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Online Supplement to: Adjustable robust treatment-length optimization in radiation therapy

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A Results exact biomarker information: out-of-sample performance

To investigate the out-of-sample performance of the methods, we assume a uniform distribution for (ρ, τ) over a larger set than Z. We can write Z as

$$Z = \{(\rho, \tau) : \rho_L \leq \rho \leq \rho_U, \tau_L \leq \tau \leq \tau_U\} = \{(\rho, \tau) : |\bar{\rho} - \rho| \leq \varepsilon_\rho,\ |\bar{\tau} - \tau| \leq \varepsilon_\tau\}, \tag{A.1}$$

where $(\varepsilon_\rho, \varepsilon_\tau)$ is the maximum deviation from the nominal scenario $(\bar{\rho}, \bar{\tau})$. This allows us to define

$$Z_c = \{(\rho, \tau) : |\bar{\rho} - \rho| \leq c\varepsilon_\rho,\ |\bar{\tau} - \tau| \leq c\varepsilon_\tau\}, \tag{A.2}$$

where $c > 0$ is a parameter. We assume a uniform distribution over the new set Z_c. If $c = 1$, we have $Z_c = Z$, so we sample exactly from Z. If $c > 1$, we sample from an interval that is c^2 times as large as Z (c times larger for both ρ and τ). For $c = 2$ we obtain the results in Table A.1. The stage-1 dose d_1 is the same as in **??** in the main manuscript for all methods except PI, because PI is the only method that is aware that the sample is not taken from uncertainty set Z but from Z_2. For NOM, the maximum violation percentage has increased slightly. All other methods are able to deal with the out-of-sample realizations and do not have any OAR constraint violations.

Due to the larger sampling space (the area of Z_2 is four times the area of Z), the difference between sample mean and sample worst-case performance is much larger than in **??** for all methods. The true worst-case objective value in Z is still lower than the sample worst-case in Z_2. The reason for this is that the true worst-case scenario can differ per patient.

NOM-FH and ARO are near optimal for the worst-case sample scenario, and are also close to PI in the sample 5% quantile and sample mean. The relative performance of the adaptive methods remains

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	Method					
	NOM	NOM-FH	RO	RO-FH	ARO	PI
Tumor BED - sample mean (Gy)	156.49	158.92	151.30	156.83	158.87	159.06
Tumor BED - sample 5% quantile (Gy)	140.14	142.11	137.55	141.11	142.11	142.32
Tumor BED - sample wc (Gy)	134.06	136.80	132.56	136.09	136.79	137.12
Tumor BED - wc over Z (Gy)	**114.72**	**116.19**	**116.19**	**116.19**	**116.19**	**116.19**
OAR violation - mean (%)	1.16	0	0	0	0	0
OAR violation - max (%)	5.30	0	0	0	0	0
Stage-1 dose d_1 (Gy)	1.50	1.50	2.29	2.29	1.51	1.73
Stage-2 dose d_2 (Gy)	3.45	3.21	2.48	2.92	3.21	3.15
Stage-2 fractions N_2	20	22.9	27.2	22.9	22.9	22.9

Table A.1: Results for experiments with exact biomarker information and uniform sampling of (ρ, τ) over Z_2. For each scenario, results are averaged over 20 patients*. All methods optimize for worst-case tumor BED in Z, which is displayed in bold.
*: the maximum OAR violation is computed over all patients and scenarios

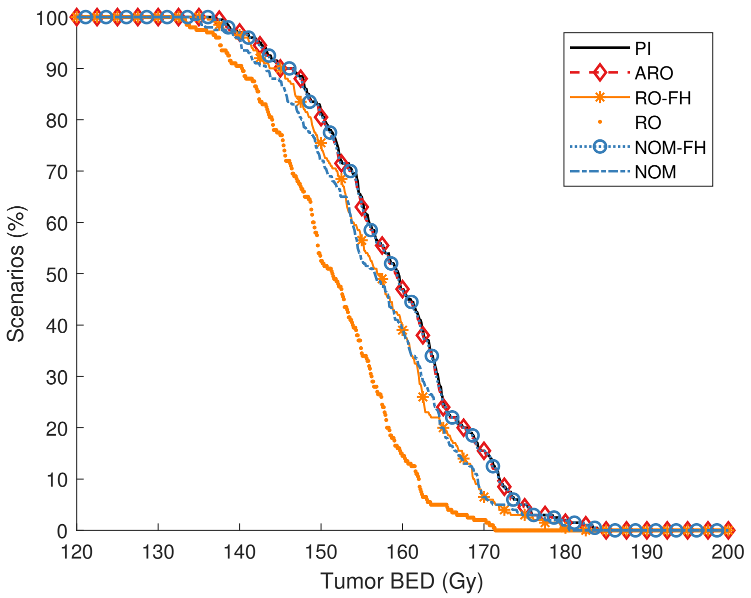


Fig. A.1: Cumulative scenario-tumor BED graph for experiments with exact biomarker information and uniform sampling of (ρ, τ) over Z_2 (200 scenarios). A point (x, y) indicates that in y% of scenarios the tumor BED (averaged over 20 patients) is at least x Gy. ARO and NOM-FH are very close to PI.

mostly unchanged, RO-FH performs slightly worse than NOM-FH and ARO, similar to **??**. Compared to **??**, NOM and RO have poor performance across the sample. This indicates bad performance of the static methods on scenarios outside of Z.

Figure A.1 shows the complete cumulative scenario-tumor BED graph for the 'average patient'. Compared to **??** in the main manuscript, the main difference is the decrease in performance of NOM. Naturally, the performance of static nominal optimization is directly related to the magnitude of possible deviations from the nominal scenario, which is higher in Z_2 than in Z.

B Extra analyses and proofs

For convenience, we repeat the definitions of functions B, g and f:

$$B(d',N';\rho) = \varphi D\Big(1+\frac{\varphi D}{T}\rho\Big) - \sigma d_1 N_1 - \sigma^2 \rho d_1^2 N_1 \tag{B.1a}$$

$$g(d',N',N'';\rho) = \frac{-1+\sqrt{1+\frac{4\rho}{N''}B(d',N';\rho)}}{2\sigma\rho} \tag{B.1b}$$

$$f(d_1,N_2;\rho,\tau) = \begin{cases} N_1 d_1 + N_2 g(d_1,N_1,N_2;\rho) + \tau\big(N_1 d_1^2 + N_2 g(d_1,N_1,N_2;\rho)^2\big) & \text{if } d_1 \in [0, g(0,0,N_1;\rho)] \\ -\infty & \text{otherwise,} \end{cases} \tag{B.1c}$$

see (**??**), (**??**) and (**??**).

B.1 Proof **??**

First, we show that for fixed d_1, feasible to (**??**), and given (ρ,τ), it is optimal to minimize the number of stage-2 fractions if $\tau \geq \sigma\rho$, and it is optimal to maximize the number of stage-2 fractions otherwise. After that, we show that with stage-2 dose d_2 such that (**??**) holds with equality, $N_2(\rho,\tau) = N_2^{\min}$ is feasible if $\tau \geq \sigma\rho$ and $N_2(\rho,\tau) = N_2^{\max}$ is feasible otherwise.

Consider problem (**??**). At the start of stage 2, we have delivered N_1 fractions with dose d_1 per fraction. Let (ρ,τ) be the realization of the uncertain parameters. The stage-2 problem reads

$$N_1 d_1 + \tau N_1 d_1^2 + \max_{d_2,N_2} \; N_2 d_2 + \tau N_2 d_2^2 \tag{B.2a}$$

$$\text{s.t.} \;\; \sigma N_2 d_2 + \rho\sigma^2 N_2 d_2^2 \leq B(d_1,N_1,\rho) \tag{B.2b}$$

$$d_2 \geq d_{\min} \tag{B.2c}$$

$$N_2 \in \{N_2^{\min},\dots,N_2^{\max}\}. \tag{B.2d}$$

This is a static fractionation problem. Constraint (B.2b) will hold with equality at the optimum, because it is the only dose-limiting constraint. This yields

$$d_2^*(d_1,N_2;\rho) = g(d_1,N_1,N_2;\rho). \tag{B.3}$$

Secondly, this allows us to rewrite the objective to

$$\max_{d_2,N_2} \; N_2 d_2\Big(\frac{\sigma\rho-\tau}{\sigma\rho}\Big) + \frac{\tau B(d_1,N_1,\rho)}{\sigma^2\rho}, \tag{B.4}$$

which implies that if $\tau > \sigma\rho$ it is optimal to minimize d_2N_2. If $\tau < \sigma\rho$ it is optimal to maximize d_2N_2, and if $\tau = \sigma\rho$ the objective value is independent of the value of N_2. Similar results are obtained in Mizuta et al. (2012); Bortfeld et al. (2015). As given in **??**, at the optimum

$$N_2 d_2^*(d_1,N_2;\rho) = N_2 g(d_1,N_1,N_2;\rho) = \frac{-N_2+\sqrt{N_2^2+4N_2\rho B(d_1,N_1;\rho)}}{2\sigma\rho}, \tag{B.5}$$

and it is straightforward to show that

$$\frac{\partial N_2 g(d_1,N_1,N_2;\rho)}{\partial N_2} \geq 0. \tag{B.6}$$

Hence, if $\tau > \sigma\rho$, it is optimal to minimize the number of fractions, and if $\tau < \sigma\rho$ it is optimal to maximize the number of fractions. If $\tau = \sigma\rho$, every feasible number of fractions is optimal.

