Puzzles in quantum gravity: what can black hole microstates teach us about quantum gravity?
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Various arguments in the literature \cite{70,9,98,79} have suggested that scaling solutions carry vastly more entropy than their non-scaling cousins and may even account for a large fraction of the black hole entropy. This coincides nicely with the fact that scaling solutions are those that can mimic the black hole geometry to arbitrary accuracy. An immediate application of the technology developed in Chapters 3 and 4, relevant to the question of black hole entropy and information loss, is the determination of the entropy coming from scaling solutions.

In this chapter we compute the entropy of a large class of scaling solutions: the Dipole Halo configurations of Section 6.1 in both the scaling ($N \geq I/2$) and non-scaling regime ($N < I/2$). This is almost the most general class accessible using the tools we’ve developed so far. Unfortunately we will see that the resultant entropy is parametrically smaller than that of black hole with the same total charge.

One might imagine that there are much larger classes of scaling solutions, inaccessible using the technology developed here (or not yet even discovered), that would account for this discrepancy. However, as we will point out, the (leading) entropy coming from these solutions matches that of free gravitons in $\text{AdS}_3 \times \text{S}^2$. This suggests that the solutions we study here constitute the leading contribution to the black hole entropy from supergravity modes and, as a consequence, it is likely that generic black hole states will not be representable entirely in terms of supergravity modes.

While investigating this issue we will encounter some interesting surprises. The change in the leading degeneracy between the non-scaling and scaling regime seems to precisely
take into account the stringy exclusion principle [28], which for a chiral primary in the NS sector states that $\tilde{L}_0 \leq c/12$. Moreover there seems to be a phase transition in the restricted ensemble of BPS supergravity states at $N \approx I$ that is somewhat reminiscent of the phase transition of Section 5.3.1 of the fully, unrestricted ensemble of BPS states. Whether deep meaning is to be ascribed to this new phase transition and its mimicry of the full theory is not clear.

### 6.1 Counting Dipole Halo States

In Section 3.4.4 the “Dipole Halo” system, consisting of a D6-D6$^-$ pair orbited by a “Halo” of D0s, was introduced as well as a convenient coordinate system on the solution space of such configurations. In this section we will count states using techniques of geometric quantization of the supergravity solution spaces developed in Chapter 4; in the next section we will compare this to the calculation of free supergravity states on AdS$^3$ and see that the two results agree beautifully.

#### 6.1.1 Symplectic Form

Let us review the construction of the symplectic form on the solution space. Recall from Chapters 4 and 5 that once the symplectic form on the solution space (parameterized by the locations of centers satisfying (3.123)-(3.124)) has been found it can be used to quantize the system using methods of geometric quantization.

Using the explicit coordinatization in Section 3.4.4 and eqn. (4.9) the symplectic form turns out to be:

$$\Omega = -\frac{1}{4} d \left[ 2 j \cos \theta \, d\phi + 2 \sum_{a} q_{a} \cos \theta_{a} \, d\phi_{a} \right]$$

with $d$ denoting the exterior derivative.

The symplectic form (6.1) is non-degenerate on the BPS solution space parameterized by the locations of the centers implying that the latter is in fact a phase space. By virtue of arguments in Section 2.3 and Chapter 4, this space can be quantized in its own right, ignoring the much larger non-BPS solution space in which it is embedded, and from this treatment one might hope to extract information about the BPS states of the full theory (including at least the number of such states).

Note that, as is manifest from our angular coordinatization, the phase space is actually toric with a $U(1)^{n+1}$ action coming from $\phi$ and the $n \phi_a$'s. This is a consequence of the fact that the D0's are mutually non-interacting; their sole interaction is via the D6D6$^-$. As
previously mentioned this toric structure is a technical (but not conceptual) requirement for quantization using the methods of Chapter 4.

### 6.1.2 Physical Picture

As much of the subsequent presentation will be a rather technical treatment of the phase space we would like to lend the reader some intuition. We begin by recalling that the angular momentum carried by these solutions is

\[
\vec{J} = \sum_{i<j} \vec{J}_{ij}, \quad \vec{J}_{ij} := \frac{\langle \Gamma_i, \Gamma_j \rangle \vec{x}_{ij}}{2r_{ij}} \quad (6.2)
\]

where now \(i, j\) run over all centers, including the D6s. Thus each pair of centers contributes angular momentum \(\vec{J}_{ij}\) to the total. The length of these vectors is fixed to \(\langle \Gamma_i, \Gamma_j \rangle / 2\) but their direction is not fixed. The dependence on the intersection product \(\langle \Gamma_i, \Gamma_j \rangle\), pairing electric and magnetic sources, reflects the fact that this angular momentum is carried by the electromagnetic field and is due to crossed electric and magnetic fields. Since the D0’s have vanishing intersection product with each other there are only \((2n + 1)\) momenta vectors: \(\vec{J}_{6\bar{6}}, \vec{J}_{6a}, \text{ and } \vec{J}_{6\bar{a}}\).

