Estimation methods for statistical process control

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Preface

The thesis you have in front of you is the result of four years of hard work. But I could never have completed it without the help of many others. In this final phase, there are some people I would like to thank in particular.

The subject of the thesis is process control. I can't think of anyone better able to control the research process than Ronald Does, my promoter. Ronald, I want to thank you for motivating me and accelerating the process at the right moments. I am sure that without you this process would not have led to so many concrete results.

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Alongside hard work, it is good to relax. Here, I would like to thank my friends and family, especially my parents Hans and Annemiek as well as Ilse and Niels, for always being there and always showing interest in my work.

Finally, I would like to thank Auke. Auke, you really helped me in all aspects of this thesis, not least with the technical Latex part, but mainly with your love and all the pleasant moments we shared in the past four years. I look forward to the future together filled with many more challenges.

Marit Schoonhoven
November 2011

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Chapter 1

Introduction

This chapter gives a short introduction to Shewhart control charts, an overview of new developments and an outline of the thesis.

1.1 Shewhart control charts

Processes are subject to variation. Whether or not a given process is functioning normally can be evaluated with control charts. Such charts show whether the variation is entirely due to common causes or whether some of the variation is due to special causes. Variation due to common causes is inevitable: it is generated by the design and standard operations of the process. When the process variation is due to common causes only, the process is said to be in statistical control. In this case, the process fluctuates within a predictable bandwidth. Special causes of process variation may consist of such factors as extraordinary events, unexpected incidents, or a new supplier for incoming material. For optimal process performance, such special causes should be detected as soon as possible and prevented from occurring again. Control charts are used to signal the occurrence of a special cause. The power of the control chart lies partly in its simplicity: it consists of a graph of a process characteristic plotted through time. The control limits in the graph provide easy checks on the stability of the process (i.e. no special causes present). The concept of control charts originates with Shewhart (1931) and has been extensively discussed and extended in numerous textbooks (see e.g. Duncan (1986), Does et al. (1999) and Montgomery (2009)).

In the standard situation, 20-30 samples of about five units are taken initially to construct a control chart. When a process characteristic is a numerical variable, it is standard practice to control both the mean value of the characteristic and its spread. The control limits of the statistic of interest are calculated as the average of the sample mean or standard deviation plus or minus a multiplier times the standard deviation of the statistic. The spread parameter of the process is controlled first, followed by the location parameter. An example of such a combined standard deviation and location chart is given in Figure 1.1.

The general set-up of a Shewhart control chart for the dispersion parameter is as follows. Let Y_{ij}, $i = 1, 2, 3, ...$ and $j = 1, 2, ..., n$, denote samples of size n taken in sequence of the process variable to be monitored. We assume the Y_{ij}'s to be independent and $N(\mu, (\lambda\sigma)^2)$ distributed, where λ is a constant. When $\lambda = 1$, the standard deviation of the process is in control; otherwise the standard deviation has changed. Let $\hat{\sigma}_i$ be an estimate of $\lambda\sigma$ based on the i-th sample Y_{ij}, $j = 1, 2, ..., n$. Usually, $\lambda\sigma$ is estimated by the sample standard deviation S. When the in-control σ is known, the process standard deviation can be monitored by plotting $\hat{\sigma}_i$ on a standard deviation control chart with respective upper and lower control limits

$$UCL = U_n\sigma, \; LCL = L_n\sigma, \tag{1.1}$$

where U_n and L_n are factors such that for a chosen type I error probability α we have

$$P(L_n\sigma \leq \hat{\sigma}_i \leq U_n\sigma) = 1 - \alpha.$$

When $\hat{\sigma}_i$ falls within the control limits, the spread is deemed to be in control.

For the location control chart, the Y_{ij}'s, $i = 1, 2, 3, ...$ and $j = 1, 2, .., n$, again denote samples of the process variable to be monitored. In this case, we assume the Y_{ij}'s to be independent and $N(\mu + \delta\sigma, \sigma^2)$ distributed, where δ is a constant. When $\delta = 0$, the mean of the process is in control; otherwise the process mean has changed. Let $\bar{Y}_i = \frac{1}{n}\sum_{j=1}^{n} Y_{ij}$ be an estimate of $\mu + \delta\sigma$ based on the i-th sample Y_{ij}, $j = 1, 2, ..., n$. When the in-control μ and σ are known, the process mean can be monitored by plotting $\bar{Y}_i$ on a location control chart with respective upper and lower control limits

$$UCL = \mu + C_n\sigma/\sqrt{n}, \; LCL = \mu - C_n\sigma/\sqrt{n}, \tag{1.2}$$

where C_n is the factor such that for a chosen type I error probability α we have

$$P(LCL \leq \bar{Y}_i \leq UCL) = 1 - \alpha.$$

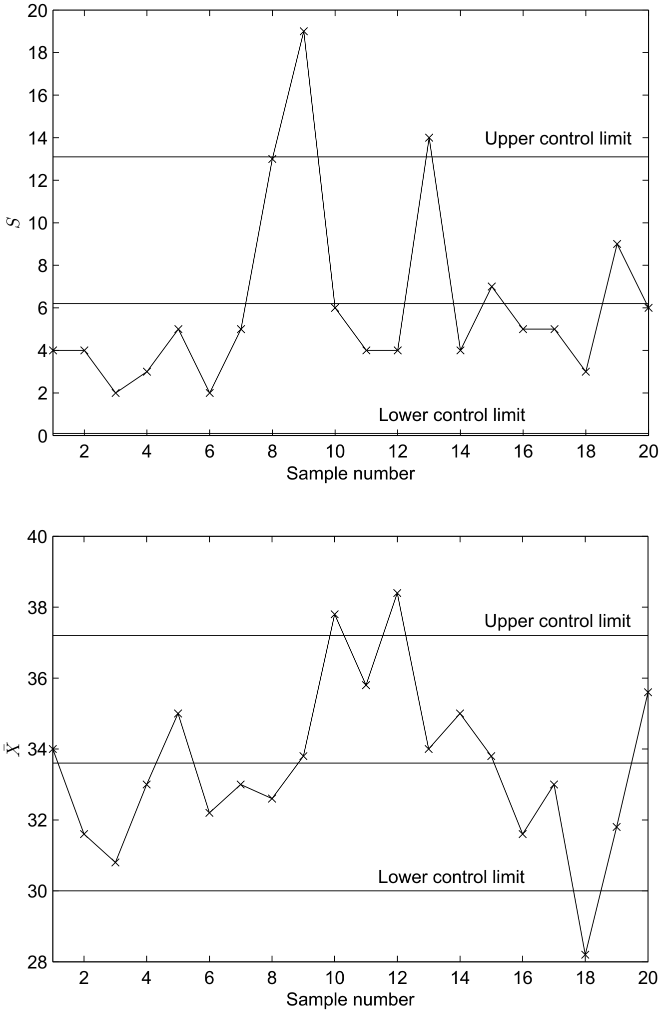


Figure 1.1: Standard deviation and location control chart

When $\bar{Y}_i$ falls within the control limits, the location of the process is deemed to be in control.

The performance of the spread control chart is evaluated in the same way as that of the location control chart. We define E_i as the event that $\hat{\sigma}_i$ ($\bar{Y}_i$) falls beyond the control limits, $P(E_i)$ as the probability that $\hat{\sigma}_i$ ($\bar{Y}_i$) falls beyond the limits and RL as the run length, i.e. the number of samples drawn until the first $\hat{\sigma}_i$ ($\bar{Y}_i$) falls beyond the limits. When σ (μ, σ) is known, the events E_i are independent, and therefore RL is geometrically distributed with parameter $p = P(E_i) = \alpha$. It follows that the average run length (ARL) is given by $1/p$ and that the standard deviation of the run length (SDRL) is given by $\sqrt{1-p}/p$.

In practice, the in-control process parameters are usually unknown. Therefore, they must be estimated from k samples of size n taken when the process is assumed to be in control. This stage in the control charting process is called Phase I (cf. Woodall and Montgomery (1999) and Vining (2009)). The monitoring stage is denoted by Phase II. The samples used to estimate the process parameters are denoted by X_{ij}, $i = 1, 2, ..., k$ and $j = 1, 2, ..., n$. Define $\hat{\sigma}$ and $\hat{\mu}$ as the unbiased estimates of σ and μ respectively, based on the X_{ij}. The control limits are estimated by

$$\widehat{UCL} = U_n\hat{\sigma}, \ \widehat{LCL} = L_n\hat{\sigma} \tag{1.3}$$

for the standard deviation control chart and

$$\widehat{UCL} = \hat{\mu} + C_n\hat{\sigma}/\sqrt{n}, \ \widehat{LCL} = \hat{\mu} - C_n\hat{\sigma}/\sqrt{n} \tag{1.4}$$

for the location control chart.

