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Loss Sharing between Non-negligent Parties*

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Abstract

In this paper, we study the effects and desirability of legal rules that allow the sharing of an accident loss between a nonnegligent injurer and his nonnegligent victim. In order to identify the virtues and limits of loss-sharing rules, we begin by considering the effect of a loss-sharing regime on parties’ incentives. We address an unresolved issue in the literature, exploring whether loss-sharing in equilibrium undermines the parties’ primary care incentives. We establish the conditions under which loss-sharing may be desirable and characterize the regime providing the best overall incentives to minimize the social cost of accidents. Our results indicate that loss-sharing may indeed be desirable in a vast range of situations. The results are later extended to consider the effect of parties’ uncertainty in a loss-sharing regime and reveal that loss-sharing may at the same time be desirable and unnecessary in real-life accident law.

JEL classification: K13, K32.

Keywords: tort, loss-sharing, negligence, strict liability, comparative fault.

1 Introduction

The question of how to allocate losses between non-negligent parties provides a theoretically appealing and practically relevant setting for the scholar of tort law. Tort liability—the most common system for the internalization of negative externalities—is largely dominated by the notion that the party deemed negligent in court should bear the accident loss.1 Although several legal rules exist

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1For a recent critical review of the literature on comparative negligence see Bar-Gill and Ben-Shahar (2003).
to allocate or split an accident loss between two negligent parties through *comparative negligence*, legal systems do not generally provide ways for sharing an accident loss between an injurer and his victims when neither party is at fault for the accident.

In a well-functioning liability system, parties are induced to adopt optimal care in equilibrium. In-equilibrium accidents are thus likely to be caused by non-negligent parties. The adoption of due care reduces the probability and severity of an accident loss, but the probability of an accident is not completely eliminated in such situations. The current literature devotes only limited attention to the study of accident losses that are not attributable to negligence. Though frequently observed in real life, these accident losses are viewed as less problematic, inasmuch as they can be regarded as the necessary by-product of valuable and otherwise desirable human activities.

Calabresi (1965) noted that tort systems that apportion liability based on fault only deter accidents that are caused through negligent behavior and ignore the value of deterring accidents that are faultless. Calabresi explicitly suggested dividing the costs of an accident pro rata between the sub-activities involved, irrespective of legal notions of fault. In Calabresi’s own example, if a walker, a bicyclist and an automobile are all involved in an accident without fault on any of these parties, the accident loss could be divided among these three activities. Calabresi (1996) and Calabresi and Cooper (1996) recently returned to this issue, lamenting the lack of attention of current scholars to the apportionment of liability between non-negligent parties.

When neither party is negligent, existing legal rules tend to adopt an all-or-nothing approach by either leaving the entire loss on the victim or shifting it entirely to the injurer, with no intermediate alternatives. Rules that burden the victim are commonly called negligence rules, whereas strict liability rules exclusively burden the injurer. We invite the reader to notice that in all cases,

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2Both Common law and Civil law systems frequently, but not universally, address the issue of dual party negligence, through *comparative negligence*, according to which an accident loss should be shared between the parties.

3Calabresi suggested that if accidents of this sort occurred on a recurring basis, liability should be apportioned on the basis of the cumulative effect, assigning greater liability to those activities that result in more frequent and more severe accidents (Calabresi, 1965, p. 740-741).

4We know three specific incarnations of negligence rules, differing in the way in which they allocate the loss when both parties are negligent: *simple negligence*, allocating the entire loss to the victim; *contributory negligence*, allocating the entire loss to the injurer; and *comparative negligence* where both negligent parties bear a portion of the loss.

5Strict-liability-based rules are those that burden the injurer when neither party is negligent. The simplest form of strict liability considers the injurer liable under all circumstances, but courts more frequently recognize some form of negligence defense. Thus, we can again distinguish among three versions of strict-liability-based rules, each differing in how they allocate loss when both parties are negligent: *strict liability with defense of dual contributory negligence*, leaving the entire loss with the victim; *strict liability with defense of contributory negligence*, shifting the entire loss to the injurer; and *strict liability with defense of comparative negligence*, under which both negligent parties bear the loss, in some proportion. Under strict liability with defense of dual contributory negligence, the injurer can claim that the victim was negligent, but the victim can in turn claim that the injurer was also negligent. It is easy to see that if both are negligent the injurer pays, as under simple negligence. However,
a party that is unilaterally negligent pays for the entire loss. These rules can be distinguished from each other only by looking at the allocation of the loss when both parties adopt the same behavior. The figure below provides a graphical representation of this taxonomy. In particular, the upper-right box in each graph indicates who bears the loss when both parties are nonnegligent.

Figure 1: Who bears the accident loss under different liability rules

In this paper we consider the possibility of a sharing of the loss between non-negligent parties—a possibility which is excluded by traditional liability rules. The rule which we consider can be seen as a continuum of possible sharing alternatives of which negligence and strict liability constitute the two extremes. A loss-sharing approach was first advocated by Hugo Grotius (1625) who called it the "compensation principle", in contrast to the already established fault principle. According to the fault principle, fault is what justifies the shifting of the loss from an innocent victim to his injurer. An injurer is liable only if he is found negligent, and when no party is at fault, the loss should rest where it falls. Hence, faultless victims bear the loss in such cases. An opposite result is reached by application of the compensation principle, calling for the need to provide damage compensation to a faultless victim. Under this principle, compensation is due in all cases except when the victim is at fault.

Classic literature on tort law pays little attention to the idea of sharing an accident loss between an injurer and his victims when neither party is at fault for under strict liability with defense of contributory negligence, the victim does not have such defense, and hence a negligent victim pays even if the injurer was also negligent. When both are negligent, the victim pays, as under contributory negligence.

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6See Jain and Singh (2002).
7The figure does not account for the two more extreme cases of no liability (the victim always pays) and strict liability with no defenses (the injurer always pays). These rules are not interesting for our purposes as they fail to provide one of the parties with incentives.
the accident.\textsuperscript{8} Traditional wisdom suggests that loss-sharing may be desirable as a form of implicit insurance for risk-averse parties.\textsuperscript{9} Absent such an insurance function, loss-sharing would have no reason to exist. Much of the interest for loss-sharing rules was further obfuscated by the belief that a loss-sharing rule was likely to undermine the parties’ care incentives.

We consider the incentive properties of loss-sharing rules with respect to care and activity levels. We show that even in the absence of risk-aversion, loss-sharing may be a valuable instrument for the reduction of the cost of accidents. We specifically consider loss-sharing as a policy control variable independent of the parties’ degree of negligence or causal contribution to the loss. Unlike other rules that use all-or-nothing solutions in the apportionment of residual liability (thereby concentrating all activity level incentives on one party), rules that create loss-sharing in equilibrium spread both the threat of residual liability and the resulting activity level incentives between the parties. Both victims and injurers face some incentives to optimize their respective activity levels. Given these properties, we consider if and when it may be desirable to introduce a loss-sharing rule.

In Section ??, we begin the analysis by addressing an unresolved question in the existing literature: whether the adoption of a loss-sharing rule between non-negligent parties undermines the parties’ incentives to adopt a socially optimal level of due care. Contrary to conventional wisdom, we show that loss-sharing does not undermine care incentives. Parties will always have incentives to comply with an optimally chosen standard of due care, irrespective of the sharing rule implemented. This irrelevance result mirrors a similar result proven by Landes and Posner (1980, note 51; Haddock and Curran, 1985) with respect to sharing between negligent parties. Our finding, combined with Landes and Posner’s (1980), shows that sharing the loss between negligent or non-negligent parties does not affect compliance with optimally set negligence standards.

Having established the compatibility of loss-sharing rules with optimal care incentives, in Section ?? we consider the related question of which sharing rule would most effectively promote a reduction of the social cost of accidents. Loss-sharing spreads the incentives to reduce activity levels between the parties. The desirability, or lack thereof, of spreading such incentives depends on the relationship between the parties’ efforts. This paper reveals that under standard assumptions of increasing marginal costs and decreasing marginal benefits of activity level, loss-sharing may induce greater overall reduction of “inefficient” activity levels than any of the traditional liability rules. The relative effectiveness of alternative liability rules is also affected by the possible returns to scale and the synergies and complementarities of activity level reduction by the two

\textsuperscript{8}In recent legal and economic literature, the issue of loss sharing has been considered by Calabresi (1996) and Parisi and Fon (2004) under the name of comparative causation. Parisi and Fon consider a comparative causation rule used in conjunction with negligence and strict liability rules. In their model, loss-sharing follows from a damage-apportionment rule based on causality. When neither party is at fault, the parties share damages based on their causal contribution to the loss. Dari-Mattiacci and De Goest (2005) discuss loss-sharing irrespective of fault and show that it might filter out the most harmful accidents. See also Singh (2005).

\textsuperscript{9}See the discussion in Cooter (1991).
parties.

In Section 2, we extend the analysis to consider the impact of parties’ role-uncertainty. When parties are faced with role-uncertainty—uncertainty as to whether they will find themselves as victims or injurers in a future accident—the law is incapable of affecting activity level incentives. At the limit, strict liability and negligence rules have the same effects on the parties’ activity levels. Role-uncertainty leads to an expected loss-sharing, which renders loss-sharing rules superfluous. We suggest that our results may explain why loss-sharing rules, although in principle desirable, are seldom utilized by legal systems.

