



## UvA-DARE (Digital Academic Repository)

### Real balance effects, timing and equilibrium determination

Stoltenberg, C.

**Publication date**  
2008

[Link to publication](#)

#### **Citation for published version (APA):**

Stoltenberg, C. (2008). *Real balance effects, timing and equilibrium determination*. Afdeling Algemene Economie.  
<http://www1.fee.uva.nl/toe/content/people/content/stoltenberg/downloads/timing.pdf>

#### **General rights**

It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

#### **Disclaimer/Complaints regulations**

If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: <https://uba.uva.nl/en/contact>, or a letter to: Library of the University of Amsterdam, Secretariat, Singel 425, 1012 WP Amsterdam, The Netherlands. You will be contacted as soon as possible.

# REAL BALANCE EFFECTS, TIMING AND EQUILIBRIUM DETERMINATION\*

Christian Stoltenberg<sup>†</sup>

October, 2008

## Abstract

This paper examines whether the existence and the timing of real balance effects contribute to the determination of the absolute price level, as suggested by Patinkin (1949, 1965). As the main novel result, I show that there exists a unique price level sequence that is consistent with an equilibrium under interest rate policy, if beginning-of-period money yields transaction services. Predetermined real money balances can then serve as a state variable, implying that interest rate setting must be passive – a violation of the Taylor-principle – for unique, stable, and non-oscillatory equilibrium sequences.

**JEL classification:** E32, E41, E52.

**Keywords:** Real balance effects, predetermined money, price level determination, real determinacy, monetary policy rules, flexible prices.

---

\*I'm especially thankful to Harald Uhlig, Jordi Galí, Andreas Schabert and Mark Weder for helpful discussions. Further, I would like to thank Michael Burda, Fabio Canova, Bartosz Mackowiak, Emanuel Mönch, Marco Airaudo and Matthias Paustian for suggestions and comments. This research was supported by the Deutsche Forschungsgemeinschaft through the SFB 649 'Economic Risk'.

<sup>†</sup>Department of Economics, Universiteit van Amsterdam, Roeterstraat 11, 1018 WB Amsterdam, The Netherlands, email: C.A.Stoltenberg@uva.nl, tel: +31 20 525 3913.

# 1 Introduction

The conduct of monetary and fiscal policy is known to affect the determination of the price level and, under non-neutrality, the real equilibrium allocation. While previous contributions to this line of research have primarily considered monetary policy regimes that are characterized by constant money growth (see, e.g., Obstfeld and Rogoff, 1983; Matsuyama, 1990, 1991), recent studies mainly focus on policy regimes summarized by interest rate feedback rules, such as Taylor (1993), Benhabib, Schmitt-Grohé, and Uribe (2001a,b, 2003), Woodford (1994), Carlstrom and Fuerst (2001) or Bénassy (2000). Correspondingly, researchers nowadays pay less attention to the role of monetary aggregates and increasingly employ money demand specifications that allow to neglect money for the analysis of equilibrium determination (see Dupor (2001); Woodford (2003a) or Carlstrom and Fuerst (2005)). There are two prominent results in this literature (Benhabib, Schmitt-Grohé, and Uribe, 2001a). First, whether an equilibrium allocation is neutral with respect to the absolute price level (nominal indeterminacy) or not depends exclusively on the stance of fiscal policy and not on monetary policy, preferences or technology. Second, standard assumptions on preferences and technology imply that the Taylor-principle ensures stability and uniqueness of equilibrium sequences if fiscal solvency is guaranteed under all possible circumstances. According to the Taylor-principle (activeness), monetary policy should aggressively fight inflation by raising the nominal interest rate more than the increase in inflation.

In this paper, I challenge these prominent findings. I do this by revisiting the role of real balance effects and their timing for equilibrium determination as suggested by Patinkin (1949, 1965). Throughout my analysis I assume that the intertemporal government budget constraint is satisfied under all possible path of endogenous variables. This implies that fiscal policy is not capable to pin down the price level as suggested by the fiscal theory of the price level (see e.g. Woodford, 1994, 1995, 1996; Sims, 1994; Leeper, 1991).

As my main novel result, I show that if the beginning-of-period stock of money facilitates transactions, predetermined real money balances can serve as an endogenous state variable of the economy under interest rate policy – a key role of real money which has been disregarded in the literature. In this case, a perfect foresight equi-

librium displays nominal determinacy: it is associated with a unique price level sequence. To put it differently, if real money balances and not only nominal balances are a relevant state variable, the determination of prices is a monetary not a fiscal phenomenon. Interest rate policy should then rather be passive than active – a violation of the Taylor-principle – to avoid oscillatory or explosive equilibrium sequences, such that a perfect foresight equilibrium is uniquely determined (real determinacy). Notably, the unique determination of the price level and the uniqueness of equilibrium sequences are two sides of the same coin. If real money is a state variable, then the whole set of equilibrium sequences is indexed with a particular value for initial real money balances, which results in a particular initial value for the price level, since initial nominal balances are given. Working forward, this mechanism pins down uniquely the complete set of sequences for the absolute price level and nominal balances under interest policy. Whether real money balances are relevant for equilibrium behavior depends on both, households' preferences and on the stance of monetary policy under interest rate policy.

I set up a discrete time general equilibrium model with flexible prices, where real money balances and consumption enter the utility function in a non-separable way, that is consistent with a shopping time technology (McCallum and Goodfriend, 1987). I apply two different specifications about the particular stock of money, that enters the utility function: Either the stock of money at the beginning or at the end of the period is assumed to yield transaction services. The idea of the former specification can be interpreted as the money-in-the-utility-function version of Svensson (1985)'s timing of markets within one period, where the goods market is closed, before the asset market is opened. Then, households rely on the stock of money carried over from the previous period for transactions in the goods market. This formulation is applied for example in Woodford (1990), McCallum and Nelson (1999) or more recently in Persson and Svensson (2006). The second specification – when the end-of-period-stock of money yields utility – can be found in Woodford (2003a) or Ljungqvist and Sargent (2004). It can be interpreted as a money-in-the-utility-function version for a reverse timing of markets, i.e. households can always adjust their money holdings within one period to facilitate transactions.<sup>1</sup> The resulting real balance effects are commonly neglected, since they are typically found to be very small (Lucas, 2000 or Ireland, 2004). I show, that the existence and the timing of real

---

<sup>1</sup>See figure 1 for a graphical illustration of first and figure 2 for the second specification.

balance effects (not the magnitude) can have substantial implications for equilibrium determination.

Under real balance effects and interest rate policy, a uniquely determined price level is associated with real money being a relevant state variable and, thus, with a history dependent evolution of equilibrium sequences, which crucially affects the conditions for macroeconomic stability. The main principles for equilibrium determination under simple interest rate feedback rules and following Sargent and Wallace (1975) for a constant money growth rule can be summarized as follows.

- If beginning-of-period money provides transaction services and interest rate policy responds to current inflation, the perfect foresight equilibrium displays nominal determinacy. Thus, price-level determinacy becomes a monetary instead of a fiscal phenomenon. Neither an interest rate peg nor a forward looking interest rate rule lead to this result.
- Under the beginning-of-period specification, an interest rate policy that reacts to changes in current inflation has to be passive for equilibrium sequences to be uniquely determined and to converge to the steady state in a non-oscillatory way.
- Under a constant money growth regime, a perfect foresight equilibrium displays nominal determinacy, but real money does not serve as a relevant state variable. Equilibrium sequences are, in any case, locally stable and uniquely determined.

Remarkably, for the economy to evolve in a history dependent way, it does not suffice, that monetary policy is history dependent. While these results are derived for the case where the labor supply elasticity is finite, I further show that the assumption of an infinitely elastic labor supply, which is for example made in Dupor (2001), or Carlstrom and Fuerst (2005), for a related purpose, is not harmless for the local equilibrium properties under interest rate policy. For example, I find that an equilibrium under interest rate policy and flexible prices is then consistent with any initial price level, and that the well-established principles for equilibrium uniqueness for a separable utility function (see Woodford, 2003a) apply when end-of-period money provides utility.

## Related Literature

I now turn to the related literature. Most closely related to my paper is the work by Benhabib, Schmitt-Grohé, and Uribe (2001a) and Carlstrom and Fuerst (2001) who analyze equilibrium determination under flexible prices. Benhabib, Schmitt-Grohé, and Uribe (2001a) were among the first to show that conditions for local stability and uniqueness under interest policy are highly sensitive to changes in preferences and technology. However, since they employ a specification that corresponds to my end-of-period formulation they find that nominal determinacy is a purely fiscal phenomenon. Carlstrom and Fuerst (2001) also examine whether the particular stock of money that enters the utility function matters for local stability and uniqueness. They allow for two specifications, either the money stock after agents leave the asset market (which opens first) or the amount of money balances after agents leave the goods market is assumed to enter the utility function. The crucial difference to my approach is that in their model, only the stock of money but not the (implicit) timing of markets changes across the specifications. In particular, since the financial market always opens first, agents do not rely on the stock of money carried over from the previous period to purchase consumption. Correspondingly, predetermined real money balances can not serve as a state variable, and the timing conventions analyzed in Carlstrom and Fuerst (2001) affect only conditions for local stability and uniqueness but not the determination of the absolute price level.

Brückner and Schabert (2005) and Kurozumi (2006) analyze local stability and uniqueness of equilibrium sequences in stochastic maximizing economies under sticky prices, when beginning-of-period money yields transaction services. Employing a shopping-time specification and an infinitely elastic labor supply, Brückner and Schabert (2005) find that interest policy should react passively to changes in inflation to ensure local stability and uniqueness of equilibrium sequences. Kurozumi (2006) studies determinacy and expectational stability of Taylor-rules in a non-separable money-in-the-utility-function framework. He finds that conditions that ensure real determinacy and expectational stability are highly sensitive to assumption on the stock of money that is assumed to deliver transaction services. My contribution is to show that predetermined real money balances play not only a role for learning, local stability and uniqueness of equilibrium sequences under sticky prices. Moreover, predetermined real money balances crucially affect the determination of the absolute price level

under flexible prices – a key role for money which has been disregarded in the aforementioned studies.

The remainder of the paper is organized as follows. Section 2 develops the model. Section 3 analyzes nominal and real determinacy under flexible prices. In the first part, I consider the case where the beginning-of-period stock of money provides utility, while the results for the end-of-period specification are briefly summarized in the second part.<sup>2</sup> For both specifications, I derive the implications for equilibrium determination and local stability under current and forward looking interest rate rules, and for money growth rules. The last part of section 3 discusses my findings and compares them to results in related studies. In section 4 I list the main results when prices are set in a staggered way. Section 5 concludes.

## 2 The model

In this section an infinite horizon general equilibrium model with representative agents and perfectly flexible prices is developed. I consider a money in the utility function specification that leads to real balance effects and assume either that the stock of money at the beginning or at the end of the period yields transaction services. Monetary policy is either specified in form of an interest rate feedback rule or constant money growth. To check for the robustness of the results for the former policy regime, I apply contemporaneous and forward looking interest rate rules.

