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el Showk, S.

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G2 M A N I F O L D S A N D N O N - P E R T U R B A T I V E S T R I N G T H E O R Y

The first part of this thesis focused quite concretely on particular kinds of states in quantum gravity related to black holes and also the relation of the latter to the full Hilbert space of string theory. Although great attention was given to the physics of black holes it should be recalled that the latter are mostly interesting because of their potentially important role in elucidating the fundamental principles of quantum gravity and it is along these lines that we made some interesting progress in the first half of this work.

In the second half of this thesis we will step back somewhat and consider seemingly unrelated questions about the structure of topological string theory. Although the works involved in the two parts of this thesis were conceived quite independently there has, in fact, been a strong interplay between black hole physics and topological string theory as the latter provides a very important window into various foundational questions in quantum gravity which are often deeply interwoven with the questions posed in Part I at least for BPS black holes. An obvious instance of this was [10] which, in fact, was part of the initial motivation for [3] on which Part I of this thesis is partially based.

In Part II our focus will be topological theories on G2 manifolds. This part of the thesis is based on [1][2]. Specifically, we will define and explore open topological strings on G2 manifolds in order to extend the work done in [12]. We will also very briefly review our work in [2] on topological field theories on G2 manifolds; we eschew a more detailed treatment, however, and refer the reader to the original paper for all details.

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Although [1] and [2] may seem very far removed from black hole physics or even BPS states in physical string theory, a central aspect of this part will be how these theories potentially relates the A/B model topological string theories to each other. The latter have of course played an essential role in understanding the BPS partition function of \( \mathcal{N} = 2 \) string theory which plays an essential role in understanding BPS black holes. In [10] the BPS partition function of \( \mathcal{N} = 2 \) was argued to be given (to leading order) by the square of the topological string partition function, with the latter being thought of as some kind of a wavefunction. An essential goal of studying topological string/membrane theory on \( G_2 \) manifolds is to better understand the nature of the CY topological string partition function. For instance in [8] it is claimed that the wavefunction nature of the latter can be derived by uplifting to a \( G_2 \) manifold, namely CY\(_3 \times S^1 \).

### 7.1 \( G_2 \) HITCHIN FUNCTIONAL AT ONE LOOP

Here we will very briefly review the content of [2]. We eschew a more detailed treatment of this material for reasons of length as this thesis is already somewhat extensive. The results of this work are nonetheless interesting and relate to those discussed below of [1]. We will this introduce the main ideas and results and refer the reader to the original paper for more details.

Topological string theory on Calabi-Yau manifolds has been the source of many recent insights in the structure of gauge theories and black holes. The traditional construction for topological strings is in terms of topologically twisted worldsheet A- and B-models, computing Kähler and complex structure deformations. The topological information these theories compute is encoded in Gromov-Witten invariants.

More recently a target space quantum foam reformulation of the A-model in terms of the Kähler structure has emerged [135, 14]. The topological information computed are the Donaldson-Thomas invariants, providing a powerful reformulation of Gromov-Witten invariants. For topological string theories on Calabi-Yau manifolds there are additional well-developed computational tools using open-closed duality such as the topological vertex or matrix models.

In comparison, topological theories on \( G_2 \) manifold target spaces are much less explored. One motivation to consider such theories is that \( G_2 \) structure couples Kähler and complex structure naturally so such a theory might couple topological A- and B-models non-perturbatively, a coupling which we expect to exist following recent work on topological string theory. A recent proposal for topological theories on \( G_2 \) manifolds that goes under the name of topological M-theory was given in [8].

The classical effective description of topological M-theory is in terms of a Hitchin func-
Alternative topological theories on $G_2$ manifolds employing quantum world-sheet/worldvolume formulations have been proposed in terms of topological strings $\text{[12]}$ and topological membranes $\text{[137, 138, 139, 140]}$. The topological $G_2$ string and topological membrane theories $\text{[139]}$ have the same structure of local observables associated to the de Rham cohomology of $G_2$ manifolds. The full quantum worldvolume formulation of these theories, especially the computation of the complete path integral is much more difficult though than for the usual topological string theories on Calabi-Yau target spaces $\text{[2]}$.