Note that U_n, L_n and C_n in (1.3) and (1.4) are not necessarily the same as in (1.1) and (1.2) and might be different even when the probability of signaling is the same. Below, we describe how we evaluate the standard deviation control chart with estimated parameters. The location chart with estimated parameters is evaluated in the same way. Let F_i denote the event that $\hat{\sigma}_i$ is above $\widehat{UCL}$ or below $\widehat{LCL}$. We define $P(F_i|\hat{\sigma})$ as the probability that sample i generates a signal given $\hat{\sigma}$, i.e.

$$P(F_i|\hat{\sigma}) = P(\hat{\sigma}_i < \widehat{LCL} \text{ or } \hat{\sigma}_i > \widehat{UCL}|\hat{\sigma}). \tag{1.5}$$

Given $\hat{\sigma}$, the distribution of the run length is geometric with parameter $P(F_i|\hat{\sigma})$. Consequently, the conditional ARL is given by

$$E(RL|\hat{\sigma}) = \frac{1}{P(F_i|\hat{\sigma})}. \tag{1.6}$$

In contrast with the conditional RL distribution, the unconditional RL distribution takes into account the random variability introduced into the charting procedure through parameter estimation. It can be obtained by averaging the conditional RL distribution over all possible values of the parameter estimates. The unconditional p is

$$p = E(P(F_i|\hat{\sigma})), \tag{1.7}$$

the unconditional average run length is

$$ARL = E(\frac{1}{P(F_i|\hat{\sigma})}) \tag{1.8}$$

and the unconditional standard deviation of the run length is determined by

$$\begin{aligned} SDRL &= \sqrt{Var(RL)} \\ &= \sqrt{E(Var(RL|\hat{\sigma})) + Var(E(RL|\hat{\sigma}))} \\ &= \sqrt{2E(\tfrac{1}{p(F_i|\hat{\sigma})})^2 - (E\tfrac{1}{p(F_i|\hat{\sigma})})^2 - E\tfrac{1}{p(F_i|\hat{\sigma})}}. \end{aligned} \tag{1.9}$$

Quesenberry (1993) showed that for the $\overline{X}$ and X control charts the unconditional ARL is higher than in the (μ,σ)-known case. Furthermore, a higher in-control ARL is not necessarily better because the RL distribution will reflect an increased number of short RL's as well as an increased number of long RL's. He concluded that, if limits are to behave like known limits, the number of samples (k) in Phase I should be at least $400/(n\text{-}1)$ for $\overline{X}$ control charts and 300 for X control charts. Chen (1998) studied the unconditional RL distribution of the standard deviation control chart under normality. He showed that if the shift in the standard deviation in Phase II is large, the impact of parameter estimation is small. In order to achieve a performance comparable with known limits, he recommended taking at least 30 samples of size 5 and updating the limits when more samples become available. For permanent limits, at least 75 samples of size 5 should be used. Thus, the situation is somewhat better than for the $\overline{X}$ control chart with both process mean and standard deviation estimated.

1.2 Contributions and thesis outline

Jensen et al. (2006) conducted a literature survey of the effects of parameter estimation on control chart properties and identified the following issue for future research: *"The effect of using robust or other alternative estimators has not been studied thoroughly. Most evaluations of performance have considered standard estimators based on the sample mean and the standard deviation and have used the same estimators for both Phase I and Phase II. However, in Phase I applications it seems more appropriate to use an estimator that will be robust to outliers, step changes and other data anomalies. Examples of robust estimation methods in Phase I control charts include Rocke (1989), Rocke (1992), Tatum (1997), Vargas (2003) and Davis and Adams (2005). The effect of using these robust estimators on Phase II performance is not clear, but it is likely to be inferior to the use of standard estimates because robust estimators are generally not as efficient"* (Jensen et al. 2006, p. 360). This recommendation is the main subject of the thesis. In particular, we will study alternative estimators in Phase I and we will study the impact of these estimators on the performance of the Phase II control chart.

Chen (1998) studied the standard deviation control chart when σ is estimated by the pooled sample standard deviation ($\tilde{S}$), the mean sample standard deviation ($\bar{S}$) or the mean sample range ($\bar{R}$) under normality. He showed that the performance of the charts based on $\tilde{S}$ and $\bar{S}$ is almost identical, while the performance of the chart based on $\bar{R}$ is slightly worse. Rocke (1989) proposed robust control charts based on the 25% trimmed mean of the sample ranges, the median of the sample ranges and the mean of the sample interquartile ranges in contaminated Phase I situations. Moreover, he studied the use of a two-stage procedure whereby the initial chart is constructed first and then subgroups that seem to be out of control are excluded. Rocke (1992) gave the practical details for the construction of these charts. Wu et al. (2002) considered three alternative statistics for the sample standard deviation, namely the median of the absolute deviation from the median (MDM), the average absolute deviation from the median (ADM) and the median of the average absolute deviation (MAD), and investigated their effect on $\overline{X}$ control chart performance. They concluded that, if there are no or only a few contaminations in the Phase I data, ADM performs best. Otherwise, MDM is the best estimator. Riaz and Saghir (2007 and 2009) showed that the statistics for the sample standard deviation based on Gini's mean difference and the ADM are robust against non-normality. However, they only considered the situation where a large number of samples is available in Phase I and did not consider contaminations in Phase I. Tatum (1997) clearly distinguished two types of disturbances: diffuse and localized. Diffuse disturbances are outliers that are spread over multiple samples whereas localized disturbances affect all observations in a single sample. He proposed a method, constructed around a variant of the biweight A estimator, that is resistant to both diffuse and localized disturbances. A result of the inclusion of the biweight A estimator is, however, that the method is relatively complicated in its use. Apart from several range-based methods, Tatum did not compare his method with other methods for Phase I estimation. Finally, Boyles (1997) studied the dynamic linear model estimator for individuals charts (see also Braun and Park (2008)).

In Chapter 2 we compare an extensive number of Phase I estimators that have been presented in the literature and a number of variants of these statistics. We study their effect on the Phase II performance of the standard deviation control chart. The estimators considered are $\tilde{S}$, $\bar{S}$, the 25% trimmed mean of the sample standard deviations, the mean of the sample standard deviations after trimming the observations in each sample, $\bar{R}$, the sample interquartile range, Gini's mean difference, the MDM, the ADM, the MAD, and the robust estimator of Tatum (1997). Moreover, we propose a robust estimation method based on the mean absolute deviation from the median supplemented with a simple screening method. The performance of the estimators is evaluated by assessing the mean squared error (MSE) of the estimators under normality and in the presence of various types of contaminations. Finally, we assess the Phase II performance of the control charts by means of a simulation study.

Most of the standard deviation estimators presented in Chapter 2 are robust against *either* diffuse disturbances, i.e. outliers spread over the samples, *or* localized disturbances, which affect an entire sample. In Chapter 3 we therefore propose an algorithm that is robust against *both* types of disturbances. The method is compared with the pooled standard deviation (because this estimator is most efficient under normality), the robust estimator of Tatum (1997) and several adaptive trimmers. The performance of the estimators is evaluated by assessing the MSE of the estimators in several situations. Furthermore, we derive factors for the Phase II limits of the standard deviation control chart and assess the performance of the Phase II control charts by means of a simulation study.

As noted earlier, the dispersion parameter of the process is controlled first, followed by the location parameter.

So far the literature has proposed several alternative robust location estimators. Rocke (1989) proposed the 25% trimmed mean of the sample means, the median of the sample means and the mean of the sample medians. Rocke (1992) followed with the practical details for the construction of the corresponding charts. Alloway and Raghavachari (1991) constructed a control chart based on the Hodges-Lehmann estimator. Tukey (1997) and Wang et al. (2007) developed the trimean estimator, which is defined as the weighted average of the median and the two other quartiles. Finally, Jones-Farmer et al. (2009) proposed a rank-based Phase I location control chart. Based on this control chart, they define the in-control state of a process and identify an in-control reference dataset to estimate the location parameter.

In Chapter 4 we consider several robust location estimators as well as several estimation methods based on a Phase I analysis, whereby a control chart is used to study a historical dataset retrospectively and thus identify disturbances. In addition, we propose a new type of Phase I analysis. The methods are evaluated in terms of their MSE and their effect on $\overline{X}$ Phase II control chart performance. We consider situations where the Phase I data are uncontaminated and normally distributed, as well as various types of contaminated Phase I situations.

The results of Chapter 4 indicate that the $\overline{X}$ Phase II control chart (with σ known) based on the new estimation method performs well under normality and outperforms the other charts when contaminations are present in Phase I. However, the results indicate that the effect of estimating the process location on the performance of the $\overline{X}$ Phase II control chart is more limited than the effect of the standard deviation estimator. Chapter 5 therefore looks at the effect of alternative standard deviation estimators under various Phase I scenarios.

In Chapter 5 we develop an estimation method to derive the standard deviation for the $\overline{X}$ control chart when both μ and σ are unknown. Apart from the new method, several alternative estimation methods are included in the comparison. The methods are evaluated in terms of their MSE and their effect on $\overline{X}$ Phase II control chart performance. We again consider the situation where the Phase I data are uncontaminated and normally distributed, as well as various types of contaminated Phase I situations.

The material presented in Chapters 2-5 has led to four papers in various stages of publication. The analysis in Chapter 2 has been published in the *Journal of Quality Technology* (Schoonhoven et al. (2011b)). A follow-up paper based on Chapter 3 has been accepted for publication in *Technometrics*, with minor revisions (Schoonhoven and Does (2011a)). The work in Chapter 4 has been published in the *Journal of Quality Technology* (Schoonhoven et al. (2011a)) and a follow-up article based on the material in Chapter 5 has been submitted to the *Journal of Quality Technology* (Schoonhoven and Does (2011b)).

Chapter 2

Standard Deviation Control Charts

2.1 Introduction

This chapter concerns the design and analysis of the standard deviation control chart with estimated limits. We consider an extensive range of statistics to estimate the in-control standard deviation (Phase I) and design the control chart for real-time process monitoring (Phase II) by determining the factors for the control limits. The Phase II performance of the design schemes is assessed when the Phase I data are uncontaminated and normally distributed as well as when the Phase I data are contaminated. We propose a robust estimation method based on the mean absolute deviation from the median supplemented with a simple screening method.

The chapter is structured as follows. The next section introduces various estimators of the standard deviation and assesses the MSE of the estimators. We then derive the Phase II control limits. Next, we describe the simulation procedure and simulation results. Furthermore, we discuss a real-world example implementing the various charts created. The chapter ends with some concluding remarks.

2.2 Proposed Phase I estimators

In practice the same statistic is generally used to estimate both the in-control standard deviation σ in Phase I and the standard deviation $\lambda\sigma$ in Phase II. Since the requirements for the estimators differ between the two phases, this is not always the best choice. In Phase I, an estimator should be efficient in uncontaminated situations and robust against disturbances, whereas in

Phase II the estimator should be sensitive to disturbances (cf. Jensen et al. (2006)). In the next two sections, we present the Phase I estimators considered in our study (Section 2.2.1) and evaluate the estimators by comparing their MSE (Section 2.2.2).

