Excess covariance and dynamic instability in a multi-asset model

Anufriev, M.; Bottazzi, G.; Marsili, M.; Pin, P.

Excess Covariance and Dynamic Instability in a Multi-Asset Model*

Mikhail Anufriev[†], Giulio Bottazzi[‡], Matteo Marsili[§] and Paolo Pin[¶]

December 15, 2011

Abstract

The presence of *excess covariance* in financial price returns is an accepted empirical fact: the price dynamics of financial assets tend to be more correlated than their fundamentals would justify. We propose an intertemporal equilibrium multi–assets model of financial markets with an explicit and endogenous price dynamics. The market is driven by an exogenous stochastic process of dividend yields paid by the assets that we identify as market *fundamentals*. The model is rather flexible and allows for the coexistence of different trading strategies. The evolution of assets price and traders' wealth is described by a high-dimensional stochastic dynamical system. We identify the equilibria of the model consistent with a baseline assumption of procedural rationality. We show that these equilibria are characterized by excess covariance in prices with respect to the dividend process. Moreover, we show that in equilibrium there is a positive expected marginal profit in choosing more risky portfolios. As a consequence, the evolutionary pressure generates a trend towards more remunerative strategies, which, in turn, increase the variance of prices and the dynamic instability of the system.

JEL classification codes: D81, G11, G12

*G.B. acknowledges financial support from the Institute for New Economic Thinking, INET inaugural grant #220

†CeNDEF, Department of Quantitative Economics, University of Amsterdam, Roetersstraat 11, NL - 1018 WB Amsterdam, The Netherlands

‡Corresponding author: Scuola Superiore Sant'Anna, Piazza Martiri della Libertá 33, 56127 Pisa, Italy

§The Abdus Salam International Centre for Theoretical Physics, Strada Costiera 11, 34014 Trieste, Italy

¶Dipartimento di Economia Politica e Statistica, Università degli Studi di Siena, Piazza San Francesco 7, 53100 Siena, Italy

1 Introduction

A huge econometric literature, started in the first part of the '80, suggests that price returns in financial markets are characterized by volatility levels too high to be explained by corresponding movements in the fundamental value of assets or by macroeconomic variability in terms of aggregate consumption or money supply (LeRoy, 2008). The high "unexplained" volatility of prices of financial securities has been taken by many as suggestive of a intrinsic instability in the functioning of speculative markets. Along similar lines, several recent studies investigate whether the observed high degree of covariance in price returns can be explained by a similar degree of correlation among the economic values of the traded assets (Bouchaud and Potters, 2003). As discussed in Marsili et al. (2011), a remarkable finding is that this covariance exhibits different behaviors at different timescales. In practice, however, the phenomenon of *excess covariance* is difficult to measure, because of the vagueness of what can be identified as fundamentals in the real world: what is the original covariance above which we find an additional excess covariance? There are attempts in the literature to identify proxies of these fundamentals. Shiller (1989), Froot and Dabora (1999) and Brealey et al. (2010), each for a different historical period, have shown that the same asset traded in different markets tend to behave differently, following in each location the trend of the whole market. The appropriate way of disentangling this empirical fact from arbitrage opportunities and from differences due to the spread of information is still an open question. Kallberg and Pasquariello (2008) propose a purely empirical approach. They analyze the assets' prices of 82 firms, from different sectors, traded on the NYSE: filtering the 1976–2001 data from aggregate shocks they obtain idiosyncratic data that, they show, are much less correlated than the global financial ones. Regardless of any possible objection that could be made towards the choice of the *fundamentals*, all these papers confirm the evidence of an excess of covariance in the financial markets, compared to what is normally observed in the real economy.

Several models have been proposed to explain the appearance of excess covariance. Most of them focus on the behavior of agents: traders tend to be correlated in their activity and consequently induce an analogous correlation in assets' prices. Kyle and Xiong (2001) propose a model in which the different wealth effects of traders induce some homogeneity in portfolio choices. Kodres and Pritsker (2002) assume that the correlation is instead due to the re–balancing activity of risk averse agents. Yuan (2005) assumes it is the effect of financial constraints, while Veldkamp (2006) imagines that information is costly and introduces a herding behavior. Finally, Marsili and Raffaelli (2006) describe a stochastic behavioral model in which a single type of agents adjusts dynamically a single portfolio with a mean–variance strategy.

In this paper we show that it is possible to obtain the same result in a very general setting as a direct effect of trading, independently on any collinearity in the strategies of different traders. We do it by formalizing a stylized model of financial markets with a multiplicity of coexisting trading strategies, and a multiplicity of traded assets. We assume an exogenous underlying *real* economy, that we simply identify with a stream of random dividends payed by every asset. The dividend process is governed by a multivariate stationary distribution which is described by a constant variance-covariance matrix. This matrix represents the covariance structure of the fundamental value of the traded assets. The model assumes a Walrasian endogenous asset pricing for all risky assets through market clearing and an inelastic supply of a riskless security which act as the numeraire of the economy.

The model we propose fits in the rich literature of Heterogeneous Agents Model (HAM) (see Hommes (2006) for a fairly recent survey) but is much less restrictive in terms of agents trading behavior and does not constraint the analysis to a subset of boundedly rational trading strategies. There are only a few attempts that deal with multiple-assets framework within the HAM literature (see for instance Westerhoff (2004), Chiarella et al. (2005) and Chiarella et al. (2010)) but none of them, to the best of our knowledge, analyze the emergence of excess covariance.

The dynamics of our model is described by a high-dimensional stochastic dynamical system. In the HAM literature these systems are typically analyzed by "switching the noise off", i.e., by replacing the driving stochastic process with its expected values and considering the corresponding deterministic skeleton (see for instance the analysis in Anufriev et al. (2006) and Anufriev and Bottazzi (2010)). The argument supporting the use of the deterministic skeleton for the analysis of the stochastic dynamics runs typically as follows. If the deterministic skeleton converges to an asymptotically stable fixed point, the stochastic processes are also "close" to the corresponding fixed values, provided that the noise is reasonably small (e.g., when the dividend yields are i.i.d. and their support belong to the basins of attraction of the fixed point). On the other hand, if the deterministic skeleton exhibits a bifurcation in which the fixed point loses its stability, the associated random system is analogously perturbed away from the fixed point. In a model with multiple assets, however, the previous deterministic skeleton approach has the important drawback of not allowing for the analysis of higher moments of the fluctuation around equilibria. Indeed the correlations between the price returns of different assets is always zero in the deterministic approximation. Clearly, an alternative approach is needed. One possible solution is to directly characterize the stochastic or deterministic fixed points of the complete random dynamical system and study their stability using stochastic version of the Hartman-Grobman theorem. This is the approach followed by the evolutionary finance literature, recently surveyed in Blume and Easley (2009) and Evstigneev et al. (2009). These methods becomes however very complicated if one wants to adopt more realistic dividend processes and abandon the assumption of a finite number of states of the world. Indeed we are not aware of any study of markets with long-lived multiple assets, continuous support of the dividend structure, and investment strategies with a feedback from the past returns. Bottazzi and Dindo (2010) partially fills the gap introducing direct price feedback in agents' strategy, but only discuss the case of short-lived assets. In this paper we try to follow and intermediate approach between the full-fledged analysis of the stochastic system, which is unfeasible, and the study of the deterministic skeleton, which is too crude. We exploit the notion of Procedurally Consistent Equilibria (PCE) already introduced in Anufriev et al. (2006) Anufriev and Bottazzi (2010) and Anufriev and Dindo (2010) for single-asset markets. In PCE the unique requirement is that in equilibrium the investment shares of agents are constant. The model is then closed by the rationality assumption that realized prices are consistent with the assumption on which the investment shares were chosen.

The structure of the paper is as follows. In Section 2 we present the model, introducing the exogenous dividend process, the market clearing mechanism and discussing the range of admissible strategies. We look at price dynamics and growth wealth in the more general setting, without specializing to any particular behavioral rule. In Section 3 we study the PCE of the model and we derive a system describing the co-evolution of the agent's wealth and the assets' returns. We show that the returns' correlation matrix can be decomposed in two terms: the first is the correlation matrix of dividend process, while the second accounts for the excess correlation. This excess correlation is endogenously created and depends, in general, on the agents' investment strategies. Analogously, we show that there exists endogenous excess return for different assets, which implies that the equity premium can simply be the result of endogenous investment process. In Section 3.3 we study the evolutionary stability of PCE equilibria with respect to an invasion by a different strategy. We show that, under the most general conditions, in this model the evolutionary pressure goes in the direction of increasing the dynamic instability of the system. In Section 4 we investigate more in depth the aggregate market dynamics generated by the adoption of a specific investment rule, based on mean-variance expected utility and exponentially weighted moving average estimators. Section 5 concludes.

2 The model

Consider a market populated by N traders where the shares of I long-lived financial assets paying risky dividends are traded. Moreover a riskless security exists paying a constant and exogenous interest rate $r_f > 0$ per period. Traders have an opportunity to lend the wealth not invested in the risky assets at that rate, or borrowing at the same rate if they want to invest more in the assets. The riskless bond serves as the numéraire of the economy (prices are fixed with respect to it) and is provided in total inelastic supply. For every risky asset the amount of shares is fixed.

Time is discrete. The economy is evolving through a sequence of "temporary equilibria" (Grandmont, 1985), where prices of the risky assets are fixed through Walrasian mechanism of market clearing equating demand and supply. In the rest of this section we present all the features of the model and derive general results for its solvability. The assumptions about traders' demand at each time step are presented first. Then in Section 2.2 the price and wealth dynamics are derived. Section 2.3 completes our formalization of the traders' behavior by showing how the individual demands are changing in time. In Section 2.4 the model is summarized as a high-dimensional stochastic dynamical system. Finally, in Section 2.5 we introduce a simple example which will be used for illustrative purposes throughout Section 3.

2.1 Traders' demand

At each time step each trader decides the fraction of individual wealth to be invested in each risky asset and in the riskless bond. The fraction of wealth invested by agent n at time t in the risky asset i is denoted by $x^i_{t,n}$. The residual fraction $x^0_{t,n} = 1 - \sum_{i=1}^{I} x^i_{t,n}$ is invested in the riskless security. Throughout the paper the vector notation is used. The (column) vector of individual wealth fractions invested in the risky assets is denoted by $\mathbf{x}_{t,n}$.

In a dynamical context the individual investment fractions are, in general, changing as a new information becomes available. In this sense, we think of agent's behavior as about a mapping from available past data to the present wealth fractions $\mathbf{x}_{t,n}$. We will postpone a formalization of this idea to Section 2.3.

Notice that at time t the investment decision of agent n is completely described by the vector of investment fractions, $\mathbf{x}_{t,n}$. It is assumed that this vector is independent of the agents' contemporary wealth and prices. The individual demand function for the risky asset i is given by

$$Z^i_{t,n}\left(p^i_t\right) = \frac{x^i_{t,n} w_{t,n}}{p^i_t}\,, \tag{2.1}$$

where p^i_t denote the price of asset i at time t and $w_{t,n}$ denote the wealth of agent n at time t. A demand function like the one in (2.1) with $x^i_{t,n}$ independent of $w_{t,n}$ and p^i_t implies a specific dependence of agent's demand on wealth and prices and is equivalent to assuming constant relative risk aversion (CRRA) preferences over final wealth.[1] It has been often used in models with heterogeneous agents and in the evolutionary finance literature[2]. Moreover, it mimics the financial decisions taken in real world, where portfolio are often designed as fractions of wealth to be split across different category of assets.

Notice that the model is laid down without including consumption in agent's behavior. In this way we have a pure exchange economy, where traders decision are plausibly driven by expectations about their future wealth. While the inclusion of consumption in the model is rather straightforward,[3] we believe that the simpler "pure-exchange" framework we present is more suitable to study the issues we are interested here. Our next step is to derive the prices and individual wealth at time t using intertemporal budget constraints and market clearing conditions.

2.2 Pricing of risky assets

Every asset i pays non-negative random dividend d^i_t at the beginning of period t. The dividend yield of asset i, $e^i_t = d^i_t/p^i_{t-1}$, is defined as the ratio of the dividend payed by the asset over the last price before the payment. With usual vector notation, the column vectors of dividends, prices, and yields are written as $\mathbf{d}_t$, $\mathbf{p}_t$ and $\mathbf{e}_t$ respectively. Table 1 summarizes the notation used in this paper.

[1]Alternatively, it characterizes an investor who maximizes mean-variance utility of return, see Section 4.

