \Diamond K\varphi$? We give various semantic results and show completeness for a Hilbert-style axiomatization of this logic. There is a natural generalization to a logic for arbitrary events.

1. Introduction. One motivation to formalize the dynamics of knowledge is to characterize how truth or knowledge conditions can be realized by new information. From that perspective, it seems unfortunate that in public announcement logic (Plaza, 1989; Gerbrandy & Groeneveld, 1997; van Ditmarsch *et al.*, 2007), a true formula may become false because it is announced. The prime example is the Moore sentence “atom p is true and you do not know that”, formalized by $p \wedge \neg Kp$ (Moore, 1942; Hintikka, 1962), but there are many other examples (van Ditmarsch & Kooi, 2006). After the Moore sentence is announced, you know that p is true, so $p \wedge \neg Kp$ is now false. This is formalized as $\langle p \wedge \neg Kp\rangle Kp$ and $\langle p \wedge \neg Kp\rangle\neg(p \wedge \neg Kp)$, respectively. The part ‘$\langle p \wedge \neg Kp\rangle$’ is a diamond-style dynamic operator representing the announcement. Therefore, the way to make something known may not necessarily be to announce it. Is there a different way to get to know something?

Received: 27 January 2008

doi:10.1017/S1755020308080210

The realization of knowledge (or truth) by new information can be seen as a specific form of what is called 'knowability' in philosophy. Fitch (1963) addressed the problematic question of whether what is true can become known. It is considered problematic (paradoxical even) that the existence of unknown truths is inconsistent with the requirement that all truths are knowable. Again, the Moore sentence $p \wedge \neg Kp$ provides the prime example: It cannot become known because $K(p \wedge \neg Kp)$ entails an inconsistency under the standard interpretation of knowledge. For an overview of the literature on Fitch's paradox, see Brogaard & Salerno (2004); we later discuss some of that in detail, mainly Tennant's proposal on cartesian formulas (Tennant, 1997). The suggestion to interpret 'knowable' as 'known after an announcement' was made by van Benthem (2004).

Of course, some things *can* become known. For example, true facts p can always become known by announcing them, formalized as $p \rightarrow \langle p \rangle Kp$ ('if the atom p is true, then after announcing p, p is known')—The aforementioned paradox involves announcement of epistemic information. One has to be careful with what one wishes for: Some things can become known that were not true in the first place. Consider factual knowledge again: After announcing a fact, you also know that you know it. In other words, 'knowledge of p' is knowable in the sense that there is an announcement that makes it true: We now have that $\langle p \rangle KKp$. But Kp was not true before that announcement, so this formula is not a knowable truth, except in the trivial sense when it was already true before the announcement.

Consider an extension of public announcement logic wherein we can express what becomes true, whether known or not, without explicit reference to announcements realizing that. Let us work our way upward from a concrete announcement. When p is true, it becomes known by announcing it. Formally, in public announcement logic,

$$\langle p \rangle Kp,$$

which stands for 'the announcement of p can be made and after that the agent knows p'. More abstractly, this means that there is a announcement ψ, namely, $\psi = p$, that makes the agent know p, slightly more formal:

$$\text{there is a formula } \psi \text{ such that } \langle \psi \rangle Kp.$$

We introduce a dynamic modal operator that expresses that

$$\Diamond Kp.$$

Obviously, the truth of this expression depends on the model: p has to be true. In case p is false, we can achieve $\Diamond K \neg p$ instead. The formula $\Diamond(Kp \vee K \neg p)$ is valid. Actually, we were slightly imprecise when suggesting that $\Diamond$ means 'there is a ψ such that'. In fact, a restriction on ψ to purely epistemic formulas is required in the semantics, for a technical reason. The resulting logic is called arbitrary public announcement logic, *APAL*, or in short, *arbitrary announcement logic*.

Unlike the introductory examples so far, we present the logic as a *multiagent* logic, wherein all knowledge operators are labeled with the knowing agent in question. For example, we write the validity above as $\Diamond(K_a p \vee K_a \neg p)$, indicating that this concerns what agent a can get to know. There are both conceptual and technical reasons for this multiagent perspective: (i) Various paradoxical situations involving knowledge—that we can in principle also address in arbitrary announcement logic—require more than one agent (such as the Hangman Paradox, also known as the Surprise Examination; for a dynamic epistemic analysis, see van Ditmarsch & Kooi, 2006). (ii) One technical reason is that arbitrary announcement logic for more than one agent is strictly more expressive than public announcement logic, but that for a single agent, it is equally expressive. (iii) We present interesting multiagent formulations of knowability, such as knowledge transfer between agents and how to make distributive knowledge common knowledge.

1.1. Overview of contents. In Section 2, we define the logical language $\mathcal{L}_{apal}$ and its semantics. This section also contains some technical tools repeatedly used in later sections. Section 3 shows various semantic results, including a 'knowable' fragment of the language (we do not fully characterize the knowable formulas) and an expressivity result: Indeed, our logic can express more than the public announcement logic on which it is based. In Section 4, we provide a Hilbert-style axiomatization of arbitrary announcement logic. Section 5 discusses the generalization to a logic for arbitrary events.

2. Syntax and semantics. For both the language and the structures, we assume as background parameters a finite set of agents A and a countably infinite set of atoms P.

2.1. Syntactic notions.

DEFINITION 2.1 **(Language)**. *The language $\mathcal{L}_{apal}$ of arbitrary public announcement logic is inductively defined as:*

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_a\varphi \mid [\varphi]\varphi \mid \Box\varphi$$

where $a \in A$ and $p \in P$. Additionally, $\mathcal{L}_{pal}$ is the language without inductive construct $\Box\varphi$, $\mathcal{L}_{el}$ the language without as well $[\varphi]\varphi$, and $\mathcal{L}_{pl}$ the language without as well $K_a\varphi$. The language with only $\Box$ as modal operator is $\mathcal{L}_{\Box}$.

The languages $\mathcal{L}_{pal}$, $\mathcal{L}_{el}$, and $\mathcal{L}_{pl}$ are those of *public announcement logic*, *epistemic logic*, and *propositional logic*, respectively. A formula in $\mathcal{L}_{el}$ is also called an *epistemic formula*, and a formula in $\mathcal{L}_{pl}$ is also called a *propositional formula* or a *boolean*. For $K_a\varphi$, read 'agent a knows that φ'. For $[\varphi]\psi$, read '(if φ is true, then) after announcement of φ, ψ (is true)'. (Announcements are supposed to be public and truthful, and this is common knowledge among the agents.) For $\Box\psi$, read 'after every announcement, ψ is true'. Other propositional and epistemic connectives are defined by usual abbreviations. The dual of K_a is $\widehat{K}_a$, the dual of $[\varphi]$ is $\langle\varphi\rangle$, and the dual of $\Box$ is $\Diamond$. For $\widehat{K}_a\varphi$, read 'agent a considers it possible that φ'; for $\langle\varphi\rangle\psi$, read '($\varphi$ is true and) after announcement of φ, ψ (is true)'; and for $\Diamond\psi$, read 'there is an announcement after which ψ (is true)'. Write P_φ for the set of atoms occurring in the formula φ (and similarly for necessity and possibility forms, below). Given some $P' \subseteq P$, $\mathcal{L}_x(P')$ is the logical language $\mathcal{L}_x$ ($\mathcal{L}_{apal}, \mathcal{L}_{el}, \dots$) restricted to atoms in P'.

2.1.1. Necessity and possibility forms. A *necessity form* (Goldblatt, 1982) contains a unique occurrence of a special symbol $\sharp$. If $\boldsymbol{\psi}$ is such a necessity form (we write boldface Greek letters for arbitrary necessity forms) and $\varphi \in \mathcal{L}_{apal}$, then $\boldsymbol{\psi}(\varphi)$ is obtained from $\boldsymbol{\psi}$ by substituting φ for $\sharp$ in $\boldsymbol{\psi}$. Necessity forms are used to formulate the axiomatization of the logic (in Section 4) and in the proofs of several semantic results (in Section 3).

DEFINITION 2.2 (**Necessity forms**). *Let $\varphi \in \mathcal{L}_{apal}$. Then,*

- *$\sharp$ is a necessity form*
- *if $\boldsymbol{\psi}$ is a necessity form, then $(\varphi \rightarrow \boldsymbol{\psi})$ is a necessity form*
- *if $\boldsymbol{\psi}$ is a necessity form, then $[\varphi]\boldsymbol{\psi}$ is a necessity form*
- *if $\boldsymbol{\psi}$ is a necessity form, then $K_a\boldsymbol{\psi}$ is a necessity form.*

We also use the dual notion of *possibility form*. It can be defined by the dual clauses to a necessity form: $\sharp$ is a possibility form, and if $\varphi \in \mathcal{L}_{apal}$ and $\boldsymbol{\psi}$ is a possibility form, then $\varphi \wedge \boldsymbol{\psi}$, $\langle\varphi\rangle\boldsymbol{\psi}$, and $\hat{K}_a\boldsymbol{\psi}$ are possibility forms. To distinguish necessity forms from possibility forms, we use different bracketing: Write $\boldsymbol{\psi}\{\varphi\}$ for the possibility form with a unique occurrence of φ. For each necessity form $\boldsymbol{\psi}(\sharp)$, there is a possibility form $\boldsymbol{\psi}'\{\sharp\}$ such that for all φ, $\neg\boldsymbol{\psi}(\varphi)$ is logically equivalent to $\neg\boldsymbol{\psi}'\{\neg\varphi\}$.

2.2. Structural notions.

DEFINITION 2.3 (**Structures**). *An* epistemic model $M = (S, \sim, V)$ *consists of a* domain *S of (factual)* states *(or 'worlds');* accessibility $\sim : A \to \mathcal{P}(S \times S)$*, where each* $\sim(a)$ *is an equivalence relation; and a* valuation $V : P \to \mathcal{P}(S)$. *For* $s \in S$, (M, s) *is an* epistemic state *(also known as a pointed Kripke model). An* epistemic frame S *is a pair* $(S, \sim)$. *For a model, we also write* (S, V) *and for a pointed model also* (S, V, s).

For $\sim(a)$, we write $\sim_a$, and for $V(p)$, we write V_p; accessibility $\sim$ can be seen as a set of equivalence relations $\sim_a$, and V as a set of valuations V_p. Given two states s, s' in the domain, $s \sim_a s'$ means that s is indistinguishable from s' for agent a on the basis of its knowledge. We adopt the standard rules for omission of parentheses in formulas, and we also delete them in representations of structures such as (M, s) whenever convenient and unambiguous. Given a domain S of a model M, instead of $s \in S$, we also write $s \in M$.

2.2.1. Bisimulation. Bisimulation is a well-known notion of structural similarity (Blackburn *et al.*, 2001) that we frequently use in examples and proofs, for example, to achieve our expressivity results.

DEFINITION 2.4 (**Bisimulation**). *Let two models* $M = (S, \sim, V)$ *and* $M' = (S', \sim', V')$ *be given. A nonempty relation* $\mathfrak{R} \subseteq S \times S'$ *is a bisimulation between* M *and* M' *iff for all* $s \in S$ *and* $s' \in S'$ *with* $(s, s') \in \mathfrak{R}$*:*

atoms: *for all* $p \in P$*:* $s \in V_p$ *iff* $s' \in V'_p$

forth: *for all* $a \in A$ *and all* $t \in S$*: if* $s \sim_a t$*, then there is a* $t' \in S'$ *such that* $s' \sim'_a t'$ *and* $(t, t') \in \mathfrak{R}$

back: *for all* $a \in A$ *and all* $t' \in S'$*: if* $s' \sim'_a t'$*, then there is a* $t \in S$ *such that* $s \sim_a t$ *and* $(t, t') \in \mathfrak{R}$.

We write $(M, s) \leftrightarrows (M', s')$ iff there is a bisimulation between M and M' linking s and s', and we then call (M, s) and (M', s') bisimilar. The maximal bisimulation $\mathfrak{R}^{\mathsf{max}}$ between M and itself is an equivalence relation, and the result of identifying all $\mathfrak{R}^{\mathsf{max}}$ bisimilar worlds is a *minimal model* (also known as bisimulation contraction or strongly extensional model) (Aczel, 1988). The construction preserves equivalence relations: If M is an epistemic model, its minimal model is also an epistemic model.

2.3. Semantics.

DEFINITION 2.5 (**Semantics**). *Assume an epistemic model* $M = (S, \sim, V)$*. The interpretation of* $\varphi \in \mathcal{L}_{apal}$ *is defined by induction. Note the restriction to epistemic formulas in the clause for* $\Box\varphi$.

$$
\begin{array}{lll}
M, s \models p & \textit{iff} & s \in V_p \\
M, s \models \neg\varphi & \textit{iff} & M, s \not\models \varphi \\
M, s \models \varphi \wedge \psi & \textit{iff} & M, s \models \varphi \textit{ and } M, s \models \psi
\end{array}
$$

$$M, s \models K_a\varphi \quad \textit{iff} \quad \textit{for all } t \in S : s \sim_a t \textit{ implies } M, t \models \varphi$$
$$M, s \models [\varphi]\psi \quad \textit{iff} \quad M, s \models \varphi \textit{ implies } M|\varphi, s \models \psi$$
$$M, s \models \Box\varphi \quad \textit{iff} \quad \textit{for all } \psi \in \mathcal{L}_{el} : M, s \models [\psi]\varphi.$$

In clause $[\varphi]\psi$ for public announcement, epistemic model $M|\varphi = (S', \sim', V')$ is defined as:

$$\begin{aligned} S' &= \{s' \in S \mid M, s' \models \varphi\} \\ \sim'_a &= \sim_a \cap (S' \times S') \\ V'_p &= V_p \cap S'. \end{aligned}$$

Formula φ is valid in model M, notation $M \models \varphi$, iff for all $s \in S$: $M, s \models \varphi$. Formula φ is valid, notation $\models \varphi$, iff for all M: $M \models \varphi$.

The dynamic modal operator $[\varphi]$ is interpreted as an epistemic state transformer. Announcements are assumed to be truthful and public, and this is commonly known to all agents. Therefore, the model $M|\varphi$ is the model M restricted to all the states where φ is true, including access between states. Similarly, the dynamic model operator $\Box$ is interpreted as an epistemic state transformer. Note that in the definiendum of $\Box\varphi$, the announcements ψ in $[\psi]\varphi$ are restricted to purely epistemic formulas $\mathcal{L}_{el}$. This is motivated in depth below. For the semantics of the dual operators, we have that $M, s \models \Diamond\psi$ iff there is a $\varphi \in \mathcal{L}_{el}$ such that $M, s \models \langle\varphi\rangle\psi$. In other words, $M, s \models \langle\varphi\rangle\psi$ iff $M, s \models \varphi$ and $M|\varphi, s \models \psi$. Write $[\![\varphi]\!]_M$ for the denotation of φ in M: $[\![\varphi]\!]_M := \{s \in S \mid M, s \models \varphi\}$. Given a sequence $\vec{\psi} = \psi_1, \dots, \psi_k$ of announcements, write $M|\vec{\psi}$ for the model $(\dots(M|\psi_1)\dots|\psi_k)$ that is the result of the successive model restrictions.