As we will see, our quantization can essentially be understood as quantizing the direction of these vectors, or more precisely the size of their projection on a given “z-axis”, yielding familiar angular momentum multiplets. Naively the phase space of these angular momentum vectors is the direct product of \((2n + 1)\) two-spheres and the number of states is just the product of the factors \((2|\vec{J}_{ij}| + 1)\) from each multiplet. The geometric origin of the momenta (i.e. endpoint of multiple vectors fixed to be the same center), however, as well as the constraint equations (3.123)-(3.124) fix the possible relative orientations of the different angular momentum vectors. As a result not all states of the full free angular momentum multiplets are allowed. Rather, the correct phase space is now a more complicated fibration of spheres of varying size and, although intuitively it is still insightful to think of the states as part of “angular momentum multiplets”, they now only fill out a constrained subspace of the product of the full multiplets. For instance, since \(\vec{J}_{6a}\) and \(\vec{J}_{\bar{6}a}\) always end at the same point, their orientation relative to the \(w\)-axis is not independent so, rather than two angular momentum multiplets, these vectors yield only a single multiplet (the diagonal multiplet in their free product).

The best way to get some intuition for this is to consider the symplectic form on the phase space of our system. Using the coordinate system of Section 3.4.4 and introducing the notation

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we can cast the dipole halo symplectic form in a more suggestive form

$$\Omega = -\frac{1}{2} \left[ dJ_z \wedge d\phi + \sum_a dJ^a_w \wedge d\phi_a + \sum_a dJ^\pi_w \wedge d\phi_\pi \right].$$

(6.6)

with $|\vec{J}^a| = |\vec{J}\pi| = q_a/2$. Because $\vec{J}^a$ and $\vec{J}\pi$ are related by the location of the D0 they end on the last two terms above can be combined yielding

$$\Omega = -\frac{1}{2} \left[ dJ_z \wedge d\phi + 2 \sum_a dJ^a_w \wedge d\phi_a \right].$$

(6.7)

If there were no other constraints, the $J^a_w$ would independently be able to take values between $\pm |\vec{J}^a|$. However, as we will discuss below, they have to satisfy the bounds $J^a_w > 0$ and $2 \sum_a J^a_w \leq I/2$, leading to a more intricate phase space with a Hilbert space that is no longer a product of “free” angular momentum multiplets. There is also another angular momentum multiplet, coming from the total angular momentum $\vec{J}$, and this gives rise to a full multiplet with $-|J| < J_z < |J|$. The size of $\vec{J}$, however, depends on the $J^a$ (even classically). Thus, each state in the Hilbert state labelled by $J^a$ quantum numbers will be tensored with a $J$ multiplet corresponding to the total $\vec{J}$ associated to its $J^a$ quantum numbers (via $J = I/2 - 2 \sum_a J^a_w$).

This is the intuitive physical picture for which we develop a precise mathematical treatment in the next subsection, using the observation above that the phase space is a toric manifold. The upshot is, however, that we are doing nothing more than quantizing angular momentum variables, but ones that are non-trivially connected and constrained.

There is a second physical phenomena that only appears when quantizing the system, which we would like to highlight here. As stressed above, we are in essence quantizing the classical angular momentum of the system. However, when we quantize we need to take into account the intrinsic spin of the particles involved, as was beautifully explained in [17]. As pointed out there, the centers are superparticles containing excitations in various spin states. Due to the presence of magnetic fields, however, the lowest energy BPS state is a spin half state, where energy is gained by aligning the intrinsic magnetic dipole moment with the magnetic field. The situation is sketched for our dipole halo system in figure 6.1. Including these quantum corrections the size of the total angular momentum is given by

$$J = \frac{I - 1}{2} - \sum_a \left( q_a \cos \theta_a + \frac{1}{2} \right).$$

(6.8)
These spins are especially important when considering classical scaling solutions.

Figure 6.1: In this figure all contributions to the total angular momentum are shown. The large arrows denote the classical angular momenta carried by the electromagnetic field. They are proportional to the intersection products of the charges, D6 (red), D6 (blue) and D0 (green). The small arrows extending from the D0’s represent their spin aligning with the dipole magnetic field sourced by the D6D6-pair, the small arrow in the bottom is the spin of the center of mass multiplet of the D6D6-pair aligning with the magnetic fields.

6.1.3 States and Polytopes

In Section 3.4.4 we showed that for the D6D6D0 system the solution space is a toric manifold. This allows us, in principle, to construct all the normalizable quantum states explicitly. Here, however, we will be less interested in the explicit form of the wavefunctions than in their number. In appendix G we show how the number of states can be easily obtained from the combinatorics of the toric polytope. For more information on the technology of geometric quantization of toric manifolds we refer the reader to Appendices E and G.

From the symplectic form (6.1) we read off the coordinates on the polytope

\[ y = j \cos \theta, \quad y_a = q_a \cos \theta_a \geq 0 \]  

(6.9)

So we see that the polytope is bounded by the inequalities

\[ -j \leq y \leq j, \quad 0 \leq y_a \leq q_a \]  

(6.10)

and furthermore the requirement that the angular momentum is positive

\[ j = \frac{I}{2} - \sum_a y_a \geq 0. \]  

(6.11)
It is this last condition that differentiates the non-scaling regime $N = \sum_a q_a < I/2$ from the scaling regime $N \geq I/2$. In the former range the condition (6.11) is redundant in the definition of the polytope as it is automatically satisfied for all values of $x_a$ allowed by the other constraints (6.10). In case $N > I/2$ the constraint (6.11) actually becomes essential and can make some of the constraints (6.10) redundant, although this depends on the specific values of the $q_a$. What is shared by all the solution spaces in the scaling case is that it is possible to approach the point where all centers coincide arbitrarily closely, which automatically implies that $j$ has to approach zero. When this happens, an infinitely deep scaling throat forms in space-time [17, 70]. For more than a single D0 center there are however different types of solution spaces with a scaling point, depending on the specific values of the charges $q_a$. We show all the different possible polytope topologies for the case with two D0 centers in figure 6.2, clearly the number of topologies grows very fast with the number of D0-centers.