Lower (upper) case letters denote real (nominal) variables. There is a continuum of identical and infinitely lived households. At the beginning of period  $t$ , households' financial wealth comprises money  $M_{t-1}$  and nominally non-state contingent government bonds  $B_{t-1}$  carried over from the previous period. The households' budget constraint reads

$$M_t + B_t + P_t c_t \leq R_{t-1} B_{t-1} + M_{t-1} + P_t w_t l_t - P_t \tau_t, \quad (1)$$

$c_t$  denotes consumption,  $P_t$  the aggregate price level,  $w_t$  the real wage rate,  $l_t$  work-

---

<sup>2</sup>My findings for the latter case relate to the results in Benhabib, Schmitt-Grohé, and Uribe (2001a), Carlstrom and Fuerst (2001), Woodford (2003a)

ing time,  $\tau_t$  a lump-sum tax, and  $R_t$  the gross nominal interest rate on government bonds. Further, households have to fulfill the no-Ponzi game condition,  $\lim_{t \rightarrow \infty} (m_t + b_t) \prod_{i=1}^t \pi_i / R_{i-1} \geq 0$ , where  $b_t = B_t / P_t$  and  $m_t = M_t / P_t$  denote real bonds and real money balances. The objective of the representative household is

$$\sum_{t=0}^{\infty} \beta^t u(c_t, l_t, A_t / P_t), \quad \beta \in (0, 1), \quad (2)$$

$\beta$  denotes the subjective discount factor and  $A_t$  nominal balances, which will be defined below. The instantaneous utility function is assumed to satisfy

$$u_c > 0, u_l < 0, u_a > 0, u_{cc} < 0, u_{aa} < 0, u_{ll} \leq 0, \quad (3)$$

$$u_{ca} > 0, u_{cl} = u_{al} = 0, u_{cc}u_{aa} - u_{ca}^2 > 0, \quad (4)$$

and the usual Inada-conditions, where  $a_t = A_t / P_t$ . According to (4) the cross derivative  $u_{ca}$  is (strictly) positive, such that marginal utility of consumption rises with real money balances. The resulting properties, i.e., non-separability and real balance effects, typically emerge under more explicit specifications of transaction frictions. As, for example, shown by Brock (1974) or Feenstra (1986), a money-in-the-utility (MIU) function specification, which is equivalent to a specification where purchases of consumption goods are associated with transaction costs that are either measured by shopping time or real resources, is usually characterized by these properties. To be more precise, introducing these transaction frictions in a corresponding model with a utility function  $v(c_t, 1 - l_t)$  would lead to real balance effects, which are equivalent to a MIU specification with  $u_{ca} > 0$ , if (but not only if) the labor supply elasticity is finite (see appendix A.1). It should be noted that an infinite labor supply elasticity will lead to be of particular interest in what follows.

To avoid additional complexities, I assume that the respective cross derivatives are equal to zero  $u_{lc} = u_{la} = 0$ .<sup>3</sup> The last assumption in (4),  $u_{cc}u_{aa} - u_{ca}^2 > 0$ , is imposed to ensure – together with (3) – the utility function to be strictly concave. The conditions in (3)-(4) further ensure that real money balances and consumption are normal goods, i.e. that the utility function exhibits increasing expansion paths with respect to money and consumption.

---

<sup>3</sup>This implies that the instantaneous utility function  $u(c_t, a_t, l_t)$  can be written as  $f(c_t, a_t) - g(l_t)$ .





The public sector consists of a fiscal and a monetary authority. I consider two widely applied specifications for the monetary policy regime. The first regime is characterized by the central bank setting the nominal interest rate contingent on current or on future inflation.

$$R_t = \rho(\pi_t), \quad \text{or} \quad R_t = \rho(\pi_{t+1}), \quad \text{with } \rho' \geq 0, \quad R_t \geq 1. \quad (9)$$

I further assume that the steady state condition  $R = \pi/\beta$  has a unique solution for  $R > 1$ . According to the interest rate feedback rule (9), the response of the interest rate to changes in inflation,  $\rho_\pi$ , is non-negative. The second regime, is characterized by the central bank holding the money growth constant  $M_t/M_{t-1} = \mu$ , where  $\mu \geq 1$ :

$$m_t \pi_t / m_{t-1} = \mu. \quad (10)$$

The fiscal authority issues risk-free one period bonds, receives lump-sum taxes from households, and transfers from the monetary authority. I assume that tax policy guarantees government solvency (Ricardian fiscal policy), i.e. it ensures that  $\lim_{t \rightarrow \infty} (m_t + b_t) \prod_{i=1}^t \pi_i / R_{i-1} = 0$ .

### 3 Equilibrium determination under flexible prices

In this section, I assess how real balance effects, the timing of markets and monetary policy affect the determination of the price level and of the perfect foresight equilibrium. As described in the previous section, I consider two versions of the model which differ with regard to the stock of money that enters the utility function, i.e., the **B**-version and the **E**-version, and I consider three types of monetary policy rules described by (9) or (10). The equilibrium for a positive interest rate ( $R_t > 1$ ) for both versions can then be summarized as follows.

**Definition 1** *Given an initial money endowment  $M_{-1}$ , a Ricardian fiscal policy  $\tau_t \forall t \geq 0$  and a monetary policy (9) or (10), a perfect foresight equilibrium (PFE) consists of set of sequences  $\{c_t, \pi_t, m_t, R_t\}_{t=0}^\infty$  and a price level  $P_0$ , satisfying  $\forall t \geq 0$  the transversality condition  $\lim_{t \rightarrow \infty} (m_t + b_t) \prod_{i=1}^t \pi_i / R_{i-1} = 0$ , and either  $u_1(c_t) = -u_c\left(c_t, \frac{m_{t-1}}{\pi_t}\right)$ ,*

$u_c \left( c_t, \frac{m_{t-1}}{\pi_t} \right) = \beta R_t u_c \left( c_{t+1}, \frac{m_t}{\pi_{t+1}} \right) / \pi_{t+1}$ , and  $(R_t - 1)u_c \left( c_{t+1}, \frac{m_t}{\pi_{t+1}} \right) = u_a \left( c_{t+1}, \frac{m_t}{\pi_{t+1}} \right)$  for the **B**-version or  $u_l(c_t) = -u_c(c_t, m_t)$ ,  $u_c(c_t, m_t) = \beta R_t u_c(c_{t+1}, m_{t+1}) / \pi_{t+1}$ , and  $(R_t - 1)u_c(c_{t+1}, m_{t+1}) / \pi_{t+1} = u_a(c_t, m_t) / \beta$  for the **E**-version.

Notably, in contrast to initial nominal balances  $M_{-1}$ , which shows up in the condition for optimal intra-temporal substitution  $u_l(c_0) = -u_c \left( c_0, \frac{M_{-1}}{P_0} \right)$  for  $t = 0$  in the **B**-version, there is no need to include an initial price level  $P_{-1}$  in the set of relevant state variable under flexible prices: neither resources nor the optimal actions of private agents or the government depend on the initial price level. This implies that initial nominal balances and not initial real balances  $m_{-1}$  are the relevant state variable.

The dependence of a given allocation on a particular absolute price level in the first period  $P_0$  is often summarized by the notion “nominal determinacy”. It is crucial to note that the role of the price level in the first period does not relate to the unique determination of equilibrium sequences (including the inflation sequence) which is summarized by the notion “real determinacy”. These properties are summarized in the following Definition, which corresponds to the Definition applied in Benhabib, Schmitt-Grohé, and Uribe (2001a).

**Definition 2** *The equilibrium displays real determinacy if there exists a unique set of equilibrium sequences  $\{c_t, \pi_t, m_t, R_t\}_{t=0}^{\infty}$ . Given  $M_{-1}$ , the equilibrium displays nominal indeterminacy if for any particular set of equilibrium sequences  $\{c_t, \pi_t, m_t, R_t\}_{t=0}^{\infty}$ , there exist infinite many price levels  $P_0$  consistent with a perfect foresight equilibrium.*

In the following Proposition I show that the two version, **B** and **E**, differ substantially with respect the determinacy of the price level under interest rate policy.

**Proposition 3** *Consider that consumption and real money balances enter non-separably into the utility function and that monetary policy targets the nominal interest rate according to (9). Then the equilibrium displays nominal determinacy in the **B**-version and nominal indeterminacy in the **E**-version.*

**Proof.** In the **B**-version, for a given sequence  $\{c_t\}_{t=0}^{\infty}$ , if  $u_{ca} \neq 0$ , the condition for optimal intra-temporal substitution between consumption and leisure,  $u_l(c_t) =$

$-u_c\left(c_t, \frac{m_{t-1}}{\pi_t}\right)$ , defines implicitly a monotone function  $M_{t-1}/P_t = f(c_t)$  for all  $t \geq 0$ . Thus a given sequence  $\{c_t\}_{t=0}^{\infty}$  results in a unique sequence  $\{M_{t-1}/P_t\}_{t=0}^{\infty}$ . Given  $M_{-1}$ ,  $P_0$  is uniquely determined. In the **E**-version and interest policy, the equilibrium conditions determine just real money balances  $m_t$ , and do not disentangle real money balances  $m_0$  into its components  $M_0$  and  $P_0$ . It follows that the perfect foresight equilibrium is consistent with infinitely many pairs  $P_0, M_0$ , and the equilibrium displays nominal indeterminacy. ■

If consumption and real money balances enter non-separably into the utility function and the **B**-version applies, real determinacy is sufficient for the determination of  $P_0$ , such that nominal determinacy applies. However, in the **E**-version, the equilibrium under interest policy displays nominal indeterminacy even if the equilibrium is characterized by real determinacy.<sup>4</sup> The latter result corresponds to the main result in Sargent and Wallace (1975). Since in the **E**-version there is an infinite number of equilibrium pairs for nominal money balances and the price level, sunspot equilibria may occur. If the assumed welfare measure punishes fluctuations in prices, Sargent and Wallace conclude that monetary policy should not target interest rates but monetary aggregates. I show that their conclusion depends on the implicit timing of markets: it applies only for the **E**-version, when end of period is assumed to yield transaction services, but not in the **B**-version when beginning of period money delivers utility.

A *PFE*, which is characterized by real determinacy and, thus, a unique inflation sequence, can be associated with multiple price level sequences, even if beginning-of-period money enters the utility function. If there are no real balance effects ( $u_{ca} = 0$ ), real money balances are determined residually by the forward-looking money demand equation  $(R_t - 1)u_c(c_{t+1}) = u_a\left(\frac{m_t}{\pi_{t+1}}\right)$  without any relation to initial nominal balances. This implies that nominal balances and the price level can not be determined separably, and the price level is neutral with regard to the determination of equilibrium sequences  $\{c_t, \pi_t, R_t\}_{t=0}^{\infty}$  under interest rate policy. Thus, two different values for the initial price level together with an equilibrium inflation sequence lead to two different price level sequences consistent with the *PFE*. Evidently, one cannot uniquely determine a unique price level sequence if there are infinitely many equilibrium inflation sequences.

---

<sup>4</sup>By construction, the equilibrium in the **E**-version would be nominally determinate, if the interest rate policy were to react exclusively on the current price level.

In the next Proposition I show that the equilibrium under a constant money growth rate is associated with a particular price level in the first period for both versions.