For the second part, we must show that for any $(\rho,\tau)\in Z\cap\{\tau\geq\sigma\rho\}$ resp. $(\rho,\tau)\in Z\cap\{\tau<\sigma\rho\}$, it is indeed possible to deliver $N_2^{\min}$ resp. $N_2^{\max}$ fractions with dose according to (B.3) in stage 2. That is, we must show

$$g(d_1,N_1,N_2^{\min};\rho)\geq d^{\min},\ \ \forall(\rho,\tau)\in Z\cap\{\tau\geq\sigma\rho\} \tag{B.7a}$$
$$g(d_1,N_1,N_2^{\max};\rho)\geq d^{\min},\ \ \forall(\rho,\tau)\in Z\cap\{\tau<\sigma\rho\}, \tag{B.7b}$$

which is equivalent to

$$d_1\leq g(d^{\min},N_2^{\min},N_1;\rho),\ \ \forall(\rho,\tau)\in Z\cap\{\tau\geq\sigma\rho\} \tag{B.8a}$$
$$d_1\leq g(d^{\min},N_2^{\max},N_1;\rho),\ \ \forall(\rho,\tau)\in Z\cap\{\tau<\sigma\rho\}. \tag{B.8b}$$

Lemma 3a states that g is increasing or decreasing in ρ for a fixed first argument. Hence, it is sufficient to consider only the largest and smallest value of ρ in either subset of Z. Therefore, (B.8) is equivalent to

$$d_1\leq g(d^{\min},N_2^{\min},N_1;\rho_L) \tag{B.9a}$$
$$d_1\leq g(d^{\min},N_2^{\min},N_1;\min\{\frac{\tau_U}{\sigma},\rho_U\}) \tag{B.9b}$$
$$d_1\leq g(d^{\min},N_2^{\max},N_1;\frac{\tau_L}{\sigma}) \tag{B.9c}$$
$$d_1\leq g(d^{\min},N_2^{\max},N_1;\rho_U). \tag{B.9d}$$

From (B.1b) we see that function g is decreasing in its second argument, so (B.9b) is redundant. The remaining three conditions in (B.9) hold true due to **??**. Hence, an optimal decision rule for $N_2(\cdot)$ is given by

$$N_2^*(\rho,\tau)=\begin{cases}N_2^{\min} & \text{if } \tau\geq\sigma\rho\\ N_2^{\max} & \text{otherwise,}\end{cases} \tag{B.10}$$

and

$$d_2^*(d_1;\rho,\tau)=\begin{cases}g(d_1,N_1,N_2^{\min};\rho) & \text{if } \tau\geq\sigma\rho\\ g(d_1,N_1,N_2^{\max};\rho) & \text{otherwise.}\end{cases} \tag{B.11}$$

are optimal decision rules for $N_2(\cdot)$ and $d_2(\cdot)$, respectively. For $\tau\neq\sigma\rho$, these give the unique optimal decisions. For $\tau=\sigma\rho$ any $N_2\in\{N_2^{\min},\dots,N_2^{\max}\}$ is optimal, and the corresponding optimal d_2 follows according to (B.3).

B.2 Proof **??**

Due to **??** a stage-1 decision d_1 is $\mathcal{PARO}$ according to **??** if conditions (**??**) hold with $(d_2^*(\cdot),N_2^*(\cdot))$ plugged in. Thus, we must show that for any $d_1\in X^{\text{PARO}}$ there is no $\mathcal{ARO}$ $\bar{d}_1$ such that

$$f(d_1,N_2^*(\rho,\tau);\rho,\tau)\leq f(\bar{d}_1,N_2^*(\rho,\tau);\rho,\tau)\ \ \forall(\rho,\tau)\in Z \tag{B.12a}$$
$$f(d_1,N_2^*(\bar{\rho},\bar{\tau});\bar{\rho},\bar{\tau})< f(\bar{d}_1,N_2^*(\bar{\rho},\bar{\tau});\bar{\rho},\bar{\tau})\quad \text{for some } (\bar{\rho},\bar{\tau})\in Z. \tag{B.12b}$$

If $|X^{\text{PARO}}|=1$, then the single element yields a strictly better objective value than all other elements in X^{ARO} in either scenario $(\rho^{\text{aux-min}},\tau^{\text{aux-min}})$ or $(\rho^{\text{aux-max}},\tau^{\text{aux-max}})$ or both, so it is $\mathcal{PARO}$. For the remainder of this proof we assume $|X^{\text{PARO}}|\geq 2$.

Consider $X^{\text{aux-min}}$. By construction of $(\rho^{\text{aux-min}},\tau^{\text{aux-min}})$ it holds that $\tau^{\text{aux-min}}\neq\sigma\rho^{\text{aux-min}}$. Hence, according to Lemma 4, there can be at most two values for d_1 in $X^{\text{aux-min}}$ that yield the same objective value f in scenario $(\rho^{\text{aux-min}},\tau^{\text{aux-min}})$. Hence, $|X^{\text{aux-min}}|=|X^{\text{PARO}}|=2$. Denote the two elements of X^{PARO} by d_1' and d_1'', let $d_1'<d_1''$. Solutions d_1' and d_1'' are both optimal to (**??**) and (**??**). Hence, according to Lemma 4, it holds that

$$d_1''=t(d_1';\rho^{\text{aux-min}},\tau^{\text{aux-min}}) \tag{B.13a}$$
$$d_1''=t(d_1';\rho^{\text{aux-max}},\tau^{\text{aux-max}}). \tag{B.13b}$$

From the definition of t (see (C.21)) we derive for $\sigma\rho \neq \tau$:

$$\frac{\partial t(d_1;\rho,\tau)}{\partial \rho} = \frac{2N_2^*(\rho,\tau)}{N_1+N_2^*(\rho,\tau)} \frac{\partial g(d_1,N_1,N_2^*(\rho,\tau);\rho)}{\partial \rho}, \tag{B.14}$$

because $N_2^*(\rho,\tau)$ is constant in ρ unless $\sigma\rho = \tau$. According to Lemma 3a, if for given N_2 it holds that $d_1 \neq d_1^-(N_2)$ and $d_1 \neq d_1^+(N_2)$ (defined in (C.3)), then function $g(d_1,N_1,N_2,\rho)$ is strictly increasing or decreasing in ρ. By construction, it holds that $d_1^+(N_2) = t(d_1^-(N_2);\rho,\tau)$ for any ρ. According to Lemma 2b, we have $d_1^-(N_2^{\min}) \neq d_1^-(N_2^{\max})$, so d_1' cannot be equal to both. Additionally, it cannot hold that $d_1' = d_1^+(N_2^{\min})$ or $d_1' = d_1^+(N_2^{\max})$, because it would imply $d'' \leq d'$. Hence, either $d_1' \notin \{d_1^-(N_2^{\min}), d_1^+(N_2^{\min})\}$ or $d_1' \notin \{d_1^-(N_2^{\max}), d_1^+(N_2^{\max})\}$ holds (or both).

We show that in either case, we can construct two new scenarios where d_1' outperforms d_1'' in one scenario, and vice versa in the other. Suppose $d_1' \notin \{d_1^-(N_2^{\min}), d_1^+(N_2^{\min})\}$. In this case, it holds that

$$\frac{\partial t(d_1';\rho^{\text{aux-min}},\tau^{\text{aux-min}})}{\partial \rho} \neq 0. \tag{B.15}$$

We consider two new scenarios. Let $\varepsilon > 0$ be a sufficiently small number and define

$$(\rho_1,\tau_1) = (\rho^{\text{aux-min}} - \varepsilon, \tau^{\text{aux-min}}) \qquad (\in \text{int}(Z^{\min})) \tag{B.16a}$$

$$(\rho_2,\tau_2) = (\rho^{\text{aux-min}} + \varepsilon, \tau^{\text{aux-min}}). \qquad (\in \text{int}(Z^{\min})) \tag{B.16b}$$

This is visualized in Figure B.1. Due to (B.15) and (B.13a), it holds that

$$\big(t(d_1';\rho_1,\tau_1) > d'' \wedge t(d_1';\rho_2,\tau_2) < d''\big) \vee \big(t(d_1';\rho_1,\tau_1) < d'' \wedge t(d_1';\rho_2,\tau_2) > d''\big). \tag{B.17}$$

If the first clause is is true, we obtain

$$f(d_1',N_2^{\min};\rho_1,\tau_1) > f(d_1'',N_2^{\min};\rho_1,\tau_1) \tag{B.18a}$$

$$f(d_1',N_2^{\min};\rho_2,\tau_2) < f(d_1'',N_2^{\min};\rho_2,\tau_2), \tag{B.18b}$$

where we used convexity of $f(d_1,N_2^{\min};\rho,\tau)$ for $(\rho,\tau) \in \text{int}(Z^{\min})$. Similarly, if the second clause of (B.17) is true, we obtain

$$f(d_1',N_2^{\min};\rho_1,\tau_1) < f(d_1'',N_2^{\min};\rho_1,\tau_1) \tag{B.19a}$$

$$f(d_1',N_2^{\min};\rho_2,\tau_2) > f(d_1'',N_2^{\min};\rho_2,\tau_2). \tag{B.19b}$$

In either case, there is a scenario in $Z^{\min}$ where d_1' outperforms d_1'' and a scenario in $Z^{\min}$ where d_1'' outperforms d_1'. Hence, both d_1' and d_1'' are $\mathcal{PARO}$. Using similar arguments, we can show that in case $d_1' \notin \{d_1^-(N_2^{\max}), d_1^+(N_2^{\max})\}$ also both d_1' and d_1'' are $\mathcal{PARO}$.

B.3 Proof ??