[2]See the models in Levy et al. (2000), Chiarella and He (2001), Anufriev et al. (2006), Evstigneev et al. (2009) and Subbotin and Chauveau (2011), among others. It is worth to notice that many HAMs are built in an alternative framework, where demand does not depend on current wealth, see, e.g., Brock and Hommes (1998)

[3]One can think about the trading activity on the market as a notionally infinite repeated game, where at every time step there is a positive probability $1-\sigma$ that the market ends and all agents consume their wealth, so enjoying the associated utility. In this case we would introduce a homogeneous discount factor σ for the individual wealth of each agent, as in Gale (1987), which after appropriate rescaling of prices would leave the dynamics of the economy invariant.

The inter–temporal relationship of the wealth of agent n reads

$$w_{t,n} = w_{t-1,n} \left(\sum_{i=1}^{I} \frac{x^i_{t-1,n}}{p^i_{t-1}} \left(d^i_t + p^i_t\right) + x^0_{t-1,n} \left(1 + r_f\right) \right) . \tag{2.2}$$

This simple accounting relation shows two sources of change in individual wealth given the investment decision at time $t-1$. First, investment in risky assets returns random dividend yield and capital gain. Second, investment in the riskless security pays a constant exogenous rate.

The price of asset i at time t is determined via an equilibrium relation between the endogenous demand of traders and a fixed supply, normalized to 1. Using (2.1), market-clearing condition $\sum_n Z^i_{t,n} = 1$ allows us to derive the price as

$$p^i_t = \sum_{n=1}^{N} w_{t,n} x^i_{t,n} . \tag{2.3}$$

The system of equations given by (2.2) and (2.3) provides only an implicit definition of the assets prices $\mathbf{p}_t$ in terms of the past and present portfolio decisions $\mathbf{x}_{t,n}$. Indeed present prevailing prices appear both on the left and right hand sides of these equations. Our next step is to derive price evolution explicitly by solving the system (2.2) and (2.3). Theorem 2.1 shows that this is possible under reasonable restriction on traders' portfolio choices.

To present this result, some more notation is useful. Let $w_t = \sum_n w_{t,n}$ denote the total wealth at time t and $\varphi_{t,n} = w_{t,n}/w_t$ the wealth share of trader n. We also define the *market portfolio* as the wealth-weighted sum of individual portfolios of the risky assets

$$\mathbf{x}_t = \sum_{n=1}^{N} \mathbf{x}_{t,n}\, \varphi_{t,n} . \tag{2.4}$$

The components of this vector are denoted as x^i_t. The market portfolio can be considered as the portfolio of the "representative investor" describing the overall impact on the market of the collective decision of all traders.[4] In terms of the market portfolio the pricing equation (2.3) can be written simply as $\mathbf{p}_t = w_t \mathbf{x}_t$. Introducing a similar definition for the market investment in the riskless asset,

$$x^0_t = \sum_{n=1}^{N} x^0_{t,n}\, \varphi_{t,n} ,$$

[4] In particular, when the agents' investment decisions are homogeneous, the market portfolio coincides with the portfolio of each investor.

the evolution of total wealth is given by

$$w_t = w_{t-1} x^0_{t-1} (1 + r_f) + \sum_{i=1}^{I} (d^i_t + p^i_t) \,. \tag{2.5}$$

Now we are ready to present the main result of this Section, the endogenous price dynamics in a multi-asset market with arbitrary number of agents.

Theorem 2.1. *Consider the matrix*

$$H_t = \mathbb{I}_I - \sum_{n=1}^{N} w_{t-1,n} \, \mathbf{x}_{t,n} \otimes \mathbf{z}_{t-1,n} \,, \tag{2.6}$$

where $\mathbb{I}_I$ is the identity matrix of size I, $\mathbf{z}_{t-1,n}$ stands for the vector of size I with components $x^i_{t-1,n}/p^i_{t-1}$ and $\otimes$ denotes the tensor product.[5]

If the investment shares of every agent in every asset are strictly positive, i.e., $x^i_{t,n} > 0$ and $x^i_{t-1,n} > 0$ for $i \in \{0, \ldots, I\}$, then the matrix H_t is invertible and the prevailing prices are given by

$$\mathbf{p}_t = H_t^{-1} \sum_{n=1}^{N} w_{t-1,n} \, \mathbf{x}_{t,n} \left(\mathbf{x}_{t-1,n} \cdot \mathbf{e}_t + x^0_{t-1,n} (1 + r_f) \right) \,. \tag{2.7}$$

With these prices, the contemporary market portfolio is given by

$$\mathbf{x}_t = \left(w_{t-1} x^0_{t-1} (1 + r_f) + \sum_{i=1}^{I} d^i_t + \sum_{i=1}^{I} p^i_t \right)^{-1} \mathbf{p}_t$$

and

$$x^0_t = \left(w_{t-1} x^0_{t-1} (1 + r_f) + \sum_{i=1}^{I} d^i_t + \sum_{i=1}^{I} p^i_t \right)^{-1} \left(w_{t-1} x^0_{t-1} (1 + r_f) + \sum_{i=1}^{I} d^i_t \right) \,. \tag{2.8}$$

The total wealth evolves as

$$\frac{w_t}{w_{t-1}} = \left(\mathbf{x}_{t-1} \cdot \mathbf{e}_t + x^0_{t-1} (1 + r_f) \right) \frac{1}{x^0_t} \,, \tag{2.9}$$

and the evolution of the individual wealth fractions reads

$$\frac{\varphi_{t,n}}{\varphi_{t-1,n}} = x^0_t \, \frac{\mathbf{x}_{t-1,n} \cdot \mathbf{e}_t + x^0_{t-1,n} (1 + r_f)}{\mathbf{x}_{t-1} \cdot \mathbf{e}_t + x^0_{t-1} (1 + r_f)} + \sum_{i=1}^{I} x^i_{t-1,n} \frac{x^i_t}{x^i_{t-1}} \,. \tag{2.10}$$

[5] Tensor product of two vectors $\mathbf{a}$ and $\mathbf{b}$ of size I is the $I \times I$ matrix defined as $\mathbf{a} \otimes \mathbf{b} = \mathbf{a}\mathbf{b}^T$.

Proof. See Appendix A. □

This Theorem shows that when the investment fractions are positive in every asset, the economy will be maintained in a regime of positive total wealth and strictly positive prices.[6]

Further, if the investment fractions described by vector $\mathbf{x}_{t,n}$ do not depend on contemporaneous prices and wealth, as we assume in this paper (see Section 2.1 for a discussion), this Theorem provides an explicit law of evolution of the economy. From price dynamics (2.7), we can obtain the dynamics of price returns and total returns of asset i, defined, respectively, as follows

$$r_t^i = \frac{p_t^i}{p_{t-1}^i} - 1\,, \qquad \text{and} \qquad R_t^i = r_t^i + e_t^i\,.$$

The interrelation of returns between different assets will be our main focus in Sections 3 and 4.

The evolution of wealth shares in (2.10) shows that the individual wealth share is driven by *relative* return with respect to the average return earned in the market. This feature is common for many models in evolutionary game theory. In fact, the structure of (2.10) is similar to a discrete replicator dynamics (Weibull, 1997), with wealth shares playing the role of sub-population size. The relative fitness of a trader is made up of two terms. The first term on the right hand side of (2.10) represents the relative "fundamental" earning of the trader with respect to market average, coming from the dividend payments and riskless return. This term is proportional to the investment in the riskless asset. The second term represents a "speculative" earning and is proportional to the correlation of trader's risky investment decisions with the dynamics of market portfolio. This term is higher when the trader's risky portfolio is better aligned with the average re-balancing decisions of the population of traders.

2.3 Intertemporal traders' behavior

To close the model we need only to describe how the agents choose their portfolio, i.e., their investment fractions $\mathbf{x}_{t,n}$ and $x_{t,n}^0$, and to specify a dividend process. In a speculative framework, this decision is essentially based on the traders' forecasts of future returns. We do not assume that our agents

[6] If one allows for zero prices, that is, zero aggregate net demand for a given asset, this constraint can be relaxed but at a cost of a more cumbersome notation and with little advantage in terms of model flexibility. That is why we require, in the assumptions of Theorem 2.1, that the fractions $x_{t,n}^i$ and $x_{t-1,n}^i$ are bounded away from zero. This assumption is standard in this kind of models, see for instance Evstigneev et al. (2009).

$i \in \{1, \ldots, I\}$	risky assets
$\mathbf{d}_t$	$I \times 1$ vector of dividends, $\mathbf{d}_t = (d_t^1 \quad d_t^2 \quad \ldots \quad d_t^I)^T$
$\mathbf{p}_t$	$I \times 1$ vector of prices, $\mathbf{p}_t = (p_t^1 \quad p_t^2 \quad \ldots \quad p_t^I)^T$
$\mathbf{r}_t$	$I \times 1$ vector of ex–dividend returns, $\mathbf{r}_t = \left(\frac{p_t^1}{p_{t-1}^1} \quad \frac{p_t^2}{p_{t-1}^2} \quad \ldots \quad \frac{p_t^I}{p_{t-1}^I}\right)^T$
$\mathbf{e}_t$	$I \times 1$ vector of dividend yields, $\mathbf{e}_t = (e_t^1 \quad e_t^2 \quad \ldots \quad e_t^I)^T$
$\bar{\mathbf{e}}$	$I \times 1$ vector of means of the yields, $\bar{\mathbf{e}} = (e^1 \quad e^2 \quad \ldots \quad e^I)^T$
$\mathbf{R}_t$	$I \times 1$ vector of total returns, $\mathbf{R}_t = \mathbf{r}_t + \mathbf{e}_t$
D	$I \times I$ variance-covariance matrix of the yield process $\mathbf{e}_t$
$n \in \{1, \ldots, N\}$	agents
$w_{t,n}$	wealth of agent n
$x_{t,n}^i$	fraction of wealth of agent n invested in asset i
$\mathbf{x}_{t,n}$	$I \times 1$ vector of investment fractions, $\mathbf{x}_{t,n} = (x_{t,n}^1 \quad x_{t,n}^2 \quad \ldots \quad x_{t,n}^I)^T$
$x_{t,n}^0$	fraction of wealth of agent n invested in the riskless asset
w_t	total wealth at time t
$\varphi_{t,n}$	agent n fraction of the total wealth
$\mathbf{x}_t$	market portfolio $\mathbf{x}_t = (x_t^1 \quad x_t^2 \quad \ldots \quad x_t^I)^T$
x_t^0	market investment in the riskless asset

Table 1: Notation.

are able to predict future dividend realizations, or are endowed with some *a priori* knowledge of the stochastic process driving market dynamics. Conversely, we consider procedurally rational agents which dynamically build their expectations from the observed past prices.

The behavior of a procedurally rational trader can be modeled as a mapping from past available information to the current investment fractions.[7] Because of differences in preferences, beliefs or ways to process information, different agents can have different investment functions. Formally, we have the following

Assumption 1. The investment behavior (or, investment strategy) of agent n is a vector-function $\mathbf{f}_n$, called *investment function*, that maps deterministically the information set available prior to time t into portfolio choices

$$\mathbf{x}_{t,n} = \mathbf{f}_n\left(\{\mathbf{p}_\tau, \mathbf{e}_\tau\} : \tau < t\right) . \tag{2.11}$$

The components of investment function $\mathbf{f}_n$ are denoted by f_n^i.

[7]The investment function is a useful tool as previous general results can be easily applied once its functional form is specified, see Anufriev and Bottazzi (2010) for discussion and references.

In compliance with CRRA framework, neither the present prices p_t nor wealth levels $w_{t,n}$ appear among the arguments of $\mathbf{x}_{t,n}$ in (2.11). In this way one can use the result of Theorem 2.1 to study the dynamics of returns. For the same purpose we will forbid short positions, both in the risky assets and in the riskless security, by adopting the following

Assumption 2. The investment function $\mathbf{f}_n$ are such that, for any market history $\{\mathbf{p}_\tau, \mathbf{e}_\tau\}_{\tau<t}$, it is

$$f_n^i > 0 \quad \text{and} \quad \sum_{i=1}^{I} f_n^i < 1 \ .$$

For a given stochastic dividend process, the market dynamics can now be fully described. [8] The dividend process would represent a link between the financial market modeled in the paper and the real economy, which is supposed to act in the background of it. In this paper we consider a particularly simple dividend structure, under which the yield process is exogenous and stationary.

