Puzzles in quantum gravity: what can black hole microstates teach us about quantum gravity?
el Showk, S.
CHAPTER 2

BLACK HOLE PUZZLES

The first part of this thesis will focus on exploring various aspects of black hole physics. In particular we extend the technical tools available and apply them to learn qualitative lessons that may point towards resolutions of the well known paradoxes associated with black hole physics.

Before doing so we spend a chapter on some expository material. We will be relatively brief in our review of well-known material such as the information loss paradox as this thesis is not intended as a review of known results. Rather we will focus on a careful conceptual introduction to what has become known as the “fuzzball conjecture”. Because the latter has been the source of somewhat ill-deserved controversy, we spend some time clarifying the main points and disavowing various points of view that are extensions of this conjecture rather than its core. We would like to emphasize, in particular, that the conjecture, in our view, is not about the role of semi-classical or even quantum supergravity states in making up the black hole ensemble. Rather the conjecture is a statement about the spatial extent of a generic state in the ensemble, be it stringy or quantum, and how this grows with the entropy of the ensemble. The main novelty in this proposal is the claim that stringy or quantum effects may be significant in gravitational theories even in regimes where curvatures are low (i.e. where we might naively have thought we could trust effective field theory). Put another way, the claim is that a semi-classical analysis on a fixed background, such as Hawking’s analysis in [18], may be invalid in a gravitational theory because quantum gravity effects can be relevant even in regimes with low curvature. We will, in fact, demonstrate a particular instance of this phenomena in Chapter 5.

We will also spend time here providing the conceptual background for some of our main tools such as AdS/CFT and phase space quantization. Again, the idea is not to be compre-
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hensive or even detailed but rather to address conceptual issues relevant for this thesis that are perhaps unusual or novel and also to provide a modicum of background. A reader interested in more detailed background reading is referred to several other interesting theses in this field [19, 20] and to the literature survey of [6].

2.1 INFORMATION LOSS

Perhaps the most persistent and troublesome puzzle associated with black holes is their ability to “lose” information in a quantum theory. This paradox is rather technical and there has been significant effort made to show that it does not in fact exist (see [21] for a nice summary and analysis of these arguments). It nonetheless seems to persist and a large part of this thesis will be motivated by attempting to qualitatively determine which sort of resolutions may be possible. Before doing so we should naturally introduce the puzzle itself.

Black hole information loss was first proposed by Hawking in the seminal paper [18]. Here we will present a very visual, heuristic and non-technical overview of this phenomena following [22]. Information loss will play a primarily motivational role in this thesis and will not directly be dealt with. We thus forego a detailed treatment here. Rather we hope to give the reader a sense for how the paradox emerges and to hint at what elements necessarily must play a role in any resolution.

If we consider the Schwarzschild metric

\[
ds^2 = - \left( 1 - \frac{2GM}{r} \right) dt^2 + \left( 1 - \frac{2GM}{r} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \tag{2.1}
\]

we see that not only is there a coordinate singularity at \( r = r_h = 2GM \) (with \( G \) Newton’s constant and \( M \) the black hole mass) but also that this point corresponds to a “tipping” of the light-cone. Specifically the killing vector \( \frac{\partial}{\partial t} \) which becomes the standard (Minkowski) time-like killing vector asymptotically, becomes null at this point, and is space-like within the horizon. This tipping is, as we shall see, an essential feature of information loss. Note that within the horizon it is \( \frac{\partial}{\partial r} \) that is the timelike vector (though it is not a Killing vector).

To study the quantum mechanics of matter fields in this background, while neglecting quantum gravity effects, we would like to find a foliation of the spacetime corresponding to (2.1) on which the induced metric on each slice is everywhere spacelike and regular. Fixed \( t \) slices do not satisfy these requirements, as they might in a geometry without a horizon, since such slices become time-like inside the horizon. We can however, foliate spacetime as shown in figure [2.1] This foliation uses constant \( t \) slices outside the horizon,
constant $r$ slices inside the horizon and a “connector” region that crosses the horizon and is defined so that the foliation is always spatial. Because we want to keep our slices smooth and away from the singularity they only move inwards very slowly so while the external region of the slices are parameterized by values of $t$ from $t_0$, when the black hole formed, to $t = \infty$ the internal slices run, within the same parameterization, from $r = r_h$ to $r = \epsilon$ where $\epsilon > 0$ is far enough away from the singularity to keep the curvature on each slice low, even at late times. The different rates of “time-evolution” of the slices implies a stretching of the slices in the connector region as indicated via the dotted lines in the figure.