Consider problem (**??**). At the start of stage 2, we have delivered N_1 fractions with dose d_1 per fraction. Let $(\hat{\rho},\hat{\tau})$ be the observation. The resulting stage-2 problem for (**??**) reads

$$\max_{d_2,N_2} \min_{(\rho,\tau)\in Z_{(\hat{\rho},\hat{\tau})}} (N_1 d_1 + N_2 d_2) + \tau(N_1 d_1^2 + N_2 d_2^2) \tag{B.20a}$$

$$\text{s.t.} \quad \sigma N_2 d_2 + \rho\sigma^2 N_2 d_2^2 \leq B(d_1,N_1,\rho) \ \ \forall(\rho,\tau) \in Z_{\hat{\rho},\hat{\tau}} \tag{B.20b}$$

$$d_2 \geq d_{\min} \tag{B.20c}$$

$$N_2 \in \{N_2^{\min},\dots,N_2^{\max}\}. \tag{B.20d}$$

This is a static robust optimization problem. Constraint (B.20b) will hold with equality at the optimum, because it is the only dose-limiting constraint. Solving for d_2 yields the constraint

$$d_2 = g(d_1,N_1,N_2;\rho), \ \ \forall(\rho,\tau) \in Z_{(\hat{\rho},\hat{\tau})}, \tag{B.21}$$

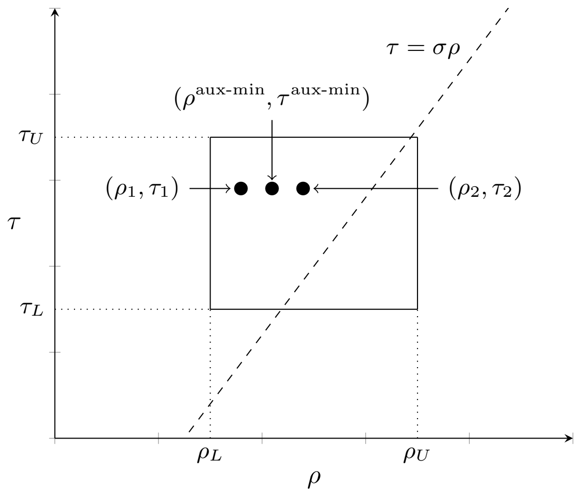


Fig. B.1: Case $d_1' \neq d_1^-(N_2^{\min})$. Construction of two new scenarios (ρ_1, τ_1) and (ρ_2, τ_2) from scenario $(\rho^{\text{aux-min}}, \tau^{\text{aux-min}})$. Solution d_1' outperforms d_1'' at one scenario, vice versa at the other.

and this is used to rewrite (B.20a) and (B.20b) in terms of functions f and g. Problem (B.20) is equivalent to

$$\max_{N_2} \min_{(\rho,\tau)\in Z_{(\hat\rho,\hat\tau)}} f(d_1, N_2, \rho, \tau) \tag{B.22a}$$

$$\text{s.t. } g(d_1, N_1, N_2; \rho) \geq d^{\min}, \ \forall(\rho,\tau) \in Z_{(\hat\rho,\hat\tau)} \tag{B.22b}$$

$$N_2 \in \{N_2^{\min}, \ldots, N_2^{\max}\}. \tag{B.22c}$$

Similar to the exact case (**??**), in any worst-case realization it will hold that τ is at its lowest value, so it is sufficient to consider only those observations $(\rho,\tau) \in Z_{(\hat\rho,\hat\tau)}$ with $\tau = \hat\tau_L$. Additionally, according to Lemma 3 functions f and g are increasing or decreasing in ρ. Hence, there are two candidate worst-case scenarios: $(\hat\rho_L, \hat\tau_L)$ and $(\hat\rho_U, \hat\tau_L)$. We can rewrite (B.22) to

$$\max_{N_2} \ \min\left\{f(d_1, N_2, \hat\rho_L, \hat\tau_L), \ f(d_1, N_2, \hat\rho_U, \hat\tau_L)\right\} \tag{B.23a}$$

$$\text{s.t. } g(d_1, N_1, N_2; \hat\rho_L) \geq d^{\min} \tag{B.23b}$$

$$g(d_1, N_1, N_2; \hat\rho_U) \geq d^{\min} \tag{B.23c}$$

$$N_2 \in \{N_2^{\min}, \ldots, N_2^{\max}\}. \tag{B.23d}$$

We distinguish three cases:

– Case $(\hat\rho,\hat\tau) \in Z_{\text{ID}}^{\max}$: Analogous to the proof of **??**, one can show that for any realization $(\rho,\tau) \in Z_{(\hat\rho,\hat\tau)}$ it is optimal to maximize the number of fractions in stage 2. We plug in $N_2^*(\rho,\tau) = N_2^{\max}$ and show that it is feasible. Constraints (B.23b) and (B.23c) reduce to

$$\min\left\{g(d_1, N_1, N_2^{\max}; \hat\rho_L), g(d_1, N_1, N_2^{\max}; \hat\rho_U)\right\} \geq d^{\min}, \tag{B.24}$$

which is equivalent to

$$d_1 \leq \min\left\{g(d^{\min}, N_2^{\max}, N_1; \hat\rho_L), g(d^{\min}, N_2^{\max}, N_1; \hat\rho_U)\right\}. \tag{B.25}$$

It holds that $\hat\rho_L \geq \frac{\hat\tau_L}{\sigma} \geq \frac{\tau_L}{\sigma}$, and $\hat\rho_U \leq \rho_U$. According to Lemma 3a function g is either increasing or decreasing in ρ for other arguments fixed. Hence, by **??** condition (B.25) holds. Hence, $N_2^*(\rho,\tau) = N_2^{\max}$ is feasible and optimal. Thus, (B.22) equals

$$\min\left\{f(d_1, N_2^{\max}, \hat\rho_L, \hat\tau_L), \ f(d_1, N_2^{\max}, \hat\rho_U, \hat\tau_L)\right\}. \tag{B.26}$$

By definition of f, this implies

$$d_2 = \min\left\{g(d_1, N_1, N_2^{\max}; \hat{\rho}_L), g(d_1, N_1, N_2^{\max}; \hat{\rho}_U)\right\}. \tag{B.27}$$

– Case $(\hat{\rho}, \hat{\tau}) \in Z_{\text{ID}}^{\min}$: Similar to the previous case. Analogous to the proof of **??**, one can show that for any realization $(\rho, \tau) \in Z_{(\hat{\rho},\hat{\tau})}$ it is optimal to minimize the number of fractions in stage 2. We plug in $N_2^*(\rho, \tau) = N_2^{\min}$ and show that it is feasible. Similar to the previous case, constraints (B.23b) and (B.23c) reduce to

$$d_1 \leq \min\left\{g(d^{\min}, N_2^{\min}, N_1; \hat{\rho}_L), g(d^{\min}, N_2^{\min}, N_1; \hat{\rho}_U)\right\}. \tag{B.28}$$

It holds that $\hat{\rho}_L \geq \rho_L$, and $\hat{\rho}_U \leq \rho_U$. Hence, by **??**, Lemma 3a and using the fact that function g is decreasing in its second argument, condition (B.28) holds. Hence, $N_2^*(\rho, \tau) = N_2^{\min}$ is feasible and optimal. Similar to the previous case, we find

$$d_2 = \min\left\{g(d_1, N_1, N_2^{\min}; \hat{\rho}_L), g(d_1, N_1, N_2^{\min}; \hat{\rho}_U)\right\}. \tag{B.29}$$

– Case $(\hat{\rho}, \hat{\tau}) \in Z_{\text{ID}}^{\text{int}}$: The optimal number of fractions in stage-2 is not known a priori. By definition of $Z_{\text{ID}}^{\text{int}}$, it holds that $\hat{\rho}_L \geq \max\{\rho_L, \frac{\tau_L}{\sigma} - 2r^{\rho})\}$ and $\hat{\rho}_U \leq \rho_U$. By **??** it holds that

$$d_1 \leq \min\left\{g(d^{\min}, N_2^{\max}, N_1; \max\{\rho_L, \frac{\tau_L}{\sigma} - 2r^{\rho})\}, g(d^{\min}, N_2^{\max}, N_1; \rho_U)\right\}. \tag{B.30}$$

Lemma 3a, the fact that function g is decreasing in its third argument and (B.30) together imply that (B.23b) and (B.23c) hold for any feasible N_2. Hence, from problem (B.23) we derive

$$N_2^*(d_1; \hat{\rho}, \hat{\tau}) = \underset{N_2 \in \{N_2^{\min}, \ldots, N_2^{\max}\}}{\arg\max} \min\left\{f(d_1, N_2, \hat{\rho}_L, \hat{\tau}_L),\ f(d_1, N_2, \hat{\rho}_U, \hat{\tau}_L)\right\}, \tag{B.31}$$

and by definition of f the corresponding value for d_2 is

$$d_2 = \min\left\{g(d_1, N_1, N_2^*(d_1; \hat{\rho}, \hat{\tau}); \hat{\rho}_L), g(d_1, N_1, N_2^*(d_1; \hat{\rho}, \hat{\tau}); \hat{\rho}_U)\right\}. \tag{B.32}$$

Combining the above three cases, we arrive at the optimal decision rules (**??**) and (**??**) for fixed d_1.

B.4 Extra analysis to ??

This analysis makes use of the lemmas in Appendix C. Consider problem (**??**). For given d_1, the optimal stage-2 decision rules are given by **??**. As stated in **??**, we split the uncertainty set Z into three subsets. This enables us to exploit the fact that depending on $(\hat{\rho}, \hat{\tau})$ the value $N_2^*(d_1; \hat{\rho}, \hat{\tau})$ may be known in advance. The split (**??**) is repeated here for convenience

$$Z_{\text{ID}}^{\min} = \{(\hat{\rho}, \hat{\tau}) \in Z : \hat{\tau}_L \geq \sigma\hat{\rho}_U\} \tag{B.33a}$$
$$Z_{\text{ID}}^{\text{int}} = \{(\hat{\rho}, \hat{\tau}) \in Z : \sigma\hat{\rho}_L < \hat{\tau}_L < \sigma\hat{\rho}_U\} \tag{B.33b}$$
$$Z_{\text{ID}}^{\max} = \{(\hat{\rho}, \hat{\tau}) \in Z : \hat{\tau}_L \leq \sigma\hat{\rho}_L\}, \tag{B.33c}$$

so that $Z = Z_{\text{ID}}^{\min} \cup Z_{\text{ID}}^{\text{int}} \cup Z_{\text{ID}}^{\max}$. The associated sets of observation-realization pairs $(\rho, \tau, \hat{\rho}, \hat{\tau})$ are given by

$$U^i = U \cap \{(\rho, \tau, \hat{\rho}, \hat{\tau}) : (\hat{\rho}, \hat{\tau}) \in Z_{\text{ID}}^i\}, \quad i \in \{\min, \text{int}, \max\}, \tag{B.34}$$