Assumption 3. Dividend yields are randomly drawn at each time steps from a joint I-dimensional distribution with mean $\mathbf{e} = (e^1 \quad e^2 \quad \dots \quad e^I)^T$ and variance-covariance matrix D.

The lack of time correlation and the constant expected dividend-price ratio will simplify the law of motion of the system. Similar assumption was used in several previous one-asset models (e.g., Chiarella and He (2001) and Anufriev et al. (2006)), and it is in line with empirical observation of stationarity of time series of yield.

Assuming that dividend payments follow the past asset prices has two major advantages. First, considered as a binding mechanism between the fundamentals of economy and the financial market, this assumption is plausible especially when the time scale is relatively large (of the order of several months). Under the assumption of competitive final and capital markets and constant return to scale in production techniques, the availability of internal and external financial resources for the firm is proportional to its current

[8] Notice that past prices, which aggregate the previous investment decisions of all traders, generally will affect the current investment fractions. We model interaction among traders through prices, which link past traders decisions to current individual choices. There are many models based on direct interactions of traders, see for instance Lux (1995) or Follmer et al. (2005), among others. On the other hand, Michael Lewis in *Liar's Poker* reminds us of the ancient Chinese proverb: "Those who know don't tell and those who tell don't know."

capitalization. Indeed, when a firm ask a loan from a bank to expand its physical capital, it is reasonable to assume that the loan is granted in proportion to the financial capitalization of the borrower, and banks rely on assets' prices as a measure of solvency of the firm or, simply, as collateral. At the same time, the cash flow generated by the activities inside the firm are a fraction of its scale of operation, measured by the firm equity. This would imply that the expected profit of firms at time $t+1$ and their expected dividends are proportional to the capitalization of the firms at time t. In this simple scenario, the expected dividend yields are constant in accordance with Assumption 3. Second, by imposing a very strong link between real and financial economic dynamics, we make our argument about their possible divergence at equilibrium even stronger.

2.4 The law of motion of the economy

Under the above assumptions and given exogenous process for dividend yield $\{\mathbf{e}_t\}$, the evolution of the economy can be formally described as the following dynamical system

$$\begin{cases} \mathbf{x}_{t,n} = \mathbf{f}_n\left(\{\mathbf{p}_\tau, \mathbf{e}_\tau\}_{\tau<t}\right)\,, \qquad n \in \{1,\dots,N\} \\ \mathbf{p}_t = H_t^{-1} \displaystyle\sum_{n=1}^{N} w_{t-1,n}\mathbf{x}_{t,n}\left(\mathbf{x}_{t-1,n}\cdot\mathbf{e}_t + x^0_{t-1,n}(1+r_f)\right) \\ x^0_t = \left(w_{t-1}x^0_{t-1}(1+r_f) + \mathbf{p}_{t-1}\cdot\mathbf{e}_t + \displaystyle\sum_{i=1}^{I} p^i_t\right)^{-1}\left(w_{t-1}x^0_{t-1}(1+r_f) + \mathbf{p}_{t-1}\cdot\mathbf{e}_t\right) \\ w_t = w_{t-1}\,\dfrac{1}{x^0_t}\,\left(\mathbf{x}_{t-1}\cdot\mathbf{e}_t + x^0_{t-1}(1+r_f)\right) \\ \varphi_{t,n} = \varphi_{t-1,n}\left(x^0_t\,\dfrac{\mathbf{x}_{t-1,n}\cdot\mathbf{e}_t + x^0_{t-1,n}(1+r_f)}{\mathbf{x}_{t-1}\cdot\mathbf{e}_t + x^0_{t-1}(1+r_f)} + \displaystyle\sum_i x^i_{t-1,n}\frac{x^i_t}{x^i_{t-1}}\right)\,, \qquad n \in \{1,\dots,N-1\} \end{cases} \tag{2.12}$$

The order of equations corresponds to the timing of our model. The first $N \times I$ equations, one for each fraction $x^i_{t,n}$ of wealth invested by each agent n in each asset i, fully represent the investment decisions of N agents. As we model an economy without consumption, the residual fraction of wealth, $x^0_{t,n}$, is invested into the riskless security. Then the trading session takes place, in which agent n demands $x^i_{t,n}w_{t,n}/p^i_t$ of shares of asset i. Of course, $w_{t,n}$ and p^i_t enter the demand functions as parameters, whose actual values should be determined during the trading session, as in a standard Walrasian equilibrium framework. Theorem 2.1 guarantees that unique positive prices exists. They are defined in the next I equations for $\mathbf{p}_t$ with matrix H_t defined in (2.6) depending on the contemporaneous investment decisions of agents,

$\mathbf{x}_{t,n}$. When the prices are found, the next equation (which is (2.8) rewritten in terms of dividend yields) provides the wealth-weighted investment x_t^0 into the riskless asset. Finally, the last N equations express the present value of total wealth and individual wealth shares as given in (2.9) and (2.10).

The market position of each agent at time t is given by demand (2.1) evaluated at equilibrium prices. The trading volume of each asset depends on how these demanded quantities vary in time. The number of shares of asset i traded at time t is

$$V_{i,t} = \frac{1}{2} \sum_{n=1}^{N} \left| \frac{w_{t+1,n} x_{t+1,n}^i}{p_{t+1}^i} - \frac{w_{t,n} x_{t,n}^i}{p_t^i} \right| , \tag{2.13}$$

where the factor 1/2 reflects the double counting of shares in each transaction. Because of the randomness of the dividend yield, in general the traders would have to adjust their portfolios along the trajectories of (2.12), generating positive volumes.

System (2.12) is a random dynamical system whose properties will be investigated in Section 3. In order to facilitate a discussion and illustrate different concepts, it will be useful to have a simple example, which we start to develop now and to which we will return several times later.

2.5 Example with two risky assets

For an illustrative purposes we consider a simple economy with one agent and two risky assets.[9] In the single agent case the dynamical system (2.12) simplifies. In fact, it is even simpler to derive the return dynamics directly from (2.3) than to use Theorem 2.1. When $N = 1$ we have $p_t^i = w_t x_t^i$, where components of market portfolio, x_t^i, are simply the agent's wealth share invested into asset i and do not depend on wealth. Recall that the price returns are defined as $r_t^i = p_t^i / p_{t-1}^i - 1$. Then from (2.9) one obtains

$$r_t^i = \frac{x_t^i}{x_{t-1}^i} \frac{w_t}{w_{t-1}} - 1 = \frac{x_t^i}{x_{t-1}^i} \frac{x_{t-1}^1 e_t^1 + x_{t-1}^2 e_t^2 + x_{t-1}^0 (1 + r_f)}{x_t^0} - 1 , \qquad i = 1, 2 . \tag{2.14}$$

Various agent's behaviors can now be modeled by specifying an appropriate investment function.

Consider an investor who uses the following rule of thumb. A fixed share of wealth, $0 < x^0 < 1$, is invested in the riskless security, and the remaining

[9] In order to exclude a possibility of strategic behavior, one can think of a large but homogeneous population. Because wealth evolution does not affect prices, it is sufficient to assume homogeneity only with respect to the investment behavior, i.e., identical investment functions.

wealth is divided between the two risky assets depending on their past return differences. Specifically, let $y_t = \mathrm{E}_{t-1}[r^1_t - r^2_t]$ denote the expectation about the return differential between the two assets at time t formed at the end of period $t-1$, and let assume that these expectations are formed in an adaptive[10] way

$$y_t = \mu(r^1_{t-1} - r^2_{t-1}) + (1-\mu)y_{t-1}\,, \tag{2.15}$$

with parameter $\mu \in [0,1]$ describing the relative weight of past observation, and impose the following investment behavior:

$$\begin{aligned} x^1_t &= a(1-x_0) + y_t \\ x^2_t &= (1-a)(1-x_0) - y_t\,, \end{aligned} \tag{2.16}$$

where $a \in [0,1]$ is an exogenous preference parameter showing a predisposed division of wealth between the two risky assets in absence of the expected return differential.[11] Notice that (2.16) is nothing else than the investment function.[12] The stochastic system governing the dynamics becomes

$$\begin{cases} r^1_t = \dfrac{1}{x^0}\,\dfrac{a(1-x_0)+y_t}{a(1-x_0)+y_{t-1}}\,\big((1-x^0)(ae^1_t + (1-a)e^2_t) + y_{t-1}(e^1_t - e^2_t) + x^0(1+r_f)\big) - 1 \\ r^2_t = \dfrac{1}{x^0}\,\dfrac{(1-a)(1-x_0)-y_t}{(1-a)(1-x_0)-y_{t-1}}\,\big((1-x^0)(ae^1_t + (1-a)e^2_t) + y_{t-1}(e^1_t - e^2_t) + x^0(1+r_f)\big) - 1 \\ y_t = \mu(r^1_{t-1} - r^2_{t-1}) + (1-\mu)y_{t-1}\,, \end{cases} \tag{2.17}$$

where the first two equations are obtained by plugging the investment function (2.16) inside Eq. (2.14), whereas the last equation is simply (2.15). This dynamics will be simulated and further studies in the next section.

3 Economic equilibria

The dynamics of multi-asset model is formally given by stochastic dynamical system (2.12) driven by the exogenous process on yields, $\mathbf{e}_t$. It is well-known

[10]Adaptive expectations are known in the economic literature at least from Irving Fisher (see Fisher, 1930). Under adaptive expectations all past observations are averaged but the weights are geometrically declining into the past.

[11]For instance if $x_0 = 0.2$ and $a = 0.6$, this investor always keeps 20% of wealth on the riskless account, then divide the remaining wealth in the proportion $6:4$ between the two risky assets, increasing (resp. decreasing) this proportion with expectation of higher (resp. lower) return of the first risky asset with respect to the second risky asset.

[12]In order to comply with Assumption 2, we should impose upper and lower bounds on given investment fractions, or, equivalently consider "cut-off" versions of y_t. For small $\varepsilon > 0$ the following should hold $\varepsilon < x^1_t < 1 - x_0 - \varepsilon$, from where the bounds on y_t can be found. We simplify our notation in the text and do not provide bounds explicitly, but we do use these bounds in the simulations.

that analytical study of such systems is a challenging task. In particular, several proposed bifurcations theories (Zeeman (1988), Arnold (1998)) have many limitations, see a discussion in Diks and Wagener (2008). For this reason, a popular approach in the literature of HAMs consists in analysis of the *deterministic skeleton* of the system. To obtain the skeleton in our model, one should replace the yield realizations by their expected values making the dynamics deterministic. The fixed points of the system are characterized then by constant returns of different assets and constant investment shares of different agents. Even if this approach often delivers important insights about the determinants of stability in the system, it has an important drawback. Since the returns of the risky assets are constant in any fixed point of the deterministic skeleton, the skeleton cannot capture possible time- and cross-correlations in the returns.

This drawback is the main motivation to explore an alternative approach. We focus on specific stochastic processes which are relatively easy to study and which often approximate the dynamics of system (2.12) quite well. We call these processes Procedurally Consistent Equilibria (PCE).

The idea leading to the PCE is simple. The high dimensional, non-linear random dynamical system (2.12) can in general display complex dynamics, along, e.g., some (noisy) chaotic or quasi-periodic attractor. Given a set of N investment functions reflecting both the preferences and the forecasting rules of agents, the circular causation behind these dynamics is clear. Agents make portfolio decisions according to the investment functions. Given these decisions, prevailing prices and returns are determined. Because of the updating of returns, the investment functions will in general dictate to revise the portfolios. This revision leads in turn to new market prices and returns. If the dynamics ultimately settle on a cyclical, i.e. periodic, quasi-periodic or chaotic attractor, the need of such revisions will never disappear. Since one can safely assume that no traders are happy to be persistently wrong in their predictions, the appearance of cycles in a trading model of such kind should lead, in the long run, to the adoption, by the part of agents, of new investment functions. This could be the result of a revision of individual preferences or beliefs, which are somehow adapted to the perceived dynamics of the market. In general, thus, cyclical attractors of (2.12) can represent an equilibrium from the purely mathematical point of view, but do not correspond to economic equilibria in the common sense. Conversely, we consider the states of the system associated with economic equilibria are those in which agents do not have incentives to change their preferences or/and expectation rules, i.e., their investment functions. For this conditions to be realized we require that: first, the realized dynamics should be consistent with agents' investment behavior and, second, given the dynamics, traders should be satisfied with their market positions and the implied realized wealth levels. These requirements lead to the notion of Procedural Consistent Equilibria (PCE)

Definition 3.1. *Procedurally Consistent Equilibria* (PCE) are the trajectories of the system defined in (2.12) with fixed investment shares $x^i_{t,n} = \overset{*}{x}{}^i_n$ and stationary wealth distribution $\varphi_{t,n} = \overset{*}{\varphi}_n$ for all n, i and t.