Performing a standard field-theoretic analysis using this foliation has several interesting consequences. First, the curving of the spatial foliations reflects the fact that null geodesics move in different directions on either side of the horizon. Within the horizon they are directed inwards towards the singularity, whereas those outside the horizon move outwards towards asymptotic infinity (see figure 2.2). Moreover, the regions of the foliation away from the horizon (i.e. the complement of the connector region) have a very low curvature in the induced metric and do not vary much from slice to slice. Near the horizon region, however, there is a “stretching” effect as discussed before. The stretching of the background acts like a time-dependent potential for e.g. a scalar field in this background.
Thus an empty region of spacetime near the horizon that is in a vacuum state at an early slice will no longer be in the vacuum state at later time slices. This can be seen by solving the wave-equation for a scalar on this spatial foliation at a particular time slice in terms of Fourier modes and comparing these modes on a different slice. Somewhat more concretely we can consider an outgoing mode of the form $e^{iky^+}$ with $y^\pm$ light-cone coordinates near the horizon at an early time. We are interested in how these coordinates evolve with time and how they are related to light-cone coordinates at a future time-slice $X^\pm(y^\pm)$. This is because we will assume modes of the form given above time-evolve by keeping a constant phase along null-geodesics.

Recall that the horizon is a splicing point for null geodesics, with geodesics within the horizon heading inwards and those outside heading outwards. Because of this fact, and the way we foliate our geometry, the coordinate transformation between $X^\pm$ and $y^\pm$ will vary over the connector region with the largest variation being near the horizon (see figure 2.2). This coordinate transformation mixes creation and annihilation operators in the basis of Fourier modes and can transform the vacuum at one time slice to an excited state at another.

A simple analog for this is the vacuum state for a harmonic oscillator in a potential parameterized, in the standard way, by a frequency $\omega$. If the potential changes fast enough to the potential associated with a harmonic oscillator with a different frequency $\omega'$ the old ground state wavefunction will not adiabatically evolve into the new one but will keep its
form and will now be given by a superposition of excited states of the new Hamiltonian. In this case it can be shown that the resultant state is heuristically of the form

$$|\psi_e\rangle = e^{\sum_k \gamma b_k^\dagger c^\dagger_k} |0\rangle$$  \hspace{1cm} (2.2)

where $b_k^\dagger$ and $c_k^\dagger$ are creation operators for localized quanta of the scalar field on the inside and outside of the horizon, respectively (of course $k$ cannot label a momentum basis here since the quanta would then not be localized so we let $k$ denote an index over some localized basis of creation operators). Thus the resultant state is an entangled state of excited quanta on either side of the horizon and, according to our previous arguments, these modes will either move inwards or outwards depending on which side of the horizon they were created.

This entangled state is at the heart of information loss. The quanta created by the $c_k^\dagger$ will move off to infinity while those created by $b_k^\dagger$ will slowly fall towards the singularity but they are in an entangled state. Generally this is not a problem as an entangled state is still a pure state. The problem arises when we realize that the black hole continues to radiate until it disappears. At this point the quanta at infinity are entangled with nothing! As a larger and larger number of quanta $c_k^\dagger$ arrive at infinity the number of quanta $b_k^\dagger$ within the horizon must also grow at the same rate to keep the state pure (each $c_k^\dagger$ quanta is entangled with one $b_k^\dagger$ quanta). At the same time, the volume within the shrinking horizon is constantly decreasing so must support a higher and higher density of states. While one can allow this to happen leading to the notion of remnants it quickly becomes clear that this leads to a host of other problems (see e.g. [21]). On the other hand, if the number of quanta within the horizon is decreasing or vanishing then we are essentially tracing over the states $b_k^\dagger$ transforming $|\psi_e\rangle$ from an entangled state to a density matrix with an associated entropy. This entropy is not a result of our ignorance of the underlying degrees of freedom in this system but represents a genuine loss of information and violation of unitarity.

Note that small corrections near the horizon will not change the entangled nature of $|\psi_e\rangle$ as this would require a relatively large change to this state. For a more detailed discussion of why this cannot be avoided by simply appealing to small quantum gravity corrections see [22]. Our review here has been rather heuristic; for more detailed and technical arguments the reader is directed to some of the original literature [18][23] and other references in [22].
2.2 FUZZBALLS

A spacetime geometry can carry an entropy in string theory via coarse graining over an underlying set of microstates. Since the initial success of string theory in accounting for the entropy of supersymmetric black holes by counting states in a field theory [24] there has been an ongoing effort to understand exactly what the structure of these microstates is in a setting where spacetime and gravity are manifest (i.e. in a closed string theory).

Recently, it was shown that, in examples with enough supersymmetry, including some extremal black holes, one can construct a basis of “coherent” microstates whose spacetime descriptions in the $\hbar \to 0$ limit approach non-singular, horizon-free geometries which resemble a topologically complicated “foam”. Conversely, in these cases the quantum Hilbert space of states can be constructed by directly quantizing a moduli space of smooth classical solutions. Nevertheless, the typical states in these Hilbert spaces respond to semiclassical probes as if the underlying geometry was singular, or an extremal black hole. In this sense, these black holes are effective, coarse-grained descriptions of underlying non-singular, horizon-free states (see [6] for a review of examples with differing amounts of supersymmetry).