2 The model of negligence-based tort liability

In this section, we consider a model of accident prevention. We assume that accidents may be prevented by taking two different types of precautionary measures. Following Shavell’s (1980) terminology, we distinguish between care levels and activity levels. Care levels are verified ex post in court and are taken into account in the determination of negligence, while activity levels are not. To use a now classical example, a motorist’s care level may be his speed—included in the determination of negligence because easily verifiable—while activity level may be the frequency with which he drives—which is generally not included in the determination of negligence.\(^{10}\)

We consider two parties: a prospective injurer \((U)\) and his victim \((V)\). They are strangers to each other, rational and risk-neutral. We consider the standard case of unilateral-risk accidents, such that the victim is the only party that suffers the loss occasioned by the injurer if an accident occurs.\(^{11}\) The expected accident loss is affected by both parties’ activity and care levels (bilateral-precaution accidents). The parties’ utilities \(U\) and \(V\) decrease in care \(x\) and \(y\) at a constant or increasing rate and increase in their levels of activity \(s\) and \(t\) at a decreasing rate. We further assume that the expected accident loss increases in the level of activity and decreases in care. Therefore, let:

\[
U = U(x, s) \text{ be the injurer’s utility function, } U_x < 0, \ U_{xx} \leq 0, \ U_s > 0, \text{ and } U_{ss} < 0;
\]

\[
V = V(y, t) \text{ be the victim’s utility, with } V_y < 0, \ V_{yy} \leq 0, \ V_t > 0, \text{ and } V_{tt} < 0;
\]

\(^{10}\)Activity levels are not utilized for the evaluation of negligence either because they yield prohibitively high verification costs (e.g., number of times that a pedestrian crosses the street in any given day), or because courts do not possess sufficient information (e.g., the private value of driving or other risk-creating activity) for establishing the socially optimal level of activity against which to compare the parties’ behavior. On the optimal setting of the scope of the negligence inquiry, see Dari-Mattiacci (2005).

\(^{11}\)See Arlen (1990) showing that results do not change when considering bilateral-risk accidents in a model without activity level.

\(^{12}\)Note that activity level could be modeled as a normal care measure (which reduces the accident loss and also the party’s utility) without changing the results of the analysis. The only crucial difference between care and activity level is the inclusion in or exclusion from the negligence inquiry.
$L = L(x, y, s, t)$ be the expected accident loss, with $L_x, L_y < 0$ and $L_{xx}, L_{yy}, L_s, L_t, L_{ss}, L_{tt} > 0$.

Furthermore, all dependent and independent variables just listed are non-negative. Social welfare is assumed to be a simple sum of the parties utilities minus the expected accident loss:

$$W = U + V - L$$

(1)

This formulation allows for any degree of interdependence between the parties’ care and activity levels. For instance, the parties’ activities are complements in the reduction of the accident loss—a reduction in one party’s activity increases the marginal effect of the other party’s activity reduction—if $L_{st} < 0$. The opposite holds true when the parties’ activities are substitutes in the reduction of the accident loss, $L_{st} > 0$.\(^{13}\) Independence between the parties’ activities would be characterized by $L_{st} = 0$. The standard model used in Shavell (1980 and 1987) and subsequent literature is a special case of the general model described above. In this literature, the expected loss is defined as $L = st(x, y)$. As it easy to see, the parties’ activities are substitutes in Shavell’s formulation of the problem, as $L_{st} = l(x, y) > 0$.\(^{14}\)

The distinction between complement and substitute cases allows us to study the problem of accident prevention within the framework of supermodular games (Topkis, 1979 and 1998; Vives, 1990; Milgrom and Roberts, 1990). As we shall show, if the parties’ activities are complements, then the game played by the parties is supermodular; conversely, if the parties’ activities are substitutes, then the game is submodular. Supermodular (and submodular) games are an appealing theoretical framework for the problem at hand in that they have at least one pure-strategy Nash equilibrium and nice comparative statics properties.

A similar distinction between complements and substitutes can be carried out for descriptive purposes with respect to the signs of $L_{xs}$ and $L_{yt}$ (determining the relationship between one party’s care and his own activity level), $L_{ys}$ and $L_{xt}$ (determining the relationship between one party’s care and the other party’s activity level), and $L_{xy}$ (determining the relationship between the parties’ care levels). Likewise, the relationship between care and activity level in the parties’ utility functions, $U_{xs}$ and $V_{yt}$, determines whether increasing the level of activity increases or reduces the cost of care.\(^{15}\)

---

\(^{13}\)Given the parties’ objective to maximize the value of risk-creating activities at the net of accident costs, $L$ enters as a negative term. Thus we look at the sign of $-L_{st}$. Therefore the case $-L_{st} > 0$ (or $L_{st} < 0$) represents the case where the parties’ activities are complements in the reduction of the accident loss. The opposite holds true when the parties’ activities are substitutes in the reduction of the accident loss, $-L_{st} < 0$ (or $L_{st} > 0$).

\(^{14}\)Note that notation is as in Shavell (1987).

\(^{15}\)Note that parties interact only in the production of the expected accident loss $L$, while a party’s utility is independent from the other party’s care and activity levels. This indicates that a party’s cost of taking precautions (be it activity or care) does not depend on the behavior of the other party. While this assumption is standard in the literature, a notable exception is Dharmapala and Hoffmann (2005), studying the performance of negligence rules in a model in which the costs of precautions are interdependent.
2.1 First-best liability rules

The first-best socially optimal levels of care and activity \((\hat{x}, \hat{y}, \hat{s}, \hat{t})\) solve:

\[
\max_{x,y,s,t} [U + V - L]
\]

A first-best liability rule should ideally set the standards of care and activity to equal these first-best levels. However, as discussed above, activity levels are too costly to verify or to implement in court as a basis of liability and hence are not included in the determination of negligence. Thus, as shown in Shavell (1980), first-best accident prevention is not attainable under the set of liability rules considered here.

2.2 Second-best liability rules

Since activity levels are not included in the determination of negligence, a judge (or a policymaker) cannot use them as policy variables. The policy instruments are restricted to the choice of standards of due care for both parties \((x_d, y_d)\) and the choice of apportionment of liability between two non-negligent parties (sharing rule \(\sigma\)) and between two negligent parties (sharing rule \(\vartheta\)), so that:

1. If both parties adopt due care, then the loss is shared according to \(\sigma\) in the interval \([0, 1]\);
2. If neither party adopts due care, then the loss is shared according to \(\vartheta\) in the interval \([0, 1]\);
3. If only one party adopts due care, then the unilaterally negligent party bears the entire loss.

Figure 2: An overview of liability rules

As shown in Figure 2, this general framework encompasses all possible liability rules described above in Figure 1 and accompanying text. Optimal care incentives are created by the negligence standards \((x_d, y_d)\), while the incentives to undertake optimal activity levels depend both on the standards of due care \((x_d, y_d)\) and on the allocation of the residual loss in equilibrium, as determined by the sharing rule \(\sigma\). More specifically, negligence rules shift the entire residual...
loss to the victim ($\sigma = 0$), while strict liability rules shift it entirely to the injurer ($\sigma = 1$). Unlike these standard rules, loss-sharing rules ($0 < \sigma < 1$) share the residual burden between the injurer and the victim, hence spreading the incentives to moderate activity levels between them.

Therefore, in the second best scenario, the policymaker’s problem can be expressed as follows:

$$
\max_{x^d, y^d, \sigma} [U + V - L]
$$

s.t. (1) : \( s = s (x^d, y^d, \sigma) \) and \( t = t (x^d, y^d, \sigma) \)

s.t. (2) : \( x = x^d \) and \( y = y^d \)

The policy variables only indirectly influence the parties’ choices of care and activity levels. The first restriction indicates that the parties’ choice of activity levels depends in some way on all policy variables. We devote Section ?? to the study of how \( s \) and \( t \) are determined. The second restriction states that the desired policy outcome occurs only if the parties abide by the due care standards, a problem we shall tackle in Section ?? . If both restrictions are satisfactorily verified, the policymaker can set \((x^d, y^d, \sigma)\) as to attain the second best level of social welfare.

### 2.2.1 The parties’ choice of activity levels

Here we hold the assumption that both parties take levels of care equal to due-care standards. We analyze the parties’ choice of their activity levels given the policymaker’s choice of \( x^d, y^d \) and \( \sigma \). Thus, \( x^d, y^d \) and \( \sigma \) are parameters of the non-cooperative game \( \Gamma(x^d, y^d, \sigma) \) played by the parties. The parties choose activity levels \( s^* \) and \( t^* \) in order to maximize their payoffs as follows:

$$
\max_s \left[ U (x^d, s) - \sigma L (x^d, y^d, s, t) \right]
$$

$$
\max_t \left[ V (y^d, t) - (1 - \sigma) L (x^d, y^d, s, t) \right]
$$

Supermodular (and submodular) games provide the appropriate framework to study games where the best response of a player is a monotonic function of its rival, as it is the case in our framework. It is shown in the appendix that if \( L_{st} \geq 0 \) (the parties’ activities are substitutes), the parties play a submodular game; likewise, if \( L_{st} \leq 0 \) (the parties’ activities are complements), the parties play a supermodular game. From this observation, it follows that the game has at least one Nash equilibrium in pure strategies. More specifically, since we have assumed \( L_{ss}, L_{tt} > 0 \), the Nash equilibrium is also unique. Thus, we can conclude that parties will choose unique levels of activity \( s^* = s (x^d, y^d, \sigma) \) and \( t^* = t (x^d, y^d, \sigma) \), which are functions of the parameters under which the game is played and which are generally different from the first best.

Another powerful property of submodular games is that we can use monotonicity arguments to prove comparative statics properties. In our settings, if
we have \( L_{st} \geq 0 \) (the parties’ activities are substitutes, the game is submodular), then \( s^* \) is a decreasing function of \( \sigma \) and \( t^* \) is an increasing function of \( \sigma \).