**Proposition 4** *Under a constant money growth rule the equilibrium displays nominal indeterminacy for both versions **B** and **E**.*

**Proof.** Given  $M_{-1}$ , a constant money growth rule uniquely pins down a whole sequence for nominal balances  $\{M_t\}_{t=0}^{\infty}$ . Given a sequence  $\{m_t\}_{t=0}^{\infty}$ , the whole sequence for the price level  $\{P_t\}_{t=0}^{\infty}$  is uniquely determined. ■

Independent of the existence of real balance effects, the *PFE* under a constant money growth rule is associated with a unique price level sequence, whenever  $\{m_t\}_{t=0}^{\infty}$  is uniquely determined.

To summarize, under interest rate policy and if there are no real balance effects, the equilibrium displays nominal indeterminacy in both versions. Given real determinacy and the presence of real balance effects, the equilibrium in the **B**-version exhibits nominal determinacy, while in the **E**-version the equilibrium is in any case associated with multiple price level sequences under interest rate policy. Under a constant money growth policy, the equilibrium is characterized by nominal determinacy if real determinacy applies. Whether real determinacy is ensured or not depends on monetary policy.

In the following analysis, I apply Blanchard and Kahn's (1980) approach to the analysis of a perfect foresight equilibrium. For this, I focus on the model's behavior in the neighborhood of the steady state, and apply a linear approximation of the set of non-linear equilibrium conditions. Throughout, I restrict my attention to *locally stable* equilibrium sequences, i.e., to equilibrium sequences that converge to the steady state.

### 3.1 Beginning-of-period money

I start with the case where the beginning-of-period stock of money enters the utility function. The deterministic steady state is then characterized by the following properties:  $\bar{R} = \bar{\pi}/\beta$ ,  $-u_l(\bar{c}) = u_c(\bar{c}, \bar{m}/\bar{\pi})$ , and  $u_c(\bar{c}, \bar{m}/\bar{\pi})(\bar{R} - 1) = u_a(\bar{m}/\bar{\pi}, \bar{c})$ . A discussion of the existence and uniqueness of a steady state for  $\bar{R} > 1$  can be

found in appendix A.2 . Log-linearizing the model at the steady state, leads to the following set of equilibrium conditions:

$$\varepsilon_{ca}\widehat{m}_{t-1} - \varepsilon_{ca}\widehat{\pi}_t = (\sigma_l + \sigma_c)\widehat{c}_t, \quad (11)$$

$$\sigma_c\widehat{c}_t - \varepsilon_{ca}\widehat{m}_{t-1} + \varepsilon_{ca}\widehat{\pi}_t = \sigma_c\widehat{c}_{t+1} - \varepsilon_{ca}\widehat{m}_t + (\varepsilon_{ca} + 1)\widehat{\pi}_{t+1} - \widehat{R}_t, \quad (12)$$

$$(\varepsilon_{ca} + \sigma_a)\widehat{m}_t = -z\widehat{R}_t + (\sigma_c + \phi_{ac})\widehat{c}_{t+1} + (\varepsilon_{ca} + \sigma_a)\widehat{\pi}_{t+1}, \quad (13)$$

where  $z \equiv \bar{R}/(\bar{R} - 1) > 1$ ,  $\sigma_l \equiv \frac{\bar{u}_{ll}}{\bar{u}_l} \geq 0$ ,  $\sigma_c \equiv -\frac{\bar{c}u_{cc}}{\bar{u}_c} > 0$ ,  $\sigma_a \equiv -\frac{\bar{a}u_{aa}}{\bar{u}_a} > 0$ ,  $\varepsilon_{ca} \equiv \frac{\bar{a}u_{ca}}{\bar{u}_c} > 0$ , and  $\phi_{ac} \equiv \frac{\bar{c}u_{ac}}{\bar{u}_a} > 0$ , and  $\widehat{f}_t$  denotes the percent deviation of a generic variable  $f_t$  from its steady state value  $\bar{f}$  :  $\widehat{f}_t = \log(f_t) - \log(\bar{f})$ . These conditions (and the transversality condition) have to be satisfied by the equilibrium sequences for the steady state deviations of consumption, real balances, the inflation rate, and of the nominal interest rate,  $\{\widehat{c}_t, \widehat{\pi}_t, \widehat{m}_t, \widehat{R}_t\}_{t=0}^{\infty}$  and a monetary policy regime satisfying

$$\widehat{R}_t = \rho_{\pi}\widehat{\pi}_t, \quad \text{or} \quad \widehat{R}_t = \rho_{\pi}\widehat{\pi}_{t+1}, \quad \text{or} \quad \widehat{m}_t = \widehat{m}_{t-1} - \widehat{\pi}_t, \quad (14)$$

where  $\rho_{\pi}$  denotes the steady state inflation elasticity  $\rho_{\pi} \equiv \rho'(\bar{\pi})(\bar{\pi}/\bar{R}) \geq 0$ . Following Benhabib, Schmitt-Grohé, and Uribe (2001a), interest rate policy is called *active* or according to the Taylor-principle if  $\rho_{\pi} > 1$ , and *passive* if  $\rho_{\pi} < 1$ . An active (passive) interest rate setting leads to an increase (decrease) in the real interest rate in response to an increase in the inflation measure. It should be noted that concavity of the utility function implies:  $Y \equiv \sigma_c\sigma_a - \varepsilon_{ca}\phi_{ac} > 0$ ,<sup>5</sup> which restricts the magnitude of real balance effects. A closer look at the equilibrium conditions (11) and (12) reveals that the private sector behavior is not independent of the beginning-of-period value for real balances  $\widehat{m}_{t-1}$ , as they are (implicitly) assumed to lower households' transactions costs. Given that  $\widehat{m}_{t-1}$  is predetermined, the households' behavior can induce the economy to evolve in a history dependent way, i.e. predetermined real money balances can be a state variable. Defining  $[\widehat{m}_t, \widehat{c}_t, \widehat{\pi}_t, \widehat{R}_t]' \equiv \widehat{x}_t$ , the following Definition summarizes this property.

**Definition 5** Consider the fundamental solution for the equilibrium sequences  $\{\widehat{x}_t\}_{t=0}^{\infty}$ , that satisfies the equilibrium conditions (11)-(13) and one monetary policy rule (14). If there

<sup>5</sup>I view this as a realistic implication, given that estimates of  $\varepsilon_{ca}$  and  $\phi_{ac}$ , are usually found to be small. According to US estimates reported in Woodford (2003),  $\varepsilon_{ca}$  does not exceed 0.005 and  $\phi_{ac} \leq 2$ .

exists a locally stable and unique fundamental solution of the linear functional form

$$\hat{x}_t = \begin{pmatrix} \eta_m \\ \eta_c \\ \eta_\pi \\ \eta_R \end{pmatrix} \hat{m}_{t-1} = \Lambda \hat{m}_{t-1} \quad \forall t \geq 0 \quad (15)$$

with  $\eta_i \neq 0$  for  $i = m, c, \pi, R$ , then predetermined real money balances are an endogenous state variable.

It is crucial to note, that if real money balances are a state variable, then not only first period values  $\hat{x}_0$  are associated with a particular first period price level. Instead, the whole set of equilibrium variables is indexed with a specific value for  $m_{-1}$  at each point in time, i.e.  $\hat{x}_t = \Lambda \cdot \hat{m}_{-1}$ ,  $\forall t > 0$ . For a given initial value  $M_{-1}$ , the set of equilibrium sequences relies on a particular initial price level  $P_{-1}$ . Since this mechanism applies to each period, the complete set of sequences for the absolute price level  $\{P_t\}_{t=0}^\infty$  and nominal balances  $\{M_t\}_{t=0}^\infty$  is uniquely determined. Evidently, if real money balances are a state variable, the equilibrium displays nominal determinacy. But as will become clear below, the reverse must not be true.

Yet,  $\hat{m}_{t-1}$  enters the equilibrium conditions jointly with the current inflation rate. Thus, predetermined real money serves as a relevant endogenous state variable, only if the current inflation  $\hat{\pi}_t$  rate is uniquely determined, which implies real determinacy. Given real determinacy, nominal determinacy applies, whenever the beginning-of-period stock of money enters the utility function. But monetary policy is decisive for real determinacy, i.e. for the possibility to uniquely determine a price level sequence. In the subsequent analysis, I will show that this requires the central bank to set the nominal interest rate contingent on current inflation. Under an interest rate peg,  $\rho_\pi = 0$ , an inflation sequence and, therefore, a price level sequence cannot be uniquely determined.<sup>6</sup>

I start with the case where the central bank sets the nominal interest rate according to an interest rate feedback rule. At first, I consider current inflation as the policy indicator,  $\hat{R}_t = \rho_\pi \hat{\pi}_t$ . The following Proposition summarizes the equilibrium prop-

---

<sup>6</sup>It should further be noted that a PFE displays nominal indeterminacy if there are no real balance effects,  $\varepsilon_{ca} = \phi_{ac} = 0$ . Nevertheless, one can always compute a price level sequence for a particular initial price level and a sequence of inflation rates.

erties for the cases where the labor supply elasticity  $1/\sigma_l$  takes a finite value or is infinite.

**Proposition 6** *Consider that beginning-of-period money enters the utility function and that the nominal interest rate is set contingent on changes in current inflation  $\widehat{R}_t = \rho_\pi \widehat{\pi}_t$ .*

1. *When the labor supply elasticity is finite,  $\sigma_l > 0$ , predetermined real money balances serve as a state variable. The equilibrium displays real determinacy and local stability if and only if*

(a)  $\overline{\rho_{\pi 1}} < \rho_\pi < 1$  for  $\varepsilon_{ca} > \frac{\sigma_a}{2z-1}$  and  $\sigma_l > \overline{\sigma}_l$ , leading to non-oscillatory equilibrium sequences, or  $\rho_\pi \in (1, \overline{\rho_{\pi 1}})$ , leading to oscillatory equilibrium sequences,

(b)  $\rho_\pi > 1$  for  $\varepsilon_{ca} < \frac{\sigma_a}{2z-1}$  or  $\sigma_l < \overline{\sigma}_l$ , leading to oscillatory equilibrium sequences,

where  $\overline{\rho_{\pi 1}} \equiv \frac{\sigma_l(\varepsilon_{ca} + \sigma_a) + Y}{\sigma_l(2z-1)\varepsilon_{ca} - \sigma_l\sigma_a - Y}$  and  $\overline{\sigma}_l \equiv \frac{Y}{(2z-1)\varepsilon_{ca} - \sigma_a}$ .