The set of validities in our logic is called *APAL*. Formally, this is relative to given sets of agents and atoms, but we also use *APAL* more informally to refer to arbitrary public announcement logic, similarly for *PL* (propositional logic), *EL* (epistemic logic, a.k.a. $S5_n$ where $|A| = n$), and *PAL* (public announcement logic).

Bisimilar states satisfy the same epistemic formulas. This extends to *APAL*. The reader may easily verify that if the epistemic states (M, s) and (M', s') are bisimilar, then for all $\varphi \in \mathcal{L}_{apal}$: $(M, s) \models \varphi$ iff $(M', s') \models \varphi$.

EXAMPLE 2.6. *A valid formula of the logic is $\Diamond(K_a p \vee K_a\neg p)$. To prove this, let (M, s) be arbitrary. Either $M, s \models p$ or $M, s \models \neg p$. In the first case, $M, s \models \Diamond(K_a p \vee K_a\neg p)$ because $M, s \models \langle p\rangle(K_a p \vee K_a\neg p)$—the latter is true because ($M, s \models p$ and) $M|p, s \models K_a p \vee K_a\neg p$ and because $M|p, s \models K_a p$. $M, s \models p$ and $M|p, s \models K_a p$; in the second case, we analogously derive $M, s \models \Diamond(K_a p \vee K_a\neg p)$ because $M, s \models \langle\neg p\rangle(K_a p \vee K_a\neg p)$.*

This example also nicely illustrates the order in which arbitrary objects come to light. The meaning of $\models \Diamond\varphi$ is:

(i) for all (M, s), there is an epistemic ψ such that $M, s \models \langle\psi\rangle\varphi$.

This is really different from:

(ii) there is an epistemic ψ such that for all (M, s), $M, s \models \langle\psi\rangle\varphi$,

which might on first sight be appealing to the reader, when extrapolating from the incorrect *reading of $\models \Diamond\varphi$ as 'there is an epistemic ψ such that $\models \langle\psi\rangle\varphi$'. For example, there is no* *epistemic formula* ψ *such that* $\langle\psi\rangle(K_a p \vee K_a\neg p)$ *is valid. (Suppose there were. The,n* ψ *would be valid, so an announcement of* ψ *would not be informative. Then,* $\langle\psi\rangle(K_a p \vee K_a\neg p)$ *would be equivalent to* $K_a p \vee K_a\neg p$*. But in any model where it is not known whether p the latter is false, so it is not valid. Contradiction.) In other words,* (i) *may be true, even when* (ii) *is false.*

2.3.1. Motivation for the semantics of $\Box$. We now compare the given semantics for $\Box\varphi$ to two infelicitous alternatives, thus hoping to motivate our choice. The three options are (infelicitous alternatives are *-ed):

$$M, s \models \Box\varphi \quad \text{iff} \quad \text{for all } \psi \in \mathcal{L}_{el} : M, s \models [\psi]\varphi \qquad \text{(Definition 2.5)}$$

$$*M, s \models \Box\varphi \quad \text{iff} \quad \text{for all } \psi \in \mathcal{L}_{apal} : M, s \models [\psi]\varphi \qquad \text{(intuitive)}$$

$$*M, s \models \Box\varphi \quad \text{iff} \quad \text{for all } S' \subseteq S \text{ containing } s : M|S', s \models \varphi. \qquad \text{(structural)}$$

The 'intuitive' version for the semantics of $\Box\varphi$ more properly corresponds to its intended meaning 'φ is true after arbitrary announcements'. This would be a circular definition, as $\Box\varphi$ is itself one such announcement. It is not clear whether this is well defined, but a restriction to announcements that are epistemic sentences seems at least reasonable in a context of knowledge and belief change.

The 'structural' version for the semantics of $\Box\varphi$ is more in accordance with one of the proposals of Fine (1970) for quantification over propositional variables in modal logic; his work strongly inspired our approach. This structural version is undesirable for our purposes, as it does not preserve bisimilarity of structures: Two bisimilar states can now be separated because they may be in different subdomains. In dynamic epistemic logics, it is considered preferable that action execution preserves bisimilarity; this is because bisimilarity implies logical equivalence, and we tend to think of such actions as changing the *theories* describing those structures, just as in belief revision. For an example, consider the following epistemic state $(M, 1)$—It consists of two states 1 and **1** where p is true and two states 0 and **0** where p is false; *linking* two states means that they are *indistinguishable* for the agent labeling the link; and the *underlined* state is the *actual* state.

$$\begin{array}{ccc}
\mathbf{0} & \text{—}\ a\ \text{—} & \mathbf{1} \\
| & & | \\
b & & b \\
| & & | \\
0 & \text{—}\ a\ \text{—} & \underline{1}
\end{array}$$

We have that $M, 1 \models \Diamond(K_a p \wedge \neg K_b K_a p)$ for the structural $\Box$-semantics, as $M|\{1, \mathbf{1}, \mathbf{0}\}, 1 \models K_a p \wedge \neg K_b K_a p$. On the other hand, for the $\Box$-semantics as defined, $M, 1 \not\models \Diamond(K_a p \wedge \neg K_b K_a p)$, which can be easily seen as that formula is also false in the two-state structure $(M', 1')$ depicted as:

$$0' \text{ — } a \text{ — } \underline{1'},$$

where agent b can distinguish 0 from 1 but agent a cannot. Epistemic state $(M, 1)$ is bisimilar to $(M', 1')$, via the bisimulation $\mathfrak{R} = \{(0, 0'), (\mathbf{0}, 0'), (1, 1'), (\mathbf{1}, 1')\}$. We make two further observations concerning our preferred semantics '$\Box\varphi$ (is true) iff $[\psi]\varphi$ for all $\psi \in \mathcal{L}_{el}$'. First, given that truth is relative to a model, this semantics for $\Box$ amounts to '$\Box\varphi$ is true in (M, s) iff φ is true in all epistemistically definable submodels of M'. Second, note that public announcement logic is equally expressive as multiagent epistemic logic (Plaza, 1989), so '$\Box\varphi$ (is true) iff $[\psi]\varphi$ for all $\psi \in \mathcal{L}_{el}$' corresponds to '$\Box\varphi$ (is true) iff $[\psi]\varphi$ for all $\psi \in \mathcal{L}_{pal}$.' So in fact, we can replace boxes by announcements of any formula, except those containing boxes—which comes fairly close to the intuitive interpretation again.

A theoretically quite justifiable and felicitous version of the 'structural' semantics for $\Box$ above would equate truth of $\Box\varphi$ with truth for all subsets of the minimal model (see page 308) of a model M that contain the actual state s (in other words, a subset must not separate states that are in the maximal bisimulation relation on M). We did not explore this alternative semantics for $\Box$ in depth. For a given model, there may be more such subsets than are epistemically definable, for example, there may be uncountably many such subsets, whereas the epistemically definable subsets are countable.

3. Semantic results.

3.1. Validities.

3.1.1. Validities only involving $\Box$: S4. The following validities demonstrate the 'S4' character of $\Box$. These validities do not, as usual, straightforwardly translate to frame properties because we interpret $\Box$ as an epistemic state transformer and not by way of an accessibility relation.[1] It is also unclear if the set of validities only involving $\Box$ (i.e., $\mathcal{L}_\Box \cap APAL$) satisfies *uniform substitution* (replacing propositional variables by arbitrary formulas is validity preserving). See further research in Section 6.

PROPOSITION 3.1 (**S4 character of $\Box$**). *Let $\varphi, \psi \in \mathcal{L}_{apal}$. Then:*

1. $\models \Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$
2. $\models \Box\varphi \rightarrow \varphi$
3. $\models \Box\varphi \rightarrow \Box\Box\varphi$
4. $\models \varphi$ *implies* $\models \Box\varphi$.

Proof.

1. Obvious.
2. Assume $M, s \models \Box\varphi$. Then in particular, $M, s \models [\top]\varphi$ and therefore (as $M, s \models \top$) $M, s \models \varphi$.
3. Let M and $s \in M$ be arbitrary. Assume $M, s \models \Diamond\Diamond\neg\varphi$. Then, there are epistemic χ and χ' such that $M, s \models \langle\chi\rangle\langle\chi'\rangle\neg\varphi$. Using the validity (for arbitrary formulas) $[\varphi][\varphi']\varphi'' \leftrightarrow [\varphi \wedge [\varphi]\varphi']\varphi''$, we therefore have $M, s \models \langle\chi \wedge [\chi]\chi'\rangle\neg\varphi$, from which follows $M, s \models \Diamond\neg\varphi$.
4. Let M, s be arbitrary. We have to show that for $\psi \in \mathcal{L}_{el}$: $M, s \models [\psi]\varphi$. From the assumption $\models \varphi$ follows $\models [\psi]\varphi$ by necessitation for $[\psi]$. Therefore, also $M, s \models [\psi]\varphi$. As ψ is arbitrary, also $M, s \models \Box\varphi$. $\Box$

[1] It is possible to associate an accessibility relation to $\Box$. Given an model M, consider the union of its epistemically definable submodels, where we label copies of states (in order to distinguish them from their original) with an epistemic formula ψ representing (the class of formulas logically equivalent to ψ, namely) $[\![\psi]\!]_M$. If $M|\varphi|\psi = M|\chi$, now add pair (s_φ, s_χ) to the accessibility relation R_ψ for announcement operator $[\psi]$. Let $R_\Box = \bigcup_{\psi \in \mathcal{L}_{el}} R_\psi$. If we do this just for announcements that correspond to sequences of announcements of a *single* epistemic formula ψ, the result is known as the *forest* for (M, s) and ψ (van Benthem *et al.*, 2007).

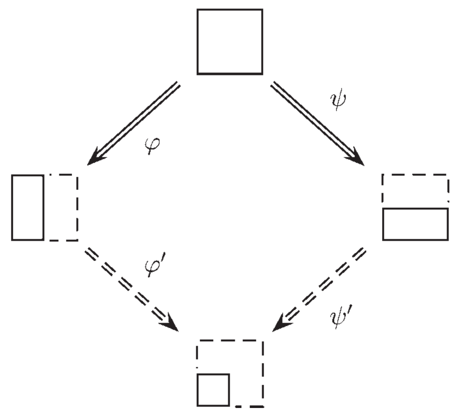


Fig. 1. Church–Rosser for announcements: Given two announcements φ, ψ in some epistemic state (M, s), there are subsequent announcements φ', ψ' such that $(M|\varphi|\varphi', s)$ is bisimilar to $(M|\psi|\psi', s)$.

3.1.2. Validities only involving $\Box$: MK and CR. Also valid are $\models \Box\Diamond\varphi \to \Diamond\Box\varphi$ (McKinsey—MK) and $\models \Diamond\Box\varphi \to \Box\Diamond\varphi$ (Church–Rosser—CR). Axiom CR corresponds to the well-known frame property of confluence: $\forall xyz(Rxy \land Rxz \to \exists w(Ryw \land Rzw))$. In our terms, this can be formulated as follows. Given two distinct (and true) announcements φ, ψ in some epistemic state (M, s), then there are subsequent announcements φ', ψ' such that $(M|\varphi|\varphi', s)$ is bisimilar to $(M|\psi|\psi', s)$ (Figure 1). The proofs of MK and CR are both somewhat involved and include lemmas and such—The first two lemmas take us to Proposition 3.4 showing validity of McKinsey and a subsequent trio of a lemma and two propositions takes us to Proposition 3.8 showing validity of Church–Rosser.

LEMMA 3.2. *Let $\varphi \in \mathcal{L}_{apal}$. Consider the set P_φ of atoms occurring in φ. Let M be a model where all states correspond on the valuation of P_φ. Then, $M \models \varphi$ or $M \models \neg\varphi$, that is, either φ or its negation is a model validity.*

Proof. Let $\varphi(\psi/p)$ be the substitution of ψ for all occurrences of p in formula φ. (Note the difference with the notation for necessity and possibility forms on page 307.) If p is true on M, then $M \models \varphi \leftrightarrow \varphi(\top/p)$, otherwise $M \models \varphi \leftrightarrow \varphi(\bot/p)$. The result of successively substituting $\top$ or $\bot$ for all atoms in φ in that way is the formula $\varphi^\emptyset$. Clearly, $M \models \varphi \leftrightarrow \varphi^\emptyset$. As $\varphi^\emptyset$ does not contain atomic propositions, and given that $\models K_a\top \leftrightarrow \top$, $\models K_a\bot \leftrightarrow \bot$, $\models \Box\top \leftrightarrow \top$, and $\models \Box\bot \leftrightarrow \bot$, we have that $\models \varphi^\emptyset \leftrightarrow \top$ or $\models \varphi^\emptyset \leftrightarrow \bot$. Therefore, $M \models \varphi \leftrightarrow \top$ or $M \models \varphi \leftrightarrow \bot$, that is, $M \models \varphi$ or $M \models \neg\varphi$. $\Box$

The characteristic formula δ_s^φ of the restriction of the valuation in a state s to the finite set P_φ of atoms occurring in φ is defined as follows:

$$\delta_s^\varphi = \bigwedge\{p \mid p \in P_\varphi \text{ and } M, s \models p\} \land \bigwedge\{\neg p \mid p \in P_\varphi \text{ and } M, s \not\models p\}.$$

LEMMA 3.3. *Let $\varphi \in \mathcal{L}_{apal}$ be arbitrary. Let M be a model, and s a world in M. Then, $M|\delta_s^\varphi, s \models \varphi \to \Box\varphi$.*

Proof. As δ_s^φ is boolean, we have that δ_s^φ is true in the model $M|\delta_s^\varphi$, that is, $M|\delta_s^\varphi \models \delta_s^\varphi$, and remains true in any further restriction of M: For any formula $\psi \in \mathcal{L}_{el}$, we have that $M|\delta_s^\varphi|\psi \models \delta_s^\varphi$. As δ_s^φ is a conjunction of literals determining the values of all the atoms of φ, we have that for arbitrary epistemic formulas ψ, all states in models $M|\delta_s^\varphi|\psi$ correspond on the valuation of P_φ. By Lemma 3.2, we therefore have either $M|\delta_s^\varphi|\psi \models \varphi$ for any ψ or $M|\delta_s^\varphi|\psi \models \neg\varphi$ for any ψ. In the former case, $M|\delta_s^\varphi \models \Box\varphi$, and in the latter case, $M|\delta_s^\varphi \models \Box\neg\varphi$. Hence, $M|\delta_s^\varphi, s \models \varphi \rightarrow \Box\varphi$. $\Box$

PROPOSITION 3.4 (**MK is valid**). $\models \Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$.