Note also that the injurer’s activity level \( s^* \) is decreasing in his due level of care \( x^d \) if \( U_{xx} - \sigma L_{xx} < 0 \), and is increasing otherwise. Likewise, the victim’s activity level \( t^* \) is decreasing in his due level of care \( y^d \) if \( V_{yt} - (1 - \sigma) L_{yt} < 0 \), and is increasing otherwise.

### 2.2.2 The policymaker’s choice of due care and sharing

The policymaker defines the socially optimal levels of care \( x^d^* \) and \( y^d^* \) and the sharing \( \sigma^* \) that maximize:

\[
\max_{\sigma, x^d^*, y^d^*} \left[ U + V - L \right]
\]

s.t. (1) : \( s = s^* \) and \( t = t^* \)

s.t. (2) : \( x = x^d \) and \( y = y^d \)

where the parties’ activity levels are determined as in the previous section:

\[
s^* \in \text{arg max}_s \left[ U(x, s) - \sigma L(x, y, s, t^*) \right]
\]

\[
t^* \in \text{arg max}_t \left[ V(y, t) - (1 - \sigma) L(x, y, s^*, t) \right]
\]

### 2.2.3 Parties’ compliance with the negligence standards

In the preceding analysis, optimal standards of care and optimal sharing were identified under the working assumption that parties would comply with the chosen standard of due care in equilibrium. In this section, we verify whether this rather critical assumption holds in the case under examination, where \( \sigma^* \), \( x^d^* \), and \( y^d^* \) are the optimal sharing rule and the optimal standards of care.

We consider a general liability rule characterized by sharing rules \( \sigma \) (for the sharing between non-negligent parties) and \( \vartheta \) (for the sharing between negligent parties) and given standards of care \( x^{d\sigma} \) and \( y^{d\sigma} \), which give rise to a new game \( \Gamma(x^{d\sigma}, y^{d\sigma}, \sigma, \vartheta) \) where the injurer’s and the victim’s payoffs are, respectively, as follows:

\[
\Pi^U(x, y) = \begin{cases} 
U - \sigma L & \text{if } x \geq x^{d\sigma} \text{ and } y \geq y^{d\sigma} \quad \text{I (both non-negligent)} \\
U - L & \text{if } x < x^{d\sigma} \text{ and } y \geq y^{d\sigma} \quad \text{II (injurer negligent)} \\
U - \vartheta L & \text{if } x < x^{d\sigma} \text{ and } y < y^{d\sigma} \quad \text{III (both negligent)} \\
U & \text{if } x \geq x^{d\sigma} \text{ and } y < y^{d\sigma} \quad \text{IV (victim negligent)}
\end{cases}
\]

\[
\Pi^V(x, y) = \begin{cases} 
V - (1 - \sigma) L & \text{if } x \geq x^{d\sigma} \text{ and } y \geq y^{d\sigma} \quad \text{I (both non-negligent)} \\
V & \text{if } x < x^{d\sigma} \text{ and } y \geq y^{d\sigma} \quad \text{II (injurer negligent)} \\
V - (1 - \vartheta) L & \text{if } x < x^{d\sigma} \text{ and } y < y^{d\sigma} \quad \text{III (both negligent)} \\
V - L & \text{if } x \geq x^{d\sigma} \text{ and } y < y^{d\sigma} \quad \text{IV (victim negligent)}
\end{cases}
\]
The numbering (I through IV) in the last columns to the right refers to the quadrants in the left-hand side graph in Figure 2, where the numbering starts from the upper-right cell and continues counter-clockwise. We are interested in determining whether the parties choose \((x = x^{d\sigma}, y = y^{d\sigma})\). In order for compliance to obtain in a Nash equilibrium, it must be impossible for the injurer to improve his utility by choosing levels of care and activity different from \(x^{d\sigma}\) and \(s^{\sigma}\), given that the victim is complying with the rule. We will analyze the parties’ incentives to comply with the chosen standard of due care and examine whether this equilibrium is unique.

We investigate whether a sharing rule undermines the parties’ care incentives in equilibrium, under the plausible assumption that the injurer’s (optimally chosen) activity level \(s\) is decreasing in \(x^{d}\) and the victim’s (optimally chosen) activity level \(t\) is decreasing in \(y^{d}\). This is for instance the case if \(U_{x,s} \leq 0\) and \(L_{x,s} \geq 0\) for the injurer and \(V_{y,t} \leq 0\) and \(L_{y,t} \geq 0\) for the victim, as explained at the end of Appendix I. Under this assumption, we can prove the following proposition:

**Proposition 1** If the standards of due care and the loss-sharing rule are set at the (second best) socially optimal levels, \(\sigma^*, x^{d*}\) and \(y^{d*}\), and a party’s (optimally chosen) activity level decreases in his due-level of care, both parties comply with the negligence standards in a unique Nash equilibrium of the game \(\Gamma(x^{d\sigma}, y^{d\sigma}, \sigma, \vartheta)\), irrespective of the sharing rule \(\vartheta\) applied when both parties are negligent.

**Proof** In order for compliance to obtain in a Nash equilibrium, the following must hold true for the injurer:

\[
(U - \sigma L) (x^{d\sigma}, y^{d\sigma}, s^{\sigma}, t^{\sigma}) > (U - L) (x, y^{d\sigma}, s, t^{\sigma})
\]

for all \((x < x^{d\sigma}, s)\) This condition implies that, if the victim is taking due care, the injurer should not be able to increase his payoff by deviating from due care. Let us tackle this question by showing first that the injurer has no incentive to increase precautions above the required standard of care. That is, we need to show that if \(x = x^{d\sigma}\) a complying injurer cannot increase his payoff by deviating upwards from due care.

The maximum of \((U - \sigma L) (x, y^{d\sigma}, s, t^{\sigma})\) is reached for a value \(s\) which depends continuously on \(\sigma\) and \(x\). Thus, we can write:

\[
s = s(\sigma, x)
\]

By hypothesis, \(x^{d\sigma} = x^{d*}\) maximizes the social welfare when \(y^{d\sigma} = y^{d*}\), \(\sigma = \sigma^*\), and \(t^{\sigma} = t^*\) are given. Thus, we have:

\[
(U - L) (x^{d\sigma}, y^{d\sigma}, s^{\sigma}, t^{\sigma}) = (U - L) (x^{d\sigma}, y^{d\sigma}, s(\sigma, x^{d\sigma}), t^{\sigma}) \\
\geq (U - L) (x, y^{d\sigma}, s(\sigma, x), t^{\sigma}) = (U - L) (x, y^{d\sigma}, s, t^{\sigma})
\]
Accordingly, since we have assumed $L_x < 0$, $L_s > 0$, and that $s = s(\sigma, x)$ is a decreasing function of $x$, we have:

\[
(U - \sigma L) \left( x^{d\sigma}, y^{d\sigma}, s^\sigma, t^\sigma \right) = (U - L) \left( x^{d\sigma}, y^{d\sigma}, s^\sigma, t^\sigma \right) + (1 - \sigma) L(x^{d\sigma}, y^{d\sigma}, s^\sigma, t^\sigma) \\
\geq (U - L) \left( x, y^{d\sigma}, s, t^\sigma \right) + (1 - \sigma) L(x^{d\sigma}, y^{d\sigma}, s^\sigma, t^\sigma) \\
\geq (U - L) \left( x, y^{d\sigma}, s, t^\sigma \right) + (1 - \sigma) L(x, y^{d\sigma}, s, t^\sigma)
\]

Therefore:

\[
(U - \sigma L) \left( x^{d\sigma}, y^{d\sigma}, s^\sigma, t^\sigma \right) > (U - \sigma L) \left( x, y^{d\sigma}, s, t^\sigma \right)
\]

(5)

for all $x > x^{d\sigma}$ and with $s = s(\sigma, x)$.

We can now continue to tackle the question of whether compliance can be obtained in a Nash equilibrium by studying whether the injurer has incentives to reduce precautions below the required standard of care. By hypothesis $x^{d\sigma} = x^{d\sigma}$ and $\sigma = \sigma^*$ maximize social welfare when $y^{d\sigma} = y^{d\sigma}$ and $t^\sigma = t^\sigma$ are given. Thus we have:

\[
(U + V - L) \left( x^{d\sigma}, y^{d\sigma}, s^\sigma, t^\sigma \right) = (U - L) \left( x^{d\sigma}, y^{d\sigma}, s(\sigma, x^{d\sigma}), t^\sigma \right) + V(y^{d\sigma}, t^\sigma) \\
\geq (U - L) \left( x, y^{d\sigma}, s(1, x), t^\sigma \right) + V(y^{d\sigma}, t^\sigma) \\
= (U - L) \left( x, y^{d\sigma}, s, t^\sigma \right) + V(y^{d\sigma}, t^\sigma)
\]

hence,

\[
(U - L) \left( x^{d\sigma}, y^{d\sigma}, s^\sigma, t^\sigma \right) \geq (U - L) \left( x, y^{d\sigma}, s, t^\sigma \right)
\]

(6)

and, since $\sigma \in [0, 1]$,

\[
(U - \sigma L) \left( x^{d\sigma}, y^{d\sigma}, s^\sigma, t^\sigma \right) \geq (U - L) \left( x^{d\sigma}, y^{d\sigma}, s^\sigma, t^\sigma \right) \\
\geq (U - L) \left( x, y^{d\sigma}, s, t^\sigma \right)
\]

Therefore:

\[
U \left( x^{d\sigma}, s^\sigma \right) - \sigma L \left( x^{d\sigma}, y^{d\sigma}, s^\sigma, t^\sigma \right) > U \left( x, s \right) - L \left( x, y^{d\sigma}, s, t^\sigma \right)
\]

(7)

for all $x < x^{d\sigma}$ and with $s = s(x, \sigma)$.