In $\text{[2]}$ we attempted to understand the moduli space of topological M-theory in terms of a $G_2$ target space description. Our strategy was similar to the A-model quantum foam, where one considers fluctuations around a fixed background Kähler form. Here the quantum path integral is computed in terms of a topologically twisted six-dimensional abelian gauge theory.

Analogously, the stable closed 3-form encoding the $G_2$ structure in seven dimensions can be understood as a perturbation around a fixed background associative 3-form. Locally the fluctuation can be regarded as the field strength of an abelian 2-form gauge field. Unlike the A-model quantum foam, however, expanding the Hitchin functional to quadratic order around this fixed background gives a seven-dimensional gauge theory that is not quite topological but which is only invariant under diffeomorphisms of the $G_2$ manifold.

We analyzed the quantum structure of this theory by taking the 2-form gauge field to be topologically trivial. In practise this means we neglected certain ‘total derivative’ terms in the expansion of the Hitchin functional involving components of the bare 2-form gauge field $\text{[3]}$. This allowed us to generalize to seven dimensions the approach used by Pestun and Witten $\text{[13]}$ to quantize the Hitchin functional for a stable 3-form in six dimensions to 1-loop order. This approach is based on the powerful techniques developed by Schwarz $\text{[142]}$ for evaluating the partition function of a degenerate quadratic action functional. The structure of the partition function here is most naturally understood by fixing the gauge symmetry of the action using the antifield-BRST method of Batalin and Vilkovisky $\text{[143]}$. See also $\text{[8, 144]}$ for possible alternatives to the perturbative quantization we considered in $\text{[2]}$.

We first computed the 1-loop partition function of the ordinary $G_2$ Hitchin functional and

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$^1$A topological version of F-theory on $\text{Spin}(7)$ manifolds which are trivial torus fibrations over Calabi-Yau spaces was also considered in $\text{[141]}$.

$^2$The topological $G_2$ string partition function is only well-understood below genus two. At genus zero it computes the Hitchin functional while its genus one contribution was calculated in $\text{[2]}$. The topological membrane partition function is written only formally.

$^3$For more conventional gauge theories such local ‘total derivative’ terms usually correspond to topological invariants computing certain characteristic classes for the gauge bundle from the patching conditions. Unlike in conventional abelian gauge theory where the gauge field corresponds to a connection on a line bundle over the base space, the 2-form gauge field we have here corresponds to a connection on a gerbe.
found agreement between the local degrees of freedom for the reduction of this theory on a circle and the corresponding theory of Pestun and Witten [13], obtained from the Hitchin functional for a stable 3-form in 6 dimensions. The calculation was repeated for the generalized $G_2$ Hitchin functional and a certain truncation of the circle reduction of this theory was related to the extended Hitchin functional in 6 dimensions, whose 1-loop partition function was equated with the topological B-model in [13].

The 1-loop partition function for the topological $G_2$ string [12] was also computed here and found to agree with the generalized $G_2$ theory only up to a power of the Ray-Singer torsion of the background $G_2$ manifold. It is not clear to us whether precise agreement could have been obtained by a more careful analysis incorporating the global topological structure of the local total derivative terms we dropped. Nonetheless, it seems that the topologically structure of such terms could potentially give rise to non-trivial 1-loop determinants which we ignored.

Our 1-loop quantization of the generalized $G_2$ Hitchin functional was in terms of linear variations of a closed stable odd-form in seven dimensions. However, the odd-form can be parameterized non-linearly in terms of other fields, that would be related to the dilaton, B-field, metric and RR flux moduli in compactifications of physical string theory on generalized $G_2$ manifolds. Hence an additional question is if we were using the appropriate degrees of freedom to describe the quantum theory. It would be interesting to see if our results could be checked by comparison with the couplings appearing in effective actions for generalized $G_2$ compactifications of physical string and M-theory.