Equilibria of this kind have been analyzed in Anufriev and Bottazzi (2010) and Anufriev and Dindo (2010), but only within the single asset framework.[13] In PCE the portfolio choices of every agent are constant. When the investment fractions are assumed *a priori* to be constant over time,[14] the PCE coincides with the dynamics of the original system. When the investment fractions depend on past behavior of prices and dividends, PCE provide a stochastic approximation to (2.12). This approximation is expected to be good when this dynamics is stationary, i.e., when the expected behavior of each variable is fixed.[15]

From (2.4) it follows that in the PCE the market portfolio is constant over time. Let us denote the PCE market portfolio as $\overset{*}{\mathbf{x}}$. This notation should not be confused with $\overset{*}{\mathbf{x}}_n$, which is, of course, the vector of investment fractions in the risky assets by agent n in PCE. It turns out, however, that, in some sense these are the same quantities, as we will show below.

We define agent n with strictly positive wealth share at equilibrium, $\overset{*}{\varphi}_n > 0$, as *surviving* in the PCE. In this terminology, the market portfolio does depend only on the investment fractions of surviving agents. The existence of multiple survivors is not ruled out in the PCE. However, all survivors must have the same investment portfolio. Indeed, from the last equation of (2.12) setting $\varphi_{t,n} = \varphi_{t-1,n}$ and assuming constant investment shares for each agent $\mathbf{x}_{t-1,n} = \mathbf{x}_n$, with a little algebra one obtains that

$$\frac{\mathbf{x}_n \cdot \mathbf{e}_t}{x^0_n} = \frac{\overset{*}{\mathbf{x}} \cdot \mathbf{e}_t}{\overset{*}{x}{}^0}$$

for each surviving agent n. If the PCE investment shares of some survivors would be different, one could find a realization of random shocks, $\mathbf{e}_t$, such

[13]Bottazzi and Dindo (2010) extend the analysis of PCE to a multi-asset general equilibrium framework quite different from the present model.

[14]This case of constant investment functions is often analyzed in the evolutionary finance literature, see, e.g., Blume and Easley (1992).

[15]Considering an auto-correlation structure in the yields process would not change the definition nor the nature of the PCE as long as the generating process remains stationary.

that the quantities in the left-hand side of this equality are different. But this contradicts to the fact that the right-hand side does not depend on agent. Since only surviving agents affect the market portfolio and all these agents must have the same investment shares in the PCE, we conclude that $\overset{*}{\mathbf{x}}_n \equiv \overset{*}{\mathbf{x}}$ for every survivor. This equivalence does not mean that all survivors have the same investment function, indeed the investment behavior of agents out of equilibrium is not restricted,[16] nor that their demand functions are the same, as these investors may have different wealth levels.

Finally, notice that whereas under the actual dynamics of (2.12) the volume as given by (2.13) is in general strictly positive, it is always zero along the PCE trajectories. Indeed, setting constant the values for investment and wealth shares of different agents in (2.13), we obtain that at every period and for every survivor, the demanded amount of assets is the same as the initial endowment kept after the previous period.

Example with two risky assets (continued). Before proceeding further it will be useful to illustrate the notion of PCE within the example introduced in Section 2.5. Recall that in case of a single investor described by investment function (2.16), the dynamics of the returns of two risky assets is given by (2.17).

First of all, notice that the deterministic skeleton approach leads us to study the fixed point of (2.17). It is easy to see that the returns in this fixed point[17] are given by

$$r^1 = r^2 = \frac{1 - x^0}{x^0} \left(a\bar{e}^1 + (1-a)\bar{e}^2 \right) + r_f \, . \tag{3.1}$$

However, as argued above, an absence of fluctuations in the fixed point prevents us from studying an impact of investment functions on correlation structure.

Consider now the PCE. Requirement of constant investment shares implies that $y_t = y_{t-1}$, see (2.16). Plugging this relationship to the dynamics equations (2.17) we find that $r_t^1 = r_t^2$ for every t, and conclude that $y_t \equiv 0$.

[16]Thus, the investment functions should not be functionally identical on their entire domain of definition. If the investment functions are different, then their identity at equilibrium implies a non-generic restriction on traders' behavior. The situation is similar to the requirement that several curves intersect in one specific point. Clearly any infinitesimal perturbation of one curve breaks the requirement.

[17]A local stability of this unique fixed point will depend on the behavioral parameter μ. A typical and intuitive result is that the smaller the value μ is, the larger the stability region of parameter space is, see Anufriev et al. (2006) for instance.

It implies that in the PCE the returns follow the process

$$r_t^1 = r_t^2 = r_f + \frac{1 - x^0}{x^0} \left(a e_t^1 + (1-a) e_t^2 \right) . \tag{3.2}$$

An important difference with (3.1) is that now the returns are random processes, and, in particular, the correlations between the returns of different assets can be computed.

Now let us contrast the PCE dynamics (3.2) with the actual dynamics given by (2.17). Obviously, the PCE is a stochastic approximation of the actual dynamics. The quality of this approximation depends on whether assumptions underlying the PCE are satisfied, i.e., whether the PCE return dynamics is consistent with the assumption of constant investment share. In our example the investment shares are given by (2.16), where y_t is time-varying. Hence, the PCE would be a good approximation only if variation of y_t would be negligible along the PCE dynamics. Intuitively, it might be the case when μ is sufficiently small, because then the effect of new returns on the expectations disappears with time. To test this intuition we turn to the simulations.

We compare the actual dynamics generated by model (2.17) and the PCE dynamics of (3.2) in Fig. 1 for two values of μ, relatively low in the left panel and relatively large in the right panel. The PCE, which is shown by line for better visibility, depends only on the two exogenous yield processes e_t^1 and e_t^2 and does not depend on μ. The realizations of the yield processes are the same in these two pictures. These are drawn independently (also from each other) from two normal distributions with means $e^1 = 0.01$ and $e^2 = 0.02$ and standard deviation $\sigma_e = 0.01$. The returns of the two assets are shown by dots and squares. We can see that for both values of μ the dynamics of PCE approximates the actual return dynamics quite nicely already after 50 periods. However, it takes much longer to reach the same goodness of approximation when μ is relatively large. In fact, for larger values of μ the dynamics of returns become non-stationary and then the PCE does not approximate it at all. Finally, notice that the deterministic skeleton approach would predict both returns to be constant in time and given by the right-hand side of (3.1), which is 0.11 for the considered values of the parameters.

The rest of this section is devoted to the analysis of the equilibria defined in Definition 3.1. We will characterize their locus in the high-dimensional space of system variables and investigate some interesting emerging properties.

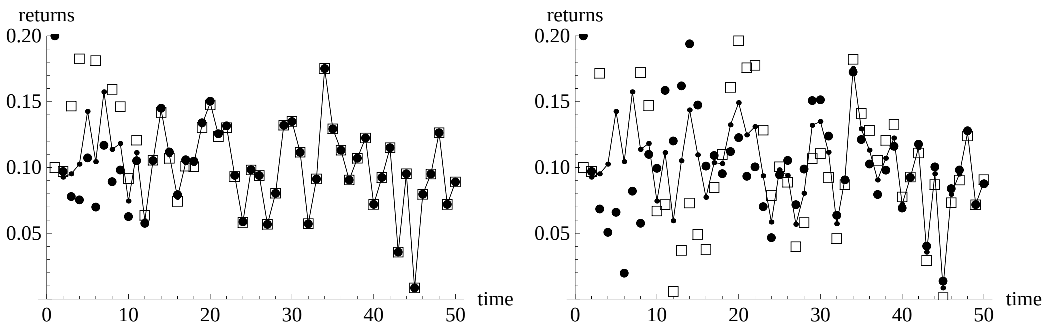

Figure 1: Return dynamics and its approximation by the PCE dynamics in the example of a single agent - two assets example. When $\mu = 0.1$ (left panel) the PCE approximates the return dynamics reasonably well already after 20 periods. When $\mu = 0.15$ (right panel) the PCE also approximates the dynamics but after longer transitory period. Simulations performed with $r_f = 0.05$, $x_0 = 0.2$, $a = 0.6$, $e^1 = 0.01$, $e^2 = 0.02$ and $\sigma_e = 0.01$.

3.1 Equilibria Market Surface

We are now back to the general case with N agents and I assets. The notion of PCE from Definition 3.1 requires us to look at the trajectories of (2.12) with constant investment and wealth shares. We start by deriving the implications of this assumption concerning the structure of price returns. Using (2.9) the asset price return $r_t^i = p_t^i / p_{t-1}^i - 1$ reads

$$r_t^i = \frac{x_t^i}{x_{t-1}^i \, x_t^0} \left(\mathbf{x}_{t-1} \cdot \mathbf{e}_t + x_{t-1}^0 (1 + r_f) \right) - 1 \,. \tag{3.3}$$

Substituting for fixed investment shares one obtains the expression of price return $\overset{*}{r}{}^i$ in PCE

$$\overset{*}{r}{}^i = r_f + \frac{\sum_i \overset{*}{x}{}^i e_t^i}{\overset{*}{x}{}^0} = r_f + \frac{\overset{*}{\mathbf{x}} \cdot \mathbf{e}_t}{\overset{*}{x}{}^0} \ , \tag{3.4}$$

where $\overset{*}{\mathbf{x}}$ is the market portfolio at the PCE.

First notice that at equilibrium all prices grow with the same growth rate. This fact is not derived from some peculiar structure in agents' preferences, but is simply a direct consequence of Definition 3.1. The growth rate of assets, however, is not constant but depends on the actual realization of the dividend yield process. Because of Assumption 3 the market growth rates follow a stationary process and are uncorrelated in time. Applying expectation operator to both sides of (3.4) one obtains

$$\overset{*}{r}_E = r_f + \frac{\overset{*}{\mathbf{x}} \cdot \bar{\mathbf{e}}}{\overset{*}{x}{}^0} \,, \tag{3.5}$$

where the subscript E stands for "expected". This last equation is an algebraic relation between the expected market return at equilibrium, $\overset{*}{r}_E$, and the equilibrium market portfolio, $\overset{*}{\mathbf{x}}$. Notice however that the return is not really a function of all the I coordinates of the market portfolio. Indeed, it only depends on the projection of the market portfolio along the vector of expected yield $\overset{*}{\mathbf{x}} \cdot \bar{\mathbf{e}}$ and on the total investment in riskless security $\overset{*}{x}{}^0 = 1 - \sum_i \overset{*}{x}{}^i$ at equilibrium. Let $\bar{e} = ||\bar{\mathbf{e}}||$ be the norm of the vector of expected dividends and $\overset{*}{x} = \sum_i \overset{*}{x}{}^i$ be the total fraction of wealth invested in the risky market. Let us define

$$C_x = \frac{\overset{*}{\mathbf{x}} \cdot \bar{\mathbf{e}}}{\bar{e}\,\overset{*}{x}} \,,$$

as a coefficient proportional to the inner product of the equilibrium market portfolio and the vector of expected dividends. Then (3.5) can be rewritten as follows

$$\overset{*}{r}_E = r_f + \bar{e}\, C_x \, \frac{\overset{*}{x}}{1 - \overset{*}{x}} \,. \tag{3.6}$$

For exogenously given parameters r_f and $\bar{e}$, the right-hand side of this equation defines a surface in coordinates C_x and $\overset{*}{x}$. This surface is shown in the left panel of Fig. 2. Since two parameters are sufficient to identify the expected return in every possible PCE, we will call the locus of these points the "equilibrium market surface".

Equation (3.6) shows that the effect of aggregate investment decision on the resulting market returns depends on two factors: how much is invested in the risky market (as given by $\overset{*}{x}$) and to what extent the invested portfolio is aligned with the endogenously given expected yields (as given by coefficient C_x). Under Assumptions 1-3 the factor C_x is positive. If $\bar{e}^H$ and $\bar{e}^L$ denote the highest and the lowest expected yields, respectively, it is immediate to see that $C_x \in (\bar{e}^L/\bar{e}, \bar{e}^H/\bar{e})$. Notice that C_x is defined in such a way as to be insensitive to a rescaling of the overall risky investment. The latter, on the other hand, can have a dramatic effect of the return of the assets. Indeed when $\overset{*}{x} \to 1$ the price return diverges and the market displays extremely high return rates. The behavior of the equilibrium return in the PCE, $\overset{*}{r}_E$, as a function of $\overset{*}{x}$ is shown in the right panel of Fig. 2.