Such results suggest the idea, first put forward by Mathur and collaborators [25, 26], that all black hole geometries in string theory, even those with finite horizon area, can be seen as the effective coarse-grained descriptions of complex underlying horizon-free states which have, essentially, an extended spacetime structure. Thus the main claim of the proposal is

Conjecture (form 1): The generic state in the black hole ensemble, realized in a closed string theory, differs from the naive geometry up to the horizon scale.

While this form captures the essence of the proposal it is perhaps somewhat imprecise as it is not immediately clear what the relation between a state and a geometry is so let us restate this somewhat differently (closely mirroring [22]).

Conjecture (form 2): Quantum gravity effects are not confined to within a Plank scale and in particular may be relevant for describing the physics up to the horizon scale.

The importance of this claim is that it would invalidate the implicit assumption made in

1The idea here is that a single microstate does not have an entropy, even if its coarse-grained description in gravity has a horizon. Thus the spacetime realization of the microstate, having no entropy, should be in some sense horizon-free, even though the idea of a horizon, or even a geometry, may be difficult to define precisely at a microscopic level.
Section 2.1 that in the region near the horizon quantum gravity effects are subleading (or highly suppressed) and effective field theory can be trusted. This idea seems initially unlikely because one might expect that the quantum effects that correct the classical black hole spacetime would be largely confined to regions of high curvature near the singularity, and would thus not modify the horizon structure.

While we are still quite far from a demonstration of this conjecture in the context of large black holes we do manage, in this thesis, to demonstrate the existence of large scale quantum or stringy structures that extend into a region of spacetime that standard reasoning would suggest is perfectly smooth and classical. More precisely we show that certain classical solutions to the equations of motion will, if embedded in a fully quantum string theory, receive quantum and stringy corrections over a large region of spacetime even though the associated solutions are smooth and of low curvature everywhere. This demonstrates that the precept underlying the conjecture may holds but it certainly does not imply the full conjecture.

Let us mention an important caveat at this point. While we claim (and demonstrate in some instances) that quantum or stringy effects can be “important” up to the horizon scale it is not clear exactly how an in-falling observer would perceive this. The idea that the black hole geometry is an averaged effective description of many underlying microstates, not unlike a thermal ensemble, would suggest that a macroscopic observer would have to make impractically precise measurements to distinguish the actual microstate from the ensemble average. There is some friction, however, between this philosophy and the apparent need for rather significant violations of the assumptions implicit in Section 2.1 (as promulgated in e.g. [22]) in order to resolve the information loss paradox. Thus, at this point it seems unclear, even within the context of the fuzzball proposal, whether the resolution to information loss will involve drastic violations of classical reasoning, immediately evident to an infalling observer, or if a more subtle resolution, wherein part of the classical picture survives, will emerge. A more complete understanding of this issue is essential if we are to have any hope of demonstrating the fuzzball proposal.

2.2.1 BACKGROUND

In string theory, black holes can often be constructed by wrapping $D$-branes on cycles in a compact manifold $X$ so they appear as point like objects in the spatial part of the non-compact spacetime, $\mathbb{R}^{1,d-1}$. As the string coupling is increased, these objects back-react on spacetime and can form supersymmetric spacetimes with macroscopic horizons. The entropy associated with these objects can be determined “microscopically” by counting BPS states in a field theory living on the branes and this has been shown in many cases to match the count expected from the horizon area (see [24] [27] for the prototypical calculations). Although the field theory description is only valid for very small values
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of the string coupling $g_s$, the fact that the entropy counting in the two regimes coincides can be attributed to the protected nature of BPS states that persist in the spectrum at any value of the coupling unless a phase transition occurs or a wall of stability is crossed. The fact that the (leading) contribution to the entropy of the black hole could be reproduced from counting states in a sector of the field theory suggests that the black hole microstates dominate the entropy in this sector.

While it is already very impressive that these states can be counted at weak coupling, understanding the nature of these states in spacetime at finite coupling remains an open problem. As $g_s$ is increased the branes couple to gravity and we expect them to start backreacting on the geometry. The main tools we have to understand the spacetime or closed string picture of the system are the AdS/CFT correspondence and the physics of D-branes.