Finally, since general background $G_2$ metric variations contain complex structure variations in 6 dimensions, it is natural to ask whether the wavefunction behaviour of B-model has a nice interpretation in 7 dimensions? Indeed this was one of the original motivations for the proposal of topological M-theory in [8]. It is possible that this could be understood from the structure of partition functions we calculated in [2] and this was one motivation for this work. Unfortunately the complexity of the 7-dimensional Hitchin functional makes such a relationship rather difficult to determine.

### 7.2 Topological $G_2$ Strings

As mentioned above, topological strings have been studied quite intensively as a toy model of ordinary string theory. Besides displaying a rich mathematical structure, they partially or completely control certain BPS quantities in ordinary string theory, and as such have found applications e.g. in the study BPS black holes and non-perturbative contributions to superpotentials.

Unfortunately, a full non-perturbative definition of topological string theory is still lack-
ing, but it is clear that it will involve ingredients from both the A- and B-model, and that both open and closed topological strings will play a role. Since M-theory is crucial in understanding the strong coupling limit and nonperturbative properties of string theory, one may wonder whether something similar is true in the topological case, i.e. does there exist a seven-dimensional topological theory which reduces to topological string theory in six dimensions when compactified on a circle? And could such a seven-dimensional theory shed light on the non-perturbative properties of topological string theory?

In order to find such a seven-dimensional theory one can use various strategies. One can try to directly guess the spacetime theory, as we described in Section 7.1 (see also [8, 145]), or one can try to construct a topological membrane theory as in [146, 138, 139, 140, 147] (after all, M-theory appears to be a theory of membranes, though the precise meaning of this sentence remains opaque). Here and in the next section we will describe a different approach involving studying a topological version of strings propagating on a manifold of $G_2$ holonomy, following [12] (for an earlier work on $G_2$ sigma-models see [148]).

In [12] the topological twist was defined using the extended worldsheets algebra that sigma-models on manifolds with exceptional holonomy possess [149]. For manifolds of $G_2$ holonomy the extended worldsheets algebra contains the $c = 7/10$ superconformal algebra [148] that describes the tricritical Ising model, and the conformal block structure of this theory was crucial in defining the twist. In [12] it was furthermore shown that the BRST cohomology of the topological $G_2$ string is equivalent to the ordinary de Rham cohomology of the seven-manifold, and that the genus zero three-point functions are the third derivatives of a suitable prepotential, which turned out to be equal to the seven-dimensional Hitchin functional of [150]. The latter also features prominently in [8, 145], suggesting a close connection between the spacetime and worldsheets approaches.

In Chapter 8 we will study open topological strings on seven-manifolds of $G_2$ holonomy, using the same twist as in [12]. There are several motivations to do this. First of all, we hope that this formalism will eventually lead to a better understanding of the open topological string in six dimensions. Second, some of the results may be relevant for the study of realistic compactifications of M-theory on manifolds of $G_2$ holonomy [3] for a recent discussion of the latter see e.g. [151]. Third, by studying branes wrapping three-cycles we may establish a connection between topological strings and topological membranes in seven dimensions. And finally, for open topological strings one can completely determine the corresponding open string field theory [134], from which one can compute arbitrary higher genus partition functions and from which one can also extract highly non-trivial all-order results for the closed topological string using geometric transitions [152]. Repeating such an analysis in the $G_2$ case would allow us to use open $G_2$ string field theory

\[^4\] This will require an extension of our results to singular manifolds which is an interesting direction for future research.
to perform computations at higher genus in both the open and closed topological $G_2$ string. This is of special importance since the definition and existence of the topological twist at higher genus has not yet been rigorously established in the $G_2$ case.

Along the way we will run into various interesting mathematical structures and topological field theories in various dimensions that may be of interest in their own right.

7.3 The closed topological $G_2$ string

Let us briefly review the definition of the closed topological $G_2$ string found in [12] as we will need this background when defining the open theory in Chapter 8. We will cover only essential points. For further details we refer the reader to [12].