Given the defining inequalities (6.10) and (6.11) we can use eqn. (G.6) from the appendix (see also the example containing eqn. (G.2)) to see that there is a unique quantum state corresponding to each set of integers $(m, m_a)$ satisfying

$$0 \leq m^a \leq q_a - 1, \quad \sum_a (m^a + \frac{1}{2}) \leq \frac{I - 1}{2}, \quad (6.12)$$

$$- \left[ \frac{I - 1}{2} - \sum_a \left( m^a + \frac{1}{2} \right) \right] \leq m + \frac{1}{2} \leq \left[ \frac{I - 1}{2} - \sum_a \left( m^a + \frac{1}{2} \right) \right] \quad (6.13)$$

The $(m, m_a)$ above are simply quantized angular momenta corresponding to quantizing the angles $(\theta, \theta_a)$ appearing in (6.10)-(6.11). The half-integral shifts are related to the fermionic nature of the centers as discussed in Section 4.3.4 and the coupling to the extrinsic spin, as explained at the end of Section 6.1.2.

To be precise the constraints above only hold under the assumption that all D0 centers carry different charges, $q_a$. To relax this assumption we introduce integer multiplicities, $n_a$, for each charge $q_a$ so that $N = \sum_a n_a q_a$ and $n = \sum_a n_a$. We now have to take into account the quantum indistinguishability of these (fermionic) particles. This translates to taking the appropriate orbifold of the polytope (see [4] for a detailed explanation) or, in terms of (6.13), augmenting the $m_a$ by an additional label $i_a$ running from 1, \ldots, $n_a$ and requiring them to satisfy

$$0 \leq m_1^a < m_2^a < \ldots < m_{n_a}^a < q_a , \quad \sum_{a,i} \left( m_{i_a}^a + \frac{1}{2} \right) \leq \frac{I - 1}{2} \quad (6.14)$$

$$- \left[ \frac{I - 1}{2} - \sum_{a,i} \left( m_{i_a}^a + \frac{1}{2} \right) \right] \leq m + \frac{1}{2} \leq \left[ \frac{I - 1}{2} - \sum_{a,i} \left( m_{i_a}^a + \frac{1}{2} \right) \right] \quad (6.15)$$

These constraints are fermionic, enforcing Pauli exclusion of indistinguishable centers. Note also that they reduce to (6.13) if all the $n_a = 1$. 

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Figure 6.2: These are the different types of polytopes corresponding to a $D6\overline{D6}D0$ system with 4 centers. On the left the ‘base’ polytope determined by the coordinates $y_1$ and $y_2$ with $y = 0$ is shown, on the right also the fiber spanned by the coordinate $y$ is included, the edges along this direction are drawn in red, the surface $y = 0$ is shown in blue. See (6.9) for a definition of the coordinates. The different cases correspond respectively to: Case A (non-scaling) $q_1 + q_2 < \frac{I}{2}$, Case B (scaling) $q_1 + q_2 \geq \frac{I}{2}$ and $q_1, q_2 \leq \frac{I}{2}$, Case C (scaling) $q_2 \leq \frac{I}{2} \leq q_1$, and Case D (scaling) $\frac{I}{2} \leq q_1, q_2$. So we see that from 4 centers onward there are different types of scaling polytopes, a feature that was absent for three scaling centers.


6.1.4 **The D6D6D0 Partition Function**

In this section we will count the combined number of supergravity states \( d_N \) of all D6D6D0 systems with total charge \((0, p^A, 0, \frac{p^3}{24} - N)\). More precisely we will calculate the leading term of \( S(N) = \log d_N \) in a large \( N \) expansion. We will notice there are two phases depending on the relative value of \( I = p^3/6 \) and \( N \), separated by a transition at \( N = I \). For larger \( N \) the appearance of scaling solutions slightly complicates the counting but can still be performed as shown below. What is interesting is that the existence of scaling solutions seems only to become dominant at \( N = I \) where a phase transition occurs.

As we previously pointed out, in the scaling regime there is an additional constraint that complicates the polytope and makes the counting of integer points inside slightly more difficult. We will find it convenient not to calculate a fully explicit generating function \( Z \) but rather, since we are only interested in the large \( N \) regime, it will be sufficient for us to find the leading term of \( \log Z \) in a large \( N \) expansion.

The complication in the scaling regime arises because of the second constraint in equation (6.14). To proceed let us introduce the quantity

\[
M = \sum_{a,i} \left( m_{ia}^a + \frac{1}{2} \right), \tag{6.16}
\]

As the \( m_{ia}^a \) are the discrete analogues of the classical \( q_a \cos \theta_a \) the interpretation of \( M \) is as the amount of angular momentum carried by the D0 centers (which, by the integrability constraints (3.123)-(3.124), is always opposite in direction to the angular momentum carried by the D6D6 pair):

\[
M = \frac{I}{2} - \frac{1}{2} - J. \tag{6.17}
\]

Both the \( \frac{1}{2} \) in the above formula and in (6.16) arise due to the spin contributions to the quantum mechanical angular momentum (see the end of section 6.1.2).