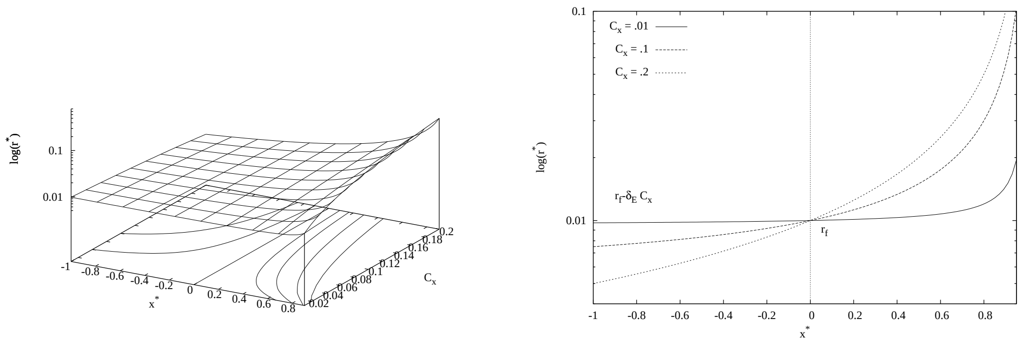

Figure 2: **Left panel:** The equilibrium market surface as a function of x and C_x. **Right panel:** The equilibrium market return $\overset{*}{r}$ as a function of the aggregate total investment in risky markets $\overset{*}{x}$ for different values of the correlation–like coefficient between expected dividends and shares of risky portfolio C_x.

Example with two risky assets (continued). We return to the example with two risky assets introduced in Section 2.5. Above we have derived the PCE for that example, which is a stochastic process for equilibrium returns of the two assets given by (3.2). We observe that this process is a special case of the general formula (3.4), for the specific investment behavior considered in the example.

The expected return in the PCE, $\overset{*}{r}_E$, coincides with the return in the fixed point of the deterministic skeleton given by (3.1). The equilibrium market surface gives this return as a function of two quantities, which for the investment behavior in (2.16) are given by

$$\overset{*}{x} = 1 - x_0 \qquad \text{and} \qquad C_x = \frac{a\bar{e}^1 + (1-a)\bar{e}^2}{||\bar{\mathbf{e}}||}\,.$$

The equilibrium market surface allows us to see how the expected return in the PCE changes with parameters of the investment behavior and the dividend yield process. For instance, the mean of the yield process and behavioral parameter a are enough to fix one of the curves shown in the right panel of Fig. 2. For this curve we can now immediately see the effect of the wealth fraction invested to the riskless security for the expected return in the PCE. In particular, when this fraction $x_0 \to 0$, i.e., when $\overset{*}{x} \to 1$, the equilibrium return diverges.

3.2 Properties of PCE

Using Definition 3.1, in the previous section we identified a locus of feasible equilibria and derived the implied relations between traders' collective investment choices and price dynamics. This section present some interesting properties of PCE's. These are, so to speak, generic properties, which do not require any further assumption about the actual form of the agents' investment functions $\mathbf{f}$. The aggregate effect of specific behavioral assumptions will be explored in the next section.

We have specified the total return of asset i, defined as the sum of price return and dividend yield, in equation (2.2). Using the algebraic relation in (3.4) and the parametrization in (3.6) it is immediate to compute its expected value at equilibrium

$$\mathrm{E}[\overset{*}{R}{}_t^i] = r_f + \bar{e}\, C_x \, \frac{\overset{*}{x}}{1-\overset{*}{x}} + \bar{e}^i \; . \tag{3.7}$$

As long as aggregate short positions are avoided ($0 < \overset{*}{x} < 1$), the total return of any risky asset is always greater than the riskless interest rate r_f. Moreover, the "equity premium" is proportional to the amount of wealth invested in the risky market. Consider now the variance-covariance matrix

$$C = \mathrm{E}\left[(\mathbf{R}_t - \mathrm{E}[\mathbf{R}_t]) \otimes (\mathbf{R}_t - \mathrm{E}[\mathbf{R}_t]) \right] \tag{3.8}$$

where $\mathbf{R}_t$ is the vector with components R_t^i and $\otimes$ the tensor product. Substituting the expression above, at equilibrium one has

$$\overset{*}{C} = \mathrm{E}\left[\left(\frac{\overset{*}{\mathbf{x}} \cdot (\mathbf{e}_t - \bar{\mathbf{e}})}{\overset{*}{x}{}^0} + \mathbf{e} - \bar{\mathbf{e}} \right) \otimes \left(\frac{\overset{*}{\mathbf{x}} \cdot (\mathbf{e}_t - \bar{\mathbf{e}})}{\overset{*}{x}{}^0} + \mathbf{e} - \bar{\mathbf{e}} \right) \right] \; .$$

Recall from Assumption 3 that D denotes the variance-covariance matrix of the dividend yields and let a generic element of this matrix be D_{ij}. Using Assumption 3, we have that a generic element $\overset{*}{C}_{ij}$ reduces to

$$\overset{*}{C}_{ij} = \frac{1}{\left(\overset{*}{x}{}^0\right)^2} \left(\sum_{k,l=1}^{I} \overset{*}{x}{}^k \overset{*}{x}{}^l D_{kl} \right) + \frac{1}{\overset{*}{x}{}^0} \left(\sum_{k=1}^{I} \overset{*}{x}{}^k \left(D_{ki} + D_{kj}\right) \right) + D_{ij} \; ,$$

and in matrix notation

$$\overset{*}{C} = \frac{1}{(1-\overset{*}{x})^2}\, \overset{*}{\mathbf{x}}{}' D \overset{*}{\mathbf{x}}\, \mathbf{1} \otimes \mathbf{1} + \frac{1}{1-\overset{*}{x}} \left(D\overset{*}{\mathbf{x}} \otimes \mathbf{1} + \mathbf{1} \otimes D\overset{*}{\mathbf{x}} \right) + D \; , \tag{3.9}$$

where $\mathbf{1}$ is a vector of ones, so that $(D\overset{*}{\mathbf{x}} \otimes \mathbf{1})_{i,j} = \sum_h D_{i,h} \overset{*}{x}^h$. In (3.9) the original variance–covariance matrix of yields, D, is augmented by two new elements. These elements are the result of the traders' activity when re–balancing their portfolios, and reflect their preferences and expectations through the representative portfolio $\overset{*}{\mathbf{x}}$. The more the market is exposed toward risky security, that is the nearer $\overset{*}{x}$ is to 1, the larger the generated extra terms.

Example with two risky assets (continued). Let us again illustrate the previous result by means of the special example of investment behavior introduced in Section 2.5. Let us assume that the dividend yield are generated from the process with the var–covar matrix given by

$$D = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} .$$

For this general yield structure we can use (3.9) and write the excess covariance as follows

$$\begin{aligned} \overset{*}{C} - D = & \left(\frac{1-x_0}{x_0}\right)^2 \begin{pmatrix} a & 1-a \end{pmatrix} D \begin{pmatrix} a \\ 1-a \end{pmatrix} \mathbf{1} \otimes \mathbf{1} + \\ & + \frac{1-x_0}{x_0} \left(D \begin{pmatrix} a \\ 1-a \end{pmatrix} \otimes \mathbf{1} + \mathbf{1} \otimes D \begin{pmatrix} a \\ 1-a \end{pmatrix} \right) = \\ = & \left(\frac{1-x_0}{x_0}\right)^2 \left(a^2\sigma_1^2 + 2a(1-a)\rho\sigma_1\sigma_2 + (1-a)^2\sigma_2^2\right) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \\ & \frac{1-x_0}{x_0} \begin{pmatrix} 2a\sigma_1^2 + 2(1-a)\rho\sigma_1\sigma_2 & a\sigma_1^2 + \rho\sigma_1\sigma_2 + (1-a)\sigma_2^2 \\ a\sigma_1^2 + \rho\sigma_1\sigma_2 + (1-a)\sigma_2^2 & 2(1-a)\sigma_2^2 + 2a\rho\sigma_1\sigma_2 \end{pmatrix} . \end{aligned}$$

In this example the excess variance-covariance becomes especially high when the investment to the riskfree security, x_0, is low.

We are now back to the general case. In order to understand better the effect of the representative portfolio at equilibrium, consider the case in which dividend yields are independent and equal, that is $D_{i,j} = \sigma_i^2 \delta_{i,j}$. Notice that this situation can be always obtained through an orthogonal transformation of the dividend distribution.[18] The diagonal element in (3.9) becomes

$$C_{i,i} = \sigma_i^2 \left(\frac{|\overset{*}{\mathbf{x}}|^2}{(1-\overset{*}{x})^2} + 2\,\frac{\overset{*}{x}^i}{1-\overset{*}{x}} + 1 \right) \tag{3.10}$$

[18]This transformation amounts to a change of basis and defining a new set of assets. Since the var-covar matrix is symmetric and positive definite, the budget constraint imposed on the original portfolios is preserved when expressed in terms of the new assets.

where the presence of excess variance is apparent. The fluctuation of return at equilibrium is amplified and is larger (possibly much larger) than the fluctuations associated to fundamentals. Consider now the off-diagonal term with $i \neq j$

$$C_{i,j} = \sigma_i \sigma_j \left(\frac{|\overset{*}{\mathbf{x}}|^2}{(1-\overset{*}{x})^2} + \frac{\overset{*}{x}{}^i + \overset{*}{x}{}^j}{1-\overset{*}{x}} \right) \ . \tag{3.11}$$

This expression clearly shows another properties of PCE's: even if the original fundamental process where independent and uncorrelated, the price returns at equilibrium display a strong auto–correlation, that becomes stronger the larger the share of wealth invested in risky assets. The cross correlation, introduced by the fluctuation of prices at equilibrium, is strong enough to generate a "market mode" in the variance-covariance matrix. To see it, consider the special case in which the dividends have all the same variance. In this case the following applies

Theorem 3.1. *If $D = \sigma^2\,\mathbb{I}$ then the matrix C has eigenvalues $\sigma^2, \lambda_+, \lambda_-$, where*

$$\lambda_\pm = \frac{\sigma^2}{1-x} + \frac{I\,\sigma^2\,|\overset{*}{\mathbf{x}}|^2}{2\,(1-x)^2} \left(1 \pm \sqrt{1 + 4\frac{1-x}{I\,|\overset{*}{\mathbf{x}}|^2}} \right) \tag{3.12}$$

Proof. See Section B. □

The eigenvalue λ_+ is associated with the principal component of the matrix and is proportional to I, the number of risky assets traded in the market. Also in the case of a generic matrix of D, the structure $\mathbf{1} \otimes \mathbf{1}$ in the first term of C in (3.9) is responsible for a large eigenvalue, proportional to I.

According to (3.7) and (3.9), a greater exposition to risky markets implies increased capital gains but also increased volatility. To see how these things combine, it is instructive to look at the average behavior of the aggregate portfolio. Consider the excess return of market portfolio

$$\rho_t = \sum_{i,t} x^i \left(R^i_t - r_f \right) \ . \tag{3.13}$$

Its expected return at equilibrium becomes

$$E[\overset{*}{\rho}] = \bar{e}\, C_x \, \frac{\overset{*}{x}}{1-\overset{*}{x}}$$

and its variance can be immediately obtained from (3.9)

$$\sigma_\rho = \overset{*}{\mathbf{x}}{}' C \overset{*}{\mathbf{x}}{}' = \frac{1}{(1-\overset{*}{x})^2}\, \overset{*}{\mathbf{x}}{}' D \overset{*}{\mathbf{x}} \ ,$$

so that, combining the two, the Sharpe ratio of the aggregate investment finally reads

$$S = \frac{\mathrm{E}[\overset{*}{\rho}]}{\sigma_\rho} = \frac{\bar{e} C_x \overset{*}{x}}{\sqrt{\overset{*}{\mathbf{x}}' D \overset{*}{\mathbf{x}}}} = \frac{\sum_i \overset{*}{x}^i}{\sqrt{\sum_{i,j} \overset{*}{x}^i \overset{*}{x}^j D_{i,j}}} \,. \tag{3.14}$$

The expression of S is equivalent to the Sharpe ratio computed using exclusively the component of portfolio return associated with dividend yields. In other terms, changing the fraction of total wealth invested in the risky assets does not modify the Sharpe ratio of aggregate portfolio at equilibrium. As long as the way in which the wealth is distributed across the different assets is constant, the increase in market expected return and variance neutralize each other, leaving the value of S unvaried.