7.3.1 Sigma model for the $G_2$ string

The topological $G_2$ string constructs a topological string theory with target space a seven-dimensional $G_2$-holonomy manifold $Y$. This topological string theory is defined in terms of a topological twist of the relevant sigma-model. In order to have $\mathcal{N} = 1$ target space supersymmetry, one starts with an $\mathcal{N} = (1, 1)$ sigma model on a $G_2$ holonomy manifold. The special holonomy of the target space implies an extended supersymmetry algebra for the worldsheet sigma-model [149]. That is, additional conserved supercurrents are generated by pulling back the covariantly constant 3-form $\phi$ and its hodge dual $\ast \phi$ to the worldsheet as

$$\phi_{\mu \nu \rho}(X) DX^\mu DX'^\nu DX^\rho,$$

where $X$ is a worldsheet chiral superfield, whose bosonic component corresponds to the world-sheet embedding map. From the classical theory it is then postulated that the extended symmetry algebra survives quantization, and is present in the quantum theory. This postulate is also based on analyzing all possible quantum extensions of the symmetry algebra compatible with spacetime supersymmetry and $G_2$ holonomy.

A crucial property of the extended symmetry algebra is that it contains an $\mathcal{N} = 1$ SCFT sub-algebra, which has the correct central charge of $c = 7/10$ to correspond to the tri-critical Ising unitary minimal model. Unitary minimal models have central charges in the series $c = 1 - \frac{6}{p(p+1)}$ (for $p$ an integer) so the tri-critical Ising model has $p = 4$.

The conformal primaries for such models are labelled by two integer Kac labels, $n'$ and $n$, as $\phi_{(n',n)}$ where $1 \leq n' \leq p$ and $1 \leq n < p$. The Kac labels determine the conformal weight of the state as $h_{n',n} = \frac{[pn'-(p+1)n]^2-1}{4p(p+1)}$. The Kac table for this minimal model is reproduced in [12, Table 1]. Note that primaries with label $(n',n)$ and
$(p + 1 - n', p - n)$ are equivalent. This model has six conformal primaries with weights $h_I = 0, 1/10, 6/10, 3/2$ (for the NS states) and $h_I = 7/16, 3/80$ (for the R states).

The conformal block structure of the weight 1/10, $\phi_{(2,1)}$, and of the weight 7/16 primary, $\phi_{(1,2)}$, is particularly simple,

\[
\phi_{(2,1)} \times \phi_{(n', n)} = \phi_{(n' - 1, n)} + \phi_{(n' + 1, n)},
\]
\[
\phi_{(1,2)} \times \phi_{(n', n)} = \phi_{(n', n - 1)} + \phi_{(n', n + 1)},
\]

where $\phi_{(n', n)}$ is any primary. This conformal block decomposition is schematically denoted as

\[
\Phi_{(2,1)} = \Phi^\downarrow_{(2,1)} \oplus \Phi^\uparrow_{(2,1)},
\]
\[
\Phi_{(1,2)} = \Phi^\downarrow_{(1,2)} \oplus \Phi^\uparrow_{(1,2)}.
\]

(7.1)

The conformal primaries of the full sigma-model are labelled by their tri-critical Ising model highest weight, $h_I$, and the highest weight corresponding to the rest of the algebra, $h_r$, as $|h_I, h_r\rangle$. This is possible because the stress tensors, $T_I$, of the tricritical sub-algebra and of the ‘rest’ of the algebra, $T_r = T - T_I$ (where $T$ is the stress tensor of the full algebra), satisfy $T_I \cdot T_r \sim 0$.