Now if we succeed in calculating the degeneracy \( d_{N,M} \) as a function of \( M \), then the full degeneracy will be

\[
d_N = \sum_{M = 1/2}^{(I - 1)/2} d_{N,M} \tag{6.18}
\]

The full degeneracy will clearly be less than \( I/2 \) times \( d_{N,M'} \) where \( M' \) is the value of \( M \) which maximizes \( d_{N,M} \). Thus instead of calculating the sum it will be sufficient for us to find the \( M' \) that maximizes \( d_{N,M} \) because

\[
S(N) = S(N, M') + \Delta S, \tag{6.19}
\]

where we have defined

\[
\Delta S = \log \sum_{M = 1/2}^{(I - 1)/2} e^{S(N,M)} - S(N, M') \leq \log I, \tag{6.20}
\]
so, as long as the leading entropy is a power-law (rather than a logarithm) in the charges, we can find the leading term in $S(N)$ by calculating $S(N, M)$ and maximizing over $M$.

As we will now show it is not too hard to calculate a generating function for $d_{N,M}$

$$Z(q, y) = \sum_{N,M} d_{N,M} q^N y^M .$$  \hfill (6.21)

Note that this does not reduce to a generating function for $d_N$ by setting $y = 1$ as in this generating function we sum over $M = 1, \ldots, \infty$ while in the case of interest the range of $M$ is restricted.

Let us derive an expression for (6.21) by approximating it in a few steps. A first key ingredient is that for a partition of $N = \sum_n n_a q_a$, one has

$$0 \leq m_1^a < \ldots < m_{n_a}^a < q_a, \quad M = \sum_{a,i} \left( m_i^a + \frac{1}{2} \right)$$  \hfill (6.22)

Forgetting for the moment about $N$, the above relation is just a fermionic partition of $M$. This is given by

$$Z_{\text{ferm}} = \prod_{l \geq 1} \left( 1 + y^{l-1/2} \right)$$  \hfill (6.23)

We need to reintroduce the information about $N$. To do so remember that the sole role of the partition of $N$ is to specify the number of $m_i^a$ above. Keeping this key point in mind we proceed in two steps. First assume that we have $n$ centers with the same charge $k$ only ($N = nk$), then it is easy to see that the appropriate modification of $Z_{\text{ferm}}$ (6.23) is

$$Z_{\text{int}} = \prod_{1 \leq l \leq k} \left( 1 + q^k y^{l-1/2} \right)$$  \hfill (6.24)

This comes about because in expanding the expression above the number of centers in each term is simply the number of $q^k$ that appear in it. The product over possible $l$ is then a reflection of the constraint (6.22). Now to generalize to an arbitrary partition of $N$ we take a product of the above expression over all possible $k \geq 1$. This yields the core generating function

$$Z_0 = \prod_{k \geq 1, 1 \leq l \leq k} \left( 1 + q^k y^{l-1/2} \right)$$  \hfill (6.25)

To get the actual generating function we include the contribution from $m$ in equation (6.15). The generating function is then

$$Z = (I - 2y \partial_y) Z_0 = (I - 2y \partial_y) \prod_{k \geq 1, 1 \leq l \leq k} \left( 1 + q^k y^{l-1/2} \right)$$  \hfill (6.26)

In evaluating the leading contribution to the entropy we can neglect the overall multiplicative factor because it will be subleading. Thus we focus on $Z_0$.  

6.1.5 The Entropy and a Phase Transition

As is familiar from thermodynamics we can study the large energy regime by evaluating the partition function at large temperature. We introduce the potentials \( \beta \) and \( \mu \) through

\[
q = e^{-\beta}, \quad y = e^{-\mu}
\]

and can then look for the behavior of the entropy for \( \beta, \mu \ll 1 \).

\[
\log Z_0 = \sum_{k \geq 1, 1 \leq l \leq k} \log \left( 1 + q^k y^{l-1/2} \right)
\]

\[
= \sum_{n \geq 1} \left( \frac{(-1)^{n+1}}{n} \left[ \sum_{k \geq 1} q^{nk} \left( \sum_{l=1}^{k} y^{n(l-1/2)} \right) \right] \right)
\]

\[
= \sum_{n \geq 1} \left( \frac{(-1)^{n+1}}{n} \frac{y^{n/2}}{1 - y^n} \left[ \sum_{k \geq 1} q^{nk} (1 - y^{nk}) \right] \right)
\]

\[
= \sum_{n \geq 1} \left( \frac{(-1)^{n+1}}{n} \frac{q^n y^{n/2}}{(1 - q^n) (1 - q^n y^n)} \right)
\]

\[
\sim \left( \sum_{n > 1} \frac{(-1)^{n+1}}{n^3} \right) \frac{1}{\beta (\mu + \beta)} =: \frac{\alpha}{\beta (\mu + \beta)}
\]