The inequality in (7) shows that, in the limiting case of strict liability, $\sigma = 1$, the injurer and the social planner face the same problem, and therefore a negligent injurer’s payoff is maximized by increasing precautions until the level of due care is reached. The incentives to avoid negligence and comply with the chosen standard of due care are strengthened with lower levels of $\sigma$, inasmuch as the injurer benefits from a larger reduction of expected liability by complying with the standard of care, which proves (7).

A similar analysis can be carried out for the victim’s incentives to undertake optimal precautions in equilibrium, proving that, when negligence standards are optimally set, both parties have incentives to comply with the negligence standard in equilibrium, under the socially optimal sharing rule $\sigma^*$. Note that
this result holds for any \( \vartheta \), that is, it holds regardless of the way in which accident losses are allocated when both parties are negligent.

We should now verify whether the parties' compliance with the standard of due care represents a unique Nash equilibrium of the game. The results derived above show that unilateral negligence is not an equilibrium. In order to show uniqueness, we need to further prove that bilateral negligence is not an equilibrium. In order to observe bilateral negligence in equilibrium, the following conditions should be simultaneously satisfied. These conditions state that, given that one party is negligent, the other party should also prefer to be negligent, rather than unilaterally non-negligent:

\[
U(x, s) - \vartheta L(x, y, s, t) > U(x^d, \bar{s})
\]

\[
V(y, t) - (1 - \vartheta) L(x, y, s, t) > V(y^d, \bar{t})
\]

s.t. (1): \( s = s(x, y, \vartheta) \) and \( t = t(x, y, \vartheta) \)

s.t. (2): \( x < x^d \) and \( y < y^d \)

Summing these conditions, we obtain:

\[
U(x, s) + V(y, t) - L(x, y, s, t) > U(x^d, \bar{s}) + V(y^d, \bar{t})
\]

Let us now consider that the loss-sharing rule is set at the socially optimal level, \( \sigma = \sigma^* \). Note that, the left-hand side of (8) represents the previously seen second-best maximization problem, subject to analogous conditions on the parties' choice of activity levels. It follows that the left-hand side of (8) is maximized by \( \vartheta = \sigma^* \), levels of care equal to \( x^d \) and \( y^d \), and the resulting levels of activity equal to \( s^* \) and \( t^* \). Therefore, regardless of the sharing rule applied in the case of bilateral negligence, \( \vartheta \), the parties' aggregate payoffs when bilaterally negligent cannot be larger than \( U(x^d, \bar{s}) + V(y^d, \bar{t}) \) \( \geq U(x^d, s^*) + V(y^d, t^*) \), which proves that both parties' negligence cannot be an equilibrium if the loss-sharing rule and the negligence standards are optimally set. ■

2.2.4 Discussion

Proposition 1 shows that a second-best tort law system based on optimal negligent standards and optimal loss-sharing between non-negligent parties is feasible, since compliance results in equilibrium, under the plausible assumption that an increase in a party due-care level reduces his chosen level of activity. These results also confirm that compliance is not affected by the choice of sharing when both parties are negligent \( \vartheta \). This finding extends a well established result in tort law and economics, first proven by Landed and Posner (1980, note 16).

16Consider that a unilaterally non-negligent party has no incentives to lower his activity level below the level he would choose under bilateral negligence. In fact, the non-negligent party can now rely on the full avoidance of liability in case of an accident, given the other party’s negligence, which in turn lowers the cost associated with his risk-creating activity.
in a model in which only care was considered. Here we show that the result holds also when activity level is taken into account. Our analysis further departs from the conventional framework of Shavell (1980) in which only two corner sharing arrangements were allowed: \( \sigma = 0 \) and \( \sigma = 1 \), excluding the possibility of intermediate loss-sharing solutions even when optimal.

2.3 Third-best liability rules

In this section, we will now consider an important interdependence between the loss-sharing rule \( \sigma \) applicable when both parties are non-negligent and the loss-sharing rule \( \vartheta \) applicable when both parties are instead negligent. The results that follow have important implications for the choice of optimal liability rules when loss-sharing between non-negligent parties is not feasible.

The previous analysis in which both negligence standards and sharing were set at the socially optimal level can now be extended to consider cases in which the sharing is not possible, or is otherwise set at a socially suboptimal level.\(^{17}\) Consider the case where negligence standards are optimally set, for the chosen value of \( \sigma \). This situation corresponds to the first step of the social welfare maximization problem formalized in Section ??, realizing in fact a third-best policy for the prevention of accidents.

2.3.1 Parties’ compliance with the negligence standards

In order to analyze this case we need to verify again whether compliance with the standard of due care results in a Nash equilibrium and whether the resulting equilibrium is unique. We do so by retaining the assumption that a party’s activity level decreases in his due-care level and proving the following proposition:

**Proposition 2** If social welfare is an increasing function of \( s \) and \( t \), if the levels of due care are set at the socially optimal levels, \( x^\sigma \) and \( y^\sigma \), but the loss-sharing rule is not optimally set \( \sigma \neq \sigma^* \), and if a party’s activity level decreases in his due-level of care, then both parties comply with the negligence standards in equilibrium irrespective of the sharing rule \( \vartheta \). This equilibrium is unique if \( \vartheta = \sigma \).

**Proof** The first part of the proof \((x > x^{d\sigma})\) of Proposition 1 is still valid when \( \sigma \neq \sigma^* \). However, the second part requires that both \( x^{d\sigma} \) and \( \sigma \) maximize social welfare when \( y^{d\sigma} \) and \( t^\sigma \) are given. Therefore, we need to verify the parties’ compliance with due care in the latter case. First, consider the injurer, \( x < x^{d\sigma} \).

By hypothesis, \( x^{d\sigma} \) maximizes social welfare when \( \sigma, y^{d\sigma} \) and \( t^\sigma \) are given. Since \( s \) is a decreasing function of \( \sigma \), for \( \sigma \leq 1 \), we have

\[
s(\sigma, x) \geq s(1, x)
\]

\(^{17}\)See further Section ?? on this point.
Thus we have:

\[(U + V - L) (x^{d\sigma}, y^{d\sigma}, s^{\sigma}, t^{\sigma}) = (U - L) (x^{d\sigma}, y^{d\sigma}, s^{\sigma}, x^{d\sigma}, t^{\sigma}) + V(y^{d\sigma}, t^{\sigma}) \geq (U - L) (x^{d\sigma}, s^{\sigma}, x^{d\sigma}, t^{\sigma}) + V(y^{d\sigma}, t^{\sigma}) \geq (U - L) (x^{d\sigma}, s, t^{\sigma}) + V(y^{d\sigma}, t^{\sigma}) \]

which follows from the assumption that social welfare \((U + V - L)\) increases in \(s\). Therefore, we have:

\[(U - L) (x^{d\sigma}, y^{d\sigma}, s^{\sigma}, t^{\sigma}) \geq (U - L) (x, y^{d\sigma}, s, t^{\sigma}) \]

Since \(\sigma \in [0,1]\), we have:

\[(U - \sigma L) (x^{d\sigma}, y^{d\sigma}, s^{\sigma}, t^{\sigma}) \geq (U - L) (x^{d\sigma}, y^{d\sigma}, s^{\sigma}, t^{\sigma}) \geq (U - L) (x, y^{d\sigma}, s, t^{\sigma}) \]

Concluding:

\[U (x^{d\sigma}, s^{\sigma}) - \sigma L (x^{d\sigma}, y^{d\sigma}, s^{\sigma}, t^{\sigma}) > U (x, s) - L (x, y^{d\sigma}, s, t^{\sigma}) \]

for all \(x < x^{d\sigma}\) and with \(s = s(x, \sigma)\) which proves the first part of Proposition 2. Note that this result holds for any \(\vartheta\).

The second part of Proposition 2 requires verifying whether the parties’ compliance with the standard of due care represents a unique Nash equilibrium of the game. Knowing that unilateral negligence is not an equilibrium, we should verify if bilateral negligence can be excluded as a possible equilibrium of the game. For bilateral negligence to occur in equilibrium condition (??) should be satisfied for values of \(\sigma \neq \sigma^{*}\). We will identify a sufficient condition under which (??) cannot be satisfied, hence proving the last part of Proposition 2.

Let us consider the case in which the loss-sharing rule is not optimally set, \(\sigma \neq \sigma^{*}\). When both parties are non-negligent, it is easy to see that if the loss is shared according to the same loss-sharing criterion used when both parties are negligent (that is, if \(\vartheta = \sigma\)), the left-hand side of (??) represents the second-best maximization problem. Given \(\vartheta = \sigma\), when both parties are negligent their aggregate net utilities are maximized when parties adopt levels of care equal to \(x^{d\sigma}\) and \(y^{d\sigma}\) and levels of activity equal to \(s^{\sigma}\) and \(t^{\sigma}\). We can hence conclude that, if \(\vartheta = \sigma\), the condition in (??) necessary for bilateral negligence cannot be satisfied: at least one party would have an incentive to deviate. This proves that when \(\vartheta = \sigma\), parties cannot both be negligent in equilibrium. The equilibrium where both parties comply with the due standard of care will thus be unique in this case, as for the second part of Proposition 2.

2.3.2 Discussion

Proposition 2 shows that, under some sufficient conditions, compliance with the standard of due care can be achieved in equilibrium for any \(\sigma\) and any \(\vartheta\). This
irrelevance result mirrors and extends the finding by Landes and Posner (1980). We show that the parties’ incentives to comply with the standard of due care are present not only under any sharing rule for bilateral negligence (Landes and Posner’s result), but also under any sharing rule for bilateral non-negligence. Landes and Posner (1980) used their irrelevance result with respect to $\vartheta$ to demonstrate the incentive-equivalence of different negligence rules (simple negligence, comparative negligence and contributory negligence), all of which were shown to lead to the adoption of due levels of care and identical levels of social welfare. The same equivalence result holds when alternative negligence defenses are applied to strict liability (strict liability with defense of contributory negligence, strict liability with defense of dual contributory negligence, and strict liability with defense of comparative negligence). However, the internal equivalence of these rules within each set of regimes cannot be used to compare the different allocations of the residual loss induced by the alternative negligence or strict-liability regimes—a task we have taken on in this paper.