Within the framework of the AdS/CFT correspondence black holes with near horizon geometries of the form $AdS_m \times M$ must correspond to objects in a dual conformal field theory that have an associated entropy\(^2\). A natural candidate is a thermal ensemble or density matrix, in the CFT, composed of individual pure states (see e.g. \(^{28}\)). AdS/CFT then suggests that there must be corresponding pure states in the closed string picture and that these would comprise the microstates of the black hole. There is no reason, \textit{a priori}, that such states will be accessible in the supergravity approximation. First, the dual objects should be closed string states\(^3\) and may not admit a classical description. Even if they do admit a classical description they may involve regions of high curvature and hence be inherently stringy. For BPS black holes\(^3\), however, we may restrict to the BPS sector in the Hilbert space where the protected nature of the states suggests that they should persist as we tune continuous parameters (barring phase transitions or wall crossings). We may then hope to see a supergravity manifestation of these states, and indeed this turns out to be the case for systems with sufficient supersymmetry.

The large $N$ limit\(^4\) however, which must be taken for supergravity to be a valid description, bears many similarities with the $\hbar \to 0$ limit in quantum mechanics where we know that most states do not have a proper classical limit. Thus it is quite possible that only a vanishing fraction of the black hole states can be realized within supergravity and, indeed, we give evidence that this is the case for large (four supercharge) black holes in Chapter 6.

We would like to emphasize once more that this is not a problem insofar as the conjecture stated above is concerned. The focus on quantum states supported on supergravity con-

\(^2\)More generally objects in AdS with horizons, microscopic or macroscopic, are expected to have an associated entropy which should manifest itself in the dual CFT.

\(^3\)Here “BPS” can mean either 1/2, 1/4 or 1/8 BPS states or black holes in the full string theory. The degree to which states are protected depends on the amount of supersymmetry that they preserve and our general remarks should always be taken with this caveat.

\(^4\) $N$ measures the size of the system. For black holes it is usually related to mass in the bulk and conformal weight in the CFT.
configurations or modes is purely technically motivated as the latter are under some degree of control; conceptually there is no reason to believe or even hope that we can describe black holes by restricted our attention to supergravity modes. Nevertheless, as we will describe, this approach has been largely successful for very supersymmetric black holes (eight or more supercharges) so we will ultimately begin our study of less supersymmetric black holes this way.

Despite the potential problems and caveats mentioned above, recently, a very fruitful program has been undertaken to explore and classify the smooth supergravity duals of coherent CFT states in the black hole ensemble. Smoothness here is important because if these geometries exhibit singularities we expect these to either be resolved by string-scale effects, making them inaccessible in supergravity, or enclosed by a horizon implying that the geometry corresponds, not to a pure state, but rather an ensemble with some associated entropy.

Large classes of such smooth supergravity solutions, asymptotically indistinguishable from black hole solutions, have indeed been found \cite{29, 30, 31, 32, 33, 34, 35, 36} (and related \cite{37, 38} to previously known black hole composites \cite{39, 17, 40}). These are complete families of solutions preserving a certain amount of supersymmetry with fixed asymptotic charges\(^6\) and with no (or very mild) singularities.

In constructing such solutions it has often been possible to start with a suitable probe brane solution with the correct asymptotic charges in a flat background and to generate a supergravity solution by backreacting the probe \cite{29, 32, 41}. In a near-horizon limit these back-reacted probe solutions are asymptotically AdS, and by identifying the operator corresponding to the probe and the state it makes in the dual CFT, the backreacted solution can often be understood as the spacetime realization of a coherent state in the CFT. Lin, Lunin and Maldacena \cite{29} showed that the back-reaction of such branes (as well their transition to flux) was identified with a complete set of asymptotically AdS\(_5\) supergravity solutions (as described above) suggesting that the latter should be related to 1/2 BPS states of the original D3 probes generating the geometry. Indeed, in \cite{42, 43} it was shown that quantizing the space of such supergravity solutions as a classical phase space reproduces the spectrum of BPS operators in the dual \(\mathcal{N} = 4\) superconformal Yang-Mills (at \(N \rightarrow \infty\)).

In a different setting Lunin and Mathur \cite{31} were able to construct supergravity solutions

\(^5\)Throughout this thesis we will be discussing “microstates” of various objects in string theory but the objects will not necessarily be holes (i.e. spherical horizon topology) nor will they always have a macroscopic horizon. We will, none-the-less, somewhat carelessly continue to refer to these as “microstates” of a black hole for the sake of brevity.

\(^6\)The question of which asymptotic charges of the microstates should match those of the black hole is somewhat subtle and depends on which ensemble the black hole is in. In principle some of the asymptotic charges might be traded for their conjugate potentials. Moreover, the solutions will, in general, only have the same isometries asymptotically.
related to configurations of a D1-D5 brane in six dimensions (i.e. compactified on a $T^4$) by utilizing dualities that relate this system to an F1-P system (see also [30]). The latter system is nothing more than a BPS excitation of a fundamental string quantized in a flat background. The back reaction of this system can be parametrized by a profile $F^i(z)$ in $\mathbb{R}^4$ (the transverse directions). T-duality relates configurations of this system to that of the D1-D5 system.