Proof. Let M, s be arbitrary and assume $M, s \models \Box\Diamond\varphi$. Consider the characteristic formula δ_s^φ of the valuation in s restricted to the atoms in φ. From $M, s \models \Box\Diamond\varphi$ and $M, s \models \delta_s^\varphi$ follows $M|\delta_s^\varphi, s \models \Diamond\varphi$. From that and twice Lemma 3.3, namely, in (also valid) dual form $M|\delta_s^\varphi, s \models \Diamond\varphi \rightarrow \varphi$ and original form $M|\delta_s^\varphi, s \models \varphi \rightarrow \Box\varphi$ follows that $M|\delta_s^\varphi, s \models \Box\varphi$. Therefore, $M, s \models \langle\delta_s^\varphi\rangle\Box\varphi$ and thus $M, s \models \Diamond\Box\varphi$. $\Box$

We now proceed with matters toward proving Church–Rosser. We extend the substitution notation already in use ($\varphi(\psi/p)$ is the substitution of ψ for all occurrences of p in formula φ) to simultaneous substitution for infinite sequences $\varphi(\psi_0/p_0, \psi_1/p_1, \ldots)$.

LEMMA 3.5. *Let $Q = \{q_n \mid n \in \mathbb{N}\} \subseteq P$ be an infinite set of atoms, let $\theta \in \mathcal{L}_{el}$ be an epistemic formula such that $P_\theta \cap Q = \emptyset$, and let $\varphi \in \mathcal{L}_{apal}$ with $P_\varphi \cap Q = \emptyset$. Given a frame S and a valuation V on S, there exists a valuation V' on S such that:*

1. $[\![\varphi]\!]_{\mathsf{S},V'} = [\![\varphi]\!]_{\mathsf{S},V}$
2. *for all $\theta' \in \mathcal{L}_{el}$:*

$$\begin{aligned}
[\![\theta']\!]_{\mathsf{S},V'} &= [\![\theta'(\theta/q_0, q_0/q_1, \ldots, q_n/q_{n+1}, \ldots)]\!]_{\mathsf{S},V} \\
[\![\theta']\!]_{\mathsf{S},V} &= [\![\theta'(q_1/q_0, \ldots, q_{n+1}/q_n, \ldots)]\!]_{\mathsf{S},V'}
\end{aligned}$$

3. $[\![q_0]\!]_{\mathsf{S},V'} = [\![\theta]\!]_{\mathsf{S},V'} = [\![\theta]\!]_{\mathsf{S},V}$.

Proof. The valuation V' needed is given by putting $V'(p) := V(p)$ for $p \notin Q$, $V'(q_0) := [\![\theta]\!]_{\mathsf{S},V}$, and for all $n \in \mathbb{N}$: $V'(q_{n+1}) := V(q_n)$. $\Box$

As a consequence of clause (2) of Lemma 3.5, we have that the epistemically definable subsets of (S, V) are the same as those of (S, V'). We now use the lemma to show the following.

PROPOSITION 3.6. *If $M, s \models \Diamond\psi$ and $p \notin P_\psi$, then there exists a model M' only differing from M in the valuation of atoms not occurring in ψ such that $M', s \models \langle p\rangle\psi$.*

Proof. Let $M = (\mathsf{S}, V) = (S, \sim, V)$. We use first Lemma 3.5, namely, for:

$$Q := P \setminus P_\psi,\ q_0 := p,\ \theta = \top, \text{ and } \varphi := \Diamond\psi,$$

obtaining a new valuation V' s.t. $V'(p) = S$ and $[\![\Diamond\psi]\!]_{\mathsf{S},V'} = [\![\Diamond\psi]\!]_{\mathsf{S},V}$. Therefore, there must exist some $\theta \in \mathcal{L}_{el}$ such that $(\mathsf{S}, V', s) \models \langle\theta\rangle\psi$, so $s \in [\![\langle\theta\rangle\psi]\!]_{\mathsf{S},V'}$. Furthermore, we can assume that $p \notin P_\theta$: The valuation of p has been set to $\top$ in V'; therefore, if there had been occurrences of p in θ, they could have been replaced by $\top$. We now apply Lemma 3.5 again, with:

$$Q := P \setminus (P_\theta \cup P_\psi),\ q_0 := p,\ \varphi := \langle\theta\rangle\psi, \text{ and } \theta \text{ as given},$$

obtaining V'' such that $s \in [\![\langle\theta\rangle\psi]\!]_{\mathsf{S},V'} = [\![\langle\theta\rangle\psi]\!]_{\mathsf{S},V''}$ and $[\![p]\!]_{\mathsf{S},V''} = [\![\theta]\!]_{\mathsf{S},V''} = [\![\theta]\!]_{\mathsf{S},V'}$. Hence, we obtain that $s \in [\![\langle\theta\rangle\psi]\!]_{\mathsf{S},V'} = [\![\langle\theta\rangle\psi]\!]_{\mathsf{S},V''} = [\![\langle p\rangle\psi]\!]_{\mathsf{S},V''}$. $\Box$

Proposition 3.6 can be generalized to the following.

PROPOSITION 3.7. *Given a possibility form $\boldsymbol{\eta}$. If $M, s \models \boldsymbol{\eta}\{\Diamond\psi\}$ and $p \notin (P_\eta \cup P_\psi)$, then there exists a model M' only differing from M in the valuation of p such that $M', s \models \boldsymbol{\eta}\{\langle p\rangle\psi\}$.*

Proof. The proof is straightforward and by induction on the complexity of possibility forms. The basic case is the proof of Proposition 3.6. The case 'conjunction' starts with $M, s \models \chi \wedge \Diamond\psi$ and $p \notin P_\chi \cup P_\psi$, and so forth. $\Box$

We use Proposition 3.7 to prove, below, the soundness of a derivation rule in the axiomatization of arbitrary announcement logic. For now, we only need Proposition 3.7 to show the CR property.

PROPOSITION 3.8 **(CR is valid)**. $\models \Diamond\Box\varphi \to \Box\Diamond\varphi$.

Proof. Suppose that CR fails. Then, there exist M, s and φ such that $M, s \models \Diamond\Box\varphi \wedge \Diamond\Box\neg\varphi$. By applying Proposition 3.7 twice (namely, for the possibility form 'conjunction', once for the left conjunct and once for the right conjunct), there are $p, q \notin P_\varphi$ and a model M' that is like M, except for the valuation of p and q, such that $M', s \models \langle p\rangle\Box\varphi \wedge \langle q\rangle\Box\neg\varphi$. We therefore also have $M', s \models \langle p\rangle[q]\varphi \wedge \langle q\rangle[p]\neg\varphi$ from which follows $M', s \models \langle p\rangle\langle q\rangle\varphi \wedge \langle q\rangle\langle p\rangle\neg\varphi$, and therefore as p and q are boolean (sequential announcement of booleans corresponds to the announcement of their conjunction), $M', s \models \langle p \wedge q\rangle(\varphi \wedge \neg\varphi)$, which is a contradiction. $\Box$

3.1.3. The relation between knowledge and arbitrary announcement.

PROPOSITION 3.9. *Let $\varphi \in \mathcal{L}_{apal}$. Then, $\models K_a\Box\varphi \to \Box K_a\varphi$.*

Proof. Suppose $M, s \models K_a\Box\varphi$ and $M, s \models \psi$. Assume $t \in M|\psi$ with $s \sim_a t$. We have to prove that $M|\psi, t \models \varphi$. Because state t is also in M, from the assumption $M, s \models K_a\Box\varphi$ and (in M) $s \sim_a t$ follows $M, t \models \Box\varphi$. As ψ is true in t, $M|\psi, t \models \varphi$. $\Box$

Proposition 3.9 is shown in Figure 2. Although $K_a\Box\varphi \to \Box K_a\varphi$ is valid, the other direction $\Box K_a\varphi \to K_a\Box\varphi$ is not valid. It is instructive to give a counterexample.

EXAMPLE 3.10 ($\Box K_a\varphi \to K_a\Box\varphi$ **is not valid**). *Consider the model:*

$$\mathbf{0} \;—\; b \;—\; 1 \;—\; a \;—\; \underline{0}.$$

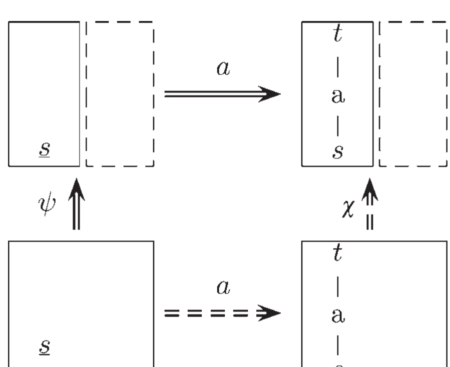


Fig. 2. Illustration of the principle $K_a\Box\varphi \to \Box K_a\varphi$. Given $\langle\psi\rangle\hat{K}_a\neg\varphi$, there is a χ such that $\hat{K}_a\langle\chi\rangle\neg\varphi$.

We now have that $M, 0 \models \hat{K}_a\langle\hat{K}_b p\rangle(K_a p \wedge \neg K_b p)$ *and hence* $M, 0 \models \hat{K}_a\Diamond(K_a p \wedge \neg K_b p)$*. On the other hand,* $M, 0 \not\models \Diamond(\hat{K}_a(K_a p \wedge \neg K_b p)$ *because* $K_a p \wedge \neg K_b p$ *is only true in the model restriction* $\{\mathbf{0}, 1\}$ *that* excludes *the actual state 0. Therefore,* $\Box K_a\varphi \rightarrow K_a\Box\varphi$ *is invalid. In simple words, it may unfortunately happen that we jump to a state where a model restriction is possible that excludes the actual state. Therefore, things that are true at that state may be impossible to realize by a reversal of that process.*

3.1.4. Validities relating booleans and arbitrary announcements. The following Proposition 3.11 will be helpful to show that in the single-agent case, every formula is equivalent to an epistemic $\mathcal{L}_{el}$-formula, as discussed in subsection 3.2.

PROPOSITION 3.11. *Let* $\varphi, \varphi_0, \ldots, \varphi_n \in \mathcal{L}_{pl}$ *and* $\psi \in \mathcal{L}_{apal}$.

1. $\models \Box\varphi \leftrightarrow \varphi$
2. $\models \Box\hat{K}_a\varphi \leftrightarrow \varphi$
3. $\models \Box K_a\varphi \leftrightarrow K_a\varphi$
4. $\models \Box(\varphi \vee \psi) \leftrightarrow (\varphi \vee \Box\psi)$
5. $\models \Box(\hat{K}_a\varphi_0 \vee K_a\varphi_1 \vee \cdots \vee K_a\varphi_n) \leftrightarrow (\varphi_0 \vee K_a(\varphi_0 \vee \varphi_1) \vee \cdots \vee K_a(\varphi_0 \vee \varphi_n))$.

Proof. In the proof, we use the dual (diamond) versions of all propositions.

1. $\models \Diamond\varphi \leftrightarrow \varphi$.
   This is valid because $\langle\psi\rangle\varphi \leftrightarrow \varphi$ is valid in *PAL*, for any ψ and boolean φ.
2. $\models \Diamond K_a\varphi \leftrightarrow \varphi$.
   Right-to-left holds because $\varphi \rightarrow \langle\varphi\rangle K_a\varphi$ is valid in *PAL* for booleans. The other way round, $\models \Diamond K_a\varphi \rightarrow \varphi$ because $\Diamond K_a\varphi \rightarrow \Diamond\varphi$ is valid in *PAL*, and $\Diamond\varphi \leftrightarrow \varphi$ is valid in *PAL* as we have seen above (φ being boolean).
3. $\models \Diamond\hat{K}_a\varphi \leftrightarrow \hat{K}_a\varphi$.
   Right-to-left holds follows from the dual form of the validity $\Box\varphi \rightarrow \varphi$ (Proposition 3.1). Left-to-right holds because $\langle\psi\rangle\hat{K}_a\varphi \rightarrow \hat{K}_a\varphi$ is valid in *PAL* for booleans φ.
4. $\models \Diamond(\varphi \wedge \psi) \leftrightarrow \varphi \wedge \Diamond\psi$.
   Left-to-right: First, $\Diamond$ distributes over $\wedge$, and second, $\models \Diamond\varphi \leftrightarrow \varphi$ as we have established above. From right-to-left: $\varphi \wedge \Diamond\psi$ is equivalent to (apply Case 1) $\Box\varphi \wedge \Diamond\psi$. From the semantics of $\Box$ now directly follows $\Diamond(\varphi \wedge \psi)$.
5. $\models \Diamond(K_a\varphi_0 \wedge \hat{K}_a\varphi_1 \wedge \cdots \wedge \hat{K}_a\varphi_n) \leftrightarrow \varphi_0 \wedge \hat{K}_a(\varphi_0 \wedge \varphi_1) \wedge \cdots \wedge \hat{K}_a(\varphi_0 \wedge \varphi_n)$.
   We show this case for $n = 1$.
   Left-to-right: Directly in the semantics. Let M, s be arbitrary and suppose $M, s \models \Diamond(K_a\varphi_0 \wedge \hat{K}_a\varphi_1)$. Let ψ be the epistemic formula such that $M, s \models \langle\psi\rangle(K_a\varphi_0 \wedge \hat{K}_a\varphi_1)$. In the model $M|\psi$, we now have that $M|\psi, s \models K_a\varphi_0$ so $M|\psi, s \models \varphi_0$. Also, $M|\psi, s \models \hat{K}_a\varphi_1$. Let t be such that $s \sim_a t$ and $M|\psi, t \models \varphi_1$. As $M|\psi, s \models K_a\varphi_0$, and $s \sim_a t$, also $M|\psi, t \models \varphi_0$. Therefore, $M|\psi, t \models \varphi_0 \wedge \varphi_1$, and therefore, $M|\psi, s \models \hat{K}_a(\varphi_0 \wedge \varphi_1)$. So $M|\psi, s \models \varphi_0 \wedge \hat{K}_a(\varphi_0 \wedge \varphi_1)$, and as φ_0 and φ_1 are booleans also $M, s \models \varphi_0 \wedge \hat{K}_a(\varphi_0 \wedge \varphi_1)$.[2]
   Right-to-left: For the other direction, suppose $M, s \models \varphi_0 \wedge \hat{K}_a(\varphi_0 \wedge \varphi_1)$. Consider the model $M|\varphi_0$. Because $M, s \models \hat{K}_a(\varphi_0 \wedge \varphi_1)$, and φ_1 is boolean, there must be a $t \in M|\varphi_0$ such that $M|\varphi_0, t \models \varphi_1$. So $M|\varphi_0, s \models \hat{K}_a\varphi_1$. Also, $M|\varphi_0, s \models K_a\varphi_0$

[2] Alternatively, one can use more straightforwardly the *S*5 validity $(K_a\varphi_0 \wedge \hat{K}_a\varphi_1) \rightarrow (K_a\varphi_0 \wedge \hat{K}_a(\varphi_0 \wedge \varphi_1))$.

because φ_0 is boolean. So $M|\varphi_0, s \models K_a\varphi_0 \wedge \hat{K}_a\varphi_1$ and therefore $M, s \models \Diamond(K_a\varphi_0 \wedge \hat{K}_a\varphi_1)$. □

3.2. Expressivity. If there is a single agent only, arbitrary announcement logic reduces to epistemic logic. But for more than one agent, it is strictly more expressive than public announcement logic. We remind the reader that in the absence of common knowledge, public announcement logic is equally expressive as epistemic logic.