3.3 Evolutionary stability

Assume that the market lays in a PCE characterized by a constant aggregate investment function $\overset{*}{\mathbf{x}}$ and price returns given by (3.4). As discussed above, the aggregate investment function can describe a single strategy, possibly adopted by a population of traders, or the convex combination of different surviving strategies. In the latter case, however, these strategies present, at equilibrium, analogous reactions to price fluctuations. Suppose that a new type of investor, with a new strategy, enters the market with initially negligible wealth or that, equivalently, a small fraction of traders decide to deviate from the original equilibrium strategy and invest according to a modified rule. We want to asses the stability of the equilibrium strategy $\overset{*}{\mathbf{x}}$ under this external "invasion". Let $\mathbf{z}$ represents the investment function of the new agents (or of the deviating sub-population). Since we are interested to study the local dynamics around the PCE we can assume the new investment function to be constant too. If the new strategy enters the market with an infinitesimally small wealth share, does it progressively disappear or does it instead gain a finite amount of wealth and, consequently, modify the dynamics of the market? The answer is provided by the following

Theorem 3.2. *The PCE associated with the investment function $\overset{*}{\mathbf{x}}$ cannot be invaded by a strategy z provided that*

$$\frac{\overset{*}{\mathbf{x}} \cdot \bar{\mathbf{e}}}{\overset{*}{x}^0} > \frac{\mathbf{z} \cdot \bar{\mathbf{e}}}{z^0} \,. \tag{3.15}$$

Proof. See Section C. □

If the condition (3.15) is violated, the new strategy gains, on average, a finite amount of wealth and, consequently, the long term dynamics and equilibria of the system are perturbed away from the PCE. The previous theorem establishes a dominance relation between strategies. Consider two strategies $\mathbf{x}_1$ and $\mathbf{x}_2$. If

$$\frac{\mathbf{x}_1 \cdot \bar{\mathbf{e}}}{x_1^0} > \frac{\mathbf{x}_2 \cdot \bar{\mathbf{e}}}{x_2^0} ,$$

then the first strategy is resilient to the second, while the second can be effectively invaded by the first. If one considers an open economy in which new strategies can develop and start to trade in the market, the previous condition generates an evolutionary pressure toward strategies x that maximize the ratio $\mathbf{x} \cdot \bar{\mathbf{e}}/x^0$. Using the notation of Section 3.1, if $x = \sum_i x^i$, the fitness measure of the constant strategy $\mathbf{x}$ is proportional to $C_x x/(1-x)$. In Fig 3 fitness isolines are reported in the (C_x, x) plane. The introduction of new strategies, irrespectively of the fact that they are developed toward a given goal or tried at random, naturally drives the system toward higher fitness levels. Notice, however, that the fitness of a constant strategy is proportional to the expected price return $\overset{*}{r}_E$ in the associated PCE equilibrium, that is the equilibrium in which the said strategy is the only survivor. One one hand, this can be understood as an efficient behavior, because the market seems to reward the strategy that produces higher returns, and, consequently, the higher nominal increase in the global wealth. On the other hand, however, the evolutionary pressure pushes the population of agents toward riskier behavior and the system toward the point $x \to 1$ where a larger share of wealth is invested in the risky portfolio. In this limit the system becomes unstable: it moves toward PCE in which the prices' fluctuations generated by the realized dividends are progressively increased to infinity, as the multiplicative factor $1/(1-\overset{*}{x})$, in equations (3.10) and (3.11) for the variance–covariance matrix $\overset{*}{C}$, grows unbounded.

4 The case of mean-variance investor

The analysis performed in the previous section is related to the notion of Procedurally Consistent Equilibria, and hence to the statics of a very general class of models. There we studied equilibria where the strategies adopted by agents result in the investment of constant fraction of individual wealth in the different securities. This notion of equilibrium is valid for a wide class of investment rules. However, to study the dynamics of the system we need more specific assumptions about the strategies of agents. In this section

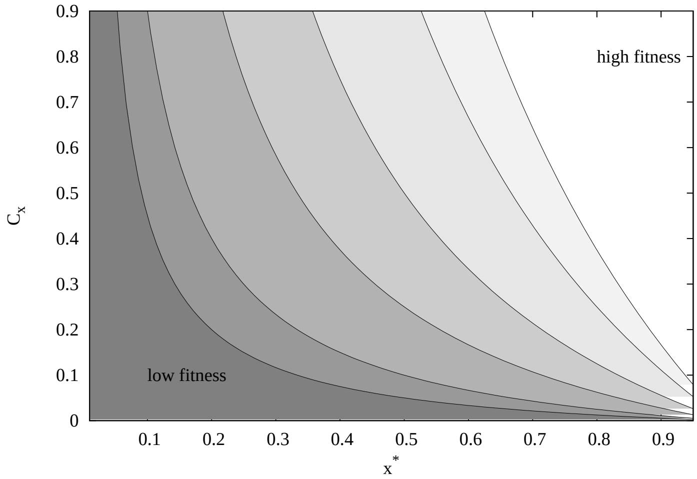

Figure 3: Isolines of equivalent fitness in the (C_x, x) plane. The evolutionary pressure pushes the market toward lighter colored areas.

we will focus on the case of a single representative agent who revise her expectations with a learning process and chooses her portfolio consequently with a simple mean-variance investment strategy. We will first analyze the PCE of this economy, and then discuss its dynamic stability under a specific learning framework.

4.1 PCE in the mean-variance case

Consider an agent adopting a myopic mean-variance investment strategy: if $\rho_{t+1} = w_{t+1}/w_t - r_f$ is the portfolio excess return between time t and $t+1$, the shares of wealth x_t invested at time t are chosen so as to maximize the utility

$$U(\mathbf{x}_t) = \mathrm{E}[\rho_{t+1}] - \gamma\,\mathrm{V}[\rho_{t+1}]\ ,$$

where γ is the risk aversion parameter and $\mathrm{V}[\rho_{t+1}]$ is the expected variance of the portfolio. Here the dependence of the utility on $\mathbf{x}_t$ is due to the fact that ρ_{t+1} depends on w_{t+1}, which in turn depends on $\mathbf{x}_t$ through the law derived in equation (2.2). Under this strategy, assuming perfect knowledge about the riskless interest rate r_f, the agent computes at each time step the total expected return and the variance covariance matrix

$$\begin{aligned} \hat{R}^i &= \mathop{\mathrm{E}}_{t-1}[R^i_t]\ , \\ \hat{C}_t &= \mathop{\mathrm{E}}_{t-1}\left[(\mathbf{R}_t - \hat{\mathbf{R}}_t) \otimes (\mathbf{R}_t - \hat{\mathbf{R}}_t)\right]\ . \end{aligned} \tag{4.1}$$

The difference from equation (3.8) lies in the fact that now expectations are updated at every time step. The notation E_{t-1} stands for an expectation based on the information revealed until time $t-1$ included. For the present discussion it is not relevant to specify which estimator the agent uses to compute the statistics above. In any case, by direct substitution of (2.2), we obtain that the maximization of the mean-variance utility dictates her investment function to be of the form

$$\mathbf{x}_t = \frac{1}{\gamma}\,\hat{C}_t^{-1}(\hat{\mathbf{R}}_t - r_f\mathbf{1}) \; . \tag{4.2}$$

If this strategy represents the aggregate behavior of the market, (4.2) implies that at each time step the following relation

$$\gamma\,\hat{C}_t\mathbf{x}_t = \hat{\mathbf{R}}_t - r_f\,\mathbf{1} \tag{4.3}$$

between expectations and prevailing prices should be satisfied. If the dividend stochastic process is stationary and the estimators adopted to compute the expected values of returns and their variance are consistent, one can assume that after sufficient statistics, they converge to an equilibrium condition so that the investment shares converge to constants, $\mathbf{x}_t \to \mathbf{x}$. We are in a PCE. From (3.7) in Section 3 we know that in this case the right hand side of equation (4.3) becomes

$$\frac{\overset{*}{\mathbf{x}}\cdot\bar{\mathbf{e}}}{\overset{*}{x}{}^0}\mathbf{1} + \mathbf{e}$$

while the left hand side, by direct substitution of (3.9), reduces to

$$\gamma\frac{\overset{*}{\mathbf{x}}{}'D\overset{*}{\mathbf{x}}}{(\overset{*}{x}{}^0)^2}\mathbf{1} + \gamma\frac{D\overset{*}{\mathbf{x}}}{\overset{*}{x}{}^0} \; .$$

Equating the product of the two previous expressions with $\overset{*}{\mathbf{x}}$ one gets

$$\overset{*}{\mathbf{x}}{}'D\overset{*}{\mathbf{x}} = \frac{\overset{*}{x}{}^0}{\gamma}\overset{*}{\mathbf{x}}\cdot\bar{\mathbf{e}}$$

which substituted back in the equation gives

$$\overset{*}{\mathbf{x}} = \frac{\overset{*}{x}{}^0}{\gamma}\,D^{-1}\,\bar{\mathbf{e}} \; .$$

The investment shares of the mean variance investor at equilibrium are proportional to the variance-corrected expected dividend $D^{-1}\,\bar{\mathbf{e}}$. In order to determine the investment share in the riskless asset, one can exploit the fact that $\overset{*}{x}{}^0 = 1 - \mathbf{1} \cdot \overset{*}{\mathbf{x}}$, finally obtaining

$$\begin{aligned} \overset{*}{\mathbf{x}} &= \frac{1}{\gamma + \mathbf{1}' D^{-1}\,\bar{\mathbf{e}}}\, D^{-1}\,\bar{\mathbf{e}} \;. \\ \overset{*}{x}{}^0 &= \frac{\gamma}{\gamma + \mathbf{1}' D^{-1}\,\bar{\mathbf{e}}} \;. \end{aligned} \tag{4.4}$$

The previous expressions provide the PCE version of the two-found separation theorem. They prove that in the case of a mean-variance investor, the optimal investment is split between a market portfolio, proportional to $D^{-1}\,\bar{\mathbf{e}}$, and the risk-less security. The risk aversion parameter γ only affects the total share of wealth invested in the risky portfolio, but not the composition of the portfolio itself.[19]

In the case of mean variance investor, the expressions of price return and variance covariance matrix, obtained by direct substitution of (4.4) in (3.4), (3.5) and (3.9), read

$$\overset{*}{r}_E = r_f + \frac{1}{\gamma} \mathbf{e}'_t D^{-1}\,\bar{\mathbf{e}} \;, \tag{4.5}$$

$$\overset{*}{r}_E = r_f + \frac{1}{\gamma} \bar{\mathbf{e}}' D^{-1}\,\bar{\mathbf{e}} \tag{4.6}$$

and

$$\overset{*}{C} = \frac{1}{\gamma^2} \bar{\mathbf{e}}' D^{-1}\,\bar{\mathbf{e}} \mathbf{1} \otimes \mathbf{1} + \frac{1}{\gamma} \left(\overset{*}{\mathbf{x}} \otimes \mathbf{1} + \mathbf{1} \otimes \overset{*}{\mathbf{x}} \right) + D \;. \tag{4.7}$$

4.2 Dynamic stability of mean-variance investor

In this Section we analyze the market dynamics generated by the presence of a representative mean-variance investor. In particular, we are interested to establish some results about the local stability of the equilibria derived in the previous Section. Analogously to what done in Section 2.5 we assume that the agent forecasts future price movements using EWMA estimators. The investor updates her expectations about the returns of the risky assets and the variance covariance matrix with a learning process governed by an updating parameter $\mu \in (0, 1)$. In this way the predicted returns and their

[19] Since the original investigation by Harry Markowitz in the '50, this kind of result has been repeatedly obtained in several frameworks. For a recent extension to the case of dynamically complete markets and perfect foresight see Schmedders (2007). Notice that in our model this property is the consequence of specific assumptions about the trader investment function and does not possess a general character.

variance-covariance matrix is computed as a weighted average of past realizations. Specifically, we have

$$\hat{\mathbf{R}}_t = \mu \sum_{\tau=0}^{+\infty} (1-\mu)^\tau (\mathbf{r}_{t-1-\tau} + \mathbf{e}_{t-1-\tau}) \tag{4.8}$$

and[20]

$$\hat{C}_t = \mu \sum_{\tau=0}^{+\infty} (1-\mu)^\tau (\mathbf{R}_{t-1-\tau} - \hat{\mathbf{R}}_{t-1-\tau}) \otimes (\mathbf{R}_{t-1-\tau} - \hat{\mathbf{R}}_{t-1-\tau}) \tag{4.9}$$

By using these estimators we assume that the agent does not know that the process is stationary and implements a strategy which can adapt to cross-sectional variation of the underlying process.[21] Using their recursive expression and the investment rule of the mean-variance investor in (4.2) the evolution of the market is described by the following system of vector equations

$$\begin{cases} \hat{\mathbf{R}}_t = (1-\mu)\hat{\mathbf{R}}_{t-1} + \mu\,(\mathbf{r}_{t-1} + \mathbf{e}_{t-1}) \\ \hat{C}_t = (1-\mu)\hat{C}_{t-1} + \mu \left(\mathbf{r}_{t-1} + \mathbf{e}_{t-1} - \hat{\mathbf{R}}_{t-1}\right) \otimes \left(\mathbf{r}_{t-1} + \mathbf{e}_{t-1} - \hat{\mathbf{R}}_{t-1}\right) \\ \mathbf{x}_t = \dfrac{1}{\gamma} \hat{C}_{t-1}^{-1} (\hat{\mathbf{R}}_{t-1} - r_f \mathbf{1}) \\ r_t^i = \dfrac{x_t^i}{x_{t-1}^i} \dfrac{\mathbf{x}_{t-1} \cdot \mathbf{e}_{t-1} + (1+r_f)(1 - \mathbf{x}_{t-1} \cdot \mathbf{1})}{(1 - \mathbf{x}_t \cdot \mathbf{1})} - 1 \end{cases} . \tag{4.10}$$

First of all, we want to assess in which sense the PCE discussed in the previous section represents an equilibrium of the stochastic dynamics. For this purpose we need a preliminary result about the exponential weighted moving average estimator EWMA.