so it holds that $U = U^{\min} \cup U^{\text{int}} \cup U^{\max}$. Set U^i can be interpreted as the set of observation-realization pairs for which the observation $(\hat{\rho}, \hat{\tau})$ is in set Z_{ID}^i. **??** illustrates the subsets Z_{ID}^i. Set $U^{\min}$ consists of those observation-realization pairs $(\rho, \tau, \hat{\rho}, \hat{\tau})$ for which $N_2^*(d_1; \hat{\rho}, \hat{\tau}) = N_2^{\max}$. If $(\rho, \tau, \hat{\rho}, \hat{\tau}) \in U^{\text{int}}$, then based on the observation $(\hat{\rho},\hat{\tau})$ it is unclear what fractionation is worst-case optimal. Last, if $(\rho,\tau,\hat{\rho},\hat{\tau}) \in U^{\max}$ we know $N_2^*(d_1;\hat{\rho},\hat{\tau}) = N_2^{\min}$. Problem (**??**) is equivalent to

$$\max_{d_1,q} \; q \tag{B.35a}$$
$$\text{s.t.} \;\; q \leq f(d_1, N_2^{\min};\rho,\tau), \;\; \forall(\rho,\tau,\hat{\rho},\hat{\tau}) \in U^{\min} \tag{B.35b}$$
$$q \leq f(d_1, N_2^*(d_1;\hat{\rho},\hat{\tau});\rho,\tau), \;\; \forall(\rho,\tau,\hat{\rho},\hat{\tau}) \in U^{\text{int}} \tag{B.35c}$$
$$q \leq f(d_1, N_2^{\max};\rho,\tau), \;\; \forall(\rho,\tau,\hat{\rho},\hat{\tau}) \in U^{\max} \tag{B.35d}$$
$$d^{\min} \leq d_1 \leq d_1^{\max}. \tag{B.35e}$$

Similar to the exact case (**??**), in any worst-case realization it will hold that $\tau = \tau_L$. Therefore, any observation with $\hat{\tau} - r^\tau > \tau_L$ cannot yield the worst-case realization. Define

$$U_L^i = U_i \cap \{(\rho,\tau,\hat{\rho},\hat{\tau}) : \hat{\tau} - r^\tau \leq \tau_L\}, \;\; i \in \{\min, \text{int}, \max\}, \tag{B.36}$$

which is the subset of U^i of observation-realization pairs for which τ_L is a possible realization of τ. Constraints (B.35b)-(B.35e) can be replaced by

$$q \leq f(d_1, N_2^{\min};\rho,\tau), \;\; \forall(\rho,\tau,\hat{\rho},\hat{\tau}) \in U_L^{\min} \tag{B.37a}$$
$$q \leq f(d_1, N_2^*(d_1;\hat{\rho},\hat{\tau});\rho,\tau), \;\; \forall(\rho,\tau,\hat{\rho},\hat{\tau}) \in U_L^{\text{int}} \tag{B.37b}$$
$$q \leq f(d_1, N_2^{\max};\rho,\tau), \;\; \forall(\rho,\tau,\hat{\rho},\hat{\tau}) \in U_L^{\max} \tag{B.37c}$$

For (B.37a) and (B.37c) it remains to find the worst-case realization of ρ for which the observation-realization pair is in $U_L^{\min}$ and $U_L^{\max}$, respectively. According to Lemma 3b, function f is increasing or decreasing in ρ for fixed d_1, so it is sufficient to check the maximum and minimum realization of ρ for which the observation-realization pair is in those sets. These are

$$\min\{\rho : (\rho,\tau,\hat{\rho},\hat{\tau}) \in U_L^{\min}\} = \rho_L, \;\; \max\{\rho : (\rho,\tau,\hat{\rho},\hat{\tau}) \in U_L^{\min}\} = \frac{\tau_L}{\sigma} \tag{B.38a}$$
$$\min\{\rho : (\rho,\tau,\hat{\rho},\hat{\tau}) \in U_L^{\max}\} = \frac{\tau_L}{\sigma}, \;\; \max\{\rho : (\rho,\tau,\hat{\rho},\hat{\tau}) \in U_L^{\max}\} = \rho_U. \tag{B.38b}$$

Plugging in $\rho = \frac{\tau_L}{\sigma}$ in (B.37a) and (B.37c) yields $q \leq K$, with K defined by (**??**). Lemma 5 provides a conservative approximation of constraint (B.37b). Putting everything together, the optimum of the following problem is a lower bound to the optimum of (B.35) (or, equivalently, (**??**)):

$$\max_{d_1,q} \; q \tag{B.39a}$$
$$\text{s.t.} \;\; q \leq f(d_1, N_2^{\min};\rho_L,\tau_L) \tag{B.39b}$$
$$q \leq f(d_1, N_2^{\max};\rho_U,\tau_L) \tag{B.39c}$$
$$q \leq K \tag{B.39d}$$
$$q \leq p(d_1) \tag{B.39e}$$
$$d_{\min} \leq d_1 \leq d_1^{\max}, \tag{B.39f}$$

with $p(d_1)$ defined by (C.32) in Appendix C. Constraint (B.39e) is the only conservative constraint, all other constraints are exact reformulations. In particular, this means that if for a solution the objective value equals K, it is certain that this is an optimal solution. It is easy to obtain other straightforward conservative approximations of (B.35c). For instance, a policy that delivers $N_2^{\min}$ or $N_2^{\max}$ fractions (or any number in between, for that matter) for any observation $(\hat{\rho},\hat{\tau}) \in Z_{\text{ID}}^{\text{int}}$ is a conservative approximation. However, these perform less good and do not use all available information, as explained in the proof of Lemma 5.

C Extra lemmas

This appendix states and proves several frequently used properties of functions g and f.

Lemma 1 **(Convexity/concavity f w.r.t. d_1)** *Let $\rho > 0$, $\tau > 0$ and $N_1, N_2 \in \mathbb{N}_+$. Let $d_1 \in [0, g(0,0,N_1;\rho)]$. The following properties hold for function f:*

- *Function $f(d_1,N_2;\rho,\tau)$ is strictly convex in d_1 if $\tau > \rho\sigma$, with unique minimizer $g(0,0,N_1+N_2;\rho)$;*
- *Function $f(d_1,N_2;\rho,\tau)$ is strictly concave in d_1 if $\tau < \rho\sigma$, with unique maximizer $g(0,0,N_1+N_2;\rho)$;*
- *Function $f(d_1,N_2;\rho,\tau)$ is constant in d_1 if $\tau = \rho\sigma$, with value $\frac{1}{\sigma}B(0,0,\frac{\tau}{\sigma})$.*

Proof The partial derivative of f w.r.t. d_1 is given by

$$\frac{\partial f(d_1,N_2;\rho,\tau)}{\partial d_1} = N_1 + N_2 \frac{\partial g(d_1,N_1,N_2;\rho)}{\partial d_1} + \tau\Big(2N_1 d_1 + 2N_2 g(d_1,N_1,N_2;\rho)\frac{\partial g(d_1,N_1,N_2;\rho)}{\partial d_1}\Big), \quad \text{(C.1)}$$

where the partial derivative of g w.r.t. d_1 is given by

$$\frac{\partial g(d_1,N_1,N_2;\rho)}{\partial d_1} = -\frac{1}{N_2}(N_1 + 2N_1 d_1 \sigma\rho)\Big(1 + \frac{4\rho}{N_2}B(d_1,N_1;\rho)\Big)^{-\frac{1}{2}}. \quad \text{(C.2)}$$

Define $h(d_1,N_2;\rho) = 1 + 4\frac{\rho}{N_2}B(d_1,N_1;\rho)$. Then, plugging (C.2) in (C.1), we obtain

$$\begin{aligned}
\frac{\partial f(d_1,N_2;\rho,\tau)}{\partial d_1} = &\ (N_1 - (N_1 + 2N_1 d_1\sigma\rho)h(d_1,N_2;\rho)^{-\frac{1}{2}} \\
&+ \tau\Big(2N_1 d_1 - \frac{2}{N_2}(N_1 + 2N_1 d_1 \sigma\rho)h(d_1,N_2;\rho)^{-\frac{1}{2}} N_2 \frac{-1 + h(d_1,N_2;\rho)^{\frac{1}{2}}}{2\sigma\rho}\Big) \\
= &\ \frac{N_1}{\sigma\rho}\big(h(d_1,N_2;\rho)^{-\frac{1}{2}}(2\sigma\rho d_1 + 1) - 1\big)(\tau - \rho\sigma).
\end{aligned}$$

Further elementary math shows that $h(d_1,N_2;\rho)^{-\frac{1}{2}}(2\sigma\rho d_1+1) - 1 = 0$ if and only if $d_1 = g(0,0,N_1+N_2;\rho)$. For the second derivative of f w.r.t. d_1 we obtain:

$$\begin{aligned}
\frac{\partial^2 f(d_1,N_2;\rho,\tau)}{\partial d_1^2} &= \Big(\frac{\tau-\rho\sigma}{\sigma\rho}N_1\Big)\frac{\partial}{\partial d_1} h(d_1,N_2;\rho)^{-\frac{1}{2}}(2\sigma\rho d_1 + 1) \\
&= \Big(\frac{\tau-\rho\sigma}{\sigma\rho}N_1\Big)\Bigg[h(d_1,N_2;\rho)^{-\frac{1}{2}}2\sigma\rho + \frac{2\rho}{N_2}(2\sigma\rho d_1+1)h(d_1,N_2,\rho)^{-\frac{3}{2}}(\sigma N_1 + 2\rho\sigma^2 d_1 N_1)\Bigg],
\end{aligned}$$

and the second part of this product is positive. Hence, its sign depends only on the term $\tau - \rho\sigma$. Combining the result for the first and second derivative, we obtain

- Function $f(d_1,N_2;\rho,\tau)$ is strictly convex in d_1 if $\tau > \rho\sigma$, with unique minimizer $g(0,0,N_1+N_2;\rho)$;
- Function $f(d_1,N_2;\rho,\tau)$ is strictly concave in d_1 if $\tau < \rho\sigma$, with unique maximizer $g(0,0,N_1+N_2;\rho)$;
- Function $f(d_1,N_2;\rho,\tau)$ is constant in d_1 otherwise.