First, we consider the single-agent case. Let $A = \{a\}$. We obtain the result by applying Proposition 3.11. We need some additional terminology as well. A formula is in *normal form* when it is a conjunction of disjunctions of the form $\varphi \vee \hat{K}_a\varphi_0 \vee K_a\varphi_1 \vee \cdots \vee K_a\varphi_n$, where $\varphi, \varphi_0, \ldots, \varphi_n$ are all formulas in propositional logic. Every formula in single-agent *S5* is equivalent to a formula in normal form (Meyer & van der Hoek, 1995). A normal form may not exist for a multiagent formula, for example, it does not exist for K_aK_bp. This explains why the result below does not carry over to the multiagent case.

PROPOSITION 3.12. *Single-agent arbitrary announcement logic is equally expressive as epistemic logic.*

Proof. We prove by induction on the number of occurrences of $\Box$ that every formula in single-agent arbitrary announcement logic is equivalent to a formula in epistemic logic. Put the epistemic formula in the scope of an innermost $\Box$ in normal form. First, we distribute $\Box$ over the conjunction (Proposition 3.1, Part (1)). We now get formulas of the form $\Box(\varphi \vee \hat{K}_a\varphi_0 \vee K_a\varphi_1 \vee \cdots \vee K_a\varphi_n)$. These are reduced by application of Propositions 3.11, Parts (4) and (5), to formulas $(\varphi \vee \varphi_0) \vee K_a(\varphi_0 \vee \varphi_1) \vee \cdots \vee K_a(\varphi_0 \vee \varphi_n)$. □

PROPOSITION 3.13. *Arbitrary announcement logic is strictly more expressive than epistemic logic.*

Proof. The proof follows an abstract argument. Suppose the logics are equally expressive, in other words, that there is some reduction rule for arbitrary announcement such that any formula can be reduced to an expression without $\Box$. Given the reduction of *PAL* to *EL*, this entails that every arbitrary announcement formula should be equivalent to an epistemic logical formula. Now the crucial observation is that this epistemic formula only contains a *finite* number of atomic propositions. We then construct models that cannot be distinguished in the restricted language but can be distinguished in a language with more atoms.

So it remains to give a specific formula and a specific pair of models. Note that the formula must involve more than one agent, as single-agent arbitrary announcement logic is reducible to epistemic logic (see Proposition 3.12).

Consider the formula $\Diamond(K_ap \wedge \neg K_bK_ap)$. Assume, toward a contradiction, that it is equivalent to an epistemic logical formula ψ. W.l.o.g., we may assume that ψ only contains the atom p.[3] We now construct two different epistemic states (M, s) and (M', s') involving a *new* atom q such that $\Diamond(K_ap \wedge \neg K_bK_ap)$ is false in the first but true in the second. We also take care that the two models are bisimilar with respect to the language without q.

[3] The alternative is that ψ contains a *finite* number of atoms. What other atoms apart from p? It does not matter: The contradiction on which the proof of Proposition 3.13 is based merely requires a 'fresh' atom not yet occurring in ψ.

Therefore, the supposed reduction is either true or false in both models. Contradiction. Therefore, no such reduction exists.

The required models are as follows. Epistemic state $(M, 1)$ consists of the well-known model M where a cannot distinguish between states where p is true and false, but b can (but knows that a cannot, etc.), that is, domain $\{0, 1\}$ with universal access for a and identity access for b, where p is only true at 1, and 1 is the actual state. Visualized as:

$$0 \text{ --- } a \text{ --- } \underline{1}.$$

Epistemic state $(M', 10)$ consists of two copies of M, namely, one where a new fact q is true and another one where q is false. In the actual state 10, q is false. We visualize this as: We now have that $(M, 1)$ is bisimilar to $(M', 10)$ with regard to the epistemic language

$$\begin{array}{ccccc}
\mathbf{01} & \text{---} & a & \text{---} & \mathbf{11} \\
| & & & & | \\
b & & & & b \\
| & & & & | \\
00 & \text{---} & a & \text{---} & 10
\end{array}$$

for atom p and agents a, b. Therefore, $M, 1 \models \psi$ iff $M', 10 \models \psi$. On the other hand, $(M, 1)$ is not bisimilar to $(M', 10)$ with regard to the epistemic language for atoms p, q and agents a, b. This is evidenced by the fact that $M, 1 \not\models \Diamond(K_a p \wedge \neg K_b K_a p)$, but instead, $M', 10 \models \Diamond(K_a p \wedge \neg K_b K_a p)$. The latter is because $M', 10 \models \langle p \vee q \rangle (K_a p \wedge \neg K_b K_a p)$: the announcement $p \vee q$ restricts the domain to the three states where it is true, and $M'|(p \vee q), 10 \models K_a p \wedge \neg K_b K_a p$ because $10 \sim_b 11$ and $M'|(p \vee q), 11 \models \neg K_a p$.[4] $\Box$

As an aside, because it departs from our assumption that all accessibility relations are equivalence relations, we have yet another result concerning expressive power. Consider the more general *multiagent models* $M = (S, R, V)$ for accessibility functions $R : A \to \mathcal{P}(S \times S)$. Unlike the corresponding relations $\sim_a$ in epistemic models, the relations R_a are not necessarily equivalence relations. We now interpret the same language on those structures, with the obvious (only) difference that $M, s \models K_a \varphi$ iff for all $t \in S : R_a(s, t)$ implies $M, t \models \varphi$. Many results still carry over to the more general logic, but the expressivity results are now different.

PROPOSITION 3.14. *With respect to the class of multiagent models for (a single) accessibility relation R_a, single-agent arbitrary announcement logic is strictly more expressive than public announcement logic.*

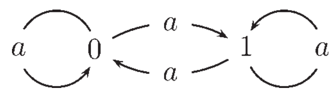
Proof. Along the same argument as in Proposition 3.13, on the assumption that a given formula φ is logically equivalent to a $\Box$-free formula ψ not containing some fresh atom q, we present two models that are bisimilar with respect to the atoms in ψ and that therefore cannot be distinguished by ψ but that have a different valuation for φ. From the contradiction follows strictly larger expressivity.

---

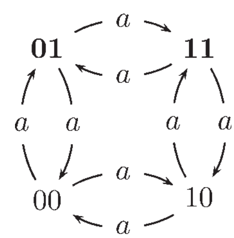
[4] Kooi, in a personal communication, suggested an interesting alternative proof of larger expressivity that does not require a fresh atom but deeper and deeper modal nesting. The proof is almost the same as the one we present here, but rather than an atom that distinguishes the worlds, there are strings of worlds of different length attached to each world of the square.

Consider the formula $\diamond(K_a p \wedge \neg K_a K_a p)$ and assume that it is equivalent to an epistemic ψ only containing atom p; and consider models M and M' as follows.

Multiagent state $(M, 1)$ consists of the familiar model M where a cannot distinguish between states 1 and 0 where p is true and false, respectively, and where 1 is the actual state. We now explicitly visualize all pairs in the accessibility relation and get: Multiagent



state $(M', 10)$ consists of two copies of M, namely, a bottom one where a new fact q is false and a top one where q is true. The actual state is 10. Accessibility relations are as shown—note that there is no reflexive access on any world. We now have that $(M, 1)$ is bisimilar to



$(M', 10)$ with regard to the epistemic language for atom p and agent a but that $(M, 1)$ is not bisimilar to $(M', 10)$ with regard to the epistemic language for atoms p, q and agent a. Therefore, $M, 1 \models \psi$ iff $M', 10 \models \psi$. On the other hand, $M, 1 \not\models \diamond(K_a p \wedge \neg K_a K_a p)$, but $M', 10 \models \diamond(K_a p \wedge \neg K_a K_a p)$, as $M', 10 \models \langle p \vee q\rangle(K_a p \wedge \neg K_a K_a p)$. $\square$

3.3. Compactness and model checking.

3.3.1. Compactness. The counterexample used in the proof of Proposition 3.13 can be adjusted to show that *APAL* is not compact.

PROPOSITION 3.15. *Arbitrary announcement logic is not compact.*

Proof. Take the following infinite set of formulas:

$$\{[\theta](K_a p \to K_b K_a p) \mid \theta \in \mathcal{L}_{el}\} \cup \{\neg\Box(K_a p \to K_b K_a p)\}.$$

By the semantics of $\Box$, this set is obviously not satisfiable. But we show that *any of its finite subsets is satisfiable*. This contradicts compactness. Let

$$\{[\theta_i](K_a p \to K_b K_a p) \mid 0 \leq i \leq n\} \cup \{\neg\Box(K_a p \to K_b K_a p)\}$$

be any such finite subset, and let q be an atomic sentence that is distinct from p and does not occur in any of the sentences θ_i $(0 \leq i \leq n)$. Take now the epistemic state $(M', 10)$ as in the proof of Proposition 3.13. As shown above, we have $M', 10 \models \diamond(K_a p \wedge \neg K_b K_a p)$, and thus, $M', 10 \models \neg\Box(K_a p \to K_b K_a p)$. On the other hand, for the epistemic state $(M, 1)$ as in the above proof, we have shown above that we have $M, 1 \not\models \diamond(K_a p \wedge \neg K_b K_a p)$, that is, $M, 1 \models \Box(K_a p \to K_b K_a p)$. By the semantics of $\Box$, it follows that $M, 1 \models [\theta_i](K_a p \to K_b K_a p)$ for all $0 \leq i \leq n$; but q does not occur in any of these formulas, so their truth values must be the same at $(M', 10)$ and $(M, 1)$ (since as shown above, the two epistemic states are bisimilar w.r.t. the language without q). Thus, we have

Table 1. *Overview of formula properties*[a]

Positive	$\varphi ::= p \mid \neg p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid K_a\varphi \mid [\neg\varphi]\varphi \mid \Box\varphi$
Preserved	$\models \varphi \to \Box\varphi$
Successful	$\models [\varphi]\varphi$
Knowable	$\models \varphi \to \Diamond K_a\varphi$

[a] A formula satisfying the condition in the right column is said to have the property in the left column.

$M', 10 \models [\theta_i](K_a p \to K_b K_a p)$ for all $0 \leq i \leq n$. Putting these together, we see that our finite set of formulas is satisfied at the state $(M', 10)$. $\Box$

3.3.2. Model checking. We preferred to keep some technical results on model checking out of the article. The model checking problem for the logic *APAL* (to determine the extension of a given formula in a given model) is PSPACE-complete (Work in progress by Balbiani *et al.*).

Let us briefly sketch why the model checking problem for *APAL* is decidable. This result is not trivial because of the implicit quantification over *all* atoms in the $\Box$-operator. Consider a finite model with a recursive valuation map (from the infinite set of atomic sentences to the powerset of the model). It is well known that determining the largest bisimulation on such a model is a decidable problem and so is finding all subsets of the model that are closed under the largest bisimulation. Given such a model and a formula, we can then replace all occurrences of $\Box\varphi$ in that formula by a finite conjunction of announcement sentences $[\theta]\varphi$, where the denotation of the announced formulas θ ranges over all the subsets that are closed under the largest bisimulation of the model. (We use here the known fact that a subset of a finite model is definable in basic modal/epistemic logic if and only if it is closed under the largest bisimulation.) To determine the truth of the resulting formula, one can then use a model checking algorithm for public announcement logic.

3.3.3. Decidability. The issue of the decidability of the logic has been resolved by French & van Ditmarsch (2008): Arbitrary announcement logic is *undecidable*. A logic is decidable iff there is a terminating procedure to determine whether a given formula is satisfiable. French and van Ditmarsch proved via a tiling argument (and an embedding) that it is co-RE complete to determine whether a given formula can be satisfied in some model.

3.4. Knowability and other semantic or syntactic fragments. A suitable direction of research is the syntactic or semantic characterization of interesting fragments of the logic. In this section, we define *positive*, *preserved*, *successful*, and *knowable* formulas, and investigate their relation (see Table 1, for an overview of definitions).

The *positive formulas* intuitively correspond to formulas that do not express ignorance; in epistemic logical ($\mathcal{L}_{el}$) terms: in which negations do not precede K_a-operators. We consider a generalization of that notion to $\mathcal{L}_{apal}$. The fragment of the *positive formulas* is inductively defined as:

$$\varphi ::= p \mid \neg p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid K_a\varphi \mid [\neg\varphi]\varphi \mid \Box\varphi.$$

Note that the truth of the announcement is a *condition* of its execution, which, when seen as a disjunction, explains the negation in $[\neg\varphi]$. Unfortunately, the negation in $[\neg\varphi]\varphi$ makes ‘positive’ somewhat of a misnomer.

The *preserved formulas* preserve truth under arbitrary (epistemically definable) model restriction, also known as relativization. They are (semantically) defined as those φ for which $\models \varphi \rightarrow \Box\varphi$.[5] There is no corresponding semantic principle in public announcement logic that expresses truth preservation.

We now prove that positive formulas are preserved. Restricted to epistemic logic without common knowledge, this was observed by van Benthem (2006). van Ditmarsch & Kooi (2006) extended van Benthem's result, with an additional clause $[\neg\varphi]\varphi$. (And also, unlike here, an additional clause $C_B\varphi$ for subgroup common knowledge operators, where $B \subseteq A$.) Surprisingly, we can further extend the notion of 'positive' to arbitrary announcement logic, by adding a clause $\Box\varphi$: In the case $\Box\varphi$ of the inductive proof below to show truth preservation, assuming the opposite easily leads to a contradiction.