Recall that the naive back-reaction of a bound state of D1-D5 branes is a singular or “small” black hole in five dimensions. These geometries have naked singularities though horizons are believed to form when $\alpha'$ corrections are incorporated. The geometries arising from the F1-P system, on the other hand, are smooth after dualizing back to the D1-D5 frame, though they have the same asymptotics as the naive solution [30]. Each F1-P curve thus defines a unique supergravity solution with the same asymptotics as the naive D1-D5 black hole but with different subleading structure. Smoothness of these geometries led Lunin and Mathur to propose that these solutions should be mapped to individual states of the D1-D5 CFT. The logic of this idea was that individual microstates do not carry any entropy, and hence should be represented in spacetime by configurations without horizons. Lunin and Mathur also conjectured that the naive black hole geometry is somehow a coarse graining over all these smooth solutions, i.e. that the black hole itself is simply an effective, coarse-grained description. In this context a lot of evidence was put forth to demonstrate that this indeed likely is the case and, due to the large amount of supersymmetry of the associated black holes, it seems that the black hole microstates can be realized directly within quantized supergravity. Although this initial success of the “fuzzball program” hinged upon using supergravity microstates it is not clear that extensions of the program to large black holes, with macroscopic horizon areas, will also be able to restrict purely to supergravity modes or if stringy modes will be essential (as argued in Chapter 6, current evidence seems to favour the latter).

2.2.2 ANSWERS TO SOME POTENTIAL OBJECTIONS

The idea that black holes are simply effective descriptions of underlying horizon-free objects is confusing because it runs counter to well-established intuition in effective field theory; most importantly the idea that near the horizon of a large black hole the curvatures are small and hence so are the effects of quantum gravity. Indeed, it is not easy to formulate a precisely stated conjecture for black holes with finite horizon area, although for extremal black holes with enough supersymmetry a substantial amount of evidence has accumulated for the correctness of the picture, as reviewed in e.g. [6]. To clarify some potential misconceptions, we transcribe below a FAQ from [6], addressing some typical objections and representing our current point of view. See also [26] [22] [44] [45].

1. How can a smooth geometry possibly correspond to a “microstate” of a black hole?
Smooth geometries do not exactly correspond to states. Rather, as classical solutions they define points in the phase space of a theory (since a coordinate and a momenta define a history and hence a solution; see section 2.3 for more details) which is isomorphic to the solution space. In combination with a symplectic form, the phase space defines the Hilbert space of the theory upon quantization. While it is not clear that direct phase space quantization is the correct way to quantize gravity in its entirety this procedure, when applied to the BPS sector of the theory, seems to yield meaningful results that are consistent with AdS/CFT (and passes other non-trivial consistency checks).

As always in quantum mechanics, it is not possible to write down a state that corresponds to a point in phase space. The best we can do is to construct a state which is localized in one unit of phase space volume near a point. We will refer to such states as coherent states. Very often (but not always, as we will see later in these notes) the limit in which supergravity becomes a good approximation corresponds exactly to the classical limit of this quantum mechanical system, and in this limit coherent states localize at a point in phase space. It is in this sense, and only in this sense, that smooth geometries can correspond to microstates. Clearly, coherent states are very special states, and a generic state will not admit a description in terms of a smooth geometry.

Let us note that, as the discussion above and throughout the thesis pertains to supergravity geometries, the relevant Hilbert space is an approximate factor, in the entire BPS Hilbert space of string theory, comprised of states that do not spread into the “stringy” directions of the full phase space of string theory.

2. How can a finite dimensional solution space provide an exponential number of states?

The number of states obtained by quantizing a given phase space is roughly given by the volume of the phase space as measured by the symplectic form $\omega$, $N \sim \int \omega^k / k!$ for a $2k$-dimensional phase space. Thus, all we need is an exponentially growing volume which is relatively easy to achieve.

3. Why do we expect to be able to account for the entropy of the black hole simply by studying smooth supergravity solutions?

Well, actually, we do not really expect this to be true. In cases with enough supersymmetry, one does recover all BPS states of the field theory by quantizing the space of smooth solutions, but there is no guarantee that the same will remain true for large black holes, and the available evidence does not support this point of view. We do however expect that by including stringy degrees of freedom we should be
able to accomplish this, in view of open/closed string duality.

4. If black hole “microstates” are stringy in nature then what is the content of the “fuzzball proposal”?

The content of the fuzzball proposal is that the closed string description of a generic microstate of a black hole, while possibly highly stringy and quantum in nature, has interesting structure that extends all the way to the horizon of the naive black hole solution, and is well approximated by the black hole geometry outside the horizon.