### 7.3.2 The $G_2$ Twist

The standard $\mathcal{N} = (2, 2)$ sigma-models can be twisted by making use of the U(1) $R$-symmetry of their algebra. Using the U(1) symmetry, the twisting can be regarded as changing the worldsheet sigma-model with a Calabi-Yau target space by the addition of the following term:

\[
\pm \frac{\omega}{2} \overline{\psi} \psi,
\]

with $\omega$ the spin connection on the world-sheet. This effectively changes the charge of the fermions under worldsheet gravity to be integral, resulting in the topological A/B-model depending on the relative sign of the twist in the left and right sector of the theory (for fermions with holomorphic or anti-holomorphic target space indices). Here $\overline{\psi}$ and $\psi$ can be either left- or right-moving worldsheet fermions and $\omega$ is the spin-connection on the worldsheet. In the topological theory, before coupling to gravity, there are no ghosts or anti-ghosts so these are the only spinors/fermions in the system.

This twist has been re-interpreted [153][154] as follows. First think of the exponentiation of (7.2) as an insertion in the path integral rather than a modification of the action. By
bosonising the world-sheet fermions we can write $\bar{\psi} \psi = \partial H$ for a free boson field so the above becomes

$$\int \frac{\omega}{2} \partial H = - \int H \frac{\partial \omega}{2} = \int HR, \quad (7.3)$$

where $R$ is the curvature of the world-sheet. We can always choose a gauge for the metric such that $R$ will only have support on a number of points given by the Euler number of the worldsheet.

For closed strings on a sphere the Euler class has support on two points which can be chosen to be at $0$ and $\infty$ (in the CFT defined on the sphere) so the correlation functions in the topological theory can be calculated in terms of the original CFT using the following dictionary:

$$\langle \ldots \rangle_{\text{twisted}} = \left\langle e^{H(\infty)} \ldots e^{H(0)} \right\rangle_{\text{untwisted}}. \quad (7.4)$$

The ‘untwisted’ theory should not be confused with the physical theory, because it does not include integration over world-sheet metrics and hence has no ghost or superghost system and also it is still not at the critical dimension. The equation above simply relates the original untwisted $\mathcal{N} = 2$ sigma-model theory to the twisted one.

In [12] a related prescription is given to define the twisted ‘topological’ sigma-model on a 7-dimensional target space with $G_2$ holonomy. Here the role of the U(1) R-symmetry is played by the tri-critical Ising model sub-algebra. However, a difference is that the topological $G_2$ sigma-model is formulated in terms of conformal blocks rather than in terms of local operators. In particular the operator $H$ in the above is replaced by the conformal block $\Phi^{+}_{(1,2)}$.

The main point of the topological twisting is to redefine the theory in such a way that it contains a scalar BRST operator. In the $G_2$ sigma model, the BRST operator is defined to be related to the conformal block of the weight $3/2$ current $G(z)$ of the super stress-energy tensor

$$Q = G^{\frac{1}{2}}_{\frac{1}{2}}. \quad (7.5)$$

The states of the twisted $G_2$ theory are defined to be in the cohomology of this operator. See [12] for a more detailed definition of this operator.

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5The super stress-energy tensor is given as $T(z, \theta) = G(z) + \theta T(z)$. The current $G(z)$ can be further decomposed as $G(z) = \Phi_{(2,1)} \otimes \Psi_{14, 14}$, in terms of the tri-critical Ising-model part and the rest of the algebra, respectively. Since its tri-critical Ising model part contains only the primary $\Phi_{(2,1)}$, it can be decomposed into conformal blocks accordingly.
It should be pointed out that in [12] it was not possible to explicitly construct the twisted stress tensor, and although there is circumstantial evidence that the topological theory does exist beyond tree level this statement remains conjectural.

### 7.3.3 The $G_2$ String Hilbert Space

In a general CFT the set of states can be generated by acting with primary operators and their descendents on the vacuum state, resulting in an infinite dimensional Fock space. In string sigma models this Fock space contains unphysical states, and so the physical Hilbert space is given by the cohomology of the BRST operator on this physical Hilbert space which is still generally infinite-dimensional.

In the topological A- and B-models a localization argument [154] implies that only BRST fixed-points contribute to the path integral and these correspond to holomorphic and constant maps, respectively. Thus the set of field configurations that when quantized, generate states in the Hilbert space is restricted to this subclass of all field configurations and so the Fock space is much smaller. Upon passing to BRST cohomology this space actually becomes finite-dimensional.