PROPOSITION 3.16. *Positive formulas are preserved.*

Proof. For 'M' *is a submodel of* M', write $M' \subseteq M$. To prove the proposition, it is sufficient to show the following:

> Given M, M' with $M' \subseteq M$, a state s in the domain of M', and a positive formula φ. If $(M, s) \models \varphi$, then $(M', s) \models \varphi$ (i).

It is sufficient because it then also holds for all *epistemically definable* submodels M'. We show (i) by proving an even slightly stronger proposition, namely:

> Given M, M', M'' with $M'' \subseteq M' \subseteq M$, state s in the domain of M'', and positive φ. If $(M', s) \models \varphi$, then $(M'', s) \models \varphi$.

This has the advantage of loading the induction hypothesis. Loading is needed for the case $[\neg\varphi]\psi$ of the proof, that is by induction on the formula. We assume most cases to be well known, except for the case $[\neg\varphi]\psi$, similarly shown in van Ditmarsch & Kooi (2006), and $\Box\varphi$, which is new.

Case $[\neg\varphi]\psi$: Given is $(M', s) \models [\neg\varphi]\psi$. We have to prove that $(M'', s) \models [\neg\varphi]\psi$. Assume that $(M'', s) \models \neg\varphi$. Using the contrapositive of the induction hypothesis, $(M', s) \models \neg\varphi$. From that and the assumption $(M', s) \models [\neg\varphi]\psi$ follows $(M'|\neg\varphi, s) \models \psi$. Because $(M', s) \models \neg\varphi$, $M''|\neg\varphi$ is a submodel of $M'|\neg\varphi$. From $(M'|\neg\varphi, s) \models \psi$ and $M''|\neg\varphi \subseteq M'|\neg\varphi \subseteq M' \subseteq M$, it follows from (the loaded version of!) induction that $(M''|\neg\varphi, s) \models \psi$. Therefore, $(M'', s) \models [\neg\varphi]\psi$.

Case $\Box\varphi$: Assume $(M', s) \models \Box\varphi$. Suppose toward a contradiction that $(M'', s) \not\models \Box\varphi$. Then, there is a ψ such that $(M'', s) \models \langle\psi\rangle\neg\varphi$, from which follows $(M''|\psi, s) \not\models \varphi$. From $M''|\psi \subseteq M'' \subseteq M'$ and contraposition of induction follows $(M', s) \not\models \varphi$. But from $(M', s) \models \Box\varphi$ follows $(M', s) \models [\top]\varphi$ which equals $(M', s) \models \varphi$ that contradicts the previous. $\Box$

van Benthem (2006) also shows that preserved formulas are (logically equivalent to) positive. This is not known for the extension of these notions to public announcement logic in van Ditmarsch & Kooi (2006), nor for arbitrary announcement logic. An answer to this question seems hard.

Another semantic notion is that of *success*. *Successful formulas* are believed after their announcement or, in other words, after 'revision' with that formula. This corresponds to the postulate of 'success' in AGM belief revision. Formally, φ is a *successful formula*

---

[5] In Moss & Parikh (1992), the same semantic condition defines the *persistent* formulas.

iff $[\varphi]\varphi$ is valid (see van Ditmarsch & Kooi, 2006, elaborating an original but slightly different proposal in Gerbrandy, 1999). The validity of $[\varphi]\varphi$ is equivalent to the validity of $\varphi \to [\varphi]K_a\varphi$: "if φ is true, then after announcing φ, φ is believed." (van Ditmarsch & Kooi, 2006). This validity describes in a dynamic epistemic setting the postulate of success for belief expansion: "if φ is true, then after expansion with φ, φ should be believed."

PROPOSITION 3.17. *Preserved formulas are successful.*

Proof. $\models \varphi \to \Box\varphi$ implies $\models \varphi \to [\varphi]\varphi$, and $\models \varphi \to [\varphi]\varphi$ iff $\models [\varphi]\varphi$. □

COROLLARY 3.18. *Positive formulas are successful.*

Fitch observed that not all unknown truths can become known (Fitch, 1963; Brogaard & Salerno, 2004), such as the well-known $p \wedge \neg Kp$. Instead of calling this a paradox (which Fitch did not do either!), we prefer to call it a fact, and the question then is what unknown truths *can* become known. For a single agent a, we can define the *knowable formulas* as those for which $\models \varphi \to \Diamond K_a\varphi$, and the most obvious multiagent version defines the *knowable formulas* as those for which, for *all* agents $a \in A$, $\models \varphi \to \Diamond K_a\varphi$. (See a paragraph below for some additional multiagent versions of knowability.) We can now observe the following.

PROPOSITION 3.19. *Positive, preserved, and successful formulas are all knowable.*

Proof. Similar to the proof of Proposition 3.17. Observe that $\models \varphi \to \Box\varphi$ implies $\models \varphi \to [\varphi]\varphi$, which is equivalent to $\models \varphi \to [\varphi]K_a\varphi$; $\models \varphi \to [\varphi]K_a\varphi$ is equivalent to $\models \varphi \to \langle\varphi\rangle K_a\varphi$; and $\models \varphi \to \langle\varphi\rangle K_a\varphi$ implies $\models \varphi \to \Diamond K_a\varphi$. □

The syntactic characterization of knowable formulas remains an open question, but we would like to emphasize that given a choice of interpretation for $\Box$ as in our logic, this has become a purely technical question. We think that this is a proper way to address knowability issues. Some knowable formulas are not positive, for example, $\neg K_a p$: If true, announce $\top$, and $K_a\neg K_a p$ (still!) holds. Therefore, $\models \neg K_a p \to \Diamond K_a \neg K_a p$.

3.4.1. Other approaches. The excellent entry in the Stanford Encyclopedia of Philosophy on Fitch's Paradox (Brogaard & Salerno, 2004) gives an overview of semantic and syntactic restrictions intended to avoid its paradoxical character.

It is relevant to mention Tennant's cartesian formulas: A formula φ is *cartesian* iff $K\varphi$ is not provably inconsistent (Tennant, 1997). A semantic correspondent of that, more in-line with semantic features of formulas that we distinguished above, would be to define φ as cartesian iff $K\varphi$ is satisfiable, or, in other terms, iff $\not\models \neg K\varphi$. van Benthem (2004) observed that cartesian formulas may not be knowable. For example, the formula $p \wedge \neg Kq$ is cartesian but not knowable:

Consider a model where the formula is satisfied in a state wherein p is true but q is false. Now announce p. This results in a state where p is now known but $\neg Kq$ is of course still true. So, with introspection for knowledge and distribution of K over $\wedge$, we have that $K(p \wedge \neg Kq)$ is true. Therefore, the formula is cartesian.

On the other hand, we have that $\not\models p \wedge \neg Kq$ because in a model where the denotations of atoms p and q are the *same*, $p \wedge \neg Kq$ is false in any model restriction. Therefore, the formula is not knowable (in our sense).

It seems reasonable that this formula should be knowable in some other sense. But it is unclear in what sense. For example, what if one characterizes the knowable formulas as those for which for all agents—returning to the multiagent situation—$\varphi \to \Diamond K\varphi$ is merely *satisfiable* and not necessarily valid? Unfortunately, *every* formula is knowable in that sense. If φ is valid, then $\Box K\varphi$ is valid, and $\varphi \to \Box K\varphi$ as well, so also $\varphi \to \Diamond K\varphi$, so a fortiori it is satisfiable. If φ is not valid, there must be an epistemic model M and a state s in that model where φ is false. But in that case, we also have, trivially, that $M, s \models \varphi \to \Diamond K\varphi$. Therefore, $\varphi \to \Diamond K\varphi$ is satisfiable. Therefore, $\varphi \to \Diamond K\varphi$ is satisfiable for all φ.

Another (rather summary) syntactic characterization, within an intuitionistic setting, is that by Dummett (2001).

Moss and Parikh's topologic (Moss & Parikh, 1992; Parikh *et al.*, 2007) has the same language combining the knowledge operator K with box $\Box$, although for a single agent only. They interpret $\Box$ not in our temporal sense but in a spatial sense. With us, $\Diamond\varphi$ means 'φ is true after a sequence of announcements', that is, 'after some time'. Moss and Parikh suggest to interpret $\Diamond\varphi$ as 'φ is true when taking some effort narrowing down the possibilities', that is, 'closer'. How they relate K and $\Box$ in their semantics is different from our approach because the structure on which they interpret their language is a topology of subsets of the domain of states. Most interestingly, an open set in a topology is characterized by a 'knowability-like' formula: $M \models \varphi \to \Diamond K\varphi$ iff $[\![\varphi]\!]_M$ is an *open set* (Moss & Parikh, 1992, p. 98). An open set is a subset of the domain of the model M with a certain property relative to the topology defined on that domain. They do not observe the relevance of their logic for knowability issues. Incidentally, Fitch leaves the question of how to interpret $\Box$ open in Fitch (1963) and explicitly says that it does not have to be interpreted temporally: "the element of time will be ignored in dealing with these various concepts [such as knowledge]" (Fitch, 1963, p.135).

3.4.2. Multiagent versions of knowability. There are various multiagent versions of knowability that can be explored. To name a few:

- $\varphi \to \Diamond C_A\varphi$: commonly knowable truths
- $K_a\varphi \to \Diamond C_A\varphi$: an individual can *publish* his knowledge
- $K_a\varphi \to \Diamond K_b\varphi$: *knowledge transfer* from a to b
- $D_A\varphi \to \Diamond C_A\varphi$: distributed knowledge can be made common.

Such notions are useful for the specification of both static and dynamic aspects of multiagent systems, including properties of communication protocols. They suffer from similar constraints as the original Fitch knowability. For example, my knowledge that p is true and that you are ignorant of p, formalized as $K_{me}(p \wedge \neg K_{you}p)$, is not transferable to you, as $K_{you}(p \wedge \neg K_{you}p)$ is inconsistent for knowledge. The question what distributed knowledge can be made common is relevant to compute the global consequences of local propagation of information through distributed networks.

4. Axiomatization.

4.1. The axiomatization APAL and its soundness. We now provide a complete axiomatization of $\mathcal{L}_{apal}$.

DEFINITION 4.1. *The axiomatization* **APAL** *is given in Table 2. A formula is a* theorem *if it belongs to the least set of formulas containing all axioms and closed under the rules. If φ is a theorem, we write $\vdash \varphi$.*

PROPOSITION 4.2 (**Soundness**). *The axiomatization* **APAL** *is sound. We only pay attention to the axiom and the derivation rule involving $\Box$.*

Table 2. *The axiomatization* **APAL**

All instantiations of propositional tautologies	
$K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$	distribution of knowledge over implication
$K_a\varphi \to \varphi$	truth
$K_a\varphi \to K_aK_a\varphi$	positive introspection
$\neg K_a\varphi \to K_a\neg K_a\varphi$	negative introspection
$[\varphi]p \leftrightarrow (\varphi \to p)$	atomic permanence
$[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$	announcement and negation
$[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$	announcement and conjunction
$[\varphi]K_a\psi \leftrightarrow (\varphi \to K_a[\varphi]\psi)$	announcement and knowledge
$[\varphi][\psi]\chi \leftrightarrow [(\varphi \wedge [\varphi]\psi)]\chi$	announcement composition
$\Box\varphi \to [\psi]\varphi$, where $\psi \in \mathcal{L}_{el}$	arbitrary and specific announcement
From φ and $\varphi \to \psi$, infer ψ	modus ponens
From φ, infer $K_a\varphi$	necessitation of knowledge
From φ, infer $[\psi]\varphi$	necessitation of announcement
From $\psi \to [\theta][p]\varphi$, infer $\psi \to [\theta]\Box\varphi$, where $p \notin P_\psi \cup P_\theta \cup P_\varphi$	deriving arbitrary announcement / $R(\Box)$

1. *$\Box\varphi \to [\psi]\varphi$, where $\psi \in \mathcal{L}_{el}$ (arbitrary and specific announcement).*
2. *From $\psi \to [\theta][p]\varphi$, infer $\psi \to [\theta]\Box\varphi$, where $p \notin (P_\psi \cup P_\theta \cup P_\varphi)$ (deriving arbitrary announcement).*

Proof.

1. The soundness of 'arbitrary and specific announcement' follows directly from the semantics of $\Box$. The restriction to epistemic formulas is important. Without that restriction, it is unclear if the axiom is sound.
2. To show the soundness of 'deriving arbitrary announcement', we first observe that the formulas $\psi \to [\theta][p]\varphi$ and $\psi \to [\theta]\Box\varphi$ are necessity forms, such that their negations are equivalent to possibility forms (see Definition 2.2 on page 307). We then use Proposition 3.7, which says that diamonds in possibility forms can be witnessed by fresh atoms.
   Suppose, toward a contradiction, that $\psi \to [\theta][p]\varphi$ is valid but that $\psi \to [\theta]\Box\varphi$ is *not* valid, that is, we have a model such that $(\mathsf{S}, V, s) \models \neg(\psi \to [\theta]\Box\varphi)$. As it is the negation of a necessity form, formula $\neg(\psi \to [\theta]\Box\varphi)$ is equivalent to a possibility form $\boldsymbol{\chi}\{\Diamond\neg\varphi\}$. (Note that $\neg(\psi \to [\theta][p]\varphi)$ is therefore equivalent to the possibility form $\boldsymbol{\chi}\{\langle p\rangle\neg\varphi\}$.) From $(\mathsf{S}, V, s) \models \boldsymbol{\chi}\{\Diamond\neg\varphi\}$ and Proposition 3.7 follows that there exists a valuation V' and an atom $p \notin (P_\psi \cup P_\theta \cup P_\varphi)$ such that $(\mathsf{S}, V', s) \models \boldsymbol{\chi}\{\langle p\rangle\neg\varphi\}$. As fresh atom p, we may choose the p in $\psi \to [\theta][p]\varphi$. So $(\mathsf{S}, V', s) \models \neg(\psi \to [\theta][p]\varphi)$. This contradicts the validity of $\psi \to [\theta][p]\varphi$.$\Box$

4.2. Example derivations.

EXAMPLE 4.3. *We show that the validity $\Box p \to \Box\Box p$ is also a theorem. In Step 4, we use that the axiomatization for public announcement logic* **PAL** *satisfies the property of 'substitution of equivalents' (see Plaza, 1989, 2007, or van Ditmarsch et al., 2007, for details). In Step 8 of the derivation, we use that $\Box p \to [q]\sharp$ is a necessity form, and in Step 9 of the derivation, we use that $\Box p \to \sharp$ is a necessity form.*