2.2.1 Standard deviation estimators

David (1998) gave a brief account of the history of standard deviation estimators. The traditional estimators are of course the pooled and the mean sample standard deviation and the mean sample range. Mahmoud et al. (2010) studied the relative efficiencies of these estimators for different sample sizes n and numbers of samples k. In deriving estimates of the in-control standard deviation, we will look at these as well as nine other estimators.

The first estimator of σ is based on the pooled sample standard deviation

$$\tilde{S} = \left(\frac{1}{k} \sum_{i=1}^{k} S_i^2 \right)^{1/2}, \tag{2.1}$$

where S_i is the i-th sample standard deviation defined by

$$S_i = \left(\frac{1}{n-1} \sum_{j=1}^{n} (X_{ij} - \bar{X}_i)^2 \right)^{1/2}.$$

An unbiased estimator is given by $\tilde{S}/c_4(k(n-1)+1)$, where $c_4(m)$ is defined by

$$c_4(m) = \left(\frac{2}{m-1} \right)^{1/2} \frac{\Gamma(m/2)}{\Gamma((m-1)/2)}.$$

The second estimator is based on the mean sample standard deviation

$$\bar{S} = \frac{1}{k} \sum_{i=1}^{k} S_i. \tag{2.2}$$

An unbiased estimator of σ is given by $\bar{S}/c_4(n)$.

Rocke (1989) proposed the trimmed mean of the sample ranges. In our study, we consider a variant of this estimator, namely the trimmed mean of the sample standard deviations because it is well known that the sample standard deviation is more robust than the sample range. The trimmed mean of the sample standard deviations is given by

$$\bar{S}_a = \frac{1}{k - \lceil ka \rceil} \times \left[\sum_{v=1}^{k-\lceil ka \rceil} \bar{S}_{(v)} \right], \tag{2.3}$$

where a denotes the percentage of samples to be trimmed, $\lceil z \rceil$ denotes the smallest integer not less than z and $\bar{S}_{(v)}$ denotes the v-th ordered value of the sample standard deviations. We consider the 25% trimmed mean of the sample standard deviations. To simplify the analysis, we trim an integer number of samples. For example, the 25% trimmed mean trims off the eight largest sample standard deviations when $k = 30$. To provide an unbiased estimate of σ for the normal case, the estimate must be divided by a normalizing constant. These constants are obtained from 100,000 simulation runs. For $n = 5$ and $k = 20, 30, 75$, the constants are 0.579, 0.585 and 0.568 respectively; for $n = 9$ and $k = 20, 30, 75$, the constants are 0.701, 0.705 and 0.693 respectively.

Because the above estimator trims off samples instead of individual observations, we expect the estimator to be robust against localized disturbances. We also consider a variant that is expected to be robust against diffuse disturbances, namely the mean sample standard deviation after trimming the observations in each sample

$$\overline{S_b} = \frac{1}{k} \sum_{i=1}^{k} S'_i, \tag{2.4}$$

where S'_i is the standard deviation of sample i after trimming the observations, given by

$$S'_i = \left(\frac{1}{n - 2\lceil nb \rceil - 1} \sum_{v=\lceil nb \rceil+1}^{n-\lceil nb \rceil} (X_{i(v)} - \bar{X}'_i)^2 \right)^{1/2},$$

where

$$\overline{X}'_i = \frac{1}{n - 2\lceil nb \rceil} \sum_{v=\lceil nb \rceil+1}^{n-\lceil nb \rceil} X_{i(v)},$$

with $X_{i(v)}$ the v-th ordered value in sample i and b the percentage of lowest and highest observations to be trimmed in each sample. In this study, we take 20% as our trimming percentage and, again, we trim an integer number of observations. The estimator trims off the smallest and largest observation for $n = 5$; it trims off the two smallest and the two largest observations for $n = 9$. The normalizing constant is 0.520 for $n = 5$ and 0.473 for $n = 9$.

The next estimator is based on the mean sample range

$$\bar{R} = \frac{1}{k}\sum_{i=1}^{k} R_i, \tag{2.5}$$

where R_i is the range of the i-th sample. An unbiased estimator of σ is $\bar{R}/d_2(n)$, where $d_2(n)$ is the expected range of a random $N(0,1)$ sample of size n. Values of $d_2(n)$ can be found in Duncan (1986), Table M.

The next estimator is based on the mean of the sample interquartile ranges

$$\overline{IQR} = \frac{1}{k}\sum_{i=1}^{k} IQR_i, \tag{2.6}$$

with IQR_i the interquartile range of sample i

$$IQR_i = X_{i(n-\lceil ne\rceil)} - X_{i(\lceil ne\rceil+1)}.$$

Thus, the same observations are trimmed off as in the calculation of S_i'. (Note that one would expect the IQR to correspond to $e = 0.25$. However, to simplify the analysis we only trim an integer number of observations.) The normalizing constant is 0.990 for $n = 5$ and 1.144 for $n = 9$.

We also consider an estimator based on the mean of the sample Gini's mean differences

$$\bar{G} = \frac{1}{k}\sum_{i=1}^{k} G_i, \tag{2.7}$$

where G_i is Gini's mean difference of sample i, defined by

$$G_i = \sum_{j=1}^{n-1}\sum_{l=j+1}^{n} |X_{ij} - X_{il}|/(n(n-1)/2),$$

representing the mean absolute difference between any two observations in the sample. This statistic was proposed by Gini (1912), although basically the same statistic had already been proposed by Jordan (1869). An unbiased estimator of σ is given by $\bar{G}/d_2(2)$. The appendix shows that the estimator based on Gini's mean difference can be rewritten as a linear function of order statistics and that Gini's mean difference is essentially the same as the so-called Downton estimator (Downton (1966)) and the probability-weighted moments estimator (Muhammad et al. (1993)). From David (1981, p. 191) it follows that the estimator derived from Gini's mean difference is highly efficient (98%) and is more robust to outliers than the estimators based on R or S.

An estimator of σ that is simpler and easier to interpret uses the mean of the sample average absolute deviation from the median, given by

$$\overline{ADM} = \frac{1}{k}\sum_{i=1}^{k} ADM_i, \tag{2.8}$$

where ADM_i is the average absolute deviation from the median of sample i, defined as

$$ADM_i = \frac{1}{n}\sum_{j=1}^{n} |X_{ij} - M_i|,$$

with M_i the median of sample i. An unbiased estimator of σ is given by $\overline{ADM}/t_2(n)$. Since it is difficult to obtain the constant $t_2(n)$ analytically, it is obtained by simulation. Extensive tables of $t_2(n)$ can be found in Riaz and Saghir (2009).

We also study the above estimator supplemented with a screening method based on control charting. Rocke (1989) proposed a two-stage procedure which first estimates σ by $\bar{R}$, then deletes any sample that exceeds the control limits and recomputes $\bar{R}$ using the remaining samples. Our approach follows a similar procedure. First, we estimate σ by $\overline{ADM}$ because $\overline{ADM}$ is expected to be more robust against outliers. For simplicity, our screening method uses the well known factors of the $S/c_4(n)$ control chart corresponding to the 3σ control limits in Phase I. Hence, the factors for the limits are 2.089 and 0 for $n = 5$ and 1.761 and 0.239 for $n = 9$ (cf. Table M in Duncan (1986)). We then chart $S/c_4(n)$, delete any sample that exceeds the control limits and recompute $\overline{ADM}$ using the remaining samples. We continue until all sample estimates fall within the limits. The normalizing constant is 0.996 for $n = 5$ and 0.998 for $n = 9$. The resulting estimator is denoted by $\overline{ADM}'$.

Next we study two other median statistics, namely the average of the sample medians of the absolute deviation from the median

$$\overline{MDM} = \frac{1}{k}\sum_{i=1}^{k} MDM_i, \tag{2.9}$$

with

$$MDM_i = median\{|X_{ij} - M_i|\},$$

and the mean of the sample medians of the average absolute deviation

$$\overline{MAD} = \frac{1}{k}\sum_{i=1}^{k} MAD_i, \tag{2.10}$$

with

$$MAD_i = median\{|X_{ij} - \overline{X}_i|\}.$$

The normalizing constant for $\overline{MDM}$ is 0.554 for $n = 5$ and 0.613 for $n = 9$. For $\overline{MAD}$, the normalizing constant is 0.627 for $n = 5$ and 0.658 for $n = 9$.

We also evaluate a robust estimator proposed by Tatum (1997). His method has proven to be robust to both diffuse and localized disturbances. The estimation method is constructed around a variant of the biweight A estimator. The method begins by calculating the residuals in each sample, which involves substracting the sample median from each value: $res_{ij} = X_{ij} - M_i$. If n is odd, then in each sample one of the residuals will be zero and is dropped. As a result, the total number of residuals is equal to $m' = nk$ when n is even and $m' = (n-1)k$ when n is odd. Tatum's estimator is given by

$$S_c^* = \frac{m'}{(m'-1)^{1/2}} \frac{(\sum_{i=1}^{k}\sum_{j:|u_{ij}|<1} res_{ij}^2(1-u_{ij}^2)^4)^{1/2}}{|\sum_{i=1}^{k}\sum_{j:|u_{ij}|<1}(1-u_{ij}^2)(1-5u_{ij}^2)|}, \tag{2.11}$$

where $u_{ij} = h_i res_{ij}/(cM^*)$, M^* is the median of the absolute values of all residuals,

$$h_i = \begin{cases} 1 & E_i \leq 4.5, \\ E_i - 3.5 & 4.5 < E_i \leq 7.5, \\ c & E_i > 7.5, \end{cases}$$

and $E_i = IQR_i/M^*$. The constant c is a tuning constant. Each value of c leads to a different estimator. Tatum showed that $c = 7$ gives an estimator that loses some efficiency when no disturbances are present, but gains efficiency when disturbances are present. We apply this value of c in our simulation study. Note that we have $h(i) = E_i - 3.5$ for $4.5 < E_i \leq 7.5$ in the equations instead of $h(i) = E_i - 4.5$ as presented by Tatum (Tatum 1997, p. 129). This was a typographical error in the formula, resulting in too much weight on localized disturbances and thus an overestimation of σ. An unbiased estimator of σ is given by $S_c^*/d^*(c, n, k)$, where $d^*(c, n, k)$ is the normalizing constant. During the implementation of the estimator we discovered that, for odd values of n, the values of $d^*(c, n, k)$ given by Table 1 in Tatum (1997) should be corrected. We use the corrected values, which are presented in Table 2.1 below. The resulting estimator is denoted by $D7$ as in Tatum (1997).