2. *When the labor supply elasticity is infinite,  $\sigma_l = 0$ , predetermined real money balances do not serve as a state variable. Consumption  $\widehat{c}_t$  cannot uniquely be determined, while the equilibrium sequences  $\{\widehat{c}_{t+1}, \widehat{\pi}_t, \widehat{m}_t, \widehat{R}_t\}_{t=0}^\infty$  are locally stable and uniquely determined if and only if  $\rho_\pi > 1$ .*

**Proof.** See appendix A.3.

Proposition 6 reveals that the requirements for local equilibrium stability and uniqueness in terms of the policy parameter  $\rho_\pi$  are not robust with regard to changes in the elasticities  $\varepsilon_{ca}$  and  $\sigma_l$ .<sup>7</sup> For finite labor supply elasticities,  $\sigma_l > 0$ , predetermined real balances serve as an endogenous state variable. Correspondingly, passiveness ( $\rho_\pi < 1$ ) – a violation of the Taylor-principle – is necessary for locally stable, unique, and non-oscillatory equilibrium sequences (see part 1a). An interest rate peg, however, violates the conditions in part 1 of Proposition 6 and, thus, implies real indeterminacy. On the contrary, if interest rate policy follows the Taylor-principle ( $\rho_\pi > 1$ ), locally stable and unique equilibrium sequences are oscillatory, which is hardly recommendable for a central bank that aims at stabilizing the economy. Thus, when

<sup>7</sup>Note that for the sets  $(\overline{\rho_{\pi 1}}, 1)$  and  $(1, \overline{\rho_{\pi 1}})$  (see part 1a. of Proposition 6) to be non-empty  $\sigma_l > Y[(z-1)\varepsilon_{ca} - \sigma_a]^{-1}$  and  $\varepsilon_{ca} > \sigma_a/(z-1)$ , and, respectively,  $\sigma_l < Y[(z-1)\varepsilon_{ca} - \sigma_a]^{-1}$  or  $\varepsilon_{ca} < \sigma_a/(z-1)$  has to be satisfied.

beginning-of-period money relates to households' consumption, interest rate policy that reacts on current inflation should rather be passive than active for macroeconomic stability and for the unique determination of the price level.

To see this, suppose that inflation exceeds its steady state value and equilibrium sequences are non-oscillatory.<sup>8</sup> Given that the inflation elasticity is positive,  $\rho_\pi > 0$ , the nominal interest rate rises, which – ceteris paribus – causes households to reduce their end-of-period real money holdings  $\hat{m}_t$ , by (13). According to (12), the expected real interest rate is further negatively related to the growth rate of real balances. Thus, an active interest rate setting – implying an increase in the real interest rate – leads to a decline in the level and the growth rate of real balances, such that the sequences of real balances and, thus, of consumption and inflation do not converge to the steady state.

Notably, the equilibrium exhibits different properties if the marginal disutility of labor is constant, i.e. if the inverse of the labor supply elasticity is zero (see part 2 of Proposition 6). Then, the amount of labor supplied by the households is not related to their consumption expenditures and the marginal utility of consumption is always identical to its steady state value (see 11). In this case, the Euler equation and money demand reduce to a constant real interest rate  $\hat{R}_t - \hat{\pi}_{t+1} = 0$ , and  $\sigma_a \hat{m}_t = -z \hat{R}_t + \phi_{ac} \hat{c}_{t+1} + \sigma_a \hat{\pi}_{t+1}$ , such that the equilibrium is not associated with a unique value for beginning-of-period real money and that current consumption  $\hat{c}_t$  cannot be determined. Correspondingly, predetermined real money balances do not serve as a state variable. The equilibrium sequences for  $\hat{c}_{t+1}$ ,  $\hat{\pi}_t$ ,  $\hat{m}_t$ , and  $\hat{R}_t$  are then locally stable and uniquely determined for an active interest rate policy, which contrasts the results for the case of finite labor supply elasticities,  $\sigma_l > 0$ , presented in part 1 of Proposition 6.

I now turn to the case where the central bank applies a forward looking rule,  $\hat{R}_t = \rho_\pi \hat{\pi}_{t+1}$ .

**Proposition 7** *Consider that beginning-of-period money enters the utility function and that the nominal interest rate is set contingent on changes in future inflation  $\hat{R}_t = \rho_\pi \hat{\pi}_{t+1}$ . Then, consumption and inflation cannot uniquely be determined and predetermined real money balances do not serve as a state variable.*

---

<sup>8</sup>The latter property implies that current and expected future inflation are not negatively related.

1. When the labor supply elasticity is finite,  $\sigma_l > 0$ , then  $\rho_\pi > 1$  is a necessary condition for uniqueness and local stability of the equilibrium sequences

$$\{\widehat{c}_{t+1}, \widehat{\pi}_{t+1}, \widehat{m}_t, \widehat{R}_t\}_{t=0}^\infty.$$

Necessary and sufficient conditions are given by:

$$(a) \quad 1 < \rho_\pi \text{ for } \sigma_l > \overline{\sigma}_{l2} \text{ and } \varepsilon_{ca} > \frac{\sigma_a}{z-1},$$

$$(b) \quad 1 < \rho_\pi < \overline{\rho}_{\pi 2}, \text{ for } \sigma_l < \overline{\sigma}_{l2} \text{ or } \varepsilon_{ca} < \frac{\sigma_a}{z-1}, \text{ or } 1 < \overline{\rho}_{\pi 2} < \rho_\pi \text{ if } \sigma_l > \overline{\sigma}_l \text{ and } \varepsilon_{ca} > \frac{\sigma_a}{2z-1}, \text{ for } \sigma_l \in (\overline{\sigma}_l, \overline{\sigma}_{l2}) \text{ or } \varepsilon_{ca} \in \left(\frac{\sigma_a}{2z-1}, \frac{\sigma_a}{z-1}\right),$$

$$(c) \quad 1 < \overline{\rho}_{\pi 2} < \rho_\pi < -\overline{\rho}_{\pi 1} \text{ for } \sigma_l < \overline{\sigma}_l \text{ or } \varepsilon_{ca} < \frac{\sigma_a}{2z-1},$$

$$\text{where } \overline{\sigma}_{l2} \equiv \frac{Y}{(z-1)\varepsilon_{ca} - \sigma_a} \text{ and } \overline{\rho}_{\pi 2} \equiv \frac{Y + \sigma_l(\varepsilon_{ca} + \sigma_a)}{Y + \sigma_l(\varepsilon_{ca} + \sigma_a) - z\varepsilon_{ca}\sigma_l}.$$

2. When the labor supply elasticity is infinite,  $\sigma_l = 0$ , then the equilibrium sequences  $\{\widehat{c}_{t+1}, E_t \widehat{\pi}_{t+1}, \widehat{m}_t, \widehat{R}_t\}_{t=0}^\infty$  are locally stable and uniquely determined if and only if  $\rho_\pi \neq 1$ .

**Proof.** See appendix A.4.

In comparison to Proposition 6 the most fundamental difference relates to the role of beginning-of-period real balances,  $\widehat{m}_{t-1}$ . If monetary policy applies a forward looking interest rate rule, households' optimal consumption decisions are not affected by predetermined real money balances. I.e. real money balances are not a state variable of the economy. The initial stock of real money balances  $m_{-1} = M_{-1}/P_{-1}$  is irrelevant for the equilibrium allocation and thus, there are multiple price level sequences. Correspondingly, current inflation can not be pinned down since it enters jointly with  $\widehat{m}_{t-1}$  and the equilibrium is consistent with infinitely many values for current inflation. Given that the current values for inflation and consumption can not be determined, households adjust  $\widehat{m}_t$  in accordance with their planned future consumption  $\widehat{c}_{t+1}$ , implying that their behavior is not history dependent. On the contrary, if current inflation serves as a policy indicator, predetermined real money balances restrict households' consumption decisions and initial real money balances are relevant for the equilibrium sequences  $\widehat{c}_t, \widehat{\pi}_t, \widehat{m}_t, \widehat{R}_t$  at each point in time: predetermined real money balances are an endogenous state variable (see Definition 3) and the perfect foresight equilibrium is characterized by nominal determinacy. Remarkably in that case, by applying an interest rate rule, the complete set of nominal

sequences, the absolute price level and nominal balances, can be uniquely determined.

Under an interest rate rule featuring current inflation, it turns out that there is no robust value for the inflation elasticity that ensures local stability and uniqueness. For example, when the real balance effect and the labor supply elasticity satisfy  $\varepsilon_{ca} > \frac{\sigma_a}{2z-1}$  and  $\sigma_l > \frac{Y}{(2z-1)\varepsilon_{ca}-\sigma_a}$ , interest rate policy should be passive,  $\rho_\pi < 1$ , while the inverse,  $\rho_\pi > 1$ , is required under  $\varepsilon_{ca} < \frac{\sigma_a}{2z-1}$  or  $\sigma_l < \bar{\sigma}_l$  (see Proposition 6). When the central bank sets the nominal interest rate contingent on expected future inflation, activeness  $\rho_\pi > 1$  is always necessary (but not sufficient) for uniqueness.<sup>9</sup> As in the previous case (see part 2 of Proposition 6), the equilibrium exhibits different properties if the labor supply elasticity is infinite  $\sigma_l = 0$  as described in part 2 of Proposition 7. With a forward looking interest rate rule, the model then reduces to a set of static equilibrium conditions characterized by unique equilibrium sequences  $\{\hat{c}_{t+1}, \hat{\pi}_{t+1}, \hat{m}_t, \hat{R}_t\}_{t=0}^\infty$  for any non-zero inflation elasticity  $\rho_\pi \neq 1$ .

Under a money growth regime equilibrium determination is less sensitive. Ruling out unreasonable parameter values, I focus, for convenience, on the case where the inverse of the elasticity of intertemporal substitution of money is not extremely large,  $\sigma_a < z = \bar{R}/(\bar{R} - 1)$ .<sup>10</sup>

**Proposition 8** *Suppose that beginning-of-period money enters the utility function and that  $\sigma_a < z$ . Under a constant money growth rule, predetermined real money balances do not serve as a state variable. The equilibrium sequences  $\{\hat{c}_{t+1}, \hat{\pi}_{t+1}, \hat{m}_t, \hat{R}_t\} \forall t \geq 0$  are locally stable and uniquely determined, and there exists a unique consistent price level  $\forall t \geq 0$ .*

**Proof.** See appendix A.5.

A comparison of the results in the propositions 6-8 shows that the PFE displays real determinacy, if and only if predetermined real money balances are an endogenous state variable. This requires an interest policy contingent on current inflation. Remarkably, the money growth regime leads to an equilibrium behavior being different

<sup>9</sup>Non-emptiness of the sets for  $\rho_\pi$  requires  $\bar{\rho}_{\pi 2} > 1$  and  $-\bar{\rho}_{\pi 1} > \bar{\rho}_{\pi 2}$ , which is fulfilled for the given restrictions on  $\sigma_l$  and  $\varepsilon_{ca}$  in part 1b and 1c.