If $\tau = \rho\sigma$, we can rewrite $f(d_1,N_2;\frac{\tau}{\sigma},\tau)$ to

$$\begin{aligned}
f(d_1,N_2;\frac{\tau}{\sigma},\tau) &= \max_{d_2}\Big\{d_1N_1 + d_2N_2 + \tau(d_1^2N_1 + d_2^2N_2) \mid \sigma(d_1N_1 + d_2N_2) + \rho\sigma^2(d_1^2N_1 + d_2^2N_2) \le B(0,0;\rho)\Big\} \\
&= \max_{d_2}\Big\{d_1N_1 + d_2N_2 + \tau(d_1^2N_1 + d_2^2N_2) \mid \sigma(d_1N_1 + d_2N_2) + \rho\sigma^2(d_1^2N_1 + d_2^2N_2) = B(0,0;\rho)\Big\} \\
&= \max_{d_2}\Big\{d_1N_1 + d_2N_2 + \tau(d_1^2N_1 + d_2^2N_2) \mid d_1N_1 + d_2N_2 + \tau(d_1^2N_1 + d_2^2N_2) = \frac{1}{\sigma}B(0,0,\frac{\tau}{\sigma})\Big\} \\
&= \frac{1}{\sigma}B(0,0,\frac{\tau}{\sigma}).
\end{aligned}$$

Define

$$d_1^-(N_2) = \begin{cases} \dfrac{\varphi D - \varphi D\Big(1+(N_1+N_2)\frac{N_2-T}{N_1 T}\Big)^{\frac{1}{2}}}{\sigma(N_1+N_2)} & \text{if } N_1+N_2 \geq T \wedge N_1 \leq T \\ -\infty & \text{otherwise} \end{cases} \tag{C.3a}$$

$$d_1^+(N_2) = \begin{cases} \dfrac{\varphi D + \varphi D\Big(1+(N_1+N_2)\frac{N_2-T}{N_1 T}\Big)^{\frac{1}{2}}}{\sigma(N_1+N_2)} & \text{if } N_1+N_2 \geq T \wedge N_2 \leq T \\ +\infty & \text{otherwise.} \end{cases} \tag{C.3b}$$

If two functions f with equal N_2 but different ρ intersect, d_1 takes value $d_1^-(N_2)$ or $d_1^+(N_2)$. The following lemma provides information on the existence and location of these intersection points. We consider only those values for d_1 where function $f(d_1,N_2;\rho,\tau)$ is finite for all $(\rho,\tau) \in Z$. Let

$$d_{\text{UB}} = \min_{(\rho,\tau)\in Z} g(0,0,N_1;\rho). \tag{C.4}$$

Lemma 2 **(Properties d_1^- and d_1^+)** *Let $N_1, T \in \mathbb{N}_+$.*

(a) Let $N_2 \in \mathbb{N}_+$. If $\rho_1 \neq \rho_2$, the equation

$$f(d_1,N_2;\rho_1,\tau) = f(d_1,N_2;\rho_2,\tau), \tag{C.5}$$

has the following real roots for d_1 on the interval $[0,d_{UB}]$:

- $d_1^-(N_2)$ *and* $d_1^+(N_2)$ — *if* $N_1+N_2 \geq T,\ N_2 \leq T$ *and* $N_1 \leq T$ (C.6a)
- $d_1^-(N_2)$ — *if* $N_1+N_2 \geq T,\ N_2 \leq T$ *and* $N_1 > T$ (C.6b)
- $d_1^+(N_2)$ — *if* $N_1+N_2 \geq T,\ N_2 > T$ *and* $N_1 \leq T$ (C.6c)
- *no roots on interval* — *if* $N_1+N_2 \geq T,\ N_2 > T$ *and* $N_1 > T$ (C.6d)
- *no real roots* — *otherwise*. (C.6e)

(b) Let $N_2^A, N_2^B \in \mathbb{N}_+$ such that $N_2^A < N_2^B$. It holds that
(i) *If $d_1^-(N_2^A)$ and $d_1^-(N_2^B)$ are both finite, then $d_1^-(N_2^A) > d_1^-(N_2^B)$;*
(ii) *If $d_1^+(N_2^A)$ and $d_1^+(N_2^B)$ are both finite, then $d_1^+(N_2^A) \leq d_1^+(N_2^B)$.*

Proof Both parts of the lemma are proved individually.
Proof Lemma 2a
By definition of f, the equation $f(d_1,N_2;\rho_1,\tau) = f(d_1,N_2;\rho_2,\tau)$ reduces to $g(d_1,N_1,N_2;\rho_1) = g(d_1,N_1,N_2;\rho_2)$ with $d_1 \in [0,\min\{g(0,0,N_1;\rho_1),g(0,0,N_1;\rho_2)\}]$. By construction of g, this means we are interested in the pairs (d_1,d_2) that solve the system

$$\sigma(N_1 d_1 + N_2 d_2) + \rho_1 \sigma^2(N_1 d_1^2 + N_2 d_2^2) = \varphi D(1+\rho_1 \frac{D}{T}\varphi) \tag{C.7a}$$

$$\sigma(N_1 d_1 + N_2 d_2) + \rho_2 \sigma^2(N_1 d_1^2 + N_2 d_2^2) = \varphi D(1+\rho_2 \frac{D}{T}\varphi) \tag{C.7b}$$

$$d_1 \geq 0,\ d_2 \geq 0. \tag{C.7c}$$

We subtract $\frac{\rho_2}{\rho_1}$ times (C.7a) from (C.7b) and solve for d_1 to obtain

$$d_1 = \frac{\varphi D - \sigma N_2 d_2}{\sigma N_1}. \tag{C.8}$$

We know that $d_2 = g(d_1, N_1, N_2; \rho_1)$. Plugging in (C.8) in this expression and simplifying yields the following roots for d_2:

$$r_2^-(N_2) = \frac{\varphi D + \varphi D\Big(1 + (N_1+N_2)\big(\frac{N_1}{T} - 1\big)/N_2\Big)^{\frac{1}{2}}}{\sigma(N_1+N_2)} \tag{C.9a}$$

$$r_2^+(N_2) = \frac{\varphi D - \varphi D\Big(1 + (N_1+N_2)\big(\frac{N_1}{T} - 1\big)/N_2\Big)^{\frac{1}{2}}}{\sigma(N_1+N_2)}. \tag{C.9b}$$

Plugging (C.9) in (C.8) and simplifying yields the following roots for d_1:

$$r_1^-(N_2) = \frac{\varphi D - \varphi D\Big(1 + (N_1+N_2)\frac{N_2-T}{N_1 T}\Big)^{\frac{1}{2}}}{\sigma(N_1+N_2)} \tag{C.10a}$$

$$r_1^+(N_2) = \frac{\varphi D + \varphi D\Big(1 + (N_1+N_2)\frac{N_2-T}{N_1 T}\Big)^{\frac{1}{2}}}{\sigma(N_1+N_2)}. \tag{C.10b}$$

These roots need not be real-valued, nor in the interval $[0, \min\{g(0,0,N_1;\rho_1), g(0,0,N_1;\rho_2)\}]$. For both $r_1^-(N_2)$ and $r_1^+(N_2)$ to be real-valued, we require that

$$1 + (N_1+N_2)\big(\frac{N_1}{T} - 1\big)/N_2 \geq 0,$$

which reduces to $N_1 + N_2 \geq T$. Furthermore, for nonnegativity of $r_1^-(N_2)$ and $r_1^+(N_2)$ it suffices to check nonnegativity of the former. This is equivalent to

$$\varphi D - \sigma N_2 d_2^- \geq 0,$$

which reduces to $N_2 \leq T$. Moreover, it needs to hold that $r_1^+(N_2) \leq g(0,0,N_1;\rho_1)$ and $r_1^+(N_2) \leq g(0,0,N_1;\rho_2)$. This is equivalent to $r_2^+(N_2) \geq 0$, which can be rewritten to

$$1 + (N_1+N_2)\big(\frac{N_1}{T} - 1\big)/N_2 \leq 1,$$

and this reduces to $N_1 \leq T$. Parameters $d_1^-(N_2)$ resp. $d_1^+(N_2)$ (see (C.3)) take the values of $r_1^-(N_2)$ resp. $r_1^-(N_2)$ if they are a root of (C.5), and $-\infty$ resp. $+\infty$ otherwise. All together, we obtain the cases in (C.6).

It remains to show that the obtained roots are in the interval $[0, d_{\mathrm{UB}}]$. It is already shown that, if they are (real-valued) roots to (C.5), then $d_1^-(N_2), d_1^+(N_2)$ are nonnegative. Furthermore, in that case $d_1^-(N_2) \leq d_1^+(N_2)$. It holds that

$$\frac{\partial g(0,0,N_1;\rho)}{\partial \rho} \leq 0 \Leftrightarrow N_1 \leq T.$$

Hence, if $d_1^+(N_2)$ is a real-valued root to (C.5) it follows that

$$d_{\mathrm{UB}} = \min_{(\rho,\tau)\in Z} g(0,0,N_1;\rho) \geq \lim_{\rho\to+\infty} g(0,0,N_1;\rho) = \frac{\varphi D}{\sigma\sqrt{N_1 T}} \geq d_1^+(N_2),$$

where the second equality follows from the definition of g. This implies that indeed $d_1^-(N_2), d_1^+(N_2) \in [0, d_{\mathrm{UB}}]$.

Proof Lemma 2b
Assume $N_2^A, N_2^B \in \mathbb{N}_+$ such that $N_2^A \leq N_2^B$, and assume $N_1 + N_2^A \geq T$. Statements (i) and (ii) are proved individually.

Proof part (i)
Assume $d_1^-(N_2^A)$ and $d_1^-(N_2^B)$ are both finite. The denominator of $d_1^-(N_2)$ (see (C.3a)) is increasing in

N_2. The derivative (w.r.t. N_2) of the part within the square root in the numerator of (C.3a) is given by $(N_1T)^{-1}(N_1+2N_2-T) \geq 0$, because $N_1+N_2 \geq T$. Hence, the numerator is decreasing in N_2, while the denominator is increasing in N_2. This implies $d_1^-(N_2^A) > d_1^-(N_2^B)$.