More precisely the naive black hole solution is argued to correspond to a thermodynamic ensemble of pure states. The generic constituent state will not have a good geometrical description in classical supergravity; it may be plagued by regions with string-scale curvature and may suffer large quantum fluctuations. These, however, are not restricted to the region near the singularity but extend all the way to the horizon of the naive geometry. This is important as it might shed light on information loss via Hawking radiation from the horizon as near horizon processes would now encode information about this state that, in principle, distinguish it from the ensemble average.

5. Why would we expect string-scale curvature or large quantum fluctuations away from the singularity of the naive black hole solution? Why would the classical equations of motion break down in this regime?

As mentioned in the answer to question 1, it is not always true that a solution to the classical equations of motion is well described by a coherent state, even in the supergravity limit. In particular there may be some regions of phase space where the density of states is too low to localize a coherent state at a particular point. Such a point, which can be mapped to a particular solution of the equations of motion, is not a good classical solution because the variance of any quantum state whose expectation values match the solution will necessarily be large.

Another way to understand this is to recall that the symplectic form effectively discretizes the phase space into $\hbar$-sized cells. In general all the points in a given cell correspond to classical solutions that are essentially indistinguishable from each other at large scales. It is possible, however, for a cell to contain solutions to the equation of motion that do differ from each other at very large scales. Since a quantum state can be localized at most to one such cell it is not possible to localize any state to a particular point within the cell. Only in the strict $\hbar \to 0$ limit will the cell size shrink to a single point suggesting there might be states corresponding to a given solution but this is just an artifact of the limit. A specific explicit example of such a scenario is discussed later in this thesis (in Chapter 5).
Thus, even though the black hole solution satisfies the classical equation of motion all the way to the singularity this does not necessarily imply that when quantum effects are taken into account that this solution will correspond to a good semi-classical state with very small (localized) quantum fluctuations.

6. So is a black hole a pure state or a thermal ensemble?

In a fundamental theory we expect to be able to describe a quantum system in terms of pure states. This applies to a black hole as well. At first glance, since the black hole carries an entropy, it should be associated to a thermal ensemble of microstates. But, as we know from statistical physics, the thermal ensemble can be regarded as a technique for approximating the physics of the generic microstate in the microcanonical ensemble with the same macroscopic charges. Thus, one should be able to speak of the black hole as a coarse grained effective description of a generic underlying microstate. Recall that a typical or generic state in an ensemble is very hard to distinguish from the ensemble average without doing impossibly precise microscopic measurements. The entropy of the black hole is then, as usual in thermodynamics, a measure of the ignorance of macroscopic observers about the nature of the microstate underlying the black hole.

7. What does an observer falling into a black hole see?

This is a difficult question which cannot be answered at present. The above picture of a black hole does suggest that the observer will gradually thermalize once the horizon has been passed, but the rate of thermalization remains to be computed. It would be interesting to do this and to compare it to recent suggestions that black holes are the most efficient scramblers in nature [46, 47, 48].

8. Does the fuzzball proposal follow from AdS/CFT?

As we have defined it the fuzzball proposal does not follow from AdS/CFT. The latter is obviously useful for many purposes. For example, given a state or density matrix, we can try to find a bulk description by first computing all one-point functions in the state, and by subsequently integrating the equations of motion subject to the boundary conditions imposed by the one-point function. If this bulk solution is unique and has a low variance (so that it represents a good saddle-point of the bulk path integral) then it is the right geometric dual description. In particular, this allows us to attempt to find geometries dual to superpositions of smooth geometries. What it does not do is provide a useful criterion for which states have good geometric dual descriptions; it is not clear that there is a basis of coherent states that all have decent dual geometric descriptions, and it is difficult to determine the way in which bulk descriptions of generic states differ from each other. In particular, it
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is difficult to show that generic microstates have non-trivial structure all the way up to the location of the horizon of the corresponding black hole.

9. To what degree does it make sense to consider quantizing a (sub)space of supergravity solutions?

In some instances a subspace of the solution space corresponds to a well defined symplectic manifold and is hence a phase space in its own right. Quantizing such a space defines a Hilbert space which sits (as a factor) in the larger Hilbert space of the full theory. Under some favourable circumstances the resulting Hilbert space may be physically relevant because a subspace of the total Hilbert space can be mapped to this smaller Hilbert space. That is, there is a one-to-one map between states in the Hilbert space generated by quantizing a submanifold of the phase space and states in the full Hilbert space whose support is localized on this submanifold.

For instance, in determining BPS states we can imagine imposing BPS constraints on the Hilbert space of the full theory, generated by quantizing the full solution space, and expect that the resulting states will be supported primarily on the locus of points that corresponds to the BPS phase space; that is, the subset of the solution space corresponding to classical BPS solutions. It is therefore possible to first restrict the phase space to this subspace and then quantize it in order to determine the BPS sector of the Hilbert space.