<sup>10</sup>It should be noted that  $\sigma_a < z$  is just a sufficient precondition for the result in Proposition 8 and hardly restrictive if one assigns values for  $\sigma_a$  that relate to reasonable magnitudes of  $\sigma_c$ .

from the behavior under both interest rate policy regimes. On the one hand, the price level can always be determined if real balances are determined, given that the value for the nominal stock of money is known in every period. On the other hand, the initial values for the inflation rate  $\hat{\pi}_0$  and real money  $\hat{m}_{-1}$  are irrelevant for equilibrium determination, implying that there are – for different initial price levels – multiple values for both which are consistent with a unique set of equilibrium sequences  $\{\hat{c}_{t+1}, \hat{\pi}_{t+1}, \hat{m}_t, \hat{R}_t\}_{t=0}^{\infty}$ . I.e. the equilibrium displays nominal determinacy but does not rely on predetermined real money balances as an endogenous state variable. Put differently, for the economy to evolve in a history dependent way, it is, therefore, not sufficient that monetary policy is conducted in a backward looking way. Instead, it is the households' consumption decision rather than a restriction on the evolution of money, which is responsible for the equilibrium sequences to depend on beginning-of-period money holdings. There is an analogy to the role of physical capital in a standard real business cycle model with a depreciation rate equal to one. Capital remains a relevant state variable, even though the model (virtually) lacks an accumulation equation.<sup>11</sup>

### 3.2 End-of-period money

Next, I will briefly summarize the requirements for equilibrium determination under the assumption that end-of-period money holdings enter the utility function. In this case, the equilibrium displays nominal indeterminacy – unless monetary policy follows a constant money growth rule. This specification has also been examined by Benhabib, Schmitt-Grohé, and Uribe (2001a) and by Woodford (2003a) for interest rate policies, and by Carlstrom and Fuerst (2003) for money growth rules. The deterministic steady state for this version is characterized by the following conditions,  $\bar{R} = \bar{\pi}/\beta$ ,  $-u_l(\bar{c}) = u_c(\bar{c}, \bar{m})$ , and  $u_c(\bar{c}, \bar{m})(R - 1) = u_a(\bar{m}, \bar{c})$ .<sup>12</sup> Log-linearizing the model summarized in Definition 1 for  $A_t = M_t$  at the steady state with  $\bar{R} > 1$  leads

---

<sup>11</sup>Consider a real version of my model, with perfect competition, a production technology satisfying  $y_t = s_t k_{t-1}^\alpha l_t^{1-\alpha}$ , where  $k_{t-1}$  denotes the beginning-of-period stock of physical capital and  $\alpha \in (0, 1)$ , and a capital depreciation rate of 100%. Nevertheless, capital serves as a relevant state variable, i.e.,  $k_{t-1}$  affects the equilibrium allocation in period  $t$ .

<sup>12</sup>A discussion of steady state uniqueness is provided in appendix A.2.

to the following set of equilibrium conditions:

$$\varepsilon_{ca}\widehat{m}_t = (\sigma_l + \sigma_c)\widehat{c}_t, \quad (16)$$

$$\sigma_c\widehat{c}_t - \varepsilon_{ca}\widehat{m}_t = \sigma_c\widehat{c}_{t+1} - \varepsilon_{ca}\widehat{m}_{t+1} - \widehat{R}_t + \widehat{\pi}_{t+1}, \quad (17)$$

$$(\varepsilon_{ca} + \sigma_a)\widehat{m}_t = (\phi_{ac} + \sigma_c)\widehat{c}_t - (z - 1)\widehat{R}_t. \quad (18)$$

The conditions (16)-(18), the transversality condition, and a monetary policy rule (14) have to be satisfied by the equilibrium sequences  $\{\widehat{c}_t, \widehat{\pi}_t, \widehat{m}_t, \widehat{R}_t\}_{t=0}^{\infty}$ . In contrast to the **B**-version, consumption and inflation are independent of beginning-of-period real balances. To put it differently, predetermined real money balances can not serve as a state variable. Instead, the private sector behavior is entirely forward-looking in the **E**-version with the consequence that the equilibrium displays nominal indeterminacy under interest rate policy.

The following Proposition summarizes the conditions for equilibrium determination under interest rate policy.

**Proposition 9** *Suppose that end-of-period money enters the utility function and that the central bank sets the nominal interest rate.*

1. *When current inflation enters the interest rate rule,  $\widehat{R}_t = \rho_\pi \widehat{\pi}_t$ , the equilibrium displays real determinacy and local stability if and only if  $\rho_\pi > 1$ .*
2. *When future inflation enters the interest rate rule,  $\widehat{R}_t = \rho_\pi \widehat{\pi}_{t+1}$ , inflation cannot uniquely be determined. The equilibrium sequences  $\{\widehat{c}_t, \widehat{\pi}_{t+1}, \widehat{m}_t, \widehat{R}_t\}_{t=0}^{\infty}$  are locally stable and uniquely determined if and only if*

$$(i) \ \rho_\pi > 1 \text{ or } \rho_\pi < \left(1 + \frac{2(z-1)\sigma_l\varepsilon_{ca}}{Y + \sigma_l(\varepsilon_{ca} + \sigma_a)}\right)^{-1} \text{ for } \sigma_l > 0, \text{ and}$$

$$(ii) \ \rho_\pi \neq 1 \text{ for } \sigma_l = 0.$$

**Proof.** See appendix A.6.

As in the **B**-version, equilibrium determination depends on the particular interest rate rule. When the nominal interest rate is set contingent on current inflation, inflation can be determined for all periods. Under a forward looking interest rate policy, one can only uniquely determine future inflation. In any case, the initial price

level and initial real balances are irrelevant for a *REE*, implying nominal indeterminacy and the absence of an endogenous state variable. Uniqueness of equilibrium sequences is further ensured by an active interest rate policy,  $\rho_\pi > 1$ , under both types of rules. For the special case, where the labor supply elasticity is infinite, any forward looking interest rate rule satisfying  $\rho_\pi \neq 1$  leads to unique equilibrium sequences  $\{\hat{c}_t, \hat{\pi}_{t+1}, \hat{m}_t, \hat{R}_t\}_{t=0}^\infty$ .

Turning to the case where the central bank holds the money growth rate constant, I find that the equilibrium behavior closely relates to the one in the **B**-version.

**Proposition 10** *Suppose that end-of-period money enters the utility function and that the money growth rate is held constant. Then, the equilibrium sequences  $\{\hat{c}_t, \hat{m}_t, \hat{R}_t\} \forall t \geq 0$  and  $\{\hat{\pi}_t\} \forall t \geq 1$  are locally stable and uniquely determined, and there exists a unique consistent price level  $\forall t \geq 0$ .*

**Proof.** See appendix A.7.

Summing up, the specification of money demand has substantial consequences for the determination of equilibrium sequences and for macroeconomic stability. The beginning-of-period value for real money balances is only relevant for equilibrium determination in the **B**-version under a non-forward looking interest rate rule. In the **E**-version, where the households' behavior lacks any backward looking element, the initial value of real balances is irrelevant for any policy regime under consideration. Whether beginning-of period real money is serving as a relevant endogenous state variable or not, is, on the one hand, decisive for a unique determination of the price level, and, on the other hand, crucially affects the conditions for local stability and uniqueness under an interest rate policy regime: policy should rather be passive than active, to avoid unstable or oscillatory equilibrium sequences. Under a constant money growth regime, however, local stability and uniqueness of equilibrium sequences and a unique price level sequence is ensured for both versions – regardless of the labor supply elasticity. Given that the stock of money is known in every period, a unique sequence for real money balances suffices to pin down uniquely the entire sequence for the absolute price level. Evidently, this does not require the economy to evolve in a history dependent way.

### 3.3 Related results

The main novel results in this section refer to the case where beginning-of-period money enters the utility function and the central bank applies an interest rate rule, while some results for the alternative cases correspond to results in related studies on real balances effects and equilibrium determinacy in flexible price models. For example, my findings for the E-version (see part 1 of Proposition 9) resemble the results in Benhabib, Schmitt-Grohé, and Uribe (2001a) and Woodford (2003a) for non-separable utility functions. They find that when current inflation serves as an indicator, active interest rate setting is necessary and sufficient for local stability and uniqueness. This, however, changes when beginning-of-period money provides utility, since equilibrium sequences are then – except for the case  $\sigma_l = 0$  – unstable or oscillatory (see Proposition 6). Thus, the literature has disregarded the role of pre-determined real balances as a relevant state variable, which substantially affects the real and nominal determinacy properties.

If the monetary authority applies a constant money growth rule, then local stability and uniqueness impose restrictions on preferences only in case where the stock of money held at the beginning of the period provides utility. In particular, the inverse of the intertemporal elasticity of substitution for real money balances should not be too large (see Proposition 8), which corresponds to the results in Brock (1974), Matsuyama (1990), Carlstrom and Fuerst (2003) and Woodford (2003a). Assuming that end-of-period money provides transaction services, Brock (1974), Matsuyama (1990) and Woodford (2003a), show that local stability and uniqueness is ensured if consumption and real balances are Edgeworth-complements, as in my framework. Furthermore, Carlstrom and Fuerst (2003) find that the intertemporal elasticity of substitution for money can matter for local stability and uniqueness is guaranteed, as in Proposition 8.

To unveil the role of non-separability for the results and to facilitate comparisons with related studies (see, e.g., Carlstrom and Fuerst (2001)), I further briefly discuss the case where money demand is separable,  $\varepsilon_{ca} = \phi_{ac} = 0$ . Then, the model reduces to

$$\widehat{R}_t = \widehat{\pi}_{t+1}, \text{ and } \sigma_a \widehat{m}_t = \begin{cases} -(z - \sigma_a) \widehat{R}_t & \text{for the } B\text{-version} \\ -(z - 1) \widehat{R}_t & \text{for the } E\text{-version} \end{cases} .$$

while consumption is exogenously determined. When utility is separable, the conditions for uniqueness under money growth policy, which are presented in Proposition 8 and 10, are unchanged. In contrast to the results for the non-separable case, the particular stock of money that enters the utility function is now irrelevant for equilibrium determination under interest rate policy: Equilibrium uniqueness requires  $\rho_\pi > 1$  for  $\hat{R}_t = \rho_\pi \hat{\pi}_t$  and  $\rho_\pi \neq 1$  for  $\hat{R}_t = \rho_\pi \hat{\pi}_{t+1}$ , which accords to the results in Carlstrom and Fuerst (2001). As in the case of non-separable utility, current inflation cannot be determined under a forward looking interest rate rule, while under a money growth rule inflation is only indetermined in the first period.

## 4 Imperfectly Flexible Prices

In a framework with monopolistic competitive firms and staggered price setting as developed by Calvo (1983), the initial price level belongs to the set of relevant state variables.<sup>13</sup> Under this specification, real balances serve as a relevant predetermined state variable for all aforementioned policy rules, when the beginning-of-period specification applies. If, however, the end-of-period stock of money enters the utility function, households are entirely forward looking, and real money serves as a relevant state variable only if monetary policy is history dependent, i.e., when the central bank applies a money growth rule. Nonetheless, the determinacy properties under constant money growth and sticky prices correspond to those under flexible prices. The main implications for equilibrium uniqueness and stability under imperfectly flexible prices are as follows:

- When beginning-of-period money provides utility, interest rate policy has to be passive to lead to locally stable, unique, and non-oscillatory equilibrium sequences, regardless whether current or future inflation enters the policy rule. An active interest rate policy is associated with locally stable and unique equilibrium sequences if and only if end-of-period money provides utility and current inflation serves as the policy indicator.<sup>14</sup>

<sup>13</sup>Please refer to Schabert and Stoltenberg (2005) for details in that case.