(6.27)

with \( \alpha = \frac{3}{4} \zeta(3) \). Using the above relation we find

\[
N = -\partial_\beta \log Z_0 \sim \frac{\alpha (\mu + 2\beta)}{\beta^2 (\mu + \beta)^2} \quad (6.28)
\]

\[
M = -\partial_\mu \log Z_0 \sim \frac{\alpha}{\beta (\mu + \beta)^2} \quad (6.29)
\]

From the equations above it follows that the approximation is valid for \( N, M \gg 1 \), which is exactly the regime we are interested in. Furthermore the relative size between \( M \) and \( N \) is determined by the ratio \( \mu/\beta \) as

\[
N/M = 2 + \frac{\mu}{\beta} \quad (6.30)
\]

The entropy in the large \( M, N \) regime then reads

\[
S(N, M) = -\log Z_0 + \beta N + \mu M \sim \frac{\alpha}{\beta (\mu + \beta)^2} \sim (\alpha M [N - M])^{1/3} \quad (6.31)
\]

Maximizing \( S(N, M) \) over \( M \) in the range \( 1/2 < M < I/2 \) we find that

\[
S(N) = \begin{cases} 
\left( \frac{\alpha N^2}{4} \right)^{1/3} & \text{if } N \leq I \\
\left( \frac{\alpha}{2} \left( N - \frac{I}{2} \right) \right)^{1/3} & \text{if } I \leq N
\end{cases} \quad (6.32)
\]

\[1\] Note that we are interested in the large charge limit \( I \gg 1 \), so throughout the paper we will often neglect quantum mechanical shifts of \( 1/2 \) to \( I \).
Chapter 6 - Spectrum and Phase Transitions

The most entropic configuration always has $M' = N/2$ until $N = I$ and then the bound \text{[6.11]} restricts $M' = I/2$. Thus most entropy is realized by low angular momentum states (remember $J \sim I/2 - M$) and, deep in the scaling regime where $N > I$, most of the entropy is given by the $j = 0$ states.

The saddle point approximation used to obtain eqn. \text{[6.32]} is only valid for charges $N \lesssim I^2$ because the discussion above shows we are interested in $M \approx I^2$ and in that regime $N \gtrsim I^2$ is not consistent with $\mu, \beta \ll 1$, as can be seen from \text{[6.28]}-\text{[6.29]}. We will presently focus on the regime $N \gg I$ which is still consistent (in the large charge regime) so long as their ratio does not scale with $I$.

For $N \gg I$ Cardy’s formula implies the leading entropy of the associated black hole grows as \text{[27]}

$$S_{BH}(N, I) \sim 4\pi \sqrt{NI}$$

Thus the D6D6D0 configurations we are considering do not exhibit the correct growth of entropy as a function of the charges to dominate the black hole ensemble, especially for large charges they are parametrically subleading.

Associated with the change from the first to the second line of \text{[6.32]} appears to be a second order phase transition occurring at $N = I$. In this phase transition we seem to move from an asymmetric phase, $\langle j \rangle \neq 0$, to a symmetric phase $\langle j \rangle = 0$. It is not immediately clear that any physical meaning should be ascribed to this “phase transition” since these configurations are not the dominant constituents of this sector of the BPS Hilbert space. Curiously, however, this seems to mirror the phase transition of Section 5.3. Although the latter was analyzed for different constituents centers, if we simply equate the total charges of the two systems then the critical point of Section 5.3 would be at $N \approx I/4$ and would correspond to a (first-order) transition from a phase with $\langle j \rangle \neq 0$ to a $\langle j \rangle = 0$ phase as $N$ increases (note that here there is a discontinuous jump in $\langle j \rangle$).

It is both curious and interesting that the set of states we obtained, while relatively sparse in the overall Hilbert space, nonetheless exhibits a non-trivial phase structure that even seems to qualitatively share some of the structure of the full theory.

In the regime $N \gg I$ of \text{[27]}, the number of states we obtained was substantially smaller than total number of BPS states of the conformal field theory. One may therefore wonder whether other solutions of supergravity exist with the same asymptotic charges and which could account for the missing states, or whether this is the best supergravity can do. Such additional solutions could look like complicated multi-centered solutions of the type we have been considering, or be of an entirely different form. To address this question we will now compute the spectrum and degeneracy of a gas of free supergravitons in $\text{AdS}_3 \times \text{S}^2$.

As we will argue, this will provide an estimate for the maximal number of states we might expect to be obtainable from supergravity. It turns out that this computation yields a result whose asymptotic expansion agrees precisely with the number of D6D6D0 states, which
supports the claim that the supergravity does not give rise to significantly more states in addition to those that we described.

### 6.2 Free Supergravity Estimate

In the previous section we calculated the number of BPS states in a given D4D0-charge sector that can be associated to configurational degrees of freedom of a D6D6D0 system of that same total charge. As we pointed out, there is an exponential number of states leading to a macroscopic statistical entropy. However the entropy scales with a different power of the charges than the D4D0 black hole entropy, making it parametrically subleading in the large charge supergravity limit. In other words, although we found very many D6D6D0 states the corresponding single center black hole still has exponentially more of them, indicating that these are not generic states of the black hole.