1. $\vdash \Box p \to [q \wedge (q \to r)]p$ *arbitrary and specific announcement*
2. $\vdash (q \to r) \leftrightarrow [q]r$ *atomic permanence*

3. $\vdash (q \wedge (q \to r)) \leftrightarrow (q \wedge [q]r)$	*2, propositionally*
4. $\vdash [q \wedge (q \to r)]p \leftrightarrow [q \wedge [q]r]p$	*substitution of equivalents for* **PAL** *(*)*
5. $\vdash [q \wedge [q]r]p \leftrightarrow [q][r]p$	*announcement composition*
6. $\vdash [q \wedge (q \to r)]p \leftrightarrow [q][r]p$	*4,5, propositionally*
7. $\vdash \Box p \to [q][r]p$	*1,6, propositionally*
8. $\vdash \Box p \to [q]\Box p$	*7, deriving arbitrary announcement*
9. $\vdash \Box p \to \Box\Box p$	*8, deriving arbitrary announcement.*

EXAMPLE 4.4. *For another example, we show that $[\Box p]p$ is a theorem. This means that regardless of the restriction in axiom $\Box\varphi \to [\psi]\varphi$ (arbitrary and specific announcement) that $\psi \in \mathcal{L}_{el}$, there are already very basic theorems of the form $[\psi]\varphi$ where ψ is* not *an epistemic formula. The restriction is therefore not* 'per se' *a reason to fear incompleteness of the logic.*

1. $\vdash \Box p \to [\top]p$	*arbitrary and specific announcement*
2. $\vdash [\top]p \to (\top \to p)$	*atomic permanence*
3. $\vdash (\top \to p) \leftrightarrow p$	*propositionally*
4. $\vdash \Box p \to p$	*1,2,3, propositionally*
5. $\vdash [\Box p]p \leftrightarrow (\Box p \to p)$	*atomic permanence*
6. $\vdash [\Box p]p$	*4,5, propositionally.*

Finally, we show that a derivation rule for necessitation of $\Box$ is derivable in **APAL**. The proof presents another, very short, example of a derivation. But as the reader might have expected this rule in the proof system, we present the result as a proposition and not as an example. In Proposition 3.1, Part (4), on page 311. we proved the soundness of this principle.

PROPOSITION 4.5. *Necessitation of arbitrary announcement is derivable in* **APAL**.

Proof.

1. $\vdash \varphi$	assumption
2. $\vdash [p]\varphi$	1, necessition of announcement; choose $p \notin P_\varphi$
3. $\vdash \Box\varphi$	2, deriving arbitrary announcement.

$\Box$

4.3. Variants of the rule for deriving arbitrary announcement. We now prove completeness for the logic *APAL*. We do this indirectly, by way of an infinitary variant of the axiomatization **APAL**, that we can show to be complete with respect to the *APAL* semantics. We apply a technique suggested by Goldblatt (1982) using the 'necessity forms' that were introduced in Definition 2.2 on page 307. Necessity forms are used in the formulation of two variants $R^1(\Box)$ and $R^\omega(\Box)$, now to follow, of the rule $R(\Box)$ ('deriving arbitrary announcement') from system **APAL**.

DEFINITION 4.6.

- *From $\varphi \to [\theta][p]\psi$, infer $\varphi \to [\theta]\Box\psi$, where $p \notin (P_\varphi \cup P_\theta \cup P_\psi)$ (already defined)* $R(\Box)$.
- *From $\boldsymbol{\varphi}([p]\psi)$, infer $\boldsymbol{\varphi}(\Box\psi)$, where $p \notin P_{\boldsymbol{\varphi}} \cup P_\psi$* $R^1(\Box)$.
- *From $\boldsymbol{\varphi}([\chi]\psi)$ for all $\chi \in \mathcal{L}_{el}$, infer $\boldsymbol{\varphi}(\Box\psi)$* $R^\omega(\Box)$.

Axiomatization **APAL**$^{\omega}$ *is the variant of* **APAL** *with the infinitary rule* $R^{\omega}(\Box)$ *instead of* $R(\Box)$. *Axiomatization* **APAL**^{1} *is the variant of* **APAL** *with the different finitary rule* $R^1(\Box)$ *instead of* $R(\Box)$.

PROPOSITION 4.7. *The rules* $R^1(\Box)$ *and* $R^{\omega}(\Box)$ *are sound.*

Proof. The reader may easily verify that the rule $R^{\omega}(\Box)$ is sound, as this directly corresponds to the semantics for $\Box$: A formula of the form $\Box\psi$ is valid, if $[\varphi]\psi$ is valid for all epistemic φ. Now replace 'valid' by 'derivable' and observe that the argument can be generalized for other necessity forms than the basic necessity form.

The soundness of rule $R^1(\Box)$ is shown exactly as the soundness of $R(\Box)$: In the soundness proof of $R(\Box)$, it was *only* essential that $\varphi \rightarrow [\theta][p]\psi$ and $\varphi \rightarrow [\theta]\Box\psi$ were in necessity form. $\Box$

Next, we show in Proposition 4.9 that every **APAL** theorem is a **APAL**^{1} theorem, and vice versa. That proposition requires a lemma.

LEMMA 4.8. *Given a necessity form* $\boldsymbol{\varphi}(\sharp)$, *there are* $\psi, \chi \in \mathcal{L}_{apal}$ *such that for all* $\theta \in \mathcal{L}_{apal}$:

$$\vdash \boldsymbol{\varphi}(\theta) \textit{ iff } \vdash \psi \rightarrow [\chi]\theta.$$

Proof. Let $\boldsymbol{\varphi}(\theta)$ be a theorem. Such an instance of a necessity form $\boldsymbol{\varphi}(\sharp)$ has the following shape: The formula θ is entirely on the right (or, if you wish, entirely on the inside); it is successively bound by, in arbitrary order and arbitrarily often, K_a-operators, announcement operators $[\chi']$, and implicative forms $\chi'' \rightarrow \ldots$. We can 'rearrange the order of these bindings', so to speak, to get the required form $\psi \rightarrow [\chi]\theta$. This, of course, is *still* a necessity form. But a fairly simple one. For these rearrangements, it does not matter whether the formula $\boldsymbol{\varphi}(\theta)$ contains *other* logical connectives (or even $\Box$ operators!) that were not used as constructors for the necessity form: These remain bound as they already were. We are only shifting around epistemic operators, announcements, and implications that were used to construct the necessity form and other subformulas remain unchanged.

First, we examine all the *public announcement modalities* occurring in $\boldsymbol{\varphi}(\theta)$. Using the reduction axioms for public announcement logic, we can push these modalities inside, past all the other components of the necessity form. To push them past the knowledge operators K_a, we use the reduction axiom 'announcement and knowledge':

$$[\varphi]K_a\psi \leftrightarrow (\varphi \rightarrow K_a[\varphi]\psi).$$

To push them past implications, we use the axioms 'announcement and negation' and 'announcement and conjunction'. So now *all* the announcement modalities are 'stacked' on the bottom of the necessity form, right in front of θ. We repeatedly apply the axiom announcement composition:

$$[\varphi][\psi]\eta \leftrightarrow [\varphi \wedge [\varphi]\psi]\eta,$$

so that we can collapse all these announcement modalities into one announcement modality.

We now take care of epistemic modalities. So far, what is left of the necessity form $\boldsymbol{\varphi}(\theta)$ is a sequence of symbols of the forms ($\varphi \rightarrow \ldots$ or $K_a \ldots$), followed by, at the bottom ('at right'), $[\chi]\theta$. We do not yet have the desired form '$\psi \rightarrow [\chi]\theta$' because, for example, the right-hand side of the status quo of our efforts may look like $\ldots K_a[\chi]\theta$. First, we get rid of all K_a-modalities in that sort of position: We push all the implication symbols $\rightarrow$ past all the K_a-modalities using that in the axiomatization **S5** theorems of form $\varphi \rightarrow K_a\psi$ can be transformed into theorems of form $\hat{K}_a\varphi \rightarrow \psi$, and vice versa. From left-to-right: apply monotonicity of $\hat{K}_a$ to both sides of $\varphi \rightarrow K_a\psi$, getting the theorem $\hat{K}_a\varphi \rightarrow \hat{K}_a K_a\psi$. In $S5$, $\hat{K}_a K_a\psi$ is equivalent to $K_a\psi$, so we get $\hat{K}_a\varphi \rightarrow K_a\psi$. Using veracity for K_a, we get $\hat{K}_a\varphi \rightarrow \psi$. From right-to-left is similar, except that we now first derive $K_a\hat{K}_a\varphi \rightarrow K_a\psi$ from $\hat{K}_a\varphi \rightarrow \psi$. In this way, we iteratively remove all K_a-modalities in wrong position.

Finally, we take care of implications. We now have a theorem of the form $(\varphi_1 \rightarrow \ldots \rightarrow (\varphi_n \rightarrow [\chi]\theta)\ldots)$. By a number of propositional steps, this gives us a theorem of form $\psi \rightarrow [\chi]\theta$, as desired.

Clearly, the argument works both ways, as all axioms applied are equivalences. □

PROPOSITION 4.9 ($\mathbf{APAL}^1 = \mathbf{APAL}$). *Every* $\mathbf{APAL}^1$ *theorem is an* **APAL** *theorem, and vice versa.*

Proof. Suppose we have a derivation involving an application of $R^1(\Box)$, such that given some $\boldsymbol{\varphi}([p]\psi)$, we infer $\boldsymbol{\varphi}(\Box\psi)$. We can now transform this into a derivation with an application of $R(\Box)$. Apply Lemma 4.8 to $\boldsymbol{\varphi}([p]\psi)$ for $\theta = [p]\psi$. From the result of form $\varphi \rightarrow [\chi][p]\psi$, we now infer $\varphi \rightarrow [\chi]\Box\psi$ by applying rule $R(\Box)$. Again using Lemma 4.8, now for $\theta = \Box\psi$, we get a derivation of $\boldsymbol{\varphi}(\Box\psi)$. Repeat this for all applications of $R^1(\Box)$. The resulting derivation does not have a single $R^1(\Box)$ application! The argument works in both directions. □

Finally, we show that every $\mathbf{APAL}^\omega$ theorem is a $\mathbf{APAL}^1$ theorem.

PROPOSITION 4.10 ($\mathbf{APAL}^\omega \subseteq \mathbf{APAL}^1$). *Every* $\mathbf{APAL}^\omega$ *theorem is an* $\mathbf{APAL}^1$ *theorem.*

Proof. Let us observe that the rule $R^1(\Box)$ is stronger than the rule $R^\omega(\Box)$: If we can prove $\boldsymbol{\varphi}([\theta]\psi)$ for all epistemic formulas θ, then we can prove in particular $\boldsymbol{\varphi}([p]\psi)$ for some atom $p \notin P_{\boldsymbol{\varphi}} \cup P_\psi$. As a result, we can derive the conclusion of the infinitary rule using only the finitary rule $R^1(\Box)$, and the axiomatization based on the infinitary rule $R^\omega(\Box)$ defines a set of theorems that is included in or equal to the set of theorems for the axiomatization based on the finitary rule $R^1(\Box)$. □

4.4. Completeness of the axiomatization $\mathbf{APAL}^\omega$**.** Let us now demonstrate that the axiomatization based on the infinitary rule $R^\omega(\Box)$ is complete with respect to the semantics. We use Goldblatt's technique applying necessity forms, where the main effect of rule $R^\omega(\Box)$ is that it makes the canonical model (consisting of all maximal consistent sets of formulas closed under the rule) standard for $\Box$.

A set x of formulas is called a *theory* if it satisfies the following conditions:

- x contains the set of all theorems
- x is closed under the rule of modus ponens and the rule $R^\omega(\Box)$.

Obviously, the least theory is the set of all theorems, whereas the greatest theory is the set of all formulas. The latter theory is called the trivial theory. A theory x is said to be consistent if $\bot \notin x$. Let us remark that the only inconsistent theory is the set of all formulas. We shall say that a theory x is maximal if for all formulas φ, $\varphi \in x$ or $\neg\varphi \in x$. Let x be a set of formulas. For all formulas φ, let $x + \varphi = \{\psi \mid \varphi \rightarrow \psi \in x\}$. For all agents a, let $K_a x = \{\varphi \mid K_a\varphi \in x\}$. For all formulas φ, let $[\varphi]x = \{\psi \mid [\varphi]\psi \in x\}$.

LEMMA 4.11. *Let x be a theory, φ be a formula and a be an agent. Then, $x + \varphi$, $K_a x$ and $[\varphi]x$ are theories. Moreover, $x + \varphi$ is consistent iff $\neg\varphi \notin x$.*

Proof. We only prove that $K_a x$ is a theory. First, let us prove that $K_a x$ contains the set of all theorems. Let ψ be a theorem. By the necessitation of knowledge, $K_a \psi$ is also a theorem. Since x is a theory, then $K_a \psi \in x$. Therefore, $\psi \in K_a x$. It follows that $K_a x$ contains the set of all theorems. Second, let us prove that $K_a x$ is closed under modus ponens. Let ψ, χ be formulas such that $\psi \in K_a x$ and $\psi \to \chi \in K_a x$. Thus, $K_a \psi \in x$ and $K_a(\psi \to \chi) \in x$. Since $K_a \psi \to (K_a(\psi \to \chi) \to K_a \chi)$ is a theorem and x is a theory, then $K_a \psi \to (K_a(\psi \to \chi) \to K_a \chi) \in x$. Since x is closed under modus ponens, then $K_a \chi \in x$. Hence, $\chi \in K_a x$. It follows that $K_a x$ is closed under modus ponens. Third, let us prove that $K_a x$ is closed under $R^\omega(\Box)$. Let $\boldsymbol{\varphi}$ be a necessity form and ψ be a formula such that $\boldsymbol{\varphi}([\chi]\psi) \in K_a x$ for all $\chi \in \mathcal{L}_{el}$. It follows that $K_a \boldsymbol{\varphi}([\chi]\psi) \in x$ for all $\chi \in \mathcal{L}_{el}$. Since x is a theory, then $K_a \boldsymbol{\varphi}(\Box\psi) \in x$. Consequently, $\boldsymbol{\varphi}(\Box\psi) \in K_a x$. It follows that $K_a x$ is closed under $R^\omega(\Box)$. $\Box$

LEMMA 4.12 (**Lindenbaum lemma**). *Let x be a consistent theory. There exists a maximal consistent theory y such that $x \subseteq y$.*

Proof. Let $\psi_0, \psi_0, \ldots$ be a list of the set of all formulas. We define a sequence $y_0, y_1, \ldots$ of consistent theories as follows. First, let $y_0 = x$. Second, suppose that, for some $n \geq 0$, y_n is a consistent theory containing x that has been already defined. If $y_n + \psi_n$ is inconsistent and $y_n + \neg\psi_n$ is inconsistent, then, by Lemma 4.11, $\neg\psi_n \in y_n$ and $\neg\neg\psi_n \in y_n$. Since $\neg\psi_n \to (\neg\neg\psi_n \to \bot)$ is a theorem, then $\neg\psi_n \to (\neg\neg\psi_n \to \bot) \in y_n$. Since y_n is closed under modus ponens, then $\bot \in y_n$: a contradiction. Hence, either $y_n + \psi_n$ is consistent or $y_n + \neg\psi_n$ is consistent. If $y_n + \psi_n$ is consistent, then we define $y_{n+1} = y_n + \psi_n$. Otherwise, $\neg\psi_n \in y_n$ and we consider two cases.