	$c = 7$				$c = 10$			
n	$k = 20$	$k = 30$	$k = 40$	$k = 75$	$k = 20$	$k = 30$	$k = 40$	$k = 75$
5	1.070	1.069	1.068	1.068	1.054	1.053	1.053	1.052
7	1.057	1.056	1.056	1.056	1.041	1.040	1.040	1.040
9	1.052	1.051	1.050	1.050	1.034	1.034	1.033	1.033
11	1.047	1.046	1.046	1.046	1.029	1.029	1.028	1.028
13	1.044	1.044	1.043	1.043	1.026	1.025	1.025	1.025
15	1.041	1.041	1.041	1.040	1.023	1.023	1.023	1.022

Table 2.1: Normalizing constants $d^*(c, n, k)$ for Tatum's estimator (S_c^*)

The estimators considered are summarized in Table 2.2.

2.2.2 Efficiency of proposed estimators

In order to compare the relative efficiency of the proposed Phase I estimators, we assess their MSE as was done in Tatum (1997). The MSE is estimated as

$$MSE = \frac{1}{N} \sum_{i=1}^{N} (\hat{\sigma}^i - \sigma)^2,$$

where $\hat{\sigma}^i$ is the unbiased estimate of the standard deviation in the i-th simulation run (note that $\hat{\sigma}^i$ differs from $\hat{\sigma}_i$, the latter denoting the Phase II estimate) and N is the number of simulation runs. We include the uncontaminated case, i.e. the situation where all the X_{ij}'s are from the $N(0, 1)$ distribution as well as four types of disturbances (cf. Tatum (1997)):

1. A model for diffuse symmetric variance disturbances in which each observation has a 95% probability of being drawn from the $N(0, 1)$ distri-

Estimator	Notation
Pooled sample standard deviation	$\tilde{S}$
Mean of sample standard deviations	$\bar{S}$
25% trimmed mean of sample standard deviations	$\bar{S}_{25}$
Mean of sample standard deviations after trimming sample observations	$\overline{S_{20}}$
Mean of sample ranges	$\bar{R}$
Mean of sample interquartile ranges	$\overline{IQR}$
Mean of sample Gini's mean differences	$\bar{G}$
Mean of sample averages of absolute deviation from median	$\overline{ADM}$
$\overline{AMD}$ after subgroup screening	$\overline{ADM}'$
Mean of sample medians of absolute deviation from median	$\overline{MDM}$
Mean of sample medians of absolute deviation from mean	$\overline{MAD}$
Tatum's robust estimator	$D7$

Table 2.2: Proposed estimators for the standard deviation

bution and a 5% probability of being drawn from the $N(0,a)$ distribution, with $a = 1.5, 2.0, ..., 5.5, 6.0$.

2. A model for diffuse asymmetric variance disturbances in which each observation is drawn from the $N(0,1)$ distribution and has a 5% probability of having a multiple of a χ_1^2 variable added to it, with the multiplier equal to $0.5, 1.0, ..., 4.5, 5.0$.

3. A model for localized variance disturbances in which observations in 3 (when $k = 30$) or 6 (when $k = 75$) samples are drawn from the $N(0,a)$ distribution, with $a = 1.5, 2.0, ..., 5.5, 6.0$.

4. A model for diffuse mean disturbances in which each observation has a 95% probability of being drawn from the $N(0,1)$ distribution and a 5% probability of being drawn from the $N(b,1)$ distribution, with $b = 0.5, 1.0, ..., 9.0, 9.5$.

The MSE is obtained for $k = 30, 75$ subgroups of sizes $n = 5, 9$. The number of simulation runs N is equal to 50,000. (Note that Tatum (1997) used 10,000 simulation runs.)

The following results can be observed (see Figures 2.1-2.4). The y-intercepts show the MSE of the estimators when there are no contaminations. In this case, $\bar{S}_{25}$, $\overline{MDM}$, $\overline{S_{20}}$, $\overline{IQR}$ and $\overline{MAD}$ are less efficient than any of the other estimators because they use less information, while the other estimators are more or less equally efficient.

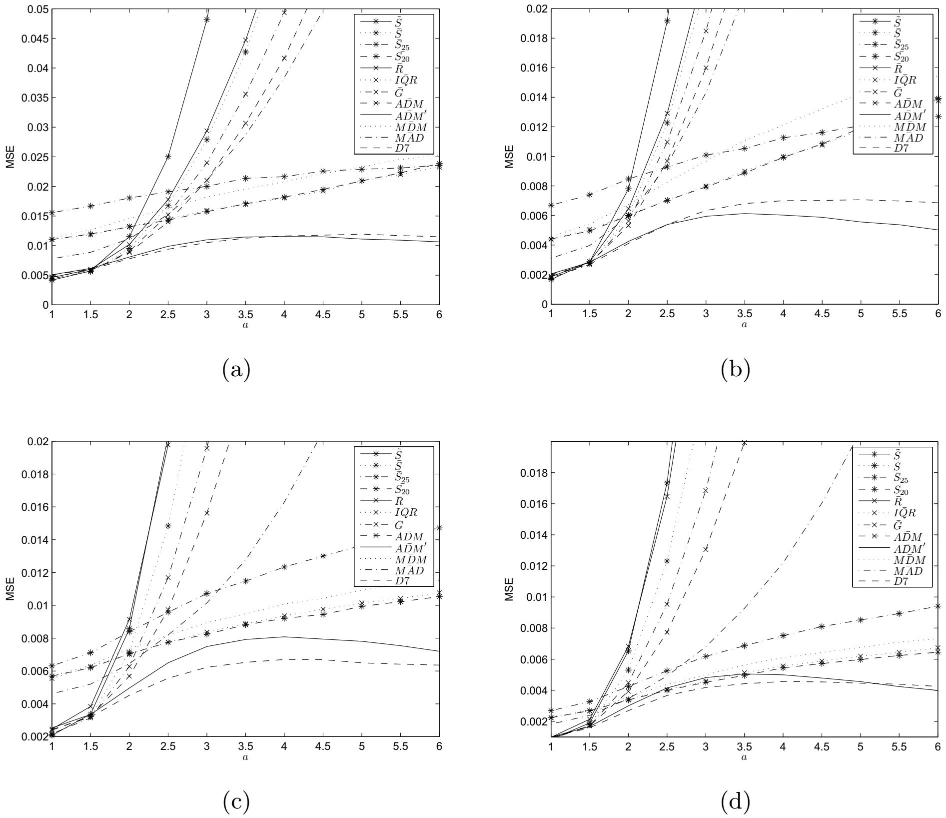
When symmetric diffuse variance disturbances are present (Figure 2.1), the best performing estimators are $D7$ and $\overline{ADM}'$. The fact that the performance of $\overline{ADM}'$ is similar to $D7$ is interesting because the former is more intuitive and the estimates are simpler to obtain. Tatum (1997) showed that the screening procedure based on the chart with σ estimated by $\bar{R}$ fails to match $D7$ in this situation. This is because R is more sensitive to outliers. Thus, using a robust statistic like $\overline{ADM}$, supplemented with subgroup screening by means of the control chart (resulting in $\overline{ADM}'$), works very well when symmetric diffuse outliers are present. The estimators $\bar{S}_{25}$, $\overline{S_{20}}$ $\overline{IQR}$ and $\overline{MDM}$ are more robust than the traditional estimators but less robust than $D7$ and $\overline{ADM}'$. Another result worth noting is that $\tilde{S}$ performs worst in this situation (comparable to the poor performance of $\bar{S}$ and $\bar{R}$). While others (e.g. Mahmoud et al. (2010)) recommend using this estimator because it is most efficient in the absence of contaminations, we see that its performance decreases most quickly when there are outliers. The estimators $\bar{G}$ and $\overline{ADM}$ are efficient when no contaminations are present and perform better than the traditional estimators ($\tilde{S}$, $\bar{S}$ and $\bar{R}$) in the case of occasional outliers. The effect is more pronounced for $n = 9$ than for $n = 5$.

When asymmetric diffuse variance disturbances are present (Figure 2.2), the same general results are found as for symmetric diffuse variance disturbances. Tatum (1997) showed that, when $n = 9$, $D7$ is superior to several other estimators, including the estimator resulting from subgroup screening based on $\bar{R}$. Our subgroup screening algorithm produces outcomes similar to Tatum's estimator. Note that, to estimate σ, we use an estimator that is less sensitive to outliers, namely $\overline{ADM}$ rather than $\bar{R}$.

In the case of localized variance disturbances (Figure 2.3), the estimator that performs best is $\overline{ADM}'$, followed by $D7$ and then by $\bar{S}_{25}$. It is interesting to see that $\overline{ADM}'$ performs substantially better than $D7$. In other words, screening based on the control charting procedure in Phase I seems more effective than using $D7$ when the data are contaminated by localized variance disturbances.

When diffuse mean disturbances are present in Phase I (Figure 2.4), $D7$ performs best, followed by $\overline{ADM}'$. The differences appear primarily for $n = 9$. When there is a possibility of this type of outlier in practice, we recommend using $D7$ or screening on the basis of an individuals chart. The latter is a subject for future research.