One might still wonder, however, if this is due to our restriction to a specific set of smooth multicenter solutions and if perhaps a larger number of states can be found by quantizing more complicated multicentered configurations. In this section we will give some non-trivial evidence that this is not the case and that the black hole degrees of freedom have to be sought outside of supergravity. An example of such states could be those of the proposal [123, 124, 125] or the possibly related setup of [126, 127]. Roughly speaking the degrees of freedom in these pictures seem to reside in non-abelian D-brane degrees of freedom; see also [35].

The approach we take to get a “bound” on the degrees of freedom coming from supergravity states is to exploit the fact that both the D4D0 black hole and the D6D6D0 system (and its generalizations) can be studied in asymptotically AdS space via the decoupling limit of Section 3.2. In this context, the counting of the previous section corresponds to counting backreacted supergravity solutions with the same asymptotics as the D4D0 BTZ black hole, whereas in this section we will simply count free supergravity modes in empty AdS. The advantage of working in this limit, where the supergravity fields become free excitations around a fixed AdS$_3 \times S^2 \times$CY$_3$ background, is that it becomes relatively easy to count them. Free supergravitons organize themselves in representations of the (0,4) superconformal isometry algebra, and we merely need to determine the quantum numbers of the highest weights of the representations. This can be done following e.g. [128, 129] by performing a KK-reduction of eleven dimensional supergravity fields on the compact S$^2 \times$CY$_3$ space to fields living on AdS$_3$. The supergravity spectrum can then be determined using pure representation theoretic methods, in terms of the massless field content

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Note that we will assume the size of the CY$_3$ to be much smaller than that of the S$^2$ so that we will only consider the massless spectrum on the CY, while keeping track of the full tower of massive harmonic modes on the sphere.
of the KK reduction of M-theory on the Calabi-Yau manifold.

6.2.1 Superconformal Quantum Numbers

We want to compare the number of states we found by counting the possible configura-
tional degrees of freedom of a D6D6D0 system to the number of chiral primaries given by
KK reduction of 5d supergravity in the free field limit. To make this comparison as clear
as possible let us first translate the conserved four dimensional charges of the solutions,
as presented in the previous section, to quantum numbers under the (0,4) superconformal
isometry algebra of the AdS3×S2 background we consider here. Such a dictionary was
derived in Section 3.2.4 and can be straightforwardly applied to the D6D6D0 case. The
map from supergravity to CFT quantum numbers is (recall that \( c = 6I \))

\[
\begin{align*}
L_0 &= N, \\
\tilde{L}_0 &= \frac{I}{4}, \\
J_3 &= -J.
\end{align*}
\]  

(6.34)

States with these quantum numbers are Ramond ground states, with minimum eigenvalues
under \( \tilde{L}_0 \), as expected for BPS states. The calculation of the KK-spectrum on AdS3,
however, is most naturally phrased in the NS sector and thus we would like to work in
this sector. Thus we relate the charges (6.34) by spectral flow \([130]\) in the right moving
sector to the charges of the corresponding states in the NS-sector. Performing the spectral
flow explicitly (as in eqn. (3.109)) we find

\[
\begin{align*}
L_0 &= N, \\
\tilde{L}_0 &= \frac{I}{2} - J, \\
J_3 &= \frac{I}{2} - J.
\end{align*}
\]  

(6.35)

As expected the BPS states manifest themselves in the NS sector as chiral primaries,
satisfying the condition \( \tilde{L}_0 = J_3 \). The well known unitarity bound \([130]\) on the R-charge
of chiral primaries implies a bound on the range of the 4d angular momentum:

\[
0 \leq J \leq \frac{I}{2}
\]  

(6.36)

From the results of the previous section it is clear that the D6D6D0 configurations sat-
ify this bound. This bound was first observed to have consequences for AdS3/CFT2 in
\([28]\), where it was called a stringy exclusion principle. As was argued there, it has to be
imposed by hand on the free supergravity principle. As was argued there, it has to be
imposed by hand on the free supergravity spectrum. What is perhaps surprising is that
in the fully interacting supergravity theory the bound seems to emerge dynamically as it
follows (at least for the D6D6D0 system) from the integrability equations (3.123) which
are essentially a consequence of the BPS equations of motion. We have no solid proof
of this, but we were unable to find other multicentered supergravity configurations that
violate the bound, even with flat space asymptotics where there is no direct connection to
the exclusion principle in the CFT.
It is interesting to note that by (6.17) and (6.35) we see that for the D6D6D0 system \( \tilde{L}_0 = M + 1/2 \) and that indeed also the bound on \( M \), observed in the previous section, follows directly from the unitarity bound discussed above. Using the identification of \( M \) and \( \tilde{L}_0 \), we can write the following analogue of the generating function (6.21):

\[
Z = \text{Tr}_{\text{NS,BPS}} (-1)^F q^{\tilde{L}_0} y^{\tilde{L}_0 - 1/2}.
\]

(6.37)

Some remarks are in order. First we would like to point out that, for computational simplicity, we will calculate, in this section, an index rather than an absolute number of states, the difference with (6.21) being an explicit insertion of \((-1)^F\). As one can see explicitly from the derivation below, the difference between the absolute number of states and the index will be only affect the numerical coefficient of the entropy, not its functional dependence on the charges. Second, note that at \( y = 0 \) the above index coincides with the standard elliptic genus for this theory.