Proof part (ii)
Assume $d_1^+(N_2^A)$ and $d_1^+(N_2^B)$ are both finite. One can show that

$$\frac{\partial d_1^+(N_2)}{\partial N_2} = \varphi D \frac{(N_1+N_2)(N_1+2N_2-T) - 2N_1T\sqrt{\frac{N_2(N_1+N_2-T)}{N_1T}} - 2N_2(N_1+N_2-T)}{2N_1T\sigma(N_1+N_2)^2\sqrt{\frac{N_2(N_1+N_2-T)}{N_1T}}}. \tag{C.11}$$

This implies

$$\begin{aligned}
&\frac{\partial d_1^+(N_2)}{\partial N_2} \geq 0 \\
\Leftrightarrow\ &(N_1+N_2)(N_1+2N_2-T) - 2N_1T\sqrt{\frac{N_2(N_1+N_2-T)}{N_1T}} - 2N_2(N_1+N_2-T) \geq 0 \\
\Leftrightarrow\ &\frac{N_1(N_1+N_2-T)+TN_2}{2N_1T} \geq \sqrt{\frac{N_2(N_1+N_2-T)}{N_1T}} \\
\Leftrightarrow\ &\frac{\big(N_1(N_1+N_2-T)+TN_2\big)^2}{4N_1^2T^2} \geq \frac{N_2(N_1+N_2-T)}{N_1T} \\
\Leftrightarrow\ &\frac{(N_1-T)^2(N_1+N_2)^2}{4N_1^2T^2} \geq 0,
\end{aligned} \tag{C.12}$$

where the fourth line is obtained by using the fact that $N_1+N_2 \geq T$, and squaring on both sides. The last line follows from simple algebraic manipulations. Condition (C.12) clearly holds, so $d_1^+(N_2^A) \leq d_1^+(N_2^B)$.

Lemma 3 (Derivative f and g w.r.t. ρ) *Let $(\rho,\tau) \in Z$.*

(a) Let $N', N'' \in \mathbb{N}_+$. Let $d' \in [0, d_{UB}]$. If $N'+N'' < T$, then

$$\frac{\partial g(d',N',N'';\rho)}{\partial \rho} < 0 \quad \text{for all } d' \in [0, d_{UB}]. \tag{C.13}$$

If $N'+N'' \geq T$, then

$$\frac{\partial g(d',N',N'';\rho)}{\partial \rho} \begin{cases} < 0 & \text{if } d' \in [0, d_1^-(N'')) \cup (d_1^+(N''), d_{UB}] \\ = 0 & \text{if } d' \in [0, d_{UB}] \cap \{d_1^-(N''), d_1^+(N'')\} \\ > 0 & \text{if } d' \in [0, d_{UB}] \cap (d_1^-(N''), d_1^+(N'')). \end{cases} \tag{C.14}$$

(b) Let $N_1, N_2 \in \mathbb{N}_+$. Let $d_1 \in [0, d_{UB}]$. If $N_1+N_2 < T$, then

$$\frac{\partial f(d_1,N_2;\rho,\tau)}{\partial \rho} < 0 \quad \text{for all } d_1 \in [0, d_{UB}]. \tag{C.15}$$

If $N_1+N_2 \geq T$, then

$$\frac{\partial f(d_1,N_2;\rho,\tau)}{\partial \rho} \begin{cases} < 0 & \text{if } d_1 \in [0, d_1^-(N_2)) \cup (d_1^+(N_2), d_{UB}] \\ = 0 & \text{if } d_1 \in [0, d_{UB}] \cap \{d_1^-(N_2), d_1^+(N_2)\} \\ > 0 & \text{if } d_1 \in [0, d_{UB}] \cap (d_1^-(N_2), d_1^+(N_2)). \end{cases} \tag{C.16}$$

Proof We first prove Lemma 3a, after that the result of Lemma 3b is easily obtained.

It holds that

$$\frac{\partial g(d',N',N'';\rho)}{\partial \rho} = \frac{\sqrt{1+\frac{4\rho}{N''}B(d',N';\rho)} - 1 + \frac{2\rho}{N''}(d'N'\sigma - \varphi D)}{2\sigma\rho^2\sqrt{1+\frac{4\rho}{N''}B(d',N';\rho)}}, \tag{C.17}$$

so

$$\frac{\partial g(d',N',N'';\rho)}{\partial \rho} \geq 0 \Leftrightarrow \sqrt{(N'')^2 + 4\rho N'' B(d',N';\rho)} \geq N'' + 2\rho(\varphi D - d'N'\sigma). \tag{C.18}$$

We distinguish 2 cases:

- $\varphi D \geq d'N'\sigma$. In this case, squaring (C.18) on both sides and simplifying results in

$$-\sigma^2 N'(N'+N'')d'^2 + 2\varphi D N'\sigma d' + \big(\frac{N''}{T} - 1\big)\varphi^2 D^2 \geq 0, \tag{C.19}$$

  which is a condition independent of ρ. If $N'+N'' < T$, this inequality has no roots for d', and (C.19) holds for all $d' \in [0, \frac{\varphi D}{N'\sigma}]$. If $N'+N'' \geq T$ one can verify that $d_1 = d_1^-(N'')$ and $d_1 = d_2^+(N'')$ are the roots of this concave parabola if they are finite. The smaller root, $d_1^-(N'')$, is finite if and only if $N'' \leq T$. The larger root, $d_1^+(N'')$ is finite if and only if $N' \leq T$.
- $\varphi D < d'N'\sigma$. In this case, $B(d',N';\rho) > 0$ only if $N' > T$. In this case, the delivered dose exceeds the dose that is used to set the BED tolerance, which is only possible if the number of fractions N' is strictly larger than the reference number of fractions T. Condition (C.18) clearly holds, so $g(d',N',N'';\rho)$ is increasing in ρ. Using the fact that $N' > T$ it is easily shown that $d_1^- < \frac{\varphi D}{\sigma N'} < d'$. Additionally, it can be shown that $d_{\mathrm{UB}} < d_1^+(N'')$. Hence this case satisfies (C.14). Putting all of the above together yields the required result for g, i.e., Lemma 3a.

The partial derivative of f w.r.t. ρ is given by

$$\frac{\partial f(d_1,N_2;\rho,\tau)}{\partial \rho} = \frac{\partial g(d_1,N_1,N_2;\rho)}{\partial \rho}\Big(N_2 + 2\tau N_2 g(d_1,N_1,N_2;\rho)\Big). \tag{C.20}$$

Hence, the sign of the partial derivative of f w.r.t. ρ is equal to the sign of the partial derivative of g w.r.t. ρ. The result of Lemma 3b immediately follows.

For given (ρ,τ) such that $\tau \neq \sigma\rho$, define the *twin point* of $d_1 \in W(\rho,\tau)$ as

$$t(d_1;\rho,\tau) = \frac{\big(N_1 - N_2^*(\rho,\tau)\big)d_1 + 2N_2^*(\rho,\tau)g\big(d_1,N_1,N_2^*(\rho,\tau);\rho\big)}{N_1 + N_2^*(\rho,\tau)}, \tag{C.21}$$

where

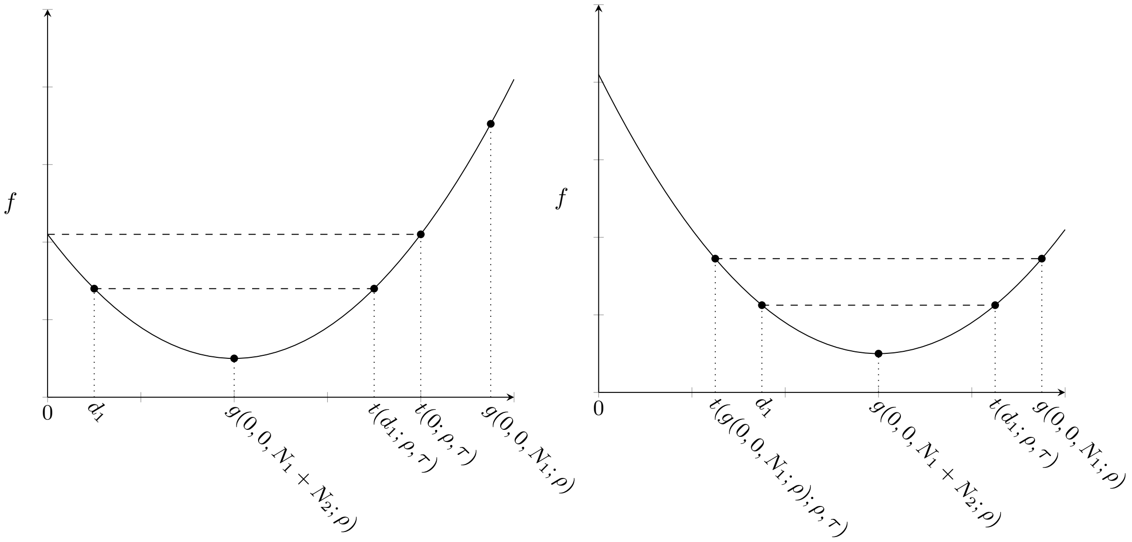
$$W(\rho,\tau) = \Big[\max\{0, t(g(0,0,N_1;\rho);\rho,\tau)\},\ \min\{t(0;\rho,\tau), g(0,0,N_1;\rho)\}\Big] \setminus \{g(0,0,N_1+N_2;\rho)\}. \tag{C.22}$$

Figure C.1 illustrates the relation between d_1 and $t(d_1;\rho,\tau)$. Set W can be interpreted as the points d_1 for which there exists another point the graph of f that has the same value, we refer to such points as twin points. The following lemma proves that for fixed (ρ,τ) any d_1 in the set $W(\rho,\tau)$ has a twin point $t(d_1;\rho,\tau)$ that is also in the set $W(\rho,\tau)$, and their objective values are equal.