In the first case, we suppose that ψ_n is not a conclusion of $R^\omega(\Box)$. Then, we define $y_{n+1} = y_n$.

In the second case, we suppose that ψ_n is a conclusion of $R^\omega(\Box)$. Let $\boldsymbol{\varphi_1}(\Box\chi_1), \ldots, \boldsymbol{\varphi_k}(\Box\chi_k)$ be all the representations of ψ_n as a conclusion of $R^\omega(\Box)$. We define the sequence $y_n^0, \ldots, y_n^k$ of consistent theories as follows. First, let $y_n^0 = y_n$. Second, suppose that, for some $i < k$, y_n^i is a consistent theory containing y_n that has been already defined. Then, it contains $\neg\boldsymbol{\varphi_i}(\Box\chi_i)$. Since y_n^i is closed under $R^\omega(\Box)$, then there exists a formula $\varphi_i \in \mathcal{L}_{el}$ such that $\boldsymbol{\varphi_i}([\varphi_i]\chi_i)$ is not in y_n^i. Then, we define $y_n^{i+1} = y_n^i + \neg\boldsymbol{\varphi_i}([\varphi_i]\chi_i)$. Now, we put $y_{n+1} = y_n^k$. Finally, we define $y = y_0 \cup y_1 \cup \ldots$. It is straightforward to prove that y is a maximal consistent theory such that $x \subseteq y$. $\Box$

The canonical model of $\mathcal{L}_{apal}$ is the structure $\mathcal{M}_c = (W, \sim, V)$ defined as follows:

- W is the set of all maximal consistent theories.
- For all agents a, $\sim_a$ is the binary relation on W defined by $x \sim_a y$ iff $K_a x = K_a y$.
- For all atoms p, V_p is the subset of W defined by $x \in V_p$ iff $p \in x$.

Note that the relations $\sim_a$ are indeed equivalence relations.

LEMMA 4.13 (**Truth lemma**). *Let φ be a formula in $\mathcal{L}_{apal}$. Then for all maximal consistent theories x and for all finite sequences $\vec{\psi} = \psi_1, \ldots, \psi_k$ of formulas in $\mathcal{L}_{apal}$ such that $\psi_1 \in x$, $[\psi_1]\psi_2 \in x, \ldots, [\psi_1]\ldots[\psi_{k-1}]\psi_k \in x$:*

$$\mathcal{M}_c|\vec{\psi}, x \models \varphi \text{ iff } [\psi_1]\ldots[\psi_k]\varphi \in x.$$

Proof. The proof is by induction on φ. The base case follows from the definition of V. The Boolean cases are trivial. It remains to deal with the modalities.

Case $K_a\varphi$: If $\mathcal{M}_c|\vec{\psi}, x \not\models K_a\varphi$, then there exists a maximal consistent theory y such that $x \sim_a y$, $\psi_1 \in y$, $[\psi_1]\psi_2 \in y, \ldots, [\psi_1]\ldots[\psi_{k-1}]\psi_k \in y$, and $\mathcal{M}_c|\vec{\psi}, y \not\models \varphi$. By induction hypothesis, $[\psi_1]\ldots[\psi_k]\varphi \notin y$. Since $x \sim_a y$, then $K_a x = K_a y$. Thus, $K_a[\psi_1]\ldots[\psi_k]\varphi \notin x$ and $[\psi_1]\ldots[\psi_k]K_a\varphi \notin x$. Reciprocally, if $[\psi_1]\ldots[\psi_k]K_a\varphi \notin x$ then $K_a[\psi_1]\ldots[\psi_k]\varphi \notin x$. Let $y = K_a x + \neg[\psi_1]\ldots[\psi_k]\varphi$. The reader may easily verify that y is a consistent theory. By Lindenbaum lemma, there is a maximal consistent theory z such that $y \subseteq z$. Hence, $K_a x \subseteq z$ and $[\psi_1]\ldots[\psi_k]\varphi \notin z$. Consequently, $x \sim_a z$, $\psi_1 \in z$, $[\psi_1]\psi_2 \in z, \ldots, [\psi_1]\ldots[\psi_{k-1}]\psi_k \in z$ and, by induction hypothesis, $\mathcal{M}_c|\vec{\psi}, z \not\models \varphi$. Therefore, $\mathcal{M}_c|\vec{\psi}, x \not\models K_a\varphi$.

Case $[\psi]\varphi$: Let x be a state in the canonical model $\mathcal{M}_c$ and let $\psi_1, \ldots, \psi_{k-1}, \psi_k$ be formulas such that $\psi_1 \in x, \ldots, [\psi_1]\ldots[\psi_{k-1}]\psi_k \in x$. If $\mathcal{M}_c|\vec{\psi}, x \not\models [\psi]\varphi$, then $\mathcal{M}_c|\vec{\psi}, x \models \psi$ and $\mathcal{M}_c|\vec{\psi}|\psi, x \not\models \varphi$. Thus, by induction hypothesis, $[\psi_1]\ldots[\psi_{k-1}][\psi_k]\psi \in x$ and $[\psi_1]\ldots[\psi_{k-1}][\psi_k][\psi]\varphi \notin x$. Reciprocally, if $[\psi_1]\ldots[\psi_{k-1}][\psi_k][\psi]\varphi \notin x$, then $[\psi_1]\ldots[\psi_{k-1}][\psi_k]\psi \in x$ and, by induction hypothesis, $\mathcal{M}_c|\vec{\psi}|\psi, x \not\models \varphi$. Thus, $\mathcal{M}_c|\vec{\psi}, x \not\models [\psi]\varphi$.

Case $\Box\varphi$: Let x be a state in the canonical model $\mathcal{M}_c$ and let $\psi_1, \ldots, \psi_{k-1}, \psi_k$ be formulas such that $\psi_1 \in x, \ldots, [\psi_1]\ldots[\psi_{k-1}]\psi_k \in x$. If $\mathcal{M}_c|\vec{\psi}, x \not\models \Box\varphi$, then there is a $\Box$-free formula ψ such that $\mathcal{M}_c|\vec{\psi}, x \not\models [\psi]\varphi$. Thus, by induction hypothesis, $[\psi_1]\ldots[\psi_{k-1}][\psi_k][\psi]\varphi \notin x$. Using the axiom $\Box\varphi \to [\psi]\varphi$ and applying k times the rule of necessitation, this implies that $[\psi_1]\ldots[\psi_{k-1}][\psi_k]\Box\varphi \notin x$. Reciprocally, if $[\psi_1]\ldots[\psi_{k-1}][\psi_k]\Box\varphi \notin x$, then using the fact that x is closed with respect to the special inference rule for $\Box$, there is a $\Box$-free formula ψ such that $[\psi_1]\ldots[\psi_{k-1}][\psi_k][\psi]\varphi \notin x$. Thus, by induction hypothesis, $\mathcal{M}_c|\vec{\psi}, x \not\models [\psi]\varphi$ and $\mathcal{M}_c|\vec{\psi}, x \not\models \Box\varphi$. $\Box$

4.5. *Completeness of the axiomatization* APAL. As a result, we now have completeness for our logic **APAL**.

PROPOSITION 4.14 (**Completeness**). *Let φ be a formula in $\mathcal{L}_{apal}$. Then, φ is a theorem (in* **APAL***) if φ is valid.*

Proof. Let φ be valid. From Lemmas 4.11, 4.12, and 4.13 follows that φ is a theorem in $\mathbf{APAL}^{\omega}$. From that, Proposition 4.9, and Proposition 4.10 follows that φ is a theorem in **APAL**. $\Box$

THEOREM 4.15 (**Soundness and completeness**). *Let φ be a formula in $\mathcal{L}_{apal}$. Then, φ is a theorem iff φ is valid.*

Proof. Soundness was proved in Proposition 4.2. Completeness was proved in Proposition 4.14. $\Box$

4.6. Further proof theoretical observations. We emphasize the rather peculiar nature of this completeness proof. Given a logic *APAL*, a finitary axiomatization **APAL**, and an infinitary version $\mathbf{APAL}^{\omega}$ of that axiomatization with the infinitary rule $R^{\omega}(\Box)$, we have been proceeding as follows. First, we showed that every theorem of **APAL** is a validity of *APAL* (soundness). Then, we showed that every validity of *APAL* is derivable in $\mathbf{APAL}^{\omega}$ by a canonical model argument (completeness). Finally, we observed that every theorem of $\mathbf{APAL}^{\omega}$ is also derivable in **APAL**, by two observations. First, an application of $R^{\omega}(\Box)$ can be adjusted to an application of $R^{1}(\Box)$: Instead of deriving the conclusion with unique occurrence $\Box\psi$ from an infinite set (namely, for all epistemic formulas φ) of premisses $[\varphi]\psi$, we pick out a premiss with a fresh atom among that infinity and derive the conclusion from $[p]\psi$ only. Second, the other observation is that we can transform derivations with

$$\begin{tikzcd}
\vdash_\omega \varphi \arrow[rr, Rightarrow, "\text{strengthening (Prop. 4.10)}"] & & \vdash \varphi \arrow[dl, Rightarrow, "\text{soundness (Prop. 4.2)}"] \\
& \models \varphi \arrow[ul, Rightarrow, "\text{completeness (Prop. 4.14)}"] &
\end{tikzcd}$$

Fig. 3. Method to prove soundness and completeness.

$R^1(\Box)$ applications into derivations with $R(\Box)$ applications. See Figure 3 for an overview of our method to prove completeness.

The crucial aspect is that the canonical model is for the infinitary version $\mathbf{APAL}^\omega$ of the proof system and not for the finitary proof system **APAL**. The infinitary version is strongly complete: From Lemmas 4.12 and 4.13 follows that every consistent theory is satisfied in a model, one of the formulations of strong completeness. But this does not imply compactness because the proof system is not finitary.

The finitary proof system **APAL** is only weakly complete: *When proving theorems*, or in other words proofs without premisses, applications of the infinitary rule $R^\omega(\Box)$ can be replaced by applications of the finitary rule $R(\Box)$, and that proof can then be transformed to one using the finitary rule in **APAL**. But in infinitary proofs, starting from infinitely many assumptions, we cannot use this trick without getting rid of our proof assumptions. So strong completeness cannot be shown for the finitary axiomatization **APAL**, and indeed, as we have seen in Proposition 3.15, *APAL* is not compact.

5. Arbitrary events. Along a common line in dynamic epistemics, one might consider more general accessibility relations on our structures (as summarily explored in Proposition 3.14), and one might expand the language with additional modal operators, in particular: with common knowledge, with actions that are not public (such as private announcements), and with assignments (actions that change the truth value of atomic propositions). Let us consider 'arbitrary events' in the sense of arbitrary action models (Baltag *et al.*, 1988).

In public announcement logic, all events are public. More complex dynamics is also conceivable, such as private messages, events involving partial observation. Action models formalize such more complex dynamics. These were proposed by Baltag *et al.* (1988). We refrain from giving sufficient technical details to understand how these action models work for a reader who has not come across them before and merely mention that an action model is a structure exactly as a Kripke model, except that elements of the domain are called 'events' u instead of 'states' s and that instead of a valuation V, that for each state determines which facts are true and false, we now have a precondition function pre, that to each event assigns a formula called a *precondition*. This formula is the precondition for the execution of that event. A singleton action model with universal access for all agents corresponds to a public announcement, and the precondition for the event 'public announcement' is the announcement formula.

Let U be a finite action model. Some possible generalizations are as follows ($*$ is arbitrary finite iteration). A sensible restriction in the semantics for arbitrary actions is that all preconditions must be epistemic formulas.

1	$M, s \models \langle U \rangle \varphi$	iff	there is a $u \in U : M, s \models \langle U, u \rangle \varphi$
2	$M, s \models \Diamond \varphi$	iff	$M, s \models \langle U \rangle^* \varphi$ for a given U
3	$M, s \models \Diamond \varphi$	iff	there is a U of a given signature $: M, s \models \langle U \rangle \varphi$
4	$M, s \models \Diamond \varphi$	iff	there is a $U : M, s \models \langle U \rangle \varphi$

In the first two proposals, a given action model U is a parameter of the language. The first was investigated by (Hoshi, 2006, p. 8). The second can be seen as a generalization of iterated relativization that was investigated in Miller & Moss (2005), and it results in undecidable logics. In the third, we allow action models of a given signature, that is, an action model frame without preconditions for action point execution. The logic *APAL* comes under this category: It is arbitrary event logic for the signature 'singleton': This sort of action model corresponds to an announcement.

The last proposal seems the end point of further generalization. From a multiagent perspective, where more complex than public events are conceivable, this also seems the most obvious perspective for multiagent knowability. Note that action model logic (without $\Box$) is again equally expressive as multiagent logic. All validities in Proposition 3.1 hold, and we conjecture that CR also holds. Axiom MK does not hold. Even in finite models, there are infinite chains of informative actions because the uncertainty of agents about each other's uncertainty can be arbitrarily complex. An example is, given initial uncertainty of two agents a, b about the value of an atom p, that a privately learns that p, after which b privately learns that a privately learnt that p, after which a privately learns *that*, and so on, thus creating an arbitrarily large finite model satisfying $K_a K_b K_a K_b \dots p$ but where b does not know *that*. Proposition 3.6 stating that the truth of $\Diamond \psi$ can always be simulated by the truth of $\langle p \rangle \psi$ for some fresh atom p has a natural generalization to replacing formula preconditions in action models by fresh atoms. In the axiomatization **APAL** (Table 2), we have to add the various axioms reducing the postconditions of updates, and we have to replace the axiom 'announcement and knowledge' by its action model reducing counterpart:

$$\Box \varphi \rightarrow [U]\varphi, \quad \text{where for all events } u \in U, \ \mathsf{pre}(u) \in \mathcal{L}_{el} \text{ (arbitrary and specific event).}$$

Unfortunately, it is unclear what derivation rule should allow the introduction of a $\Box$-formula; an inference involving announcement of a fresh atomic variable is certainly not good enough: This atom only 'witnesses' a public announcement and not other action models.