2.3 Phase Space Quantization

An important technical tool in this thesis will be the quantization of large classes of smooth BPS solutions. Quantum mechanics on the moduli space of BPS solutions is a familiar topic in string theory but the solution spaces we consider here are somewhat special as they sit within the full phase space of the theory as a symplectic submanifold (hence defining a phase space of their own) rather than as a configuration space. The latter is more familiar from standard BPS systems and leads to such notion as supersymmetric sigma models on the BPS solution space with BPS wavefunctions corresponding to the cohomology of the BPS manifold. The solution spaces in our examples are quite different, however, and require a different treatment. They are phase spaces and hence cannot be quantized via e.g. supersymmetric sigma models so we will resort to different techniques in order to understand the quantum structure of the theory. As this may be somewhat unfamiliar we devote this section to introducing the relevant notions.

The space of classical solutions of a theory is generally isomorphic to its classical phase space. Heuristically, this is because a given point in the phase space, comprised of a con-
configuration and associated momenta, can be translated into an entire history by integrating the equations of motion against this initial data; likewise, by fixing a spatial foliation, any solution can translated into a unique point in the phase space by extracting a configuration and momentum from the solution evaluated on a fixed spatial slice. This observation can be used to quantize the theory using a symplectic form, derived from the Lagrangian, on the space of solutions rather than on the phase space. This is an old idea (see also for an extensive list of references and for more recent work) which was used to quantize the LLM and Lunin-Mathur geometries. An important subtlety in these examples is that it is not the entire solution space which is being quantized but rather a subspace of the solutions with a certain amount of supersymmetry.

In general, quantizing a subspace of the phase space will not yield the correct physics as it is not clear that the resultant states do not couple to states coming from other sectors. It is not even clear that a given subspace will be a symplectic manifold with a non-degenerate symplectic pairing. As discussed in we expect the latter to be the case only if the subspace contains dynamics; for gravitational solutions we thus expect stationary solutions, for which the canonical momenta are not trivial, to possibly yield a non-degenerate phase space. This still does not address the issue of consistency as states in the Hilbert space derived by quantizing fluctuations along a constrained submanifold of the phase space might mix with modes transverse to the submanifold. When the submanifold corresponds to the space of BPS solutions one can argue, however, that this should not matter. The number of BPS states is invariant under continuous deformations that do not cross a wall of marginal stability or induce a phase transition. Thus if we can quantize the solutions in a regime where the interaction with transverse fluctuations is very weak then the energy eigenstates will be given by perturbations around the states on the BPS phase space, and, although these will change character as parameters are varied the resultant space should be isomorphic to the Hilbert space obtained by quantizing the BPS sector alone. If a wall of marginal stability is crossed states will disappear from the spectrum but there are tools that allow us to analyze this as it occurs (see section 3.1.6).

Let us emphasize that the validity of this decoupling argument depends on what questions one is asking. If we were interested in studying dynamics then we would have to worry about how modes on the BPS phase space interact with transverse modes. For the purpose of enumerating or determining general (static) properties of states, however, as we have argued, it should be safe to ignore these modes. For an example of the relation between states obtained by considering the BPS sector of the full (i.e. non-BPS) Hilbert space and those obtained by quantizing just the BPS sector the phase space compare the two center states determined in with those of section (see further discussion of this instance, in treating gravity, it is not clear if trajectories which eventually develop singularities should also be included as points in the phase space or only solutions which are eternally smooth. As we will primarily be concerned with static or stationary solutions in these notes we will largely avoid this issue.
As mentioned, the LLM and Lunin-Mathur geometries have already been quantized and the resultant states were matched with states in the dual CFTs. Our focus here will be on the quantization of \( \mathcal{N} = 2 \) solutions in four (or \( \mathcal{N} = 1 \) in five) dimensions. For such solutions, although a decoupling limit has been defined (see section 3.2), the dual \( \mathcal{N} = (0, 4) \) CFT is rather poorly understood. Thus quantization of the supergravity solutions may yield important insights into the structure of the CFT and will be important in studying the microstates of the corresponding extremal black objects.

2.4 AdS/CFT

One of the most powerful tools to study properties of black holes in string theory is the AdS/CFT correspondence [56]. This conjecture relates string theory on backgrounds of the form \( \text{AdS}_{p+1} \times \mathcal{M} \) to a CFT \( p \) that lives on the boundary of the \( \text{AdS}_{p+1} \) space. Such backgrounds arise from taking a particular decoupling limit of geometries describing black objects such as black holes, black strings, black tubes, etc. The limit amounts to decoupling the physics in the near horizon region\(^8\) of the black object from that of the asymptotically flat region by scaling the appropriate Planck length, \( l_p \), which decouples the asymptotic gravitons from the bulk (i.e. the near horizon region). At the same time the appropriate spatial coordinates are rescaled with powers of \( l_p \) to keep the energies of some excitations finite. This procedure should be equivalent to the field theory limit of the brane bound states generating the geometry under consideration.