Theorem 4.1. *Let z_t be an i.i.d. stochastic process with mean $\bar{z}$ and variance σ_z^2. Then the finite sample EWMA estimator*

$$\hat{z}_T = \mu \sum_{\tau=0}^{T} (1-\mu)^\tau z_{T-\tau}$$

is a stationary random variable with unconditional mean $\bar{z}\left(1 - (1-\mu)^{T+1}\right)$ and unconditional variance $\sigma_z^2 \left(1 - (1-\mu)^{T+1}\right)$.

---

[20] Note that $\hat{C}_t$ is computed as the covariance matrix of the prediction error $\mathbf{R}_{t'} - \hat{\mathbf{R}}_{t'}$.

[21] The expressions in (4.8) and (4.9) are good estimators for quasi-stationary, i.e. slowly moving, vector processes. They are widely adopted among practitioners, see for instance the RiskMetrics (MSCI group) methodology.

Proof. The proof simply follows from the sum of the geometric series, given that all the z_t come from the same i.i.d. process. $\square$

Now assume that the agent invests according to (4.4) at each time step so that the price return at each time step t becomes

$$r_t^i = \frac{\mathbf{e}_t D^{-1}\,\bar{\mathbf{e}}}{\gamma} + r_f \ .$$

It follows that after T time steps the agent's estimate of future return is

$$\hat{\mathbf{R}}_T = \mu \sum_{\tau=0}^{T} (1-\mu)^\tau \left(\frac{\mathbf{e}_t D^{-1}\,\bar{\mathbf{e}}}{\gamma} + r_f + \mathbf{e}_\tau \right) + (1-\mu)^T \hat{\mathbf{R}}_0$$

where $\hat{\mathbf{R}}_0$ is agent's initial estimate. So, irrespectively of the initial agent's estimate of the total return, the convergence property of the EWMA estimator guarantees that with the passing of time her estimates tend toward a value of $\hat{\mathbf{R}}$ which is, on average, the value implied by the PCE market return (4.6). The same thing happens for the estimate of the covariance matrix $\hat{C}$. Notice however that because of Theorem 4.1 the agent's estimates remain asymptotically noisy. Because the realized dividend yields are in general not equal to their expected values, the system is persistently perturbed away from the PCE. The question arises if these perturbations are sufficient to drive the system away from the PCE or if, conversely, the PCE equilibrium is stable and the dynamics fluctuate around it.

These equations can be analyzed by direct numerical simulation, which shows that for long times and small μ the system hovers around the PCE fixed point discussed above. In Fig. 4 we report the price returns obtained by simulating (4.10) in the case of a single asset. In this case the PCE investment share in the risk asset is

$$\overset{*}{\mathbf{x}} = \frac{\bar{e}}{\gamma + \bar{e}}$$

which, substituted in (4.5), gives the equilibrium price return

$$\overset{*}{r}_E = r_f + \frac{\overset{*}{x}}{1-\overset{*}{x}} e_t \ . \tag{4.11}$$

As shown in Fig. 4, for sufficiently small values of μ the expression in (4.11) provides a good approximation to the observed dynamics. Similar patterns are observed in the case of multiple assets.

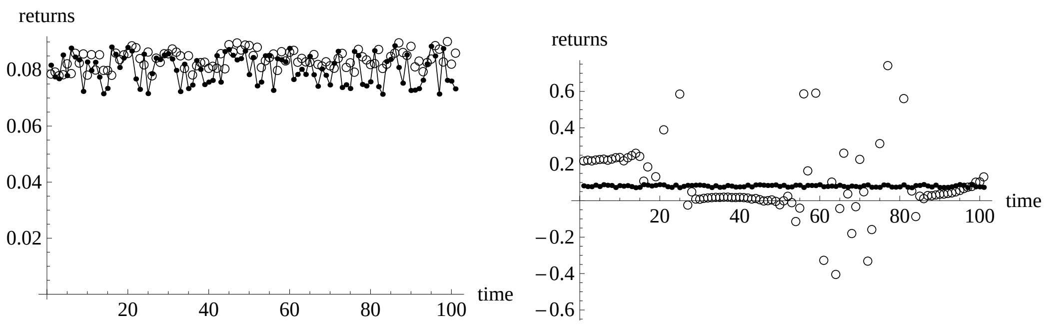

Figure 4: Return dynamics (small circles) and its approximation by the PCE dynamics (continuous line) in the case of a single mean variance investor. When $\mu = 0.05$ (left panel) the PCE approximates the return dynamics reasonably well. When $\mu = 0.15$ (right panel) the dynamics of price returns is more noisy and the PCE provides a poor approximation. Simulations performed with $\gamma = 2000$, $r_f = 0.003$, $\bar{e} = 0.05$ and $\sigma_e = 0.005$.

An alternative approach is that of studying the limit of the dynamics as $\mu \to 0$. In this limits, updates to dynamical variables from t to $t + 1$ are very small and of order μ. This implies that one can look at solutions of the dynamical equations in terms of variables expressed in a continuum time variable $\tau = \mu t$. It is possible to show that the resulting dynamical equations for $\hat{D} = D\hat{I}$ admit the equilibrium as a stationary state and that the equilibrium is stable under the dynamics. The proof, which is rather lengthy is omitted as it merely supports the results which one can already derive from numerical simulations. The interested reader may refer to the on-line appendix of this paper.

5 Conclusion

We discuss a dynamic multi–asset model in an endogenous Walrasian price formation setting. The model rests on two main assumptions: that the demand of traders is consistent with CRRA, i.e., that their invested wealth shares do not depend on their wealth level, and that the process of dividend yields is governed by a stationary distribution.

In this system we analyze a set of equilibria defined by the consistency of agents' expectations and market realizations, named Procedurally Consistent Equilibria. We show that in the PCE the actions of investors leads to two interesting results. Firstly, the market develops price volatility which is much higher than the volatility of dividends and increases the degree of covariance across assets. As such, the model explains the phenomenon of *excess covariance*, that is the tendency of price fluctuations of different financial assets to be more correlated than their fundamentals, here identified with the exogenous dividend process. The excess covariance occurs for a wide range of trading strategies adopted by the investors. The example with mean-variance investors suggests that this result can be extended to allow for short positions, even if further research is needed to generalize this claim for other behaviors. Secondly, we show that the evolutionary pressure due to the introduction of new strategies pushes the system toward an instability that is due to the explosion of the return variance–covariance matrix in the stochastic equilibrium. In order to study the convergence to PCE equilibria we restrict the analysis to a specify trading behavior. We analyze the case of a single representative agent whose investment decisions are based on the maximization of an expected mean-variance CRRA utility. The agent forecasts future assets movements based on past market realizations and using a specific recursive estimator.

Using simple and natural assumptions we show that there exists a strong trading-induced dependence in assets returns. This endogenously generated dependence is a key ingredient of the so called *systemic risk* (see the discussion about *asset price contagion* in Battiston et al. (2010) and references therein). We show that its roots are simple and rest in the selection mechanism induced by speculative trading.

References

Anufriev, M. and G. Bottazzi (2010). "Market Equilibria under Procedural Rationality". *Journal of Mathematical Economics 46 (6)*, 1140–1172.

Anufriev, M., G. Bottazzi, and F. Pancotto (2006). "Equilibria, Stability and Asymptotic Dominance in a Speculative Market with Heterogeneous Agents". *Journal of Economic Dynamics and Control 30*, 1787–1835.

Anufriev, M. and P. Dindo (2010). "Wealth-driven Selection in a Financial Market with Heterogeneous Agents". *Journal of Economic Behavior and Organization 73*, 327–358.

Arnold, L. (1998). *Random Dynamical Systems.* Springer, Heidelberg.

Battiston, S., D. D. Gatti, M. Gallegati, B. Greenwald, and J. Stiglitz (2010). "Liaisons Dangereuses: Increasing Connectivity, Risk Sharing and Systemic Risk". *Mimeo.*

Blume, L. and D. Easley (1992). "Evolution and Market Behavior". *Journal of Economic Theory 58*, 9–40.

Blume, L. and D. Easley (2009). "Market Selection and Asset Pricing". In T. Hens and K. Schenk-Hoppé (Eds.), *Handbook of financial markets: dynamics and evolution*, pp. 403–437. North-Holland (Handbooks in Finance Series).

Bottazzi, G. and P. Dindo (2010). "Evolution and Market Behavior with Endogenous Strategies". *LEM W.P. 2010/20. Scuola Superiore Sant'Anna.*

Bouchaud, J.-P. and M. Potters (2003). *Theory of Financial Risk and Derivative Pricing: from Statistical Physics to Risk Management.* Cambridge University Press.

Brealey, R., I. Cooper, and E. Kaplanis (2010). "Excess Comovement in International Equity Markets: Evidence from Cross–Border Mergers". *The Review of Financial Studies 23 (4)*, 1718–1740.

Brock, W. and C. Hommes (1998). "Heterogeneous Beliefs and Routes to Chaos in a Simple Asset Pricing Model". *Journal of Economic Dynamics and Control 22 (8–9)*, 1235–1274.

Chiarella, C., R. Dieci, and L. Gardini (2005). "the dynamic interaction of speculation and diversification". *Applied Mathematical Finance 12 (1)*, 17–52.

Chiarella, C., R. Dieci, and X. He (2010). "Time–Varying Beta: A Boundedly Rational Equilibrium Approach". *Accepted for publication on Journal of Evolutionary Economics*.

Chiarella, C. and X. He (2001). "Asset Price and Wealth Dynamics under Heterogeneous Expectations". *Quantitative Finance 1*, 509–526.

Diks, C. and F. Wagener (2008). A bifurcation theory for a class of discrete time markovian stochastic systems. *Physica D: Nonlinear Phenomena 237*(24), 3297 – 3306.

Evstigneev, I., T. Hens, and K. Schenk-Hoppé (2009). "Evolutionary Finance". In T. Hens and K. Schenk-Hoppé (Eds.), *Handbook of financial markets: dynamics and evolution*, pp. 403–437. North-Holland (Handbooks in Finance Series).

Fisher, I. (1930). *The Theory of Interest.* A. M. Kelley, New York.

Follmer, H., U. Horst, and A. Kirman (2005). Equilibria in financial markets with heterogeneous agents: a probabilistic perspective. *Journal of Mathematical Economics 41*, 123–155.

Froot, K. and E. Dabora (1999). "How are Stock Prices Affected by the Location of Trade?". *Journal of Financial Economics 53*, 189–216.

Grandmont, J.-M. (1985). "On Endogenous Competitive Business Cycles.". *Econometrica 53 (5)*, 995–1045.

Hommes, C. (2006). "Heterogeneous Agent Models in Economics and Finance". In L. Tesfatsion and K. Judd (Eds.), *Handbook of Computational Economics, Vol. 2: Agent-Based Computational Economics*. Elsevier Science.