### 6.2.2 THE SPECTRUM OF BPS STATES

To calculate the degeneracies we are interested in, we need to enumerate the possible BPS states of linearized (free) supergravity on AdS\(_3 \times \)S\(_2\). It is often easier to enumerate these states via their quantum numbers in the CFT so we will use this language.

As we only have supersymmetry in the right moving sector, there are no BPS constraints on the left moving fields and thus all descendants of highest weight states will appear. The right-moving sector has \( N = 4 \) supersymmetry and BPS states must be chiral primaries of a given weight. As a consequence, and as was shown in detail in e.g \([128, 129, 131, 132]\), the full BPS spectrum can be written in the form:

\[
\{s, \tilde{h}\} = \oplus_{n \geq 0} (L_{-1})^n |\tilde{h} + s\rangle_L \otimes |\tilde{h}\rangle_R
\]

(6.38)

where \(|h\rangle_L \) are highest weight states of weight \( h \) of the left-moving Virasoro algebra and \(|\tilde{h}\rangle_R \) are weight \( \tilde{h} \) chiral primaries of the right-moving \( \mathcal{N} = 4 \) super-Virasoro algebra.

Each field of five dimensional supergravity gives rise to a set of BPS states and their descendants after KK-reduction, where \( \tilde{h} \) essentially labels the different spherical harmonics, while \( n \) labels momentum excitations in AdS\(_3 \) and \( s \) the spin of the particle. It was shown in \([128, 129, 131, 132]\) that, given the precise field content of 5d \( \mathcal{N} = 1 \) supergravity, the reduction on a 2-sphere gives the set of quantum numbers shown in table 6.1.

Notice that the quantum numbers \( \{s, \tilde{h}\} \) are of the form \( \{s, h_{\text{min}} + m\} \), and for each

\[3\]Furthermore, to be fully precise we should point out that there remain so called singleton representations, but, for our purposes, we can ignore them as one can show they only contribute to subleading terms in the entropy in the regime studied in the last section: \( N, I \gg 1 \) and \( \mathcal{N} \ll I^2 \).
Chapter 6 - Spectrum and Phase Transitions

5d origin | number | \(\{s, \tilde{h}\}\)-towers
---|---|---
hypermultiplets | \(2h^{1,2} + 2\) | \(\{\frac{1}{2}, \frac{1}{2} + m\}\)
vectormultiplets | \(h^{1,1} - 1\) | \(\{0, 1 + m\}\) and \(\{1, m\}\)
gravitymultiplet | 1 | \(\{-1, 2 + m\}\), \(\{0, 2 + m\}\), \(\{1, 1 + m\}\) and \(\{2, 1 + m\}\)

| Table 6.1: Summary of the spectrum of chiral primaries on \(AdS_3\). The states are organized in towers of the form (6.38), the number of such towers and their characteristics are determined by the properties of the original theory and the details of the reduction. In the above table, \(m\) is an arbitrary nonnegative integer.

such set the partition function (6.37) has the following form

\[
Z_{\{s, \tilde{h}_{\min}\}} = \prod_{n \geq 0} \prod_{m \geq 0} (1 - y^{m+\tilde{h}_{\min}-1/2} q^n+m+\tilde{h}_{\min}+s)(-1)^{2n+1} \tag{6.39}
\]

with the total partition function given by a product of such factors.

To extract the large \(N\) degeneracies we proceed as in (6.27) and calculate the free energy corresponding to this partition function. We then evaluate it in the \(\beta, \mu \ll 1\) limit \((q = e^{-\beta}, y = e^{-\mu})\):

\[
F_{\{s, \tilde{h}_{\min}\}} = (-1)^{2s} \sum_{n \geq 1} \frac{q^{n(\tilde{h}_{\min}+s)} y^{n\tilde{h}_{\min}}}{n(1-q^n)(1-y^n q^n)} \tag{6.40}
\]

\[
\approx \frac{(-1)^{2s} \zeta(3)}{\beta(\beta + \mu)} \tag{6.41}
\]

Note that, as might have been expected, at high temperatures only the statistics of the particles matter, as \(\tilde{h}_{\min}\) and \(s\) only change the lowest states of the towers. The total free energy is now the sum over all different towers. Using table [6.1] we find that

\[
F \approx \left[ -(2h^{1,2} + 2) + 2(h^{1,1} - 1) + 4 \right] \frac{\zeta(3)}{\beta(\beta + \mu)} = \chi \frac{\zeta(3)}{\beta(\beta + \mu)} \tag{6.42}
\]

where we used the definition of the Euler characteristic \(\chi\) of the CY\(_3\). Finally we can do a Legendre transform to obtain the entropy. This is completely analogous to (6.31) and the result is

\[
S \approx (\chi \zeta(3) M(N - M))^{1/3} \tag{6.43}
\]

This result is equivalent to (6.31) and maximization with respect to \(M\) proceeds analogously, again leading to the result

\[
S(N) = \begin{cases} 
\left(\frac{\chi \zeta(3) N^2}{4}\right)^{1/3} & \text{if } N \leq I \\
\left(\frac{\chi \zeta(3)}{2} (N - I)\right)^{1/3} & \text{if } I \leq N
\end{cases} \tag{6.44}
\]
This might look somewhat unfamiliar when compared with other calculations of the elliptic genus, e.g \cite{15,76}. This is because those calculations were all performed in the regime $N \ll I$ where the unitarity bound on the spectrum can be ignored. It is exactly around $N \approx I$ that this bound starts to be relevant leading to a different, slower, growth of the number of states in the regime $I \ll N$. Such a behavior was also seen in the computation of the elliptic genus in \cite{133}.