In the $G_2$ string the localization argument cannot be made rigorous, because the action of the BRST operator on the worldsheet fields is inherently quantum, and so is not well defined on the classical fields. Neglecting this issue and proceeding naively, however, one can construct a localization argument for $G_2$ strings that suggests that the path integral localizes on the space of constant maps [12]. Thus we will take our initial Hilbert space to consist of states generated by constant modes $X_0^\mu$ and $\psi_0^\mu$ on the world-sheet (in the NS-sector there is no constant fermionic mode but the lowest energy mode $\psi_{-\frac{1}{2}}^\mu$ is used instead). These correspond to solutions of worldsheet equations of motion with minimal action which dominate the path integral in the large volume limit.

In [154] the fact that the path integral can be evaluated by restricting to the space of BRST fixed points is related to another feature of the A/B-models: namely the coupling-invariance (modulo topological terms) of the worldsheet path integral. Variations of the path integral with respect to the inverse string coupling constant $t \propto (\alpha')^{-1}$ are $Q$-exact, so one may freely take the weak coupling limit $t \to \infty$ in which the classical configurations dominate. This limit is equivalent to rescaling the target space metric, and so we will refer to it as the large volume limit.

Accordingly, calculations in the A- and B-model can be performed in the limit where the Calabi-Yau space has a large volume relative to the string scale, and the worldsheet theory can be approximated by a free theory (this neglects, of course, the important instanton effects in the A-model). The $G_2$ string also has the characteristics of a topological theory, such as correlators being independent of the operator’s positions, and the fact that the
BRST cohomology corresponds to chiral primaries. On the other hand since the theory is defined in terms of the conformal blocks, it is difficult to explicitly check the coupling constant independence. Based on the topological arguments, and on the postulate of the quantum symmetry algebra, in this thesis we will assume the coupling constant independence and the validity of localization arguments. Even if these arguments should fail for subtle reasons, the results derived here are still valid in the large volume limit.

### 7.3.4 The \( G_2 \) String and Geometry

As in the topological A- and B-model, for the topological \( G_2 \) string there is a one-to-one correspondence between local operators of the form \( O_{\omega_p} = \omega_{i_1...i_p} \psi^{i_1} \ldots \psi^{i_p} \) and target space \( p \)-forms \( \omega_p = \omega_{i_1...i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p} \). In \[12\] it is found that the BRST cohomology of the left (right) sector alone maps to a certain refinement of the de Rham cohomology described by the ‘\( G_2 \) Dolbeault’ complex

\[
0 \to \Lambda^0_1 \overset{\mathcal{D}}{\to} \Lambda^1_1 \overset{\mathcal{D}}{\to} \Lambda^2_1 \overset{\mathcal{D}}{\to} \Lambda^3_1 \to 0 .
\] (7.6)

The notation is that \( \Lambda^p_n \) denotes differential forms of degree \( p \), transforming in the irreducible representation \( n \) of \( G_2 \). The operator \( \mathcal{D} \) acts as the exterior derivative on 0-forms, and as

\[
\begin{align*}
\mathcal{D}(\alpha) &= \pi^2_7(d\alpha) \quad \text{if} \quad \alpha \in \Lambda^1_1 , \\
\mathcal{D}(\beta) &= \pi^3_1(d\beta) \quad \text{if} \quad \beta \in \Lambda^2 ,
\end{align*}
\]

where \( \pi^2_7 \) and \( \pi^3_1 \) are projectors onto the relevant representations. The explicit expressions for the projectors and the standard decomposition of the de Rham cohomology are included in appendix [H]. Thus, the BRST operator \( G^l_{-1/2} \) maps to the differential operator of the complex \( \mathcal{D} \). In the closed theory, combining the left- and right-movers, one obtains the full cohomology of the target manifold, accounting for all geometric moduli: metric deformations, the \( B \)-field moduli, and rescaling of the associative 3-form \( \phi \). The relevant cohomology for the open string states will be worked out in the next chapter.