To summarize, the most efficient estimators are $D7$ and $\overline{ADM}'$ when there are diffuse variance disturbances; $\overline{ADM}'$ when there are localized variance disturbances; and $D7$ when there are mean shift disturbances.



(a) (b) (c) (d)

Figure 2.1: MSE of estimators when symmetric diffuse variance disturbances are present. (a) $n = 5, k = 30$ (b) $n = 5, k = 75$ (c) $n = 9, k = 30$ (d) $n = 9, k = 75$

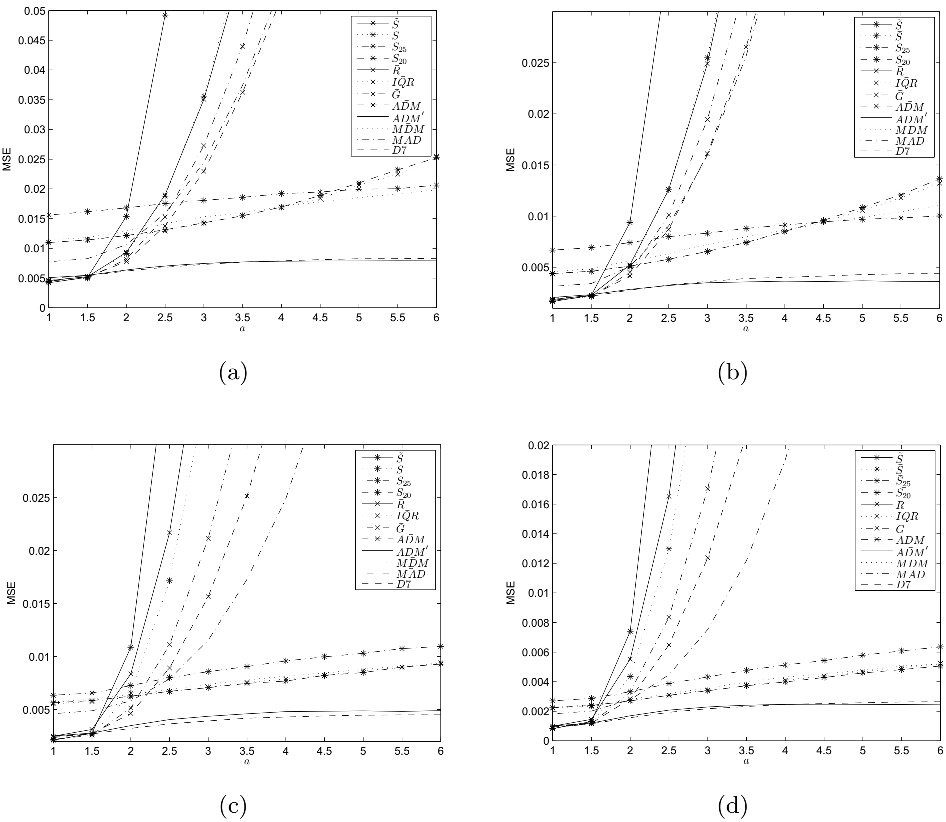
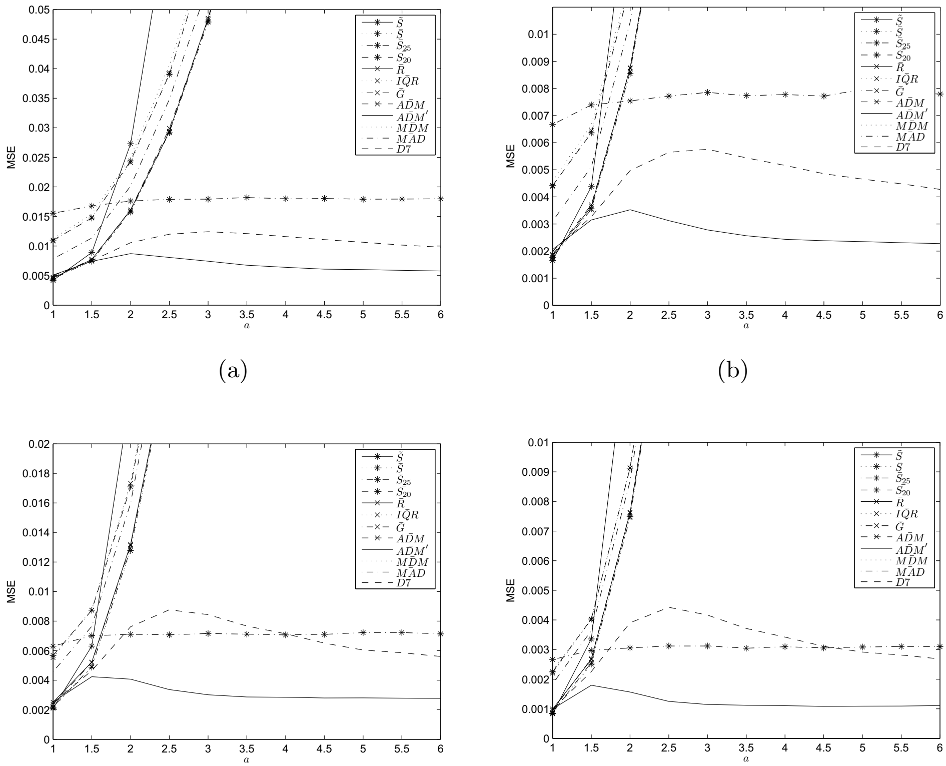
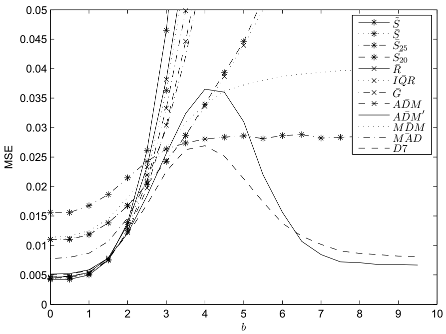


Figure 2.2: MSE of estimators when asymmetric diffuse variance disturbances are present. (a) $n = 5, k = 30$ (b) $n = 5, k = 75$ (c) $n = 9, k = 30$ (d) $n = 9, k = 75$

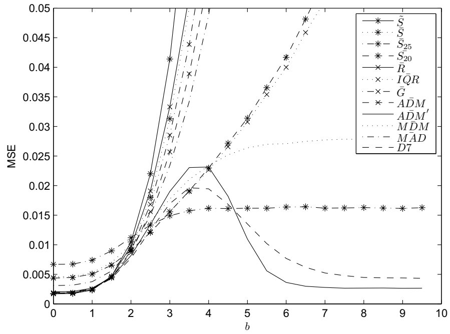


(a) (b) (c) (d)

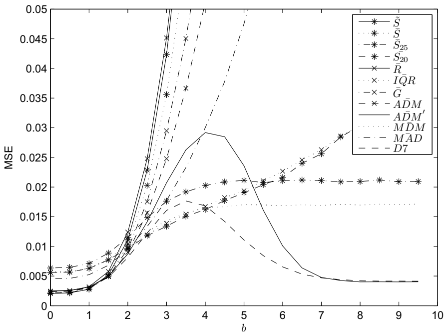
Figure 2.3: MSE of estimators when localized variance disturbances are present. (a) $n = 5, k = 30$ (b) $n = 5, k = 75$ (c) $n = 9, k = 30$ (d) $n = 9, k = 75$



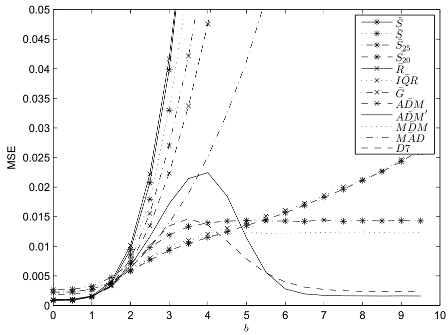
(a)



(b)



(c)



(d)

Figure 2.4: MSE of estimators when diffuse mean disturbances are present. (a) $n = 5, k = 30$ (b) $n = 5, k = 75$ (c) $n = 9, k = 30$ (d) $n = 9, k = 75$

2.3 Derivation of Phase II control limits

To control the unconditional in-control p, the design of the Phase II control charts requires a derivation of the factors U_n and L_n in (1.3). For the R chart, Hillier (1969) showed that when the limits are estimated, U_n and L_n derived for the σ-known case will not produce the desired signaling probability. To address this issue, he calculated the factors based on n, k and α in such a way that p equals α. Yang and Hillier (1970) derived correction factors for the S and $\tilde{S}$ charts. The solution suggested by Hillier (1969) is well known as a solution for short production runs. On the other hand, the ARL gives an indication of the expected run length and so is intuitively very appealing. The disadvantage of the ARL is, however, that it is determined by the occurrence of extremely long runs while in practice processes do not remain unchanged for a very long period (see also Does and Schriever (1992)). Nedumaran and Pignatiello (2001) developed an approach for constructing $\overline{X}$ control limits that attempt to match any percentile point of the run length distribution.

In this study we derive U_n and L_n so as to obtain the desired value for p. Later we will show that this issue is less important for the standard deviation control chart than for the $\overline{X}$ and X charts, because the estimation effect is less pronounced for the standard deviation control chart.

U_n and L_n depend on n, k and α. The Phase I estimators considered are the estimators presented in Table 2.2. We employ the same statistic, namely $S/c_4(n)$, as the Phase II charting statistic in each case so that any differences between the charts are entirely due to differences introduced by the Phase I estimators. We present the derivation of the factors for these charts below.