Lemma 4 *Let $(\rho,\tau) \in Z$ such that $\tau \neq \sigma\rho$, let $N_2 = N_2^*(\rho,\tau)$ and let $d_1 \in [0, g(0,0,N_1;\rho)]$. The equation*

$$f(d_1,N_2;\rho,\tau) = f(d_1',N_2;\rho,\tau) \tag{C.23}$$

has a solution $d_1' \in [0, g(0,0,N_1;\rho)]$ unequal to d_1 if and only if $d_1 \in W(\rho,\tau)$. In that case, there is a unique solution $d_1' = t(d_1;\rho,\tau) \in W(\rho,\tau)$, and it holds that $d_1 = t(t(d_1;\rho,\tau);\rho,\tau)$.



(a) Case $t(0;\rho,\tau) < g(0,0,N_1;\rho)$.

(b) Case $t(0;\rho,\tau) > g(0,0,N_1;\rho)$.

Fig. C.1: Illustration of d_1 and $t(d_1;\rho,\tau)$, for convex f.

Proof Let (ρ,τ) such that $\tau \neq \sigma\rho$, let $N_2 = N_2^*(\rho,\tau)$. We first show that if there exists a solution d_1' to (C.23) unequal to d_1, then this solution is $d_1' = t(d_1;\rho,\tau)$ and that $d_1 = t(t(d_1;\rho,\tau);\rho,\tau)$. Subsequently, we show that this solution exists only if $d_1 \in W(\tau,\rho)$ and that in that case also $t(d_1;\rho,\tau) \in W(\rho,\tau)$.

First part:
Let $L, Q \in \mathbb{R}_+$ denote the linear and quadratic contribution of d_1 to f, i.e.,

$$L(d_1,N_2,\rho) = N_1 d_1 + N_2 g(d_1,N_1,N_2;\rho) \tag{C.24a}$$
$$Q(d_1,N_2,\rho) = N_1 d_1^2 + N_2 g(d_1,N_1,N_2;\rho)^2. \tag{C.24b}$$

We show that for any $d_1 \in [0, g(0,0,N_1;\rho)]$ unequal to $g(0,0,N_1+N_2,\rho)$ there is a unique solution d_1' such that

$$L(d_1,N_2;\rho) = L(d_1',N_2;\rho) \tag{C.25a}$$
$$Q(d_1,N_2;\rho) = Q(d_1',N_2;\rho). \tag{C.25b}$$

Any such d_1' has exactly the same objective value as d_1. Plug (C.24) in (C.25), rewrite the first equation to eliminate g, and plug this in the second equation to get

$$\begin{aligned} d_1' &= \frac{L(d_1,N_2;\rho) \pm \sqrt{\frac{N_2}{N_1}\left((N_1+N_2)Q(d_1,N_2;\rho) - L^2(d_1,N_2;\rho)\right)}}{N_1+N_2} \\ &= \frac{N_1 d_1 + N_2 g(d_1,N_1,N_2;\rho) \pm N_2(d_1 - g(d_1,N_1,N_2;\rho))}{N_1+N_2}. \end{aligned} \tag{C.26}$$

The '+' solution to (C.26) returns $d_1' = d_1$, and the '–' solution returns

$$d_1' = \frac{(N_1-N_2)d_1 + 2N_2 g(d_1,N_1,N_2,\rho)}{N_1+N_2}, \tag{C.27}$$

and we denote this solution by $t(d_1;\rho,\tau)$. By construction, it holds that $d_1 = t(t(d_1;\rho,\tau);\rho,\tau)$. Because $\tau \neq \sigma\rho$, function $f(d_1,N_2;\rho,\tau)$ is a strictly convex or concave function according to Lemma 1, so $f(d_1,N_2;\rho,\tau) = z$ for some constant $z \in \mathbb{R}$ has either 0, 1 or 2 solutions. In particular, $d_1 = t(d_1;\rho,\tau)$ if and only if d_1 equals minimizer $g(0,0,N_1+N_2;\rho)$. Hence if there exists a solution d_1' to (C.23) unequal to d_1, then this solution is $d_1' = t(d_1;\rho,\tau)$.

Second part:

- Suppose $d_1 \notin W(\rho,\tau)$. We distinguish three cases. Case (i): '$d_1 = g(0,0,N_1+N_2;\rho)$'. Because this is the unique minimizer of f, there does not exist a d_1' with equal objective value. Case (ii). '$d_1 > \min\{t(0;\rho,\tau),\ g(0,0,N_1;\rho)\}$'. Because $d_1 \in [0, g(0,0,N_1;\rho)]$, this implies $d_1 > t(0,\rho,\tau)$. As shown in the first part of the proof, it holds that $d_1 = t(t(d_1;\rho,\tau);\rho,\tau)$. Hence, $d_1 > t(0;\rho,\tau)$ is equivalent to $t(t(d_1;\rho,\tau),\rho,\tau) > t(0;\rho,\tau)$. Because $t(d_1;\rho,\tau)$ is decreasing in d_1, this implies $t(d_1;\rho,\tau) < 0$, so according to (B.1c) it holds that $f(t(d_1;\rho,\tau),N_2;\rho,\tau) = -\infty$ and we have a contradiction. Case (iii): '$d_1 < \max\{0,\ t(g(0,0,N_1;\rho);\rho,\tau)\}$'. Similar to case (ii), one can show that $f(t(d_1;\rho,\tau),N_2;\rho,\tau) = -\infty$.
- Suppose $d_1 \in W(\rho,\tau)$. From (C.26) one can see that, because the term $d_1 - g(d_1,N_1,N_2,\rho)$ is increasing in d_1, the function $t(d_1;\rho,\tau)$ is decreasing in d_1. Consequently,

$$d_1 \leq \min\{t(0;\rho,\tau), g(0,0,N_1;\rho)\} \Leftrightarrow d_1' \geq \max\{0, t(g(0,0,N_1;\rho);\rho,\tau)\}. \tag{C.28}$$

  Furthermore, using the same argument,

$$d_1 \geq \max\{0, t(g(0,0,N_1;\rho);\rho,\tau)\} \Leftrightarrow d_1' \leq \min\{t(0;\rho,\tau), g(0,0,N_1;\rho)\}. \tag{C.29}$$

  Therefore, it holds that $d_1' \in W(\rho,\tau)$.

In the following lemma, let $I(\cdot|S)$ denote the indicator function for a set S:

$$I(x|S) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise.} \end{cases} \tag{C.30}$$

Lemma 5 *For given $q \in \mathbb{R}_+$ and given $d_1 \in [0, d_{UB}]$,*

$$q \leq f(d_1, N_2^*(d_1;\hat{\rho},\hat{\tau});\rho,\tau), \ \ \forall(\rho,\tau,\hat{\rho},\hat{\tau}) \in U_L^{int}, \tag{C.31}$$

holds if $\tau \leq p(d_1)$, with

$$\begin{aligned} p(d_1) = &\sum_{\eta \in \{N_2^{min},\dots,N_2^{max}\}} \max\left\{f(d_1,\eta;\rho_L^{int},\tau_L),\ f(d_1,\eta-1;\rho_U^{int},\tau_L)\right\} I(d_1|S_\eta) \\ &+ f(d_1,N_2^{min};\rho_L^{int},\tau_L) I(d_1|S_{min}) + f(d_1,N_2^{max};\rho_U^{int},\tau_L) I(d_1|S_{max}), \end{aligned} \tag{C.32}$$

where sets S_{min}, S_{max} and S_η are defined in (C.37a)*,* (C.37d) *and* (C.42)*, respectively, and ρ_L^{int}, ρ_U^{int} are defined in* (C.44)*.*

Proof By definition of $N_2^*(d_1;\hat{\rho},\hat{\tau})$ and U_L^{int}, it holds that

$$q \leq f(d_1, N_2^*(d_1;\hat{\rho},\hat{\tau});\rho,\tau), \ \ \forall(\rho,\tau,\hat{\rho},\hat{\tau}) \in U_L^{\text{int}}, \tag{C.33}$$

is equivalent to

$$q \leq \max_{\tilde{\eta} \in \{N_2^{\min},\dots,N_2^{\max}\}} \min\{f(d_1,\tilde{\eta};\hat{\rho}_L,\hat{\tau}_L), f(d_1,\tilde{\eta};\hat{\rho}_U,\hat{\tau}_L)\}, \ \ \forall(\hat{\rho},\hat{\tau}) \in Z_{\text{ID}}^{\text{int}} \cap \{(\hat{\rho},\hat{\tau}) : \hat{\tau} \leq \tau_L + r^\tau\}, \tag{C.34}$$

and because function f is increasing in τ, we need to consider only those observations $(\hat{\rho},\hat{\tau})$ with $\hat{\tau}_L = \tau_L$. For the first part of the proof, we fix the observation $(\hat{\rho},\hat{\tau})$, plug in $\hat{\tau}_L = \tau_L$, and rewrite (C.34) for this fixed observation.