We have proven a higher-level irrelevance among all combinations of sharing $\sigma$ (when parties are non-negligent) and $\vartheta$ (when parties are negligent), showing that all liability rules\footnote{Recall that no-liability and strict liability without negligence defenses are excluded from this count.} are equivalent with respect to care incentives, in the sense that they all induce both parties to comply with the negligence standards. However, unlike the sharing rule applied in the case of bilateral negligence $\vartheta$, which does not affect social welfare because it occurs out of equilibrium, the sharing rule $\sigma$ applies to non-negligent parties and is implemented in equilibrium, hence impacting social welfare.

Proposition 2 should also be related to Shavell (1987, pp. 42-43). Shavell employs a model in which the standards of due care are set at the first-best levels $\hat{x}$ and $\hat{y}$ and shows that compliance results in equilibrium. He proceeds to show that social welfare can be improved by increasing the negligence standards above those first-best levels, as long as the increase was so high as to induce parties to violate them. In this paper, we provide conditions under which the negligence standards can be increased up to the third-best standards without compromising parties’ compliance. Since second-best standards—$\sigma$ is optimally set—can be seen as a subset of third-best standards—any $\sigma$—the former result obviously carries over to second-best standards, as proven in Proposition 1.

The sufficient conditions identified in Proposition 2 guarantee that compliance with the negligence standards is the only equilibrium if the sharing $\sigma$ applied when parties are negligent is equal to the sharing $\vartheta$ applied when parties are non-negligent. This in turn is the case in three broad sets of situations: under comparative negligence, if a loss-sharing rule for the residual loss applies in the same proportion, and under two corner cases and, namely, under contributory negligence—the victim pays the whole accident loss if the parties are both negligent or if they are both nonnegligent—and under strict liability with defense of dual contributory negligence—the injurer pays in this case—as it is easy to verify in Figure 1.
3 Optimal loss-sharing between non-negligent parties

In the previous Section we have shown that the adoption of loss-sharing rules between non-negligent parties does not undermine parties’ care incentives. In this Section, we shall now consider the possible use of loss-sharing as a policy control variable, and consider if and when it may be desirable to induce loss-sharing between non-negligent parties. The previous analysis shows that the allocation of liability between non-negligent parties may take one of three possible forms, namely $\sigma = 0$ (i.e., the allocation of traditional negligence rules), $\sigma = 1$ (i.e., the allocation of strict liability rules), or $0 < \sigma < 1$ (i.e., loss-sharing between non-negligent parties). The choice of optimal sharing depends on the characteristics of the relevant accident functions.

In the following we study the optimal setting of $\sigma$ in a specification of our model, where the loss function is given an additive form. Under this formulation we have $L_{st} = 0$, that is, the parties activity levels are independent of each other in the production of the expected accident loss. Shavell (1980 and 1987), employs a multiplicative form, with $L_{st} > 0$, that is, the parties activity levels are complements. Our formulation allows us to discuss the conditions under which sharing the loss between non-negligent parties is preferable to allocating it entirely to either of them and has the advantage that the optimal level of $\sigma$ can be made explicit without having to calculate the due care standards. The analysis that follows proves that, depending on the circumstances, loss-sharing can be the socially optimal liability rule. Finally, we provide a simple example illustrating our results.

3.1 The case of a linear loss function in $s$ and in $t$.

In this Section we shall consider a special case of the general model developed above, where the activity levels have an additive effect in the production of accident losses. The parties’ utility functions are specified as in Shavell (1987); let:

$$u(x, s) = z(s) - sx,$$

with $z_s > 0$, and $z_{ss}$ an increasing and non positive function satisfying $z_{ss} \leq \alpha < 0$;

$$v(y, t) = w(t) - ty,$$

with $w_t > 0$, and $w_{tt}$ an increasing and non positive function satisfying $w_{tt} \leq \beta < 0$.

Given these assumptions it is convenient to work with the general framework of convex analysis. In particular, we can use the following result:

**Lemma 1:** $z_s(s)$ and $w_t(t)$ are continuous and strictly decreasing functions from $\mathbb{R}$ to $\mathbb{R}$. Their inverses $g = (z_s)^{-1}$ and $h = (w_t)^{-1}$ exist and are also continuous and strictly decreasing functions from $\mathbb{R}$ to $\mathbb{R}$.
In addition, it is natural to assume that \( l(x, y, s, t) \geq l(x, y, 0, 0) = 0 \), inasmuch as positive activity levels create risk, and there will be no accident losses in the absence of activity. This allows us to expand the expected accident cost function in a Maclaurin series as follows:

\[
\begin{align*}
  l(x, y, s, t) &= l(x, y, 0, 0) + l_s(x, y, 0, 0)s + l_t(x, y, 0, 0)t + (\sqrt{s^2 + t^2})\varepsilon(s, t)
  \end{align*}
\]

This gives the following linear approximation for \( l(x, y, s, t) \):

\[
  l(x, y, s, t) \to f_1(x, y)s + f_2(x, y)t
\]

Here we shall further assume that \( l(x, y, s, t) \) has exactly this explicit additive form in \( s \) and \( t \):

\[
\begin{align*}
  l(x, y, s, t) &= f_1(x, y)s + f_2(x, y)t \\
  \text{with } f_1(x, y) &> 0
\end{align*}
\]

### 3.1.1 Value of \( \sigma \)

In order to identify the optimal loss-sharing rule for non-negligent parties, we should maximize the social welfare, which is now:

\[
(z(s) + w(t) - (f_1(x, y)s + f_2(x, y)t) - sx - ty
\]

As before, let us proceed by backward induction and first analyze the parties’ choice of activity levels, \( s \) and \( t \), given legal standards of care, \( x^d \) and \( y^d \), and the sharing rule, \( \sigma \). Thus, (27) and (28) become, respectively:

\[
\begin{align*}
  \max_s \left[ z(s) - \sigma f_2(x^d, y^d)t - s(1 - \sigma) f_1(x^d, y^d) + x^d \right] \\
  \max_t \left[ w(t) - (1 - \sigma) f_1(x^d, y^d)s - t\sigma f_2(x^d, y^d) + y^d \right]
\end{align*}
\]

Let \( z^* \) and \( w^* \) be the conjugates of the proper concave functions \( z \) and \( w \) (\( z^* \) and \( w^* \) are proper convex functions).\(^{19}\)

So we can write:

\[
\begin{align*}
  z(s^*) - s^* \left[ \sigma f_1(x^d, y^d) + x^d \right] &= z^* (\sigma f_1(x^d, y^d) + x^d) \\
  w(t^*) - t^* \left[ (1 - \sigma) f_2(x^d, y^d) + y^d \right] &= w^*( (1 - \sigma) f_2(x^d, y^d) + y^d)
\end{align*}
\]

Given legal standards of due care, \( x^d \) and \( y^d \), let us set:

\[
\begin{align*}
  a(x) &= z^*(x) - \left[ f_1(x^d, y^d) - x + x^d \right] g(x) \\
  b(x) &= w^*(x) - \left[ f_2(x^d, y^d) - y + y^d \right] h(x)
\end{align*}
\]

\(^{19}\)See Rockafellar (1970, pp. 24 and 104) and appendix II.
\[
\begin{align*}
A(x) &= a(xf_1(x^d, y^d) + x^d) \\
B(x) &= b(xf_2(x^d, y^d) + y^d)
\end{align*}
\]

Then, simple calculations show that for each \((x^d, y^d)\) the social welfare function at the equilibrium levels of activity assumed to be \(\neq 0\) is:

\[
S(\sigma) = A(\sigma) + B(1 - \sigma)
\]

Since \(A(x)\) and \(B(x)\) are continuous functions on \([0, 1]\), we define the sup-convolution of \(A\) and \(B\) as:

\[
(A \square B)(x) = \sup_{y \in [0, 1]} \{A(y) + B(x - y)\}, x \in [0, 1]
\]

If equilibrium levels of activity are \(\neq 0\), it follows that the solution exists and is unique:

**Proposition 3** In the additive version of our model, sharing the loss between non-negligent parties maximizes social welfare if and only if:

\[
A(0) + B(1) \neq (A \square B)(1)
\]

and

\[
A(1) + B(0) \neq (A \square B)(1)
\]

Then for each \((x^d, y^d)\), there is a unique optimal loss-sharing rule \(\sigma \in ]0, 1]\) satisfying:

\[
A(\sigma) + B(1 - \sigma) = \max_{\tau \in [0, 1]} S(\tau)
\]

### 3.2 Example of quadratic utilities functions

As an illustration, consider this simple example, in which we have:

\[
\begin{align*}
z(s) &= s(2a - s) \\
w(t) &= t(2b - t)
\end{align*}
\]

The parties’ optimization problems become:

\[
\begin{align*}
&\max_s [s(2a - s) - s[\sigma f_1(x^d, y^d) + x^d] - \sigma tf_2(x^d, y^d)] \\
&\max_t [t(2b - t) - t[(1 - \sigma)f_2(x^d, y^d) + y^d] - (1 - \sigma)s f_1(x^d, y^d)]
\end{align*}
\]