<sup>14</sup>To be more precise, these results apply for finite labor supply elasticities. For the case of an infinite labor supply elasticity there is a related paper by Brückner and Schabert (2005). Assuming staggered

- As under flexible prices, the central bank can ensure equilibrium sequence to be uniquely determined, locally stable, and non-oscillatory under both timing specifications by holding the growth rate of money constant, provided that real balance effects are not extremely large.

## 5 Conclusion

Real balance effects typically arise when transaction costs are specified in a general equilibrium model in form of shopping time or real resource costs, which are reduced by money holdings. The fact that the equilibrium sequences for real balances and consumption can then not separately be determined, is broadly viewed as negligible for the assessment of monetary policy, given that empirical evidence suggests real balance effects to be relatively small (Ireland, 2004). In contrast to this view, it is demonstrated in this paper that the existence (not the magnitude) of real balance effects has substantial implications for the determination of a rational expectations equilibrium and of the price level under interest rate policy.

However, for real balance effects to contribute to price level determination, as for example suggested by Patinkin (1949, 1965), predetermined real money balances have to serve as a state variable. Remarkably, these properties require that the stock of money at the beginning of the period yields transaction services, which corresponds to Svensson's timing assumption (1985), that the goods market closes before the asset market opens. Hence, real money that has been acquired in the previous period restricts households' current consumption expenditures. Then, there exists a unique initial price level that is consistent with a rational expectations equilibrium, i.e., the equilibrium displays nominal determinacy. In that case, interest policy should be passive to ensure unique, non-oscillatory and locally stable equilibrium sequences – a violation of the Taylor-principle. If, on the other hand, current consumption is related to the end-of-period stock of money, then the equilibrium displays nominal indeterminacy, and the well-known principles for uniqueness and stability of equilibrium sequences of a cashless economy (roughly) apply. Remarkably, these results highlight, that the existence and timing of real balance effects jointly have substantial

---

price setting and a specific functional form for utility, they consider the implications of the timing of markets on optimal monetary policy under discretion.

implications for equilibrium determination.

If monetary policy follows a constant money growth rule, the conditions for equilibrium uniqueness are likely to be ensured. Though the economy does not evolve in a history dependent way, the entire path for the absolute price level is uniquely determined in both versions. These results suggest that a central bank that aims to avoid multiple, unstable, or oscillatory equilibrium sequences in an environment where transaction frictions are non-negligible, should rather control the supply of money than the nominal interest rate.

Yet, an optimal conduct of monetary policy will certainly require the supply of money to be state contingent (as an interest rate feedback rule), which might be associated with different determinacy implications than a constant money growth regime.

## References

- BÉNASSY, J. (2000): "Price Level Determinacy under a Pure Interest Rate Peg," *Review of Economic Dynamics*, 3, 194–211.
- BENHABIB, J., S. SCHMITT-GROHÉ, AND M. URIBE (2001a): "Monetary Policy and Multiple Equilibria," *American Economic Review*, 91, 167–185.
- (2001b): "The Perils of Taylor Rules," *Journal of Economic Theory*, 96, 40–69.
- (2003): "Backward-Looking Interest-Rate Rules, Interest Rate Smoothing, and Macroeconomic Instability," *Journal of Money, Credit, and Banking*, 35, 1379–1412.
- BLANCHARD, O., AND C. KAHN (1980): "The Solution of Linear Difference Models under Rational Expectations," *Econometrica*, 48, 1305–1313.
- BROCK, W. (1974): "Money and Growth: The Case of Long Run Perfect Foresight," *International Economic Review*, 15, 750–777.
- BRÜCKNER, M., AND A. SCHABERT (2005): "Can Money matter for Interest Rate Policy?," *Journal of Economic Dynamics and Control*, 30, 2823–2957.
- CALVO, G. A. (1983): "Staggered price setting in a utility-maximizing framework.," *Journal of Monetary Economics*, 12, 383–398.
- CARLSTROM, C., AND T. FUERST (2001): "Timing and Real Indeterminacy in Monetary Models," *Journal of Monetary Economics*, 47, 285–298.
- (2003): "Money Growth Rules and Price Level Determinacy," *Review of Economic Dynamics*, 6, 263–275.
- (2005): "Investment and Interest Rate Policy: A Discrete Time Analysis," *Journal of Economic Theory*, 123, 4–20.
- DUPOR, B. (2001): "Investment and Interest Rate Policy," *Journal of Economic Theory*, 98, 85–113.
- FEENSTRA, R. (1986): "Functional Equivalence between Liquidity Costs and the Utility of Money," *Journal of Monetary Economics*, 17, 271–291.

- IRELAND, P. (2004): "Money's Role in the Monetary Business Cycle," *Journal of Money Credit and Banking*, 36, 969–983.
- KUROZUMI, T. (2006): "Determinacy and and expectational stability of equilibrium in a monetary sticky-price model with Taylor rule," *Journal of Monetary Economics*, 53, 827–846.
- LEEPER, E. M. (1991): "Equilibria under "Active" and "Passive" Monetary and Fiscal Policies," *Journal of Monetary Economics*, 27, 129–147.
- LJUNGQVIST, L., AND T. SARGENT (2004): *Recursive Macroeconomic Theory*. The MIT Press, Cambridge Massachusetts, 2nd edn.
- LUCAS, R. E. (2000): "Inflation and welfare," *Econometrica*, 68, 247–274.
- MATSUYAMA, K. (1990): "Sunspot Equilibria (Rational Bubbles) in a Model of Money-in-the-Utility-Function," *Journal of Monetary Economics*, 25, 137–144.
- (1991): "Endogenous Price Fluctuations in an Optimizing Model of the Monetary Economy," *Econometrica*, 59, 1617–1631.
- MCCALLUM, B. T., AND M. GOODFRIEND (1987): *Demand for Money: Theoretical Studies*, in: *The New Palgrave: A Dictionary of Economics*. Macmillan, London.
- MCCALLUM, B. T., AND E. NELSON (1999): "An Optimizing IS-LM Specification for Monetary Policy and Business Cycle Analysis," *Journal of Money, Credit, and Banking*, 31, 296–316.
- OBSTFELD, M., AND K. ROGOFF (1983): "Speculative Hyperinflations in Maximizing Models: Can we rule them out?," *Journal of Political Economy*, 91, 675–687.
- PATINKIN, D. (1949): "The Indeterminacy of Absolute Prices in Classical Economic Theory," *Econometrica*, 17, 1–27.
- (1965): *Money, Interest and Prices*. Harper and Row, New York, 2nd edn.
- PERSSON, M., AND L. E. O. SVENSSON (2006): "Time Consistency of Fiscal and Monetary Policy: A Solution," *Econometrica*, 74, 193–212.

- SARGENT, T., AND N. WALLACE (1975): "'Rational' Expectations, the Optimal Monetary Instrument, and the Optimal Money Rule," *Journal of Political Economy*, 83, 1–19.
- SCHABERT, A., AND C. STOLTENBERG (2005): "Money Demand and Macroeconomic Stability Revisited," *ECB Working Paper Series*, 458.
- SIMS, C. A. (1994): "A Simple Model for Study of the Determination of the Price Level and the Interaction of Monetary and Fiscal Policy," *Journal of Economic Theory*, 4, 381–399.
- SVENSSON, L. (1985): "Money and Asset Prices in a Cash-in-advance Model," *Journal of Political Economy*, 93, 919–944.
- TAYLOR, J. (1993): "Discretion versus Policy Rules in Practice," *Carnegie-Rochester Conference Series on Public Policy*, 39, 195–214.
- WOODFORD, M. (1990): "The Optimum Quantity of Money," in *Handbook of Monetary Economics*. North Holland.
- (1994): "Monetary Policy and Price Level Determinacy in a Cash-in-Advance Economy," *Economic Theory*, 4, 345–380.
- (1995): "Price-level Determinacy without Control of s Monetary Aggregate," *Carnegie-Rochester Series*, 43, 1–46.
- (1996): "Control of the Public Debt: A Requirement for Price Stability," *NBER Working Paper*, 5684.
- (2003a): *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton University Press.

# A Appendix

## A.1 Equivalence between explicit transaction frictions and money-in-the-utility-function

In this appendix I examine the relation between the money-in-the-utility-function specification, which is applied throughout the analysis in chapter 4, and explicit specifications of transaction frictions, i.e., a shopping time specification and a specification where transactions are associated with real resource costs. For this demonstration, which relates to the analysis in Brock (1974) and Feenstra (1986), I assume for both alternative specification that the objective of the representative household is given by

$$\sum_{t=0}^{\infty} \beta^t v(c_t, x_t), \quad v_c > 0, v_{cc} < 0, v_x > 0, v_{cx} = 0, \text{ and } v_{xx} \leq 0, \quad (19)$$

where  $x$  denotes leisure.

1. First I consider a conventional *shopping time* specification which relates to the one applied in Brock (1974), McCallum and Goodfriend (1987) or Ljungqvist and Sargent (2004). For this purpose, I assume that households have to allocate total time endowment, which is normalized to equal one, to leisure  $x$ , working time  $l$ , and shopping time  $s$ , where the shopping time is assumed to depend on the consumption expenditures and on real balances

$$1 \geq x_t + l_t + s_t, \quad \text{where} \quad s_t = H(c_t, A_t/P_t).$$

Following Ljungqvist and Sargent (2004), I assume that the shopping time function  $H$  satisfies:  $H_c > 0$ ,  $H_{cc} > 0$ ,  $H_a < 0$ , and  $H_{aa} > 0$  and  $H_{ca} \leq 0$ . Using that  $x_t = 1 - l_t - s_t$  holds in the household's optimum, the utility function can be written as

$$u(c_t, l_t, a_t) = v(c_t, 1 - l_t - H(c_t, a_t)),$$

where  $u_c = v_c + v_x(-H_c) \leq 0$ ,  $u_a = v_x(-H_a) > 0$ ,  $u_l = -v_x < 0$ ,  $u_{cc} =$

$v_{cc} + v_{xx}H_c^2 - v_xH_{cc} < 0$ ,  $u_{cl} = v_{xx}H_c \leq 0$ ,  $u_{aa} = v_{xx}H_a^2 - v_xH_{aa} < 0$ ,  $u_{al} = v_{xx}H_a \geq 0$ , as well as  $u_{ll} = v_{xx}$ . Hence, the marginal utility of consumption, which is given by

$$u_{ca} = v_{xx}H_aH_c - v_xH_{ca},$$

is non-decreasing in real balances. If the shopping time function is non-separable or if leisure enters the utility function in a non-linear way, then marginal utility of consumption is strictly increasing in real balances.