We start with the factors for the chart where $\hat{\sigma}$ is estimated by $\tilde{S}/c_4(k(n-1)+1)$. Exact results for this chart can be calculated and can also be found in Yang and Hillier (1970). We derive the factor for the upper control limit; the factor for the lower control limit can be obtained in a similar way. Note that S_i and $\tilde{S}$ are independent so the factors can be chosen as the upper and lower $\alpha/2$ quantiles of the distribution $S_i/\tilde{S}$. We can write $(S_i/\tilde{S})^2$ as $\frac{(n-1)S_i^2/\sigma^2}{k(n-1)\tilde{S}^2/\sigma^2} \cdot \frac{1/(n-1)}{1/k(n-1)}$, which is distributed as $\frac{\chi^2_{n-1}/(n-1)}{\chi^2_{k(n-1)}/(k(n-1))} = F_{n-1,k(n-1)}$, where χ^2_m denotes a chi-square distribution with m degrees of freedom and $F_{v,w}$ denotes an F distribution with v numerator degrees of freedom and w

denominator degrees of freedom. Hence

$$U_n = \frac{\sqrt{F_{n-1,k(n-1)}(1-\alpha/2)}c_4(k(n-1)+1)}{c_4(n)}. \tag{2.12}$$

For the charts based on the other Phase I estimators we use the result of Patnaik (1950). Patnaik approximated the distribution of $\bar{R}/\sigma$ by $a(n,k)\chi_{\nu(n,k)}/\sqrt{\nu_{(n,k)}}$, where $\chi_{\nu(n,k)}$ is the square root of a chi-square distribution with $\nu(n,k)$ degrees of freedom and $a(n,k)$ is a scale factor. The factors $a(n,k)$ and $\nu(n,k)$ are obtained by equating the first two moments of $\bar{R}/\sigma$ to the first two moments of $a(n,k)\chi_{\nu(n,k)}/\sqrt{\nu_{(n,k)}}$. Patnaik's approach can also be applied to approximate the distribution of $\hat{\sigma}/\sigma$, where $\hat{\sigma}$ is obtained via one of the unbiased estimators of the standard deviation in Phase I. Let $M_1 = E(\hat{\sigma}/\sigma) = 1$ and $M_2 = Var(\hat{\sigma}/\sigma)$. From Patnaik (1950) it follows that the values of $\nu(n,k)$ and $a(n,k)$ are

$$\nu(n,k) = 1/(-2+2\sqrt{1+2M_2+1/(16\nu(n,k))^3}), \tag{2.13}$$

$$a(n,k) = 1 + \frac{1}{4\nu(n,k)} + \frac{1}{32\nu^2(n,k)} - \frac{5}{128\nu^3(n,k)}. \tag{2.14}$$

Since $\frac{(S_i/\sigma)^2}{(c_4(n)\hat{\sigma}/\sigma)^2}$ is distributed as

$\frac{\chi^2_{n-1}/(n-1)}{c_4^2(n)a^2(n,k)\chi^2_{\nu(n,k)}/\nu(n,k)} = F_{n-1,\nu(n,k)}/(c_4^2(n)a^2(n,k))$, it follows that

$$U_n = \sqrt{F_{(n-1),\nu(n,k)}(1-\alpha/2)}/(c_4(n)a(n,k)). \tag{2.15}$$

Table 2.3 summarizes U_n and L_n for the control charts with $k = 20, 30, 75$ subgroups of sizes $n = 5, 9$ and $\alpha = 0.0027$. For other situations, values of M_2 can be derived by simulating $\hat{\sigma}/\sigma$. Then the constants $\nu(n,k)$ and $a(n,k)$ can be readily obtained from (2.13) and (2.14).

To judge the quality of the proposed corrections, we evaluate the unconditional probabilities of a false signal (p) in Phase II. The probabilities, presented in Table 2.4, are assessed using 50,000 simulation runs. This is enough to obtain a sufficiently small relative standard error.

2.4 Control chart performance

In this section we evaluate the performance of the design schemes presented above. The schemes are set up in the uncontaminated normal situation and several contaminated situations. We consider models similar to those used to assess the MSE with a, b and the multiplier equal to 4 to simulate the contaminated cases (see Section 2.2.2).

The performance of the design schemes is assessed in terms of the unconditional p and ARL as well as the conditional ARL associated with the 2.5% and 97.5% quantiles of the distribution of $\hat{\sigma}$. We consider different shifts in the standard deviation $\lambda\sigma$ in Phase II, namely λ equal to 0.5, 1, 1.5 and 2. The performance characteristics are obtained by simulation. The next section describes the simulation method, followed by the results for the control charts constructed in the uncontaminated situation and various contaminated situations.

2.4.1 Simulation procedure

The performance characteristics p and ARL for estimated control limits are determined by averaging the conditional characteristics, i.e. the characteristics for a given set of estimated control limits, over all possible values of the control limits. Recall the definitions of $p(F_i|\hat{\sigma})$ from (1.5), $E(RL|\hat{\sigma})$ from (1.6), $p = E(p(F_i|\hat{\sigma}))$ from (1.7) and $ARL = E(\frac{1}{p(F_i|\hat{\sigma})})$ from (1.8). These expectations are obtained by simulation: 50,000 datasets are generated and for each dataset $p(F_i|\hat{\sigma})$ and $E(RL|\hat{\sigma})$ are computed. By averaging these values we obtain the unconditional values.

2.4.2 Simulation results

First we consider the situation where the process follows a normal distribution and the Phase I data are not contaminated. We investigate the impact of the estimator used to estimate σ in Phase I. Tables 2.5 and 2.6 present the unconditional probability of one sample generating a signal (p), the unconditional average run length (ARL) and the upper and lower conditional ARL values corresponding to the upper and lower 0.025 quantiles of the distribution of $\hat{\sigma}$. When $\lambda = 1$, the process is in control, so we want p to be as low as possible and ARL to be as high as possible. When $\lambda \neq 1$, i.e. in the out-of-control situation, we want to achieve the opposite. The tables show that, when the limits are estimated, the in-control ARL is higher than the desired 370 (the control limits are chosen to provide an unconditional p of 0.0027), the value which is achieved when the limits are known. Note that the increase in the unconditional ARL due to the estimation process is not as large as for the $\overline{X}$ control chart. The reason is that for the $\overline{X}$ control chart the run length distribution is very right-skewed, generating a very large unconditional ARL. This seems to be less the case for standard deviation control charts.

We also study the conditional ARL values (or, equivalently, the conditional p values, since the conditional RL distribution is simply geometric with parameter equal to the conditional p). The first value in parentheses represents the ARL for the 2.5% quantile of the distribution of $\hat{\sigma}$, while the second value represents the ARL for the 97.5% quantile of the distribution of $\hat{\sigma}$. The results show that the conditional ARL values vary quite strongly, even when k equals 75. When λ equals 0.5, we see that a lower value of $\hat{\sigma}$ gives a higher ARL and vice versa. The reason is that a smaller value of $\hat{\sigma}$ in Phase I results in a lower value for the lower control limit and hence a lower probability of detecting a decrease in the standard deviation in Phase II. In the normal uncontaminated situation, we observe a nice pattern for all the estimators: the upper and lower conditional ARL values in the in-control situation are higher than in the out-of-control situation. However, this is not always the case when there are Phase I contaminations (Tables 2.7-2.14). Confining ourselves to the conditional ARL values in the contaminated case, we judge the upper and lower conditional ARL values to be satisfactory, provided that they do not change too much from the values observed in the uncontaminated normal case.

When we compare the estimators in the uncontaminated Phase I situation (Tables 2.5 and 2.6), $\tilde{S}$, $\overline{S}$, $\overline{R}$, $\overline{G}$, $\overline{ADM}$, $\overline{ADM}'$ and $D7$ produce very similar outcomes. The estimators $\overline{S}_{25}$, $\overline{S_{20}}$, $\overline{IQR}$, $\overline{MDM}$ and $\overline{MAD}$ are less powerful under normality.

The performance of the charts in the case of contaminated data is tabulated in Tables 2.7-2.14. The same general results are found as for the MSE comparisons. The most important points are as follows.

The chart based on $\tilde{S}$ is most powerful under normality; however its performance decreases most quickly when diffuse or localized disturbances occur. In light of this risk, we do not recommend using $\tilde{S}$.

The charts based on the estimators $\overline{S_{20}}$, $\overline{IQR}$ and $\overline{MDM}$ perform relatively well in response to diffuse disturbances but not very well when there are no contaminations.

Furthermore, the charts based on the estimators $\overline{G}$ and $\overline{ADM}$ are efficient under normality and are more efficient than the traditional charts based on $\tilde{S}$, $\overline{S}$ and $\overline{R}$ when diffuse outliers are present.

Finally, the charts based on the estimators $\overline{ADM}'$ and $D7$ perform equally well as the traditional charts in the uncontaminated case and substantially better than any of the other charts in contaminated situations. When mean diffuse disturbances are likely to occur in Phase I, we recommend using $D7$ because the control chart based on this estimator is more robust against such disturbances. When localized disturbances are likely to occur, we recommend using $\overline{ADM}'$. Advantages of the latter estimator are the ease of obtaining estimates and its intuitiveness: extreme samples, and hence the root cause of any disturbances, can be readily identified.