Kallberg, J. and P. Pasquariello (2008). "Time–Series and Cross–Sectional Excess Comovement in Stock Indexes". *Journal of Empirical Finance 15*, 481–502.

Kodres, L. and M. Pritsker (2002). "A Rational Expectations Model of Financial Contagion". *The Journal of Finance 57 (2)*, 769–799.

Kyle, A. and W. Xiong (2001). "Contagion as a Wealth Effect". *The Journal of Finance 56 (4)*, 1401–1440.

LeRoy, S. (2008). Excess volatility tests. In S. Durlauf and L. Blume (Eds.), *The New Palgrave Dictionary of Economics, Second Edition*. Palgrave Macmillan.

Levy, M., H. Levy, and S. Solomon (2000). *Microscopic Simulation of Financial Markets*. Academic Press, London.

Lux, T. (1995). Herd behaviour, bubbles and crashes. *The Economic Journal 105*, 881–896.

Marsili, M., I. Mastromatteo, and P. Zoi (2011). "Financial Correlations at Ultra–High Frequency: Theoretical Models and Empirical Estimation". *Accepted for publication in The European Physical Journal B*.

Marsili, M. and G. Raffaelli (2006). "Dynamic Instability in a Phenomenological Model of Correlated Assets". *Journal of Statistical Mechanics*, L08001.

Schmedders, K. (2007). "two-found separation in dynamic general equilibrium". *Theoretical economics 2*, 135–161.

Shiller, R. (1989). "Comovements in Stock Prices and Comovements in Dividends". *The Journal of Finance 44 (3)*, 719–729.

Subbotin, A. and T. Chauveau (2011). Price dynamics in a market with heterogeneous investment horizons and boundedly rational traders. Ces working paper, Centre d'Economie de la Sorbonne.

Taussky, O. (1949). "A Recurring Theorem on Determinants". *The American Mathematical Monthly 56*, 672–676.

Veldkamp, L. (2006). "Information Markets and the Comovement of Asset Prices". *Review of Economic Studies 73*, 823–845.

Weibull, J. (1997). *Evolutionary game theory.* The MIT press.

Westerhoff, F. (2004). "Multiasset Market Dynamics". *Macroeconomic Dynamics 8*, 596–616.

Yuan, K. (2005). "Asymmetric Price Movements and Borrowing Constraints: A Rational Expectations Equilibrium Model of Crises, Contagion, and Confusion". *The Journal of Finance 60 (1)*, 379–410.

Zeeman, E. C. (1988). Stability of dynamical systems. *Nonlinearity 1*(1), 115.

Appendix

A Proof of Theorem 2.1

First, we substitute the intertemporal constraint (2.2) into the pricing equation (2.3) to obtain

$$\mathbf{p}_t = \sum_{n=1}^{N} w_{t-1,n}\,\mathbf{x}_{t,n}\left(\mathbf{x}_{t-1,n}\cdot\mathbf{e}_t + x^0_{t-1,n}(1+r_f)\right) + \sum_{n=1}^{N} w_{t-1,n}\,\mathbf{x}_{t,n}\otimes\mathbf{z}_{t-1,n}\,\mathbf{p}_t\ .$$

This leads to (2.7) as long as matrix H_t in (2.6) is invertible.

In order to show that H_t is invertible we will, first, prove that H_t is column strictly diagonally dominant, i.e.,

$$|H_t^{i,i}| > \sum_{j\neq i} |H_t^{j,i}| \qquad \text{for all } i, \tag{A.1}$$

where $H_t^{i,j}$ denotes the entry in the ith row and jth column of the matrix H_t. Under the assumption of the theorem, the off-diagonal terms of H given by

$$H_t^{i,j} = -\sum_{n=1}^{N} w_{t-1,n} x_{t,n}^i \frac{x_{t-1,n}^j}{p_{t-1}^j} \,, \qquad \text{where } i \neq j \,,$$

are strictly negative. In contrast, the diagonal elements are positive as the following inequality shows

$$H_t^{i,i} = 1 - \sum_{n=1}^{N} w_{t-1,n} x_{t,n}^i \frac{x_{t-1,n}^i}{p_{t-1}^i} > 1 - \max_n x_{t,n}^i \frac{\sum_n w_{t-1,n} x_{t-1,n}^i}{p_{t-1}^i} = 1 - \max_n x_{t,n}^i > 0 \,.$$

Given these signs of the elements of H_t, the following inequality immediately implies (A.1):

$$\begin{aligned} H_t^{i,i} + \sum_{j \neq i} H_t^{j,i} &= 1 - \sum_{j=1}^{I} \sum_{n=1}^{N} w_{t-1,n}\, x_{t,n}^j \frac{x_{t-1,n}^i}{p_{t-1}^i} = \\ &= 1 - \sum_{n=1}^{N} w_{t-1,n} (1 - x_{t,n}^0) \frac{x_{t-1,n}^i}{p_{t-1}^i} > 1 - \max_n (1 - x_{t,n}^0) > 0 \,. \end{aligned}$$

Since H_t is strictly diagonally dominant, the Levy-Desplanques theorem (Taussky, 1949) implies that H_t is invertible.

From the definition of the market portfolio, Eq. (2.4), and the pricing equation given by (2.3), it follows that the investment share in the market portfolio satisfies to $x_t^i = p_t^i / w_t$. Plugging here the evolution of the aggregate wealth (2.5), the expressions for the investment shares in the market portfolio are obtained. Using this result, analogous expressions for the investment shares in the riskless asset are obvious, since

$$\begin{aligned} x_t^0 = 1 - \sum_{i=1}^{I} x_t^i = 1 - \sum_{i=1}^{I} \frac{p_t^i}{w_{t-1} x_{t-1}^0 (1 + r_f) + \sum_j d_t^j + \sum_j p_t^j} = \\ = \frac{w_{t-1} x_{t-1}^0 (1 + r_f) + \sum_i d_t^i}{w_{t-1} x_{t-1}^0 (1 + r_f) + \sum_i d_t^i + \sum_i p_t^i} \,. \end{aligned}$$

In order to derive (2.9) we, firstly, notice that the previous result implies

$$x_t^0 \left(w_{t-1} x_{t-1}^0 (1 + r_f) + \sum_i d_t^i + \sum_i p_t^i \right) = w_{t-1} x_{t-1}^0 (1 + r_f) + \sum_i d_t^i$$

which gives

$$\sum_{i=1}^{I} p_t^i = \frac{1 - x_t^0}{x_t^0} \left(w_{t-1} x_{t-1}^0 (1 + r_f) + \sum_i d_t^i \right) .$$

Secondly,

$$\sum_{i=1}^{I} d_t^i = \sum_{i=1}^{I} p_{t-1}^i \frac{d_t^i}{p_{t-1}^i} = \mathbf{p}_{t-1} \cdot \mathbf{e}_{t-1} .$$

Summing it up, we obtain from (2.5) that

$$w_t = w_{t-1} x_{t-1}^0 (1 + r_f) + \mathbf{p}_{t-1} \cdot \mathbf{e}_{t-1} + \frac{1 - x_t^0}{x_t^0} \left(w_{t-1} x_{t-1}^0 (1 + r_f) + \mathbf{p}_{t-1} \cdot \mathbf{e}_{t-1} \right)$$

which leads to (2.9) after simplifications.

Finally, dividing both sides of (2.2) by w_t one has

$$\begin{aligned} \varphi_{t,n} &= \frac{w_{t-1}}{w_t} \cdot \varphi_{t-1,n} \left(\mathbf{x_{t-1}} \cdot \mathbf{e_t} + \sum_{i=1}^{I} x_{t-1,n}^i \frac{p_t^i}{p_{t-1}^i} + x_{t-1,n}^0 (1 + r_f) \right) = \\ &= \frac{w_{t-1}}{w_t} \cdot \varphi_{t-1,n} \left(\mathbf{x_{t-1}} \cdot \mathbf{e_t} + x_{t-1,n}^0 (1 + r_f) \right) + \varphi_{t-1,n} \sum_{i=1}^{I} x_{t-1,n}^i \frac{x_t^i}{x_{t-1}^i} , \end{aligned}$$

where we used the following equality

$$\sum_{i=1}^{I} x_{t-1,n}^i \frac{p_t^i}{p_{t-1}^i} = \sum_{i=1}^{I} x_{t-1,n}^i \frac{x_t^i}{x_{t-1}^i} \frac{w_t}{w_{t-1}} .$$

Applying the total wealth dynamics (2.9), we derive (2.10).

B Proof of Theorem 3.1

Under the hypothesis of the Theorem, the expression in (3.9) reduces to

$$\overset{*}{C} = \frac{\sigma^2 |\overset{*}{\mathbf{x}}|^2}{(1 - \overset{*}{x})^2} \mathbf{1} \otimes \mathbf{1} + \frac{\sigma^2}{1 - \overset{*}{x}} \left(\overset{*}{\mathbf{x}} \otimes \mathbf{1} + \mathbf{1} \otimes \overset{*}{\mathbf{x}} \right) + \sigma^2 \, \mathbb{I} .$$

Assume that $\mathbf{1}$ and $\overset{*}{\mathbf{x}}$ are not collinear. For any vector $\mathbf{v}$ consider the decomposition $\mathbf{v} = \alpha \mathbf{1} + \beta \overset{*}{\mathbf{x}} + \mathbf{v}^{\perp}$, where $\mathbf{v}^{\perp}$ is the component orthogonal to $\mathrm{Span}\{\mathbf{1}, \overset{*}{\mathbf{x}}\}$. First notice that all vectors with $\alpha = \beta = 0$ are eigenvectors of $\overset{*}{C}$ with eigenvalue σ^2. There are $I-2$ orthogonal vectors of this type. The other two eigenvectors belong to Span$\{\mathbf{1}, \overset{*}{\mathbf{x}}\}$. The representation of the matrix $\overset{*}{C}$ on this sub-space is

$$\sigma^2 \begin{pmatrix} \frac{|\overset{*}{\mathbf{x}}|^2 I}{(1-\overset{*}{x})^2} + \frac{1}{1-\overset{*}{x}} & \frac{N}{1-\overset{*}{x}} \\ \frac{|\overset{*}{\mathbf{x}}|^2}{(1-\overset{*}{x})^2} & \frac{1}{1-\overset{*}{x}} \end{pmatrix}$$

and the computation of eigenvalues immediately follows.

C Proof of Theorem 3.2

Let φ denote the initial wealth share associated to the new strategy. The last equation in (2.12) can be used to determine the growth rate of φ which reads

$$\frac{\Delta\varphi}{\varphi} = y^0\,\frac{\mathbf{z}\cdot\mathbf{e}_t + z^0(1+r_f)}{\mathbf{y}\cdot\mathbf{e}_t + y^0(1+r_f)} + \sum_i z^i \frac{y^i}{y^i}\ ,$$

where $\mathbf{y} = (1-\varphi)\overset{*}{\mathbf{x}} + \varphi\mathbf{z}$ and, with usual notation, $y^0 = 1 - \sum_i y^i$. Assuming a small initial wealth share for the new strategy, $\varphi \ll 1$, one can approximate the previous expression keeping only the leading term to obtain

$$\frac{\Delta\varphi}{\varphi} = \left(1 - \varphi\frac{\mathbf{z}\cdot\mathbf{e}_t + z^0(1+r_f)}{\overset{*}{\mathbf{x}}\cdot\mathbf{e}_t + \overset{*}{x}{}^0(1+r_f)}\right)\frac{\mathbf{z}\cdot\mathbf{e}_t + z^0(1+r_f)}{\overset{*}{\mathbf{x}}\cdot\mathbf{e}_t + \overset{*}{x}{}^0(1+r_f)} + o(\varphi)\ . \tag{C.1}$$

Imagine to iterate the previous equations several time. If for small values of φ it holds that $\mathrm{E}\left[\Delta\varphi/\varphi\right] < 0$, then the new strategy will progressively disappear from the market, and its impact will vanish. Conversely, if for small value of φ it holds that $\mathrm{E}[[\Delta\varphi/\varphi] > 0$, the strategy will grow to a finite size. From (C.1) we derive that

$$\lim_{\varphi\to 0} \mathrm{E}\left[\frac{\Delta\varphi}{\varphi}\right] = \frac{\mathbf{z}\cdot\bar{\mathbf{e}} + z^0(1+r_f)}{\overset{*}{\mathbf{x}}\cdot\bar{\mathbf{e}} + \overset{*}{x}{}^0(1+r_f)}\ ,$$

which proves the proposition.