2. Next, I closely follow the analysis in Feenstra (1986), and assume that purchases of consumption goods are associated with *real resource costs of transactions*  $\phi(c_t, a_t)$ , which satisfy:  $\phi \geq 0$ ,  $\phi(0, a) = 0$ ,  $\phi_c > 0$ ,  $\phi_a < 0$ ,  $\phi_{cc} \geq 0$ ,  $\phi_{aa} \geq 0$ ,  $\phi_{ac} \leq 0$ . Households' budget constraint then reads

$$M_t + B_t + P_t\phi(c_t, a_t) + P_t c_t \leq R_{t-1}B_{t-1} + M_{t-1} + P_t w_t l_t + P_t \omega_t - P_t \tau_t. \quad (20)$$

Maximizing (19) subject to (20), a no-Ponzi game condition, and  $x_t \leq 1 - l_t$ , leads – inter alia – to the following first order conditions for consumption and leisure:

$$\lambda_{rt}(1 + \phi_c(c_t, a_t)) = v_c(c_t), \quad \lambda_{rt}w_t = v_x(1 - l_t),$$

where  $\lambda_{rt}$  denotes the Lagrange multiplier on (20). Note that the aggregate resource constraint now reads  $y_t = c_t + \phi(c_t, a_t)$ . Using the linear production technology, I therefore obtain the following equilibrium condition:  $l_t = c_t + \phi(c_t, a_t)$ . Combining these conditions and using that  $w_t = 1$ , leads to the following expression for the marginal utility of consumption:

$$v_c(c_t) = v_x(1 - c_t - \phi(c_t, a_t))(1 + \phi_c(c_t, a_t)).$$

Evidently, the equilibrium sequence of consumption is in general not independent from real money balances due to the existence of transaction costs. Differentiating the latter condition gives

$$\frac{dc_t}{da_t} = \frac{v_x\phi_{ca} - v_{xx}(1 + \phi_c)\phi_a}{v_{cc} + (1 + \phi_c)^2v_{xx} - \phi_{cc}v_x}.$$

Hence, consumption is positively related to real balances even if either the cross-derivative  $\phi_{ca}$  vanishes or the labor supply elasticity is infinite, i.e.  $v_{xx} =$

0.

The corresponding properties of my MIU specification immediately show that an equivalence between the latter and the shopping time specification in 1. requires consumption and real balances to be Edgeworth-complements in the MIU version, if  $v_{xx} < 0$  or  $H_{ca} < 0$ . In order to compare the MIU specification with the transaction cost specification in 2., I apply the first order condition for consumption and labor, the aggregate resource constraint, and the production function, which imply that the equilibrium sequence of consumption under a MIU specification satisfies  $dc_t/da_t = -u_{ca}(u_{cc} + 1)u_{ll}^{-1}$ . Evidently, an equivalence between both specifications requires consumption and real balances to be Edgeworth-complements, i.e.  $u_{ca} > 0$ , if  $\phi_{ca} < 0$  or  $v_{xx} < 0$ . Thus,  $v_{xx} < 0$ , which implies a finite labor supply elasticity is sufficient for the existence of real balance effects under both specifications of transaction frictions.

## A.2 Existence and uniqueness of the steady state

In this appendix, I briefly examine the steady state properties of the model. I restrict my attention to the case where the nominal interest rate is strictly positive,  $R - 1 > 0$ . I further omit, for convenience, bars which are throughout the paper used to mark steady state values.

When the stock of money at the beginning of the period enters the utility function, the deterministic steady state is characterized by the following conditions:  $-u_l(c) = u_c(c, m/\pi)$ ,  $R = \pi/\beta$  and  $u_a(c, m/\pi)(u_c(c, m/\pi))^{-1} = R - 1$ . For an interest rate policy regime, it is assumed that the policy rule of the central bank,  $R(\pi)$ , has a unique solution for the steady state relation  $R = \pi/\beta$ , so that the inflation rate can be substituted out. The first equation implies that  $c$  is an implicit function of  $m$ ,  $c = f(m)$ , with  $f'(m) = -u_{ca}[R\beta(u_{ll} + u_{cc})]^{-1} > 0$ . Using this, the third equation can be used to determine the steady state value for  $m$  with  $u_a(f(m), m/\pi)[u_c(f(m), m/(R\beta))]^{-1} = R - 1$ . Differentiating the fraction on the left hand side reveals that

$$\frac{du_a/u_c}{dm} = \frac{u_c(u_{cc}u_{aa} - u_{ca}^2) + u_{ll}(u_{aa}u_c - u_a u_{ca})}{R\beta u_c^2(u_{ll} + u_{cc})} < 0,$$

as I assumed concavity for  $u(c, a)$ . It follows that a globally unique steady state exists if and only if:

$$\lim_{m \rightarrow 0} \frac{u_a(f(m), m/R\beta)}{u_c(f(m), m/(R\beta))} > R - 1.$$

Thus, steady state uniqueness relies on money to be essential (see Obstfeld and Rogoff, 1983): The marginal utility of real money balances should grow with a rate that is higher than the rate by which  $1/u_c$  converges to zero when  $m$  approaches zero. An analogous line of arguments in case of a money growth policy leads to the condition  $\lim_{m \rightarrow 0} u_a(g(m), m/\mu)[u_c(g(m), m/\mu)]^{-1} > \mu/\beta - 1$ , where  $c = g(m)$  is the implicit relation derived of the steady state condition  $-u_l(c) = u_c(c, m/\mu)$  with  $g'(m) = -u_{ca}[\mu(u_{ll} + u_{cc})]^{-1} > 0$ . The condition for existence and uniqueness for the interest rate policy regime if end-of-period money provides transaction services is

$$\lim_{m \rightarrow 0} \frac{u_a(f_E(m), m)}{u_c(f_E(m), m)} > \frac{R - 1}{R},$$

with  $f_E(m)' = -u_{ca}[u_{ll} + u_{cc}]^{-1} > 0$ . If the monetary authority applies a constant money growth rule then  $\lim_{m \rightarrow 0} u_a(f_E(m), m)[u_c(f_E(m), m)]^{-1} > (\mu/\beta - 1)/(\mu/\beta)$  must be satisfied.

### A.3 Proof of Proposition 6

Consider a monetary policy regime that sets the nominal interest rate contingent on changes in current inflation,  $\hat{R}_t = \rho\pi\hat{\pi}_t$ .

First, I establish the conditions for local stability and uniqueness. Second, if the labor supply supply elasticity is finite, I show that the existence of exactly one stable eigenvalue (assigned to real money balances,  $\eta_m$ ) implies non-zero coefficients  $\eta_i$ ,  $i = c, \pi, R$  of the fundamental solution.

Reducing the model in (11)-(13) leads to the following system in inflation and real

money balances:

$$\begin{aligned}
\begin{pmatrix} \hat{\pi}_{t+1} \\ \hat{m}_t \end{pmatrix} &= \begin{pmatrix} \frac{\sigma_l \varepsilon_{ca}}{\sigma_l + \sigma_c} + 1 & -\frac{\sigma_l \varepsilon_{ca}}{\sigma_l + \sigma_c} \\ \frac{Y + \sigma_l(\varepsilon_{ca} + \sigma_a)}{\sigma_c + \sigma_l} & -\frac{Y + \sigma_l(\varepsilon_{ca} + \sigma_a)}{\sigma_c + \sigma_l} \end{pmatrix}^{-1} \\
&\times \begin{pmatrix} \frac{\sigma_l \varepsilon_{ca}}{\sigma_l + \sigma_c} + \rho\pi & -\frac{\sigma_l \varepsilon_{ca}}{\sigma_l + \sigma_c} \\ z\rho\pi & 0 \end{pmatrix} \begin{pmatrix} \hat{\pi}_t \\ \hat{m}_{t-1} \end{pmatrix} \\
&= \mathbf{A} \begin{pmatrix} \hat{\pi}_t \\ \hat{m}_{t-1} \end{pmatrix}.
\end{aligned} \tag{21}$$

The characteristic polynomial of  $A$  can be simplified to

$$F(X) = X^2 - X\rho\pi \frac{Y + \sigma_l(\varepsilon_{ca} + \sigma_a) - z\sigma_l \varepsilon_{ca}}{Y + \sigma_l(\varepsilon_{ca} + \sigma_a)} - \frac{\rho\pi z\sigma_l \varepsilon_{ca}}{Y + \sigma_l(\varepsilon_{ca} + \sigma_a)}.$$

Consider the case the labor supply elasticity is finite  $\sigma_l > 0$ . In this case, the determinant of  $A$ ,  $\det(A) = F(0) < 0$ , is strictly negative, indicating that exactly one eigenvalue is negative and that real money balances are a relevant state variable. Local stability and uniqueness then requires that there exists exactly one root of  $F(X) = 0$  with modulus less than one. To examine the conditions for this, I use that  $F(X)$  further satisfies

$$\begin{aligned}
F(1) &= 1 - \rho\pi, \\
F(-1) &= \frac{(1 + \rho\pi)(Y + \sigma_l(\varepsilon_{ca} + \sigma_a)) - 2z\sigma_l \varepsilon_{ca} \rho\pi}{Y + \sigma_l(\varepsilon_{ca} + \sigma_a)}.
\end{aligned}$$

Thus, for  $F(1) < 0$  ( $> 0$ ) and  $F(-1) > 0$  ( $< 0$ ), the model is locally stable, unique and (non-)oscillatory, since the stable eigenvalue is negative (positive).

Thus, for  $F(1) > 0$  and  $F(-1) < 0$ , the model is locally stable, unique and non-oscillatory, since the stable eigenvalue is positive. If  $F(1) < 0$  and  $F(-1) > 0$ , equilibrium sequences are locally stable and unique, but oscillatory, since the stable eigenvalue has a negative sign. Suppose that the real balance effect and that the inverse of the labor supply elasticity are large enough such that  $\varepsilon_{ca} > \sigma_a(2z - 1)^{-1}$  and  $\sigma_l > \bar{\sigma}_l$ , where  $\bar{\sigma}_l \equiv \frac{Y}{(2z-1)\varepsilon_{ca} - \sigma_a}$ . Then,  $F(-1)$  can be negative if  $\rho\pi$  is sufficiently large. Local stability and uniqueness with  $F(1) > 0$  and  $F(-1) < 0$ , is then ensured by moderate inflation elasticities satisfying  $\bar{\rho}_{\pi 1} < \rho\pi < 1$ , where  $\bar{\rho}_{\pi 1} \equiv$

$\frac{\sigma_l(\varepsilon_{ca} + \sigma_a) + Y}{\sigma_l(2z-1)\varepsilon_{ca} - \sigma_l\sigma_a - Y}$ . Alternatively, local stability and uniqueness arise for  $F(1) < 0$  and  $F(-1) > 0$ , which requires  $1 < \rho_\pi < \overline{\rho_{\pi 1}}$ . Suppose that  $\varepsilon_{ca} > \sigma_a(2z-1)^{-1}$  or  $\sigma_l < \overline{\sigma_l}$ . Then,  $F(-1)$  cannot be negative and local stability and uniqueness then arise if  $\rho_\pi > 1$ .

To establish the role of predetermined real money balances as a state variable, I need to show that the coefficients  $\eta_i$ ,  $i = c, \pi, R$  are non-zero if  $\eta_m$  is non-zero and stable. Applying the method of undetermined coefficients to (11) results in the following restrictions for the coefficients  $\eta_c$  and  $\eta_\pi$ :

$$\eta_c = \frac{\varepsilon_{ca}(1 - \eta_\pi)}{\sigma_c + \sigma_l},$$

implying that for  $\varepsilon > 0$  one not both coefficients can be zero. In particular, if  $\eta_\pi$  is neither zero nor 1, predetermined real money balances are a relevant endogenous state variable for  $\rho_\pi \neq 0$ . The money demand equation (13) implies that

$$\eta_\pi = -\frac{\eta_m k}{z\rho_\pi - \eta_m k} \neq 0, 1$$

since  $k = Y + \sigma_l(\sigma_c + \sigma_a) > 0$  due to strict concavity and  $\rho_\pi \neq 0$  (no interest rate peg).