Differentiating these objective functions with respect to \(s\) and \(t\) and setting them equal to zero yields the parties’ reaction functions:

\[
\begin{align*}
2a - 2s - \sigma(f_1(x^d, y^d)) - x^d &= 0 \Rightarrow s = a - \frac{1}{2} \sigma f_1(x^d, y^d) - \frac{1}{2} x^d \\
2b - 2t - [(1 - \sigma)f_2(x^d, y^d) + y^d] &= 0 \Rightarrow t = b - \frac{1}{2} f_2(x^d, y^d) + \frac{1}{2} \sigma f_2(x^d, y^d) - \frac{1}{2} y^d
\end{align*}
\]

\(^{20}\)The first condition means that the maximum of \(S(\sigma)\) is not attained for \(\sigma = 0\) and the second condition means that the maximum of \(S(\sigma)\) is not attained for \(\sigma = 1\). It is also worthwhile to note that this approach allows us to locate the optimal \(\sigma\) without calculating \(x^d\) and \(y^d\).
The social welfare function is

\[
S(\sigma, x^d, y^d) = s(\sigma, x^d, y^d)(2a - s(\sigma, x^d, y^d)) + t(\sigma, x^d, y^d)(2b - t(\sigma, x^d, y^d))
\]

\[
- s(\sigma, x^d, y^d)f_1(x^d, y^d) - t(\sigma, x^d, y^d)f_2(x^d, y^d)
\]

\[
- s(\sigma, x^d, y^d)x^d - t(\sigma, x^d, y^d)y^d
\]

Differentiating the social welfare function with respect to \(\sigma\) and setting the derivative equal to zero gives:

\[
S_\sigma(\sigma, x^d, y^d) = 0
\]

\[
-\frac{1}{2}\sigma f_1^2(x^d, y^d) - \frac{1}{2}\sigma f_2^2(x^d, y^d) + \frac{1}{2} f_1^2(x^d, y^d) = 0,
\]

This indicates that the socially optimal sharing rule is:

\[
\sigma^* = \frac{f_1^2(x^d, y^d)}{f_1^2(x^d, y^d) + f_2^2(x^d, y^d)} \in [0, 1]
\]

In this example, without computing \(x^d, y^d\), we can see that the socially optimal allocation of the residual loss is \(\sigma^* = 0\) (the allocations of residual loss induced by traditional negligence rules), when \(f_1(x^d, y^d) = 0\) (the damage attributable to the injurer at the due level of care is equal to zero); it is \(\sigma^* = 1\) (the the allocation of residual loss induced by strict liability rules) when \(f_2(x^d, y^d) = 0\) (the damage attributable to the victim at the due level of care is equal to zero); it is \(0 < \sigma^* < 1\), when both \(f_1(x^d, y^d) > 0\) and \(f_2(x^d, y^d) > 0\). In this case, loss sharing is necessary to maximize the social net benefit function. In particular, the socially optimal loss-sharing rule requires an equal splitting of the loss between the two non-negligent parties, \(\sigma^* = \frac{1}{2}\), when \(f_1(x^d, y^d) = f_2(x^d, y^d)\).

This result is consistent with Calabresi (1965), who put forth the idea of a nonfault liability system apportioning liability according to the riskiness of the activity at the optimal level of care, irrespective of legal notions of fault. Calabresi further suggested that this criterion of (partial) nonfault liability could be implemented dividing the costs of an accident pro rata according to the cumulative effect of the non-negligent activity on accident losses.\(^{21}\)

### 4 Role-uncertainty and loss-sharing

The study of the incentive effects of liability rules carried out in the literature implicitly or explicitly assumes that parties adjust their care and activity levels as potential injurers and victims, and thus that they know with certainty whether they will play the role of injurers or victims in a potential accident. However, in many real life situations, parties face uncertainty as to whether

\(^{21}\text{Calabresi, (1965, p. 740-741).}\)
they will be victims or injurers.\footnote{Furthermore, as pointed out by Coase (1960), causation is often reciprocal and ambiguous. It is often by means of conventional legal constructs that the ambiguities are resolved with the labeling of one party as victim and the other as tortfeasor. But it is not until after the harmful event has occurred that such ambiguity is resolved. Note also that the notions of victim and injurer are defined in Section ?? with reference to where the loss initially falls and not in relation to causation, which we do not examine in this paper. Role uncertainty has also been studied by Feldman and Kim (2003) in a different setting from ours.} In traffic accidents, for instance, it is normally difficult for two motorists to know ex ante which of their vehicles involved in a collision will be (more seriously) damaged. By applying the results of the previous Sections, it is possible to study the effect of role-uncertainty on the parties’ incentives.\footnote{For simplicity we assume that role-uncertainty does not affect due care standards or else that due care levels are the same for both parties.}

Uncertainty with respect to the parties’ roles—as injurers or victims—in the event of a future accident creates a de facto sharing of the expected residual loss. With respect to incentives, this ex ante sharing of the expected residual loss occasions effects that are analogous to those produced by the ex post loss-sharing considered in this paper. The findings of this paper thus illuminate the workings of liability rules in real-life cases where parties face role-uncertainty.

Role-uncertainty can be interpreted as a form of implicit loss-sharing as follows: Given a legal sharing rule $\sigma$ and an ex ante probability $\pi$ that a given party be the injurer in the event of an accident, the expected sharing for that party is given by

$$\zeta = \pi \sigma + (1 - \pi)(1 - \sigma) \quad (9)$$

Conversely, the expected share of residual loss borne by the other party would be given by $1 - \zeta = (1 - \pi)\sigma + \pi(1 - \sigma)$. The expected sharing $\zeta$ coincides with the legally chosen sharing rule $\sigma$ only when no uncertainty exists, $\pi = 1$. At the limit, when role-uncertainty is maximal and the parties have equal probabilities of finding themselves as victims or injurers of an accident, the legal sharing rule $\sigma$ becomes irrelevant. It is in fact easy to show that if $\pi = \frac{1}{2}$ then also $\zeta = \frac{1}{2}$, irrespective of the value of $\sigma$: the legal allocation of residual liability between non-negligent parties is thus irrelevant when parties face full role-uncertainty. Because of role-uncertainty, potential injurers will internalize some of the cost associated with their activities even under a negligence rule, as they may find themselves as uncompensated victims in the event of an accident.

This yields to the interesting insight that role-uncertainty may in fact distort or even completely hamper any policy aimed at controlling the parties’ activity levels. When role uncertainty is maximal, risk-neutral parties would behave as if the loss were to be equally shared, regardless of the chosen allocation of residual liability $\sigma$. At the limit, negligence ($\sigma = 0$) and strict liability ($\sigma = 1$) would produce identical incentives with respect to activity levels, undermining the most important rationale for choosing between one or the other liability regime. Under complete role-uncertainty the allocation of residual liability is thus irrelevant. The choice of a loss-sharing regime although having no effect on the parties’ expected liability would however reduce the variance of actual
outcomes—because the sharing is actual, rather than a mere expectation—and may lead to a more desirable allocation of residual liability when risk-averse parties are involved.

When role-uncertainty is present, but less than complete, our findings suggest that legal sharing $\sigma$ could be instrumentally adapted in order to achieve the desired expected sharing $\zeta$, although not perfectly. An example will better illustrate this point. Assume that an accident might occur between two parties, A and B, and that A is the injurer in $\frac{2}{3}$ of the cases. If the desired sharing for A is $\frac{2}{3}$, setting $\sigma = \frac{2}{3}$ will not reach this outcome; in fact, in this case the expected share of party A as calculated from (??) would be $\zeta = \frac{4}{9}$, far less than aimed for. Instead, lowering the ex post sharing $\sigma = 0$ (the negligence rule) would result in an increase of the expected sharing $\zeta = \frac{2}{3}$. The reason for this apparently contradictory outcome is that, since A is more often a victim than an injurer, his expected exposure to residual liability increases if his exposure as victim increases.

From (??) we can derive a general rule for the setting of $\sigma$ in the case of role-uncertainty:

$$\sigma = \frac{\zeta + \pi - 1}{2\pi - 1}$$

From this formulation, we can note that, if $\zeta = \pi$, that is, if the desired expected sharing is equal to the role-uncertainty value, strict liability ($\sigma = 1$) is optimal, while if the desired sharing is the opposite of the role-uncertainty, $\zeta = 1 - \pi$, the negligence rule ($\sigma = 0$) is optimal. It is also important to note that not all distortions created by role-uncertainty can be corrected by an appropriate setting of $\sigma$. In general, the range of achievable ex ante sharing can be derived from (??) and is $\zeta \in [\pi, 1 - \pi]$. Role-uncertainty is largest when $\pi$ is close to $\frac{1}{2}$, which makes the constraint more binding as it restricts the set of sharing policies that can be implemented. In our example, no party can be made the residual bearer in expectation. In fact, $\zeta > 1 - \pi$. To make A the residual bearer, A should bear $\sigma = \frac{3}{2}$ of the loss when he is the injurer and $1 - \sigma = \frac{1}{2}$ when he is the victim, this implies a decoupling of liability that cannot be implemented by ordinary liability rules.\(^\text{24}\)

A second important result of our analysis is that, as previously shown for loss-sharing $\sigma$, role-uncertainty $\zeta$ does not undermine the parties’ care incentives. Far from being a problem, both loss-sharing and role-uncertainty may indeed be desirable, given the fact that the all-or-nothing allocations of the residual loss are often not optimal.