Note that once more our computation above only applies for $N \lesssim I^2$ as the asymptotic form of the free energy is essentially the same as that of the dipole halo system.

### 6.2.3 Comparison to Black Hole Entropy

As we have seen, calculating the number of free supergravity states at fixed total charge in the large $N, M$ limit proceeds rather analogously to the counting of section 6.1.5 and, more importantly, we found a precise match between the leading contributions, up to an overall prefactor.

It is not hard, however, to see that even this prefactor can be made to match. In the previous subsection we focussed on the 4 dimensional degrees of freedom of the D6D6D0-system ignoring the fact that the D0-branes bound to the D6D6 still have degrees of freedom in the internal CY$_3$ manifold. These internal degrees of freedom can be quantized via a 0+1 dimensional sigma model\footnote{In this simplistic model we neglect more complicated interactions coming from strings stretched between the D0’s and the D6’s in the CY.} on the CY. The BPS states of this sigma model correspond to the cohomology of the Calabi-Yau with even degree mapping to bosonic states and odd degree to fermionic states. Thus there are exactly $\chi$ BPS states per D0 when counted with the correct sign, $(-1)^F$. Including this extra degeneracy in the calculation of section 6.1.5 will lead to a match with (6.44), including the prefactor.

That the two calculations provide the same amount of states is non-trivial, since earlier we restricted ourselves to counting only states realized as a D6D6D0 system, while in the second calculation we count all free supergravity states in AdS$_3 \times$ S$^2$ with given momentum. This suggests that indeed the leading portion of supergravity entropy is realized as D6D6D0 configurations once backreaction is included. This is a very strong result as clearly one can think of many, much more complicated, smooth multicenter configurations with the same total charge. Furthermore, we learn from these calculations that the number of such states is parametrically smaller than the number of black hole states. This seems to strongly indicate that the generic black hole state is associated to degrees of freedom beyond supergravity.

From another perspective, however, the match between the free regime and the D6D6D0 entropy is not so surprising. If we consider first a D6D6D0 bound state we can use a coor-
coordinate transformation from \[79\] (see also Section 3.3.2) to map this to global AdS\(_3\times\)S\(_2\).

Thus we can think of the D6\(\overline{D}6\) as simply generating the empty AdS background. Recalling that D0 branes lift to gravitational shock waves in 5-dimensions one might already have anticipated that counting D0’s in the D6\(\overline{D}6\) background is closely related to counting free gravitons on an AdS\(_3\times\)S\(_2\) background. What makes the result non-trivial is that interactions are apparently not terribly relevant when counting BPS states, but then again the D0’s only interact very indirectly with each other. We might, therefore, wonder if more exotic configurations, such as the supereggs of \[79\] or the wiggling rings of \[126, 127\], are perhaps not captured by the free theory and hence not subject to the bound we find above. The problem with this is that we can compute not only the entropy but also the index in both regimes and they exhibit the same leading growth. If additional supergravity configurations are to generate parametrically more states this would either require very precise cancellations (so that the index is very different from the number of states) or a phase transition at weak coupling (a phase transition in \(g_s\), not the \(N = I\) transition discussed above). Even if many states would cancel in the index, one would still need to explain why they become invisible in the limit in which interactions are turned off.

In the above, we have only counted multiparticle BPS supergravitons in 5d supergravity. It is conceivable that additional degrees of freedom could be obtained by allowing fluctuations in the Calabi-Yau as well. For example, as we discussed, D0-branes carry an extra degeneracy corresponding to the harmonic forms on the Calabi-Yau. Though this can contribute a finite multiplicative factor to the entropy, it does not change the functional form. In addition, 5d supergravity does include all massless degrees of freedom that one gets from the reduction on the Calabi-Yau, and the other massive degrees of freedom generically do not contain any BPS states.

One might also worry that multiparticle states, which in the free theory are not BPS, become BPS once interactions are included. Though this is a logical possibility, such degrees of freedom would not contribute to the index, and therefore the estimate of the index remains unaffected by this argument.

Finally, we notice that it is possible to do similar computations for AdS\(_3\times\)S\(_3\), which leads to the result that for \(N \lesssim I, S \sim N^{3/4}\), while for \(I \ll N \ll I^2, S \sim I^{1/2} N^{1/4}\). It would be interesting to reproduce these results by counting solutions of 6d supergravity as well.

### 6.2.4 The Stringy Exclusion Principle

It is also somewhat intriguing to see that in the “free theory” we recover the phase transition noted in the previous section only after imposing (by hand) the CFT unitarity bound suggesting that the latter is taken into account by our scaling solutions. A priori this sounds somewhat mysterious as the stringy exclusion principle was argued in \[28\] to be inaccessible to perturbatively string theory. As noted earlier, however, the multicentered
solutions seem to always satisfy this bound (though there is no general proof of this). In fact, the origin of the bound in this system can simply be traced back to the fact that the size of the angular momentum equals $j = I/2 - M$, which cannot be negative, and using $M = L_0$ this then immediately implies that the unitarity bound will be satisfied by our solutions.