			Factors for control limits					
			$k = 20$		$k = 30$		$k = 75$	
n	$\hat{\sigma}$		U_n	L_n	U_n	L_n	U_n	L_n
5	$\tilde{S}$	Eq. (2.12)	2.352	0.171	2.315	0.172	2.272	0.173
	$\bar{S}$	Eq. (2.15)	2.357	0.171	2.318	0.172	2.274	0.173
	$\bar{S}_{25}$	Eq. (2.15)	2.704	0.167	2.527	0.169	2.359	0.171
	$\overline{S_{20}}$	Eq. (2.15)	2.540	0.169	2.438	0.170	2.319	0.172
	$\bar{R}$	Eq. (2.15)	2.364	0.171	2.322	0.172	2.275	0.173
	$\overline{IQR}$	Eq. (2.15)	2.541	0.169	2.439	0.170	2.318	0.172
	$\bar{G}$	Eq. (2.15)	2.359	0.171	2.320	0.172	2.275	0.173
	$\overline{ADM}$	Eq. (2.15)	2.366	0.171	2.324	0.172	2.275	0.173
	$\overline{ADM}'$	Eq. (2.15)	2.376	0.171	2.332	0.171	2.279	0.172
	$\overline{MDM}$	Eq. (2.15)	2.554	0.169	2.442	0.170	2.322	0.172
	$\overline{MAD}$	Eq. (2.15)	2.447	0.170	2.380	0.171	2.296	0.172
	$D7$	Eq. (2.15)	2.376	0.171	2.331	0.172	2.278	0.172
9	$\tilde{S}$	Eq. (2.12)	1.890	0.349	1.872	0.350	1.851	0.351
	$\bar{S}$	Eq. (2.15)	1.892	0.349	1.873	0.350	1.852	0.351
	$\bar{S}_{25}$	Eq. (2.15)	2.011	0.342	1.946	0.345	1.883	0.349
	$\overline{S_{20}}$	Eq. (2.15)	1.987	0.343	1.934	0.346	1.876	0.349
	$\bar{R}$	Eq. (2.15)	1.900	0.348	1.879	0.349	1.854	0.351
	$\overline{IQR}$	Eq. (2.15)	1.982	0.343	1.933	0.346	1.875	0.350
	$\bar{G}$	Eq. (2.15)	1.894	0.348	1.874	0.350	1.852	0.351
	$\overline{ADM}$	Eq. (2.15)	1.898	0.348	1.877	0.349	1.854	0.351
	$\overline{ADM}'$	Eq. (2.15)	1.901	0.348	1.879	0.349	1.854	0.351
	$\overline{MDM}$	Eq. (2.15)	1.987	0.343	1.936	0.346	1.876	0.349
	$\overline{MAD}$	Eq. (2.15)	1.956	0.345	1.915	0.347	1.868	0.350
	$D7$	Eq. (2.15)	1.901	0.348	1.879	0.349	1.854	0.351

Table 2.3: Factors U_n and L_n to determine Phase II control limits

			$p \times 10^2$					
			$k = 20$		$k = 30$		$k = 75$	
n	$\hat{\sigma}$		U_n	L_n	U_n	L_n	U_n	L_n
5	$\tilde{S}$	Eq. (2.12)	0.135	0.135	0.135	0.135	0.135	0.135
	$\bar{S}$	Eq. (2.15)	0.135	0.135	0.135	0.135	0.135	0.135
	$\bar{S}_{25}$	Eq. (2.15)	0.131	0.134	0.135	0.135	0.135	0.134
	$\overline{S_{20}}$	Eq. (2.15)	0.135	0.135	0.135	0.134	0.136	0.135
	$\bar{R}$	Eq. (2.15)	0.135	0.134	0.135	0.136	0.135	0.137
	$\overline{IQR}$	Eq. (2.15)	0.130	0.134	0.131	0.134	0.135	0.135
	$\bar{G}$	Eq. (2.15)	0.133	0.134	0.136	0.135	0.134	0.136
	$\overline{ADM}$	Eq. (2.15)	0.132	0.134	0.135	0.135	0.134	0.137
	$\overline{ADM}'$	Eq. (2.15)	0.136	0.135	0.137	0.133	0.134	0.134
	$\overline{MDM}$	Eq. (2.15)	0.130	0.136	0.133	0.134	0.134	0.136
	$\overline{MAD}$	Eq. (2.15)	0.131	0.135	0.130	0.135	0.134	0.135
	$D7$	Eq. (2.15)	0.135	0.135	0.134	0.136	0.136	0.133
9	$\tilde{S}$	Eq. (2.12)	0.134	0.135	0.134	0.135	0.134	0.135
	$\bar{S}$	Eq. (2.15)	0.136	0.134	0.136	0.136	0.134	0.135
	$\bar{S}_{25}$	Eq. (2.15)	0.141	0.135	0.137	0.134	0.134	0.135
	$\overline{S_{20}}$	Eq. (2.15)	0.131	0.134	0.135	0.135	0.135	0.133
	$\bar{R}$	Eq. (2.15)	0.135	0.135	0.134	0.134	0.134	0.136
	$\overline{IQR}$	Eq. (2.15)	0.134	0.134	0.133	0.135	0.133	0.137
	$\bar{G}$	Eq. (2.15)	0.133	0.134	0.134	0.136	0.134	0.136
	$\overline{ADM}$	Eq. (2.15)	0.134	0.135	0.134	0.134	0.133	0.136
	$\overline{ADM}'$	Eq. (2.15)	0.134	0.135	0.138	0.134	0.135	0.135
	$\overline{MDM}$	Eq. (2.15)	0.134	0.134	0.138	0.134	0.136	0.133
	$\overline{MAD}$	Eq. (2.15)	0.134	0.136	0.133	0.135	0.133	0.136
	$D7$	Eq. (2.15)	0.134	0.135	0.136	0.134	0.133	0.136

Table 2.4: In-control $p \times 10^2$ of control limits. The estimated relative standard error is never worse than 1%

		p				ARL			
k	$\hat{\sigma}$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$
30	$\tilde{S}$	0.019	0.0027	0.084	0.32	54.7	418	14.5	3.28
						(86.7; 33.7)	(151; 455)	(5.94; 33.0)	(2.18; 5.10)
	$\bar{S}$	0.019	0.0027	0.083	0.32	54.7	419	14.8	3.30
						(87.8; 33.4)	(150; 451)	(5.90; 34.2)	(2.18; 5.18)
	$\bar{S}_{25}$	0.019	0.0027	0.058	0.25	65.1	535	47.8	5.45
						(155; 24.7)	(91.9; 334)	(4.78; 248)	(1.95; 15.2)
	$\overline{S_{20}}$	0.019	0.0027	0.068	0.27	60.9	490	28.3	4.29
						(126; 27.1)	(104; 369)	(5.03; 113)	(2.01; 9.63)
	$\bar{R}$	0.019	0.0027	0.082	0.32	54.9	421	15.1	3.33
						(88.8; 32.9)	(147; 447)	(5.85; 35.9)	(2.17; 5.31)
	$\overline{IQR}$	0.019	0.0026	0.067	0.27	61.0	490	28.5	4.32
						(125; 27.0)	(106; 367)	(5.06; 114)	(2.01; 9.66)
	$\bar{G}$	0.019	0.0027	0.083	0.32	54.8	421	14.9	3.52
						(88.3; 33.2)	(148; 450)	(5.87; 35.1)	(2.17; 5.53)
	$\overline{ADM}$	0.020	0.0027	0.082	0.32	54.8	423	15.2	3.34
						(89.1; 32.8)	(148; 444)	(5.87; 36.6)	(2.16; 5.34)
	$\overline{ADM'}$	0.019	0.0027	0.081	0.31	56.5	434	15.7	3.39
						(95.2; 33.2)	(138; 451)	(5.69; 39.3)	(2.13; 5.50)
	$\overline{MDM}$	0.019	0.0027	0.067	0.27	60.9	490	29.2	4.37
						(129; 26.6)	(101; 365)	(5.02; 123)	(2.01; 10.0)
	$\overline{MAD}$	0.019	0.0026	0.074	0.29	57.7	457	20.4	3.77
						(107; 29.4)	(124; 404)	(5.40; 65.6)	(2.08; 7.20)
	$D7$	0.020	0.0027	0.081	0.31	55.1	427	15.7	3.38
						(92.0; 32.4)	(140; 442)	(5.72; 38.7)	(2.14; 5.49)
75	$\tilde{S}$	0.019	0.0027	0.090	0.34	52.7	391	11.9	3.02
						(70.8; 38.8)	(208; 479)	(6.98; 19.8)	(2.35; 3.92)
	$\bar{S}$	0.020	0.0027	0.090	0.34	52.7	392	12.0	3.03
						(71.2; 38.6)	(205; 478)	(6.94; 20.1)	(2.35; 3.94)
	$\bar{S}_{25}$	0.019	0.0027	0.077	0.30	57.1	446	18.2	3.60
						(101; 30.8)	(127; 423)	(5.24; 52.5)	(2.10; 6.44)
	$\overline{S_{20}}$	0.019	0.0027	0.083	0.32	54.8	420	14.8	3.30
						(88.2; 33.2)	(150; 451)	(5.90; 34.7)	(2.17; 5.20)
	$\bar{R}$	0.020	0.0027	0.090	0.34	52.3	391	12.1	3.05
						(71.1; 38.0)	(203; 475)	(6.88; 20.6)	(2.34; 4.00)
	$\overline{IQR}$	0.020	0.0027	0.083	0.32	54.7	419	14.8	3.30
						(88.0; 33.1)	(150; 450)	(5.90; 34.6)	(2.17; 5.20)
	$\bar{G}$	0.020	0.0027	0.090	0.34	52.2	391	12.0	3.04
						(70.8; 38.1)	(206; 476)	(6.93; 20.4)	(2.35; 3.98)
	$\overline{ADM}$	0.020	0.0027	0.090	0.34	52.2	391	12.1	3.04
						(71.4; 37.8)	(201; 474)	(6.87; 20.7)	(2.34; 4.00)
	$\overline{ADM'}$	0.019	0.0027	0.089	0.33	53.4	399	12.3	3.07
						(74.4; 38.2)	(194; 484)	(6.74; 21.6)	(2.32; 4.09)
	$\overline{MDM}$	0.020	0.0027	0.082	0.32	54.7	422	15.0	3.33
						(88.0; 32.8)	(150; 449)	(5.89; 36.1)	(2.17; 5.29)
	$\overline{MAD}$	0.019	0.0027	0.086	0.33	53.9	409	13.3	3.17
						(80.2; 35.4)	(174; 471)	(6.29; 27.2)	(2.25; 4.59)
	$D7$	0.019	0.0027	0.090	0.34	53.6	396	12.1	3.04
						(74.1; 38.5)	(197; 485)	(6.75; 21.1)	(2.32; 4.06)