Because $(\hat{\rho},\hat{\tau}) \in Z_{\mathrm{ID}}^{\mathrm{int}}$, it holds that $\sigma\hat{\rho}_L < \tau_L < \sigma\hat{\rho}_U$. Hence, by Lemma 1, function $f(d_1,\eta,\hat{\rho}_L,\tau_L)$ is convex and $f(d_1,\eta,\hat{\rho}_U,\tau_L)$ is concave in d_1 for any $\eta \in \mathbb{N}_+$. We make use of results of Lemma 2. Define

$$E^- = \{\eta : N_1+\eta \geq T, \eta \leq T\} \cap \{N_2^{\min},\dots,N_2^{\max}\} \tag{C.35a}$$

$$E^+ = \{\eta : N_1+\eta \geq T, N_1 \leq T\} \cap \{N_2^{\min},\dots,N_2^{\max}\}, \tag{C.35b}$$

and let $\eta^-_{\min}$, $\eta^-_{\max}$, $\eta^+_{\min}$ and $\eta^+_{\max}$, denote the smallest and largest elements of E^- and E^+, respectively. If $\eta \in E^-$ respectively $\eta \in E^+$, then, according to Lemma 2a, $d_1 = d_1^-(\eta)$ respectively $d_1 = d_1^+(\eta)$ is a nonnegative real root of

$$f(d_1,\eta;\hat{\rho}_L,\tau_L) = f(d_1,\eta;\hat{\rho}_U,\tau_L),$$

and the corresponding objective value equals K. From Lemma 2b we know that

$$d_1^-(N_2^{\max}) < \ldots < d_1^-(N_2^{\min}) \leq d_1^+(N_2^{\min}) \leq \ldots \leq d_1^+(N_2^{\max}). \tag{C.36}$$

We use this to split the domain $[0,d_{\mathrm{UB}}]$ as follows:

$$S_{\min} = \begin{cases} (d_1^-(\eta^-_{\min}), d_1^+(\eta^+_{\min})) & \text{if } E^- \neq \emptyset, E^+ \neq \emptyset \\ [0, d_1^+(\eta^+_{\min})) & \text{if } E^- = \emptyset, E^+ \neq \emptyset \\ (d_1^-(\eta^-_{\min}), d_{\mathrm{UB}}] & \text{if } E^- \neq \emptyset, E^+ = \emptyset \\ \emptyset & \text{if } N_1 + N_2^{\max} < T \\ [0, d_{\mathrm{UB}}] & \text{otherwise} \end{cases} \tag{C.37a}$$

$$S^-_\eta = \begin{cases} [d_1^-(\eta), d_1^-(\eta-1)] & \text{if } \eta^-_{\min} \leq \eta - 1 < \eta \leq \eta^-_{\max} \\ \emptyset & \text{otherwise} \end{cases} \quad \forall \eta \in \{N_2^{\min}+1,\dots,N_2^{\max}\} \tag{C.37b}$$

$$S^+_\eta = \begin{cases} [d_1^+(\eta-1), d_1^+(\eta)] & \text{if } \eta^+_{\min} \leq \eta - 1 < \eta \leq \eta^+_{\max} \\ \emptyset & \text{otherwise} \end{cases} \quad \forall \eta \in \{N_2^{\min}+1,\dots,N_2^{\max}\} \tag{C.37c}$$

$$S_{\max} = \begin{cases} [0, d_1^-(\eta^-_{\max})) \cup (d_1^+(\eta^+_{\max}), d_{\mathrm{UB}}] & \text{if } E^- \neq \emptyset, E^+ \neq \emptyset \\ (d_1^+(\eta^+_{\max}), d_{\mathrm{UB}}] & \text{if } E^- = \emptyset, E^+ \neq \emptyset \\ [0, d_1^-(\eta^-_{\max})) & \text{if } E^- \neq \emptyset, E^+ = \emptyset \\ [0, d_{\mathrm{UB}}] & \text{if } N_1 + N_2^{\max} < T \\ \emptyset & \text{otherwise.} \end{cases} \tag{C.37d}$$

We will reformulate (C.34) on each interval (set) separately, assuming it is nonempty.

1. "$S_{\min}$": If $d_1 \in S_{\min}$, then $f(d_1,\eta;\hat{\rho}_L,\tau_L) < f(d_1,\eta;\hat{\rho}_U,\tau_L)$ for all $\eta \in \{N_2^{\min},\dots,N_2^{\max}\}$ according to Lemma 3b, so it is optimal to deliver $N_2^{\min}$ fractions. Hence, on this interval (C.34) is equivalent to

$$q \leq f(d_1, N_2^{\min}; \hat{\rho}_L, \tau_L). \tag{C.38}$$

2. "S^-_η": From Lemma 2a we know that $f(d_1,\eta,\hat{\rho}_L,\tau_L) = f(d_1,\eta,\hat{\rho}_U,\tau_L)$ if $d_1 = d_1^-(\eta)$ or $d_1 = d_1^+(\eta)$. In this case, the objective value equals K. Furthermore, function $f(d_1,\eta,\hat{\rho}_L,\tau_L)$ is convex and $f(d_1,\eta,\hat{\rho}_U,\tau_L)$ is concave in d_1. Consider the interval $[d_1^-(\eta), d_1^-(\eta-1)]$. It holds that

$$f(d_1,\eta;\hat{\rho}_L,\tau_L) \leq K \leq f(d_1,\eta-1;\hat{\rho}_L,\tau_L) \tag{C.39a}$$

$$f(d_1,\eta-1;\hat{\rho}_U,\tau_L) \leq K \leq f(d_1,\eta;\hat{\rho}_U,\tau_L). \tag{C.39b}$$

This implies that if $d_1 \in [d_1^-(\eta), d_1^-(\eta-1)]$, it is optimal to deliver either η or $\eta-1$ fractions. If we deliver η fractions, the restricting worst-case scenario is $(\hat{\rho}_L,\tau_L)$ and the value f is above K for the scenario $(\hat{\rho}_U,\tau_L)$. If we deliver $\eta' > \eta$ fractions, the value for the scenario $(\hat{\rho}_L,\tau_L)$ decreases, while the value for the scenario $(\hat{\rho}_U,\tau_L)$ increases even further. Hence, delivering $\eta' > \eta$ fractions cannot be optimal. Similarly, delivering less than $\eta-1$ fractions cannot be optimal. Therefore, if $d_1 \in [d_1^-(\eta), d_1^-(\eta-1)]$ it is optimal to deliver either η or $\eta-1$ fractions. This implies that on the interval S^-_η constraint (C.34) is equivalent to

$$q \leq \max\{f(d_1,\eta;\hat{\rho}_L,\tau_L),\ f(d_1,\eta-1;\hat{\rho}_U,\tau_L)\}. \tag{C.40}$$

Note that this result does not depend on the values $\hat{\rho}_L$ and $\hat{\rho}_U$, we only use that $\hat{\rho}_L < \frac{\tau_L}{\sigma} < \hat{\rho}_U$.

3. "S_η^+": Similar to the case for S_η^+, one can show that for $d_1 \in S_\eta^+$ constraint (C.34) is equivalent to (C.40).
4. "$S_{\max}$": If $d_1 \in S_{\max}$, then $f(d_1,\eta;\hat{\rho}_L,\tau_L) > f(d_1,\eta;\hat{\rho}_U,\tau_L)$ for all $\eta \in \{N_2^{\min},\ldots,N_2^{\max}\}$ according to Lemma 3b, so it is optimal to deliver $N_2^{\max}$ fractions. Hence, on this interval (C.34) is equivalent to

$$q \le f(d_1,N_2^{\max};\hat{\rho}_U,\tau_L). \tag{C.41}$$

For sets S_η^- and S_η^+ the reformulation is the same. Therefore, define

$$S_\eta = S_\eta^- \cup S_\eta^+. \tag{C.42}$$

Putting everything together, for $d_1 \in [0,d_{\mathrm{UB}}]$ the constraint (C.34) is equivalent to

$$\begin{aligned} q \le &\sum_{\eta\in\{N_2^{\min},\ldots,N_2^{\max}\}} \max\{f(d_1,\eta;\hat{\rho}_L,\tau_L),\ f(d_1,\eta-1;\hat{\rho}_U,\tau_L)\}I(d_1|S_\eta) \\ &+ f(d_1,N_2^{\min};\hat{\rho}_L,\tau_L)I(d_1|S_{\min}) + f(d_1,N_2^{\max};\hat{\rho}_U,\tau_L)I(d_1|S_{\max}),\ \ \forall(\hat{\rho},\hat{\tau}) \in Z_{\mathrm{ID}}^{\mathrm{int}} \cap \{(\hat{\rho},\hat{\tau}) : \hat{\tau} \le \tau_L + r^\tau\}. \end{aligned} \tag{C.43}$$

In order to find a tractable conservative robust counterpart of (C.43), denote

$$\rho_L^{\mathrm{int}} = \max\{\rho_L, \frac{\tau_L}{\sigma} - 2r^\rho\} \tag{C.44a}$$

$$\rho_U^{\mathrm{int}} = \min\{\rho_U, \frac{\tau_L}{\sigma} + 2r^\rho\}, \tag{C.44b}$$

and note that $\rho_L^{\mathrm{int}} \le \hat{\rho}_L < \frac{\tau_L}{\sigma} < \hat{\rho}_U \le \rho_U^{\mathrm{int}}$. Only if $d_1 \in S_\eta$, the robust counterpart is conservative. By Lemma 3b, it holds that function f is strictly decreasing, constant or strictly increasing in ρ for fixed d_1, so

$$f(d_1,\eta;\hat{\rho}_L,\tau_L) \ge \min\{f(d_1,\eta;\rho_L^{\mathrm{int}},\tau_L), f(d_1,\eta;\frac{\tau_L}{\sigma},\tau_L)\} = \min\{f(d_1,\eta;\rho_L^{\mathrm{int}},\tau_L), K\} = f(d_1,\eta;\rho_L^{\mathrm{int}},\tau_L),$$

where the second equality follows from (C.39). A similar result holds for $f(d_1,\eta-1;\hat{\rho}_U,\tau_L)$. Furthermore, as shown before, f is increasing in ρ on $S_{\min}$ and decreasing in ρ on $S_{\max}$. Therefore, a conservative approximation of (C.31) is given by

$$\begin{aligned} q \le &\sum_{\eta\in\{N_2^{\min},\ldots,N_2^{\max}\}} \max\{f(d_1,\eta;\rho_L^{\mathrm{int}},\tau_L),\ f(d_1,\eta-1;\rho_U^{\mathrm{int}},\tau_L)\}I(d_1|S_\eta) \\ &+ f(d_1,N_2^{\min};\rho_L^{\mathrm{int}},\tau_L)I(d_1|S_{\min}) + f(d_1,N_2^{\max};\rho_U^{\mathrm{int}},\tau_L)I(d_1|S_{\max}), \end{aligned} \tag{C.45}$$

and the RHS is $p(d_1)$.

Function $p(d_1)$ is a piece-wise function. On intervals defined by $S_{\min}$ and $S_{\max}$ it is convex and concave, respectively. On any interval S_η function $p(d_1)$ is the maximum of a concave and convex function.

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