If factual change is also permitted, one has the peculiar result that $\Diamond \varphi$ is valid for all consistent φ, in other words, all satisfiable formulas are realizable (reachable) in *any information state* (subject to the restriction that the information state is finite). This applies a technical result in van Ditmarsch & Kooi (2008): Given two finite information states, there is an event transforming the first into the second. Allowing factual change seems a too drastic departure from the original Fitch question what true formulas are knowable: That seems to suggest that only informative actions are allowed to get to know things but not factual change.

6. Conclusions and further research. We proposed an extension of public announcement logic with a dynamic modal operator $\Box \varphi$ expressing that φ is true after every announcement ψ. We gave various semantic results, defined fragments of 'knowable' formulas in the Fitch sense that $\models \varphi \rightarrow \Diamond K_a\varphi$, and showed completeness for a Hilbert-style axiomatization of this logic.

We anticipate a number of further investigations, by us or others. More details on model checking and decidability would be relevant—in particular the somewhat surprising undecidability result. For that, see French & van Ditmarsch (2008).

Results on model checking and decidability are also relevant for the 'grander scheme' comparing dynamic and temporal epistemic logics, as in recent work by van Benthem *et al.* (2007) and in work in progress by Hoshi (2008). In the comparison between temporal with dynamic epistemics, if we let an announcement correspond to a tick of the clock, a dynamic announcement operator $[\varphi]$ therefore corresponds to a temporal 'next' operator, and our arbitrary announcement operator $\Diamond$ then corresponds to the temporal future operator F, for 'some time in the future'.

Given the proved validities for $\Box$, a relevant question seems where in the S4 scheme of logics, the logic *APAL* resides. It is not S5, but at least (given CR and MK), S4.1 and S4.2. Unfortunately, it is unclear whether $\Box$ is a normal modal operator, more concretely: whether the schematic validities (i.e., those employing formula variables, such as $\Box\varphi \rightarrow \Box\Box\varphi$, not those employing propositional variables, such as $\Box p \leftrightarrow p$) in $\mathcal{L}_\Box \cap APAL$ satisfy uniform substitution. Tentative evidence against it is that public announcement logic is not normal, for example, $[p]p$ is valid but $[p \wedge \neg Kp](p \wedge \neg Kp)$ is invalid. Further tentative evidence against normality is that an interpretation of $\Box$ in terms of neighborhood semantics is conceivable (Ågotnes & van Ditmarsch, 2008), which points to nonnormality. On the other hand, arbitrary announcement logic *with* $\Box$ but *without* announcements might just as well be equally expressive as *APAL*. This is the logic with language $\varphi ::= p|\neg\varphi|\varphi \wedge \varphi|K_a\varphi|\Box\varphi$ and with $\Box$-semantics: $M, s \models \Box\varphi$ iff for all $\psi \in \mathcal{L}_{el} : M, s \models \psi$ implies $M|\psi, s \models \varphi$. If so, that would be suggestive evidence for normality of $\Box$.

Because of these uncertainties about the character of $\Box$, it is sometimes difficult to interpret our results. For example, the principle MK ($\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$) in conjunction with 4 ($\Box\varphi \rightarrow \Box\Box\varphi$) correspond to the frame property of *atomicity*, defined as $\forall x\exists y(Rxy \wedge \forall z(Ryz \rightarrow z = y))$ (Blackburn *et al.*, 2001, p. 167, example 3.57). In our terms, atomicity seems to describe that one can always make a *most* informative announcement. But this is false! Consider the model consisting of $2^{|P|}$ states, namely, one for each valuation of atomic propositions, and with universal access on the domain for all agents. Every given epistemic formula contains only a finite number of atoms, so after its announcement, a further informative announcement remains possible. So a most informative announcement cannot always be made. This puzzles us.

Our $\Box$-operator is an implicit quantification over announcements. Of course, one can also make the quantification explicit. In other words, instead of $\Diamond\varphi$, we may as well write $\exists\psi\langle\psi\rangle\varphi$. This approach is currently investigated by Baltag.

Unlike public announcement logic, arbitrary announcement logic can also be used to specify *planning problems*, as in AI: We can express some initial knowledge conditions and a final desideratum in terms of knowledge, and a diamond $\Diamond$ of unknown instantiation representing a sequence of announcements supposedly realizing it. In other words, something of the form $\mathsf{init} \rightarrow \Diamond K\mathsf{final}$. Different variants of this theme are conceivable. If our logic 'works', we can reduce and manipulate such an expression so that it should ultimately deliver the concrete announcements needed to realize the final knowledge conditions: a plan. We did not pursue this matter, although a tableaux calculus for *APAL* may be relevant to mention here (Balbiani *et al.*, 2007).

7. Acknowledgments. This research started late August 2005 when Andreas, Tiago, and Hans were brainstorming in Andreas' office on quantifying over announcements to devise a logic of planning. Thus, the $\Box$ came up during the discussion as a way to express what is true after arbitrary announcements. Hans thinks Tiago came up with the idea, whereas Andreas thinks that Hans came up with the idea, and so on. As so often, it seems to have come up in a true spirit of mutual and free exchange of ideas and collaboration. Over time, others joined the collaboration. Six authors, six nationalities, only one of those (Philippe) working in his country of origin. Incidentally, the intended 'logic of planning' never materialized. We gratefully acknowledge various input from many people. We put them in alphabetical order. Even so, Johan van Benthem deserves to be thanked separately: Sometime in 2006, he pointed us toward his work on knowability and also gave detailed comments on the status quo of our research. For their input, we thank Nick Asher, Johan van Benthem, Balder ten Cate, Jan van Eijck, Tim French, Amelie Gheerbrand, Barteld Kooi, Ron van der Meyden, Larry Moss, Dung Nguyen, Eric Pacuit, Greg Restall, Joe Salerno, Hartley Slater, Yde Venema, Rineke Verbrugge, and Albert Visser. The contribution by Tiago de Lima is part of the research program Moral Responsibility in R&D Networks, which is supported by the Netherlands Organisation for Scientific Research (NWO) under grant number 360-20-160. Finally, we thank the *RSL* reviewer for the praise heaped on the article.

This journal version is based on our TARK 2007 conference contribution (Balbiani *et al.*, 2007). It is quite heavily revised. Apart from a different presentation and full proof details for all propositions, there is another expressivity result, the axiomatization is different, and the issue of knowability is addressed in much greater detail.

BIBLIOGRAPHY

Aczel, P. (1988). *Non-Well-Founded Sets* [CSLI Lecture Notes 14]. Stanford, CA: CSLI Publications.

Ågotnes, T., & van Ditmarsch, H. P. (2008). Coalitions and announcements. In Padgham, L., Parkes, D. C., Müller, J., and Parsons, S., editors. *Proceedings of the Seventh International Joint Conference on Autonomous Agents & Multiagent Systems (AAMAS 2008)*, Estoril, Portugal, May 12-16, 2008, Volume 2, IFAAMAS, 673–680.

Balbiani, P., van Ditmarsch, H. P., Herzig, A., & de Lima, T. (2007). A tableau method for public announcement logics. In Olivetti, N., editor. *Proceedings of the International Conference on Automated Reasoning with Analytic Tableaux and Related Methods (TABLEAUX)*. LNAI 4548. Berlin, Germany: Springer, pp. 43–59.

Baltag, A., Moss, L. S., & Solecki, S. (1998). The logic of public announcements, common knowledge, and private suspicions. In Gilboa, I., editor. *Proceedings of the 7th Conference on Theoretical Aspects of Rationality and Knowledge (TARK 98)*. pp. 43–56.

Blackburn, P., de Rijke, M., & Venema, Y. (2001). *Modal Logic*. Cambridge Tracts in Theoretical Computer Science 53. Cambridge: Cambridge University Press.

Brogaard, B., & Salerno, J. (2004). *Fitch's Paradox of Knowability*. Available from: http://plato.stanford.edu/archives/sum2004/entries/fitch-paradox/.

Dummett, M. (2001). Victor's error. *Synthesis*, **61**, 1–2.

Fine, K. (1970). Propositional quantifiers in modal logic. *Theoria*, **36**(3), 336–346.

Fitch, F. B. (1963). A logical analysis of some value concepts. *The Journal of Symbolic Logic*, **28**(2), 135–142.

French, T., & van Ditmarsch, H. P. (2008). Undecidability for arbitrary public announcement logic. In Areces, C., and Goldblatt, R., editors. *Proceedings of Advances in Modal Logic 2008*. Nancy, France, September 9-12, 2008. London: College Publications, pp. 23–42.

Gerbrandy, J. D. (1999). *Bisimulations on Planet Kripke*. PhD Thesis. ILLC Dissertation Series DS-1999-01. University of Amsterdam.

Gerbrandy, J. D., & Groeneveld, W. (1997). Reasoning about information change. *Journal of Logic, Language, and Information*, **6**, 147–169.

Goldblatt, R. (1982). *Axiomatising the Logic of Computer Programming*. Berlin, Germany: Springer-Verlag.

Hintikka, J. *Knowledge and Belief*. Ithaca, NY: Cornell University Press.

Hoshi, T. (2006). The logic of communication graphs for group communication and the dynamic epistemic logic with a future operator. Philosophy Department, Stanford University.

Hoshi, T. (2008). Logics of public announcements with announcement protocols (manuscript, Philosophy Department, Stanford University).

Meyer, J.-J. Ch., & van der Hoek, W. (1995). *Epistemic Logic for AI and Computer Science*. Cambridge Tracts in Theoretical Computer Science 41. Cambridge: Cambridge University Press.

Miller, J. S., & Moss, L. S. (2005). The undecidability of iterated modal relativization. *Studia Logica*, **79**(3), 373–407.

Moore, G. E. (1942). A reply to my critics. In Schilpp, P. A., editor. *The Philosophy of G.E. Moore*. The Library of Living Philosophers, vol. 4. Evanston, IL: Northwestern University, pp. 535–677.

Moss, L. S., & Parikh, R. (1992). Topological reasoning and the logic of knowledge. In Moses, Y., editor. *TARK '92: Proceedings of the Fourth Conference on Theoretical Aspects of Reasoning About Knowledge*. San Francisco, CA: Morgan Kaufmann Publishers Inc., pp. 95–105.

Parikh, R., Moss, L. S., & Steinsvold, C. (2007). Topology and epistemic logic. In Aiello, M., Pratt-Hartmann, I., and van Benthem, J. F. A. K., editors. *Handbook of Spatial Logics*. Berlin, Germany: Springer-Verlag, pp. 299–341.

Plaza, J. A. (1989). Logics of public communications. In Emrich, M. L., Pfeifer, M. S., Hadzikadic, M., and Ras, Z. W., editors. *Proceedings of the 4th International Symposium on Methodologies for Intelligent Systems: Poster Session Program*. ORNL/DSRD-24. Oak Ridge National Laboratory, pp. 201–216.

Plaza, J. A. (2007). Logics of public communications. *Synthese*, **158**(2), 165–179 (Reprint of Plaza's 1989 workshop paper).

Tennant, N. (1997). *The Taming of the True*. Oxford, UK: Oxford University.

van Benthem, J. F. A. K. (2004). What one may come to know. *Analysis*, **64**(2), 95–105.

van Benthem, J. F. A. K. (2006). 'One is a lonely number': on the logic of communication. In Chatzidakis, Z., Koepke, P., and Pohlers, W., editors. *Logic Colloquium '02*. Volume 27 of *Lecture Notes in Logic*. Poughkeepsie, NY: Association for Symbolic Logic, pp. 96–129.

van Benthem, J. F. A. K., Gerbrandy, J. D., & Pacuit, E. (2007). Merging frameworks for interaction: DEL and ETL. In Samet, D., editor. *Proceedings of TARK 2007*. Louvain-la-Neuve, Belgium: Presses Universitaires de Louvain, pp. 42–51.

van Ditmarsch, H. P., & Kooi, B. P. (2006). The secret of my success. *Synthese*, **151**, 201–232.

van Ditmarsch, H. P., & Kooi, B. P. (2008). Semantic results for ontic and epistemic change. In Bonanno, G., van der Hoek, W., and Wooldridge, M., editors. *Post-Proceedings of LOFT 2006*. Amsterdam: Amsterdam University Press. To appear in the series Texts in Logic and Games, pp. 87–117.

van Ditmarsch, H. P., van der Hoek, W., & Kooi, B. P. (2007). *Dynamic Epistemic Logic*, volume 337 of *Synthese Library*. Berlin, Germany: Springer.

CNRS INSTITUT DE RECHERCHE EN INFORMATIQUE DE TOULOUSE
UNIVERSITÉ DE TOULOUSE
118 ROUTE DE NARBONNE
31062 TOULOUSE CEDEX 9
FRANCE
E-mail: balbiani@irit.fr

OXFORD UNIVERSITY
COMPUTING LABORATORY
WOLFSON BUILDING, PARKS ROAD
OXFORD OX1 3QD
UNITED KINGDOM
E-mail: alexandru.baltag@comlab.ox.ac.uk

DEPARTMENT OF COMPUTER SCIENCE
UNIVERSITY OF OTAGO
PO BOX 56, DUNEDIN 9054,
NEW ZEALAND

CNRS INSTITUT DE RECHERCHE EN INFORMATIQUE DE TOULOUSE
UNIVERSITÉ DE TOULOUSE
118 ROUTE DE NARBONNE
31062 TOULOUSE CEDEX 9
FRANCE
E-mail: hans@cs.otago.ac.nz

CNRS INSTITUT DE RECHERCHE EN INFORMATIQUE DE TOULOUSE
UNIVERSITÉ DE TOULOUSE
118 ROUTE DE NARBONNE
31062 TOULOUSE CEDEX 9
FRANCE
E-mail: herzig@irit.fr

PHILOSOPHY DEPARTMENT
STANFORD UNIVERSITY
STANFORD, CA 94305-2155, USA
E-mail: thoshi@stanford.edu

DEPARTMENT OF INDUSTRIAL ENGINEERING AND INNOVATION SCIENCES
EINDHOVEN UNIVERSITY OF TECHNOLOGY
P.O. BOX 513
5600 MB EINDHOVEN
THE NETHERLANDS
E-mail: t.d.lima@tue.nl