This leads us to a somewhat comforting reconciliation of our results with real-life observations, in line with a positive efficiency hypothesis. The interesting intuition is that loss-sharing, although in principle desirable, is often unnecessary, and it might be for this reason that it is rarely observed in the law. Role-uncertainty de facto creates an expected loss-sharing, even in the absence of ordinary liability rules, damages are perfectly compensatory or, if punitive, are a transfer from the victim to the injurer. Decoupling could be implemented by subsidies or fines. In a different framework, decoupling is studied in Polinsky and Che (1991).\(^\text{21}\)
of a legal loss-sharing rule. Role-uncertainty might create incentives that are possibly similar to those that could have been engineered with the adoption of a loss-sharing rule. Role-uncertainty is more often observed when parties are very much alike, such as motorists, that is, in those situations in which loss sharing is possibly desirable. Instead, when parties are very different from each other, such as industrial polluters and the general population, corner solutions with respect to the allocation of the residual loss, such as strict liability, are more likely to be optimal. The implementation of those solutions is likely to be feasible, as the differences between the parties make role-uncertainty a very remote possibility. It seems that role-uncertainty creates a de facto sharing of the residual loss that, although not perfect, occurs precisely when such sharing is most likely to be desirable.

A closing note concerns the very notion of sharing. In fact, in court sharing could result under disguised form. If the victim is the only party that suffers harm—as it is in our model—sharing would provide the victim with undercompensatory damages; if both parties suffer some loss—a case to which our model applies with minor modifications—sharing might result from the denial of any compensation. Think for example of two parties suffering exactly the same amount of losses; if no compensation is awarded in court, both parties will bear half of the total accident loss. These and similar examples should suggest that again loss-sharing might indeed be at the same time desirable and unnecessary in real-life accident law.

References


APPENDIX I

Consider a two-person game \((A, B; K, L)\) with strategy spaces \(A, B\) and semi-continuous payoff functions \(K, L : A \times B \rightarrow \mathbb{R}\). This game is supermodular (Topkis, 1998) if the following three properties are satisfied:

a) \(A\) is a sublattice of \(\mathbb{R}^n\) and \(B\) is a sublattice of \(\mathbb{R}^m\) and for some \(n, m \subseteq \mathbb{R}\);

b) \(K\) and \(L\) have increasing differences on \(A \times B\), that is, for all \((a_1, a_2) \subset A^2\) and for all \((b_1, b_2) \subset B^2\) such that \(a_1 \geq a_2\) and \(b_1 \geq b_2\) we have:

\[
K(a_1, b_1) - K(a_1, b_2) \leq K(a_2, b_1) - K(a_2, b_2)
\]

\[
L(a_1, b_1) - L(a_2, b_1) \leq L(a_1, b_2) - L(a_2, b_2)
\]

c) \(K\) is a supermodular function in the first coordinate and \(L\) is a supermodular function in the second coordinate, that is, for all \((a_1, a_2) \subset A^2\) and for each \(b \subset B\) we have:

\[
K(a_1, b) + K(a_2, b) \geq K(a_1 \lor a_2, b) + K(a_1 \land a_2, b)
\]

\[
L(a, b_1) + L(a, b_2) \geq L(a, b_1 \lor b_2) + L(a, b_1 \land b_2)
\]

The game \((A, B, K, L)\) is submodular if the game \((A, B, -K, -L)\) is supermodular.

Here we hold the assumption that both parties take levels of care equal to the due-care standards. We analyze the parties’ choice of their activity levels given the policymaker’s choice of \(x^d\), \(y^d\) and \(\sigma\). Thus, \(x^d\), \(y^d\) and \(\sigma\) are parameters of the non-cooperative game \(\Gamma (x^d, y^d, \sigma) = (D, D; \Pi^U, \Pi^V)\) played by the parties. The parties choose activity levels \(s, t \in D\) in order to maximize their payoffs \(\Pi^U = U - \sigma L\) and \(\Pi^V = V - (1 - \sigma) L\).

**Remark 1:** Let \(x^d\) and \(y^d\) be two given numbers in \([0, +\infty]\) and \(\sigma\) in \([0, 1]\). If \(L_{st} \geq 0\) for every \(s, t \in D\), then \(\Gamma (x^d, y^d, \sigma)\) is a submodular game. If \(L_{st} \leq 0\) for every \(s, t \in D\), then \(\Gamma (x^d, y^d, \sigma)\) is a supermodular game.

**Proof.** Let us now verify these requirements in our setting. Let us first assume that \(L_{st} \leq 0\) for every \(s, t \in D\). Given our previous assumptions we know that the parties’ strategy space is \([0, +\infty]\), which is a convex but not compact subset of \(\mathbb{R}\). Nevertheless, we assumed that the parties’ utilities \(U\) and \(V\) increase in their levels of activity \(s\) and \(t\) but at a decreasing rate. Thus, the activity levels \(s\) and \(t\) that solve (??) are necessarily bounded and we can say that the effective set of action of \(s\) and \(t\) is in fact a compact subset \(D\) of \([0, +\infty]\). \(D\) is a sublattice of \(\mathbb{R}\), is one-dimensional and is partially ordered.\(^{25}\) These observations verify

\(^{25}\)Its natural partial ordering is denoted by \(\leq\) with \(x \lor y = \max\{x, y\}\) and \(x \land y = \min\{x, y\}\) for \(x, y\) in \(\mathbb{R}\).
a). Moreover, \( \Pi_U : D \times D \to \mathbb{R} \) and \( \Pi_V : D \times D \to \mathbb{R} \) are twice-continuously-differentiable payoff functions, which verifies b) since for every \((s, t)\) in \( D \times D \) and for every \( \sigma \) in \([0, 1]\), we have:

\[
\Pi_{st}^U = -\sigma L_{st} \geq 0 \\
\Pi_{st}^V = -(1 - \sigma) L_{st} \geq 0.
\]

c) follows automatically because \( D \) is one-dimensional.

Thus, according to Topkis (1998), the game \( \Gamma(x^d, y^d, \sigma) \) is a (smooth) supermodular game.

Likewise, if \( L_{st} \geq 0 \) for every \( s, t \) in \( D \) then \( \Pi_{st}^U \leq 0 \) and \( \Pi_{st}^V \leq 0 \) and the game \( \Gamma(x^d, y^d, \sigma) \) is a (smooth) submodular game.

\[\blacksquare\]

Remark 2: The game \( \Gamma(x^d, y^d, \sigma) \) has at least one pure strategy Nash equilibrium.

**Proof.** We know from Vives (1990) that if the game is supermodular \( (L_{st} \leq 0) \) then the set of pure strategy Nash equilibria is non-empty. Likewise, if the game is submodular \( (L_{st} \geq 0) \) then we have a similar result because it is well-known that we can reverse the natural order in player’s 2 strategy set and the transformed payoffs display increasing differences just as in the first case (see proof of remark 3). If the game is strictly supermodular (respectively submodular) \( (L_{st} < 0 \) and respectively \( L_{st} > 0) \) then the set of pure strategy Nash equilibria is a non-empty complete sub-lattice which is in fact ordered (Vives 1985).\[\blacksquare\]

Another powerful property of supermodular games is that we can use monotonicity arguments to prove comparative statics properties. Let \([A, B, K, L]\) be a supermodular game indexed by a parameter in a partially ordered set with a unique Nash equilibrium.

If each players’ payoff function has increasing differences in its own strategy and the parameter, then the Nash equilibrium is increasing in this parameter.

For given standards of care \( x^d \) and \( y^d \) in \([0, +\infty]\), optimal levels of activity \( s^* \) and \( t^* \) exist and depend on the sharing rule, \( \sigma \in [0, 1] \). When the parties’ levels of activity are complements in the reduction of the accident loss, we have the following result:

**Remark 3 (monotonicity results):** 1) We assume that \((s^*, t^*)\) is the unique Nash equilibrium of the game \( \Gamma(x^d, y^d, \sigma) \), where \((x^d, y^d)\) is given. If for every \( s, t \) in \( D, L_{st} \geq 0 \), then \( s^* = S_d(\sigma) \) is a decreasing function of \( \sigma \) and \( t^* = T_d(\sigma) \) is an increasing function of \( \sigma \).

2) Furthermore, we assume that \((s^*, t^*)\) is the unique Nash equilibrium of the game \( \Gamma(x^d, y^d, \sigma) \), where \( \sigma \) is given. If \( L_{xs} \geq 0, L_{yt} \geq 0, U_{xs} \leq 0, V_{yt} \leq 0 \), then \( s^* = S_d(x^d, y^d) \) is a decreasing function of \( x^d \) and \( t^* = T_d(x^d, y^d) \) is a decreasing function of \( y^d \).
Proof. Let $\sigma$ in $[0,1]$. If $L_{st} \geq 0$ for every $s, t \in D$, then $\Gamma(x^d, y^d, \sigma)$ is a submodular game.

For $s' = -s$ and $t' = t$ the game is supermodular in $(st, t')$ since:

$$
\Pi_{s't'}^U = -\sigma L_{s't'} = \sigma L_{st} \geq 0
$$
$$
\Pi_{st}^V = -(1 - \sigma) L_{s't'} = (1 - \sigma) L_{st} \geq 0.
$$

and for every $\sigma$ in $[0,1]$, we have:

$$
\Pi_{s's}^U = -\frac{\partial}{\partial s'} (L(s, t)) = \frac{\partial}{\partial s} (L(s, t)) = L_s \geq 0
$$
$$
\Pi_{t't}^V = L_{t'} = L_t \geq 0.
$$

Then the unique Nash equilibrium of the game $(st, t') = (-s^*, t^*)$ is increasing in $\sigma$.

We conclude that $s^* = S_d(\sigma)$ is a decreasing function of $\sigma$ and $t^* = T_d(\sigma)$ is an increasing function of $\sigma$.