Table 2.5: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values under normality for $n = 5$

		p				ARL			
k	$\hat{\sigma}$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$
30	$\tilde{S}$	0.12	0.0027	0.17	0.58	9.05	402	6.48	1.74
						(14.4; 5.64)	(173; 394)	(3.52; 11.9)	(1.42; 2.22)
	$\bar{S}$	0.12	0.0027	0.17	0.58	9.02	400	6.52	1.75
						(14.4; 5.57)	(172; 390)	(3.50; 12.1)	(1.42; 2.23)
	$\bar{S}_{25}$	0.11	0.0027	0.14	0.52	10.6	479	10.4	2.02
						(24.2; 4.52)	(108; 296)	(2.97; 32.3)	(1.34; 3.35)
	$\overline{S_{20}}$	0.11	0.0027	0.15	0.53	10.3	462	9.60	1.96
						(22.3; 4.59)	(119; 306)	(3.05; 28.4)	(1.35; 3.15)
	$\bar{R}$	0.12	0.0027	0.17	0.58	9.23	410	6.74	1.77
						(15.1; 5.48)	(163; 383)	(3.44; 13.3)	(1.42; 2.31)
	$\overline{IQR}$	0.11	0.0027	0.15	0.54	10.3	463	9.52	1.96
						(22.0; 4.59)	(118; 306)	(3.06; 28.0)	(1.35; 3.13)
	$\bar{G}$	0.12	0.0027	0.17	0.58	9.02	401	6.55	1.75
						(14.5; 5.56)	(171; 387)	(3.50; 12.2)	(1.41; 2.25)
	$\overline{ADM}$	0.12	0.0027	0.17	0.58	9.17	409	6.69	1.76
						(15.1; 5.51)	(168; 387)	(3.46; 13.0)	(1.41; 2.29)
	$\overline{ADM'}$	0.12	0.0027	0.17	0.58	9.22	409	6.75	1.77
						(15.4; 5.49)	(160; 385)	(3.40; 13.1)	(1.40; 2.32)
	$\overline{MDM}$	0.11	0.0027	0.14	0.53	10.4	465	9.68	1.97
						(22.5; 4.59)	(115; 300)	(3.03; 29.0)	(1.35; 3.18)
	$\overline{MAD}$	0.12	0.0027	0.15	0.55	9.90	444	8.48	1.89
						(19.7; 4.82)	(130; 325)	(3.13; 22.0)	(1.37; 2.85)
	$D7$	0.12	0.0027	0.17	0.58	9.22	410	6.73	1.76
						(15.3; 5.50)	(165; 385)	(3.43; 13.1)	(1.40; 2.32)
75	$\tilde{S}$	0.12	0.0027	0.18	0.60	8.67	383	5.77	1.68
						(11.7; 6.43)	(235; 429)	(3.97; 8.39)	(1.48; 1.94)
	$\bar{S}$	0.12	0.0027	0.18	0.60	8.67	383	5.78	1.68
						(11.7; 6.39)	(229; 429)	(3.96; 8.45)	(1.48; 1.94)
	$\bar{S}_{25}$	0.12	0.0027	0.17	0.57	9.27	412	6.90	1.78
						(15.8; 5.42)	(156; 380)	(3.38; 13.9)	(1.40; 2.36)
	$\overline{S_{20}}$	0.12	0.0027	0.17	0.58	9.22	408	6.58	1.75
						(15.0; 5.58)	(169; 394)	(3.46; 12.4)	(1.41; 2.26)
	$\bar{R}$	0.12	0.0027	0.18	0.60	8.70	384	5.85	1.69
						(12.0; 6.26)	(225; 425)	(3.90; 8.86)	(1.47; 1.98)
	$\overline{IQR}$	0.12	0.0027	0.17	0.58	9.02	402	6.58	1.75
						(14.6; 5.52)	(196; 386)	(3.48; 12.4)	(1.41; 2.26)
	$\bar{G}$	0.12	0.0027	0.18	0.60	8.68	384	5.80	1.68
						(11.7; 6.38)	(231; 428)	(3.95; 8.53)	(1.48; 1.95)
	$\overline{ADM}$	0.12	0.0027	0.18	0.59	8.67	386	5.86	1.69
						(11.9; 6.28)	(225; 427)	(3.95; 8.78)	(1.47; 1.97)
	$\overline{ADM'}$	0.12	0.0027	0.18	0.60	8.70	384	5.86	1.69
						(12.0; 6.28)	(219; 425)	(3.87; 8.84)	(1.46; 1.98)
	$\overline{MDM}$	0.12	0.0027	0.17	0.58	9.23	407	6.58	1.75
						(95.2; 33.2)	(138; 451)	(5.69; 39.3)	(2.13; 5.50)
	$\overline{MAD}$	0.12	0.0027	0.17	0.58	8.96	398	6.35	1.73
						(13.9; 5.73)	(185; 401)	(3.60; 11.2)	(1.43; 2.18)
	$D7$	0.12	0.0027	0.18	0.60	8.69	385	5.84	1.70
						(12.0; 6.28)	(223; 426)	(3.89; 8.81)	(1.46; 1.98)

Table 2.6: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values under normality for $n = 9$

		p				ARL			
k	$\hat{\sigma}$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$
30	$\tilde{S}$	0.055	0.0043	0.016	0.11	23.0	293	195	22.9
						(52.0; 7.68)	(475; 92.0)	(13.2; 427)	(3.22; 131)
	$\bar{S}$	0.041	0.0031	0.024	0.15	28.7	359	104	9.45
						(55.8; 12.5)	(441; 158)	(11.5; 468)	(3.02; 30.8)
	$\bar{S}_{25}$	0.024	0.0024	0.040	0.20	52.7	526	84.4	7.60
						(128; 19.4)	(164; 262)	(6.00; 478)	(2.19; 24.7)
	$\overline{S_{20}}$	0.024	0.0023	0.045	0.22	48.5	493	54.9	5.97
						(103; 20.4)	(187; 275)	(6.50; 272)	(2.29; 16.6)
	$\bar{R}$	0.041	0.0032	0.023	0.15	28.4	356	110	9.82
						(55.4; 12.2)	(445; 157)	(11.8; 483)	(3.05; 32.7)
	$\overline{IQR}$	0.024	0.0023	0.044	0.22	48.3	493	55.0	5.99
						(103; 20.4)	(191; 276)	(6.56; 274)	(2.29; 16.2)
	$\bar{G}$	0.038	0.0029	0.026	0.16	30.2	379	86.0	8.10
						(56.6; 14.0)	(429; 183)	(11.3; 388)	(2.99; 23.3)
	$\overline{ADM}$	0.035	0.0027	0.029	0.17	31.8	393	73.2	7.28
						(58.9; 15.3)	(411; 200)	(10.8; 320)	(2.92; 19.3)
	$\overline{ADM'}$	0.024	0.0025	0.060	0.26	47.2	450	27.0	4.31
						(86.6; 24.0)	(178; 330)	(6.41; 94.1)	(2.25; 8.80)
	$\overline{MDM}$	0.026	0.0023	0.041	0.21	46.6	489	63.5	6.43
						(99.6; 19.3)	(213; 259)	(6.89; 323)	(2.35; 18.4)
	$\overline{MAD}$	0.033	0.0026	0.029	0.17	34.9	423	86.3	8.02
						(69.6; 15.6)	(382; 206)	(9.68; 412)	(2.78; 23.8)
	$D7$	0.025	0.0024	0.055	0.25	44.1	452	27.8	4.41
						(76.6; 23.9)	(230; 326)	(7.30; 90.8)	(2.40; 8.50)
75	$\tilde{S}$	0.054	0.0042	0.012	0.11	20.4	270	163	13.5
						(36.1; 10.3)	(467; 128)	(23.0; 478)	(4.23; 42.5)
	$\bar{S}$	0.040	0.0030	0.022	0.16	26.7	353	68.3	7.86
						(41.7; 16.0)	(482; 210)	(17.3; 217)	(3.67; 14.8)
	$\bar{S}_{25}$	0.024	0.0023	0.053	0.25	46.4	473	29.4	4.55
						(83.1; 24.7)	(219; 339)	(7.04; 97.3)	(2.38; 9.01)
	$\overline{S_{20}}$	0.025	0.0023	0.054	0.25	43.3	459	25.1	4.25
						(27.4; 25.6)	(276; 351)	(8.08; 68.7)	(2.52; 7.49)
	$\bar{R}$	0.041	0.0030	0.022	0.16	26.2	347	70.5	7.30
						(41.2; 15.7)	(478; 205)	(17.5; 226)	(3.69; 15.2)
	$\overline{IQR}$	0.025	0.0023	0.054	0.25	43.2	459	25.0	4.25
						(70.1; 25.7)	(276; 351)	(8.07; 96.0)	(2.52; 7.48)
	$\bar{G}$	0.037	0.0028	0.026	0.17	28.1	371	55.2	6.41
						(42.7; 17.6)	(477; 233)	(16.2; 161)	(3.53; 12.2)
	$\overline{ADM}$	0.035	0.0027	0.029	0.18	29.7	387	46.9	5.89
						(44.5; 18.9)	(470; 253)	(15.1; 128)	(3.41; 10.6)
	$\overline{ADM'}$	0.023	0.0023	0.065	0.28	44.7	445	18.2	3.69
						(65.9; 29.5)	(269; 403)	(8.00; 39.3)	(2.52; 5.56)
	$\overline{MDM}$	0.026	0.0023	0.049	0.24	41.5	458	28.4	4.51
						(67.8; 24.3)	(299; 332)	(8.57; 80.6)	(2.59; 8.12)
	$\overline{MAD}$	0.033	0.0025	0.031	0.19	32.2	413	45.8	5.78
						(50.7; 20.0)	(462; 264)	(13.0; 135)	(3.18; 10.9)
	$D7$	0.024	0.0022	0.059	0.27	42.7	454	19.5	3.82
						(61.1; 29.2)	(321; 399)	(9.05; 40.3)	(2