We are interested in black objects which describe normalizable deformations in the \( \text{AdS}_{p+1} \) background. These correspond to a state/density matrix on the dual CFT according to the following dictionary

<table>
<thead>
<tr>
<th>BULK</th>
<th>BOUNDARY</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \exp (-S_{\text{on shell}}^{\text{bulk}}) )</td>
<td>( \text{Tr}[\rho \mathcal{O}_1 \ldots \mathcal{O}_n] = \langle \mathcal{O}_1 \ldots \mathcal{O}<em>n \rangle</em>{\rho} )</td>
</tr>
<tr>
<td>classical geometries</td>
<td>semiclassical states?</td>
</tr>
<tr>
<td>black hole</td>
<td>( \rho \sim \exp {- \sum_i \beta_i \mathcal{O}_i } )</td>
</tr>
<tr>
<td>entropy S</td>
<td>( S = - \text{Tr}(\rho \log \rho) )</td>
</tr>
<tr>
<td>bulk isometry D</td>
<td>( \left[ \rho, \hat{D} \right] = 0 )</td>
</tr>
<tr>
<td>ADM quantum numbers of D</td>
<td>( \text{Tr}(\rho \hat{D}) = \langle \hat{D} \rangle = D_{\text{ADM}} )</td>
</tr>
</tbody>
</table>

In the first line \( \mathcal{O}_i \) are operators dual to sources turned on in the boundary. They are included in the calculation of the on-shell bulk action, \( (-S_{\text{on shell}}^{\text{bulk}}) \). The second line can

\(^8\)In some of the cases treated in these notes the region will not be an actual near-horizon region as the original solutions may be horizon-free (or, in some cases, may have multiple horizons) but the decoupling limits are motivated by analogy with genuine black holes where the relevant region is the near horizon one.
be seen as the definition of the dual semiclassical state. More specifically, a semiclassical state is one that has an unambiguous dual bulk geometry (i.e. in the classical limit, $N \rightarrow \infty$ and $\hbar \rightarrow 0$, macroscopic observables take on a fixed expectation value with vanishing variance). In some ideal situations such semiclassical states turn out to be the analog of coherent states in the harmonic oscillator. In the third line, we describe a typical form of a density matrix that we expect to describe black holes. This form is motivated by the first law of thermodynamics: the entropy as defined in the fourth line obeys $dS = \sum_i \beta_i d\langle O_i \rangle$, and by matching this to the first law as derived from the bulk description of the black hole we can identify the relevant set of operators $O_i$ and potentials $\mu_i$ and guess the corresponding density matrix. The fourth line simply states that we expect a relation between the bulk and boundary entropies. In the fifth and the last line, $\hat{D}$ is the current/operator dual to the bulk isometry $D$.

Our use of AdS/CFT in this thesis will be quite standard and relatively basic. This is due, in part, due to the lack of control of the relevant $\mathcal{N} = (0, 4)$ CFT that we will encounter. Let us nonetheless be somewhat optimistic here and propose potentially interesting questions one might wish to explore in order to understand the spacetime structure of black hole microstates once greater control has been established over this CFT (something we hope to lay the groundwork for here).

One question relevant for understanding black holes via AdS/CFT is: “Given a density matrix $\rho$ on the CFT side, is there a dual geometry in the bulk?” On general grounds one could have expected that a general density matrix $\rho$ should be dual to a suitably weighted sum over geometries, each of which could be singular, have regions with high curvature, and perhaps not have good classical limits. As a result the dual gravitational description of a general density matrix will not generally be trustworthy. However, under suitable circumstances, it can happen that there is a dual “effective” geometry that describes the density matrix $\rho$ very well. This procedure of finding the effective geometry is what we will call “coarse graining”. In the gravity description, this amounts to neglecting the details that a classical observer cannot access anyway due to limitations associated to the resolution of their apparatus. So, one can phrase our question in the opposite direction, “What are the characteristics of a density matrix on the CFT side, so that there is a good dual effective geometry that describes the physics accurately?”.

One can try to construct the dual effective geometry following the usual AdS/CFT prescription. To do so, one should first calculate all the non-vanishing expectation values of all operators dual to supergravity modes (assuming one knows the detailed map between the two). On the CFT side, these VEVs are simply given by

$$\langle O_i \rangle = \text{Tr}(\rho O_i),$$

and they determine the boundary conditions for all the supergravity fields. The next step is to integrate the gravity equations of motion subject to these boundary conditions to
get the dual geometry. This is in principle what has to be done according to AdS/CFT prescription. The problem with this straightforward approach is that it is not terribly practical, and so alternative approaches have been sought out \[57, 58\].

\[\text{Though it would be interesting to study in some detail the connection between the two.}\]