Now, consider the case where the labor supply elasticity is infinite,  $\sigma_l = 0$ . In this case  $\det(A) = 0$ , indicating that the beginning-of-period value for real money balances is irrelevant for the determination of  $\hat{\pi}_t$  and  $\hat{m}_t$ . It follows that one eigenvalue equals zero and the other eigenvalue is larger than one, if and only if  $\rho_\pi > 1$ . Then, the equilibrium sequences for  $\hat{m}_t$ ,  $\hat{c}_{t+1}$ ,  $\hat{\pi}_t$  and  $\hat{R}_t$  for  $t \geq 0$  are locally stable and uniquely determined, while  $\hat{c}_t$  cannot be determined. ■

#### A.4 Proof of Proposition 7

Consider a monetary regime in which future inflation serves as the policy indicator,  $\hat{R}_t = \rho_\pi E_t \hat{\pi}_{t+1}$ . Substituting for consumption with (11) and inserting the forward-

looking feedback rule, the model in (11)-(13) can be reduced to

$$\begin{aligned}
\begin{pmatrix} \widehat{\pi}_{t+1} \\ \widehat{m}_t \end{pmatrix} &= \begin{pmatrix} \frac{\sigma_l \varepsilon_{ca}}{\sigma_l + \sigma_c} + 1 - \rho_\pi & -\frac{\sigma_l \varepsilon_{ca}}{\sigma_l + \sigma_c} \\ \frac{Y + \sigma_l(\varepsilon_{ca} + \sigma_a)}{\sigma_c + \sigma_l} - z\rho_\pi & -\frac{Y + \sigma_l(\varepsilon_{ca} + \sigma_a)}{\sigma_c + \sigma_l} \end{pmatrix}^{-1} \\
&\times \begin{pmatrix} \frac{\sigma_l \varepsilon_{ca}}{\sigma_l + \sigma_c} & -\frac{\sigma_l \varepsilon_{ca}}{\sigma_l + \sigma_c} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \widehat{\pi}_t \\ \widehat{m}_{t-1} \end{pmatrix} \\
&= \mathbf{B} \begin{pmatrix} \widehat{\pi}_t \\ \widehat{m}_{t-1} \end{pmatrix}.
\end{aligned} \tag{22}$$

The characteristic polynomial of  $\mathbf{B}$  is given by

$$F(X) = X \left( X - \frac{\rho_\pi z \sigma_l \varepsilon_{ca}}{(Y + \sigma_l(\varepsilon_{ca} + \sigma_a))(1 - \rho_\pi) + \rho_\pi z \sigma_l \varepsilon_{ca}} \right).$$

Evidently, real money balances are not a relevant state variable, and one can only solve for  $\widehat{m}_t$ ,  $E_t \widehat{\pi}_{t+1}$ ,  $E_t \widehat{c}_{t+1}$  and  $\widehat{R}_t \forall t \geq 0$ . For a finite labor supply elasticity,  $\sigma_l > 0$ , local stability and uniqueness requires the other eigenvalue (one is equal to zero) to be unstable. A positive unstable root arises if monetary policy is active and  $\sigma_l > \overline{\sigma}_{l2}$  or if  $1 < \rho_\pi < \overline{\rho}_{\pi2}$  for  $\sigma_l < \overline{\sigma}_{l2}$  or  $\varepsilon_{ca} < \sigma_a / (2z - 1)$ . A negative unstable root exists if  $\rho_\pi > \overline{\rho}_{\pi2}$ , given that  $\sigma_l > \overline{\sigma}_l$  and  $\varepsilon_{ca} > \sigma_a / (2z - 1)$ , for  $\sigma_l < \overline{\sigma}_{l2}$  or  $\varepsilon_{ca} < \sigma_a / (2z - 1)$ . Thus,  $1 < \overline{\rho}_{\pi2} < \rho_\pi < -\overline{\rho}_{\pi1}$  leads to a locally stable and unique equilibrium with a negative root for  $\sigma_l < \overline{\sigma}_l$  or  $\varepsilon_{ca} < \sigma_a / (2z - 1)$ . When the labor supply elasticity is infinite,  $\sigma_l = 0$ , then the Euler equation reads  $(1 - \rho_\pi) \widehat{\pi}_{t+1} = 0$ . Thus, the model displays local stability and uniqueness if and only if  $\rho_\pi \neq 1$ . ■

## A.5 Proof of Proposition 8

Under a constant money growth regime the nominal interest rate can be substituted out so that the reduced form system of the model in (11)-(13) reads (where I omitted the exogenous state)

$$\omega_1 \widehat{c}_{t+1} - (\omega_2 + 1) \widehat{m}_t + \omega_2 \widehat{\pi}_{t+1} = -\sigma_c \widehat{c}_t + \varepsilon_{ca} \widehat{m}_{t-1} - \varepsilon_{ca} \widehat{\pi}_t, \tag{23}$$

$$\varepsilon_{ca} \widehat{m}_{t-1} = (\sigma_l + \sigma_c) \widehat{c}_t + \varepsilon_{ca} \widehat{\pi}_t, \tag{24}$$

where  $\omega_1 \equiv (\sigma_c(1-z) + \phi_{ac})z^{-1}$  and  $\omega_2 \equiv (\varepsilon_{ca}(1-z) - z + \sigma_a)z^{-1}$ , and  $\widehat{m}_t = \widehat{m}_{t-1} - \widehat{\pi}_t$ . After eliminating consumption with (24) and inflation with the linearized money growth rule (14), I get the following difference equation in  $\widehat{m}_t$ :

$$\widehat{m}_{t+1} = \frac{z(\sigma_l \varepsilon_{ca} + \sigma_l + \sigma_c)}{z(\sigma_l \varepsilon_{ca} + \sigma_l + \sigma_c) - (Y + \sigma_l \varepsilon_{ca} + \sigma_l \sigma_a)} \widehat{m}_t.$$

Once  $\widehat{m}_t$  is determined, which requires an unstable root, one can solve for  $\widehat{\pi}_t$  and  $\widehat{c}_t \forall t \geq 1$ , while the initial values for consumption  $\widehat{c}_0$  and inflation  $\widehat{\pi}_0$  cannot be determined. Local uniqueness and stability of the equilibrium sequences  $\{\widehat{m}_t, \widehat{\pi}_{t+1}, \widehat{c}_{t+1}, \widehat{R}_t\}_{t=0}^\infty$  thus require  $\left| \frac{z(\sigma_l \varepsilon_{ca} + \sigma_l + \sigma_c)}{z(\sigma_l \varepsilon_{ca} + \sigma_l + \sigma_c) - (Y + \sigma_l \varepsilon_{ca} + \sigma_l \sigma_a)} \right| > 1$ . If  $z(\sigma_l \varepsilon_{ca} + \sigma_l + \sigma_c) - (Y + \sigma_l \varepsilon_{ca} + \sigma_l \sigma_a) > 0$ , then the root is positive and unstable. Rearranging and using  $Y = \sigma_c \sigma_a - \varepsilon_{ca} \phi_{ac}$  shows that this conditions is satisfied for  $z > \sigma_a$ . ■

## A.6 Proof of Proposition 9

Consider the case where the central bank sets the nominal interest rate contingent on changes in current inflation,  $\widehat{R}_t = \rho_\pi \widehat{\pi}_t$ . After substituting for consumption and eliminating  $\widehat{m}_t$  and  $\widehat{m}_{t+1}$  with the static money demand equation (18), one obtains the following difference equation (where I omitted the exogenous state):

$$(d+1)\rho_\pi \widehat{\pi}_t = (d\rho_\pi + 1)\widehat{\pi}_{t+1},$$

where  $d \equiv (z-1)\sigma_l \varepsilon_{ca}[Y + \sigma_l(\varepsilon_{ca} + \sigma_a)]^{-1} > 0$ . Therefore  $\rho_\pi > 1$  is necessary and sufficient for local stability and uniqueness of the equilibrium sequences of inflation  $\widehat{\pi}_t$ , real balances  $\widehat{m}_t$ , consumption  $\widehat{c}_t$  and the nominal interest rate,  $\widehat{R}_t \forall t \geq 0$ .

Now, consider the case where future inflation serves as the policy indicator,  $\widehat{R}_t = \rho_\pi \widehat{\pi}_{t+1}$ . When the labor supply elasticity is finite,  $\sigma_l > 0$ , then the model in (16)-(18) reduces to:

$$\widehat{\pi}_{t+2} = \frac{\rho_\pi(1+d) - 1}{d\rho_\pi} \widehat{\pi}_{t+1}.$$

Evidently, one cannot determine current inflation rate  $\widehat{\pi}_t$ . One obtains a unique and locally stable solution for expected inflation, and the current values of consumption, real money balances and the nominal interest rate, if the eigenvalue of this equation is positive and unstable, which requires  $\rho_\pi > 1$ . Alternatively,  $\rho_\pi < [1 + 2d]^{-1}$

ensures local stability and uniqueness, where one eigenvalue is smaller than  $-1$ . When the labor supply elasticity is infinite,  $\sigma_l = 0$ , then uniqueness of a equilibrium sequence for  $\hat{\pi}_{t+1} \forall t \geq 0$  is guaranteed by  $\rho_\pi \neq 1$ . ■

## A.7 Proof of Proposition 10

Under a constant money growth policy,  $\hat{m}_t = \hat{m}_{t-1} - \hat{\pi}_t$ , the model in (16)-(18) can – by eliminating the nominal interest rate – be reduced to:

$$\varepsilon_{ca}\hat{m}_t = (\sigma_l + \sigma_c)\hat{c}_t, \quad (25)$$

$$\gamma_1\hat{c}_{t+1} + \gamma_2\hat{\pi}_{t+1} + \gamma_3\hat{m}_t = \left(\gamma_1 + \frac{\sigma_c + \phi_{ac}}{z}\right)\hat{c}_t, \quad (26)$$

where  $\gamma_1 = \sigma_c(z-1)z^{-1} > 0$ ,  $\gamma_2 = (1 + \varepsilon_{ca})(z-1)z^{-1} > 0$  and  $\gamma_3 = (\varepsilon_{ca} + \sigma_a)z^{-1} > 0$ . Eliminating consumption with (25) and inflation with the linearized money growth rule leads to the following difference equation in real money balances:

$$\hat{m}_{t+1} = \frac{[\sigma_l(1 + \varepsilon_{ca}) + \sigma_c](z-1) + Y + \sigma_l(\varepsilon_{ca} + \sigma_a)}{[\sigma_l(1 + \varepsilon_{ca}) + \sigma_c](z-1)}\hat{m}_t,$$

which evidently exhibits an unstable root. Thus, one can uniquely determine end-of-period real balances  $\hat{m}_t$ , current consumption  $\hat{c}_t$ , the nominal interest rate  $\hat{R}_t \forall t \geq 0$ , while inflation  $\hat{\pi}_t$  can only be determined for  $t \geq 1$ . ■