We can prove 2) with the same method since:

$$
\frac{\partial}{\partial s \partial x^d} [U(x^d, s) - \sigma L(x^d, y^d, s, t)] \leq 0 \text{ and } \frac{\partial}{\partial t \partial y^d} [V(y^d, t) - (1 - \sigma) L(x^d, y^d, s, t)] \leq 0.
$$

.■
APPENDIX II

Proof of Lemma 1. Since the function $z$ is concave, we have:

$$\forall (a, b) \in \mathbb{R}^2 : a < b \Rightarrow z_b(b) - z_a(a) \leq \alpha (b - a) < 0$$

which shows that $z_a$ is a bijective and strictly decreasing function from $\mathbb{R}$ to $\mathbb{R}$, because we have $a = 0 \Rightarrow z_a(0) \leq a \alpha + z_a(0)$ for $0 < b$ from which we obtain $\lim_{b \to +\infty} z_a(b) = -\infty$ (recall that $\alpha < 0$).

Likewise, we have $b = 0 \Rightarrow z_a(a) \geq a \alpha + z_a(0)$ for $a < 0$, from which we obtain $\lim_{a \to -\infty} z_a(a) = +\infty$. Therefore, we conclude that $z_a$ has an inverse $g$, which is also a continuous and strictly decreasing function from $\mathbb{R}$ to $\mathbb{R}$. The same argument applies to $w_1$. ■

Derivation of $S(\sigma)$. Let $z^*$ and $w^*$ be the conjugates of the proper concave functions $z$ and $w$:

$$z^*(x) = \max_{s \geq 0} [z(s) - xs]$$
$$w^*(y) = \max_{t \geq 0} [w(t) - yt].$$

The privately optimal levels of activity $s^*$ and $t^*$ at the equilibrium verify:

$$\max_s [[z(s) - \sigma f_2(x^d, y^d)t^*] - s [\sigma f_1(x^d, y^d) + x^d]]$$
$$= z(s^*) - s^* [\sigma f_1(x^d, y^d) + x^d] - \sigma f_2(x^d, y^d)t^*$$
$$= z^* (\sigma f_1(x^d, y^d) + x^d) - \sigma f_2(x^d, y^d)t^*$$

then : $z(s^*) - s^* [\sigma f_1(x^d, y^d) + x^d] = z^* (\sigma f_1(x^d, y^d) + x^d)$ for U, and simultaneously :

$$\max_t [[w(t) - (1 - \sigma) f_1(x^d, y^d)s^*] - t [(1 - \sigma) f_2(x^d, y^d) + y^d]]$$
$$= w(t^*) - t^* [(1 - \sigma) f_2(x^d, y^d) + y^d] - (1 - \sigma) f_1(x^d, y^d)s^*$$
$$= w^* ((1 - \sigma) f_2(x^d, y^d) + y^d) - (1 - \sigma) f_1(x^d, y^d)s^*$$

then : $w(t^*) - t^* [(1 - \sigma) f_2(x^d, y^d) + y^d] = w^* ((1 - \sigma) f_2(x^d, y^d) + y^d)$ for V.

Thus, we can write :

$$\begin{cases} 
z(s^*) - s^* [\sigma f_1(x^d, y^d) + x^d] = z^* (\sigma f_1(x^d, y^d) + x^d) \\
w(t^*) - t^* [(1 - \sigma) f_2(x^d, y^d) + y^d] = w^* ((1 - \sigma) f_2(x^d, y^d) + y^d)
\end{cases}$$

Given legal standards of due care, $x^d$ and $y^d$, let us set:

$$\begin{cases} 
a(x) = z^*(x) - [f_1(x^d, y^d) - x + x^d] \sigma g(x) \\
b(x) = w^*(x) - [f_2(x^d, y^d) - y + y^d] \sigma h(x)
\end{cases}$$
Let boundary. In order to establish uniqueness, consider the following lemma:

Then let us note that for a given \( x \) the strictly concave function \( s \to [z(s) - xs] \) is maximal for \( s > 0 \) if:

\[
  z_s(s) - x = 0 \iff s = g(x) \quad \text{(cf. Lemma 1)}
\]

Now we can evaluate social welfare at the equilibrium levels of activity. If \( s^* \neq 0 \) and \( t^* \neq 0 \), then \( s^* = g(\sigma f_1(x^d, y^d) + x^d) \) and \( t^* = h((1 - \sigma) f_2(x^d, y^d) + y^d) \) where \( g \) and \( h \) are defined in lemma 1 and:

\[
  S(\sigma) = z(s^*) + w(t^*) - s^* f_1(x^d, y^d) - t^* f_2(x^d, y^d) - s^* x^d - t^* y^d
\]

\[
  = \left[ z(s^*) - s^* \left[ \sigma f_1(x^d, y^d) + x^d \right] \right] + \left[ w(t^*) - t^* \left[ (1 - \sigma) f_2(x^d, y^d) + y^d \right] \right]
\]

\[
  \quad - \sigma f_2(x^d, y^d) t^* - (1 - \sigma) f_1(x^d, y^d) s^*
\]

\[
  = z^*(\sigma f_1(x^d, y^d) + x^d) - (1 - \sigma) f_1(x^d, y^d) g(\sigma f_1(x^d, y^d) + x^d)
\]

\[
  + w^*((1 - \sigma) f_2(x^d, y^d) + y^d) - \sigma f_2(x^d, y^d) h((1 - \sigma) f_2(x^d, y^d) + y^d))
\]

\[
  = A(\sigma) + B(1 - \sigma)
\]

**Proof of Proposition 3.** For each \( (x^d, y^d) \), we need to show the existence and the uniqueness of \( \sigma \), which solves the following maximization problem:

\[
  \max_{\sigma \in [0,1]} S(\sigma) = \sup_{\sigma \in [0,1]} \{ A(\sigma) + B(1 - \sigma) \} = (A \square B) (1)
\]

The existence of \( \sigma \) is evident by noticing that \( S(\sigma) = A(\sigma) + B(1 - \sigma) \) is a continuous function on \([0,1]\) for which the maximum is not reached at the boundary. In order to establish uniqueness, consider the following lemma:

**Lemma 2:** Let \( z^* \) and \( w^* \) be the conjugates of the concave functions \( z \) and \( w \):

\[
  z^*(x) = \max_{s \geq 0} [z(s) - xs] = z(g(x)) - xg(x)
\]

\[
  w^*(y) = \max_{t \geq 0} [w(t) - yt] = w(h(y)) - yh(y)
\]

then \( z^* \) and \( w^* \) are convex functions on \( R \) and:

\[
  \left\{ \frac{d(z^*)}{dx}(x) = -g(x) \quad \frac{d^2(z^*)}{dx^2}(x) = \frac{-1}{z'(g(x))} \right\}
\]

\[
  \text{et} \left\{ \frac{d(w^*)}{dy}(y) = -h(y) \quad \frac{d^2(w^*)}{dy^2}(y) = \frac{-1}{w'(h(y))} \right\}
\]

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Proof of Lemma 2. Since 
\[ z^*(x) = \max_{s \geq 0} [z(s) - xs] = z(g(x)) - xg(x), \]
we have 
\[ z'(g(x)) = x, \]
but 
\[ z''(x) \neq 0 \]
and then the function \( g \) is differentiable on \( \mathbb{R} \):

\[ \forall x \in \mathbb{R} : g'(x) = \frac{1}{z''[g(x)]}. \]

Furthermore, we have:

\[
\frac{dz^*}{dx}(x) = g'(x)z'(g(x)) - g(x) - xg'(x)
\]

\[ = g'(x)x - g(x) - xg'(x) \quad \text{ (par définition de } g) \]

\[ = -g(x) \]

and thus:

\[ \frac{d^2(z^*)}{dx^2}(x) = -g'(x) = \frac{-1}{z''(g(x))} > 0 \]

Therefore, \( z^* \) is a convex function on \( \mathbb{R} \). ■

Now we can be back to the proof of Proposition 3. The condition \( A(\sigma) + B(1 - \sigma) = (A \boxplus B)(1) \) applies when \( \sigma \) is an internal critical point:

\[ A'(\sigma) - B'(1 - \sigma) = 0 \]

That is:

\[ f_1(x^d, y^d)a'(\sigma f_1(x^d, y^d) + x^d) + (1 - \sigma)b'((1 - \sigma)f_2(x^d, y^d) + y^d) = 0 \]

with

\[
a'(x) = z^*(x) - \left( (f_1(x^d, y^d) - x + x^d) g(x) \right)'
\]

\[ = -g(x) + g(x) - (f_1(x^d, y^d) - x + x^d) g'(x) \]

\[ = \frac{-1}{z''(g(x))} \left( f_1(x^d, y^d) - x + x^d \right) \]

and

\[
a'(\sigma f_1(x^d, y^d) + x^d) = \frac{-1}{z''(g(\sigma f_1(x^d, y^d) + x^d))} \left( f_1(x^d, y^d) - (\sigma f_1(x^d, y^d) + x^d) + x^d \right)
\]

\[ = \frac{-f_1(x^d, y^d)}{z''(s^*)} (1 - \sigma) \]

Likewise:

\[ b'((1 - \sigma)f_2(x^d, y^d) + y^d) = -\frac{f_2(x^d, y^d)}{w''(t^*)} \sigma \]

So that:

\[ -\frac{f_2^2(x^d, y^d)}{z''(s^*)} (1 - \sigma) - \frac{f_2^2(x^d, y^d)}{w''(t^*)} \sigma = 0 \]
Given that \( s^* = g(\sigma f_1(x^d, y^d) + x^d) \) is a decreasing function of \( \sigma \) and given the assumptions on \( z \), we have that \( \frac{-1}{z(s^*)} \) is an increasing function of \( \sigma \).

Likewise, \( t^* \) is an increasing function of \( \sigma \) and, hence, \( \frac{-1}{w(t^*)} \) is a decreasing function of \( \sigma \).

Since \( \frac{-f_1^2(x^d, y^d)}{s(z(s^*))}(1 - \sigma) - \frac{f_2^2(x^d, y^d)}{w(t^*)}\sigma \) is a monotonic function of \( \sigma \), it has a unique solution on \([0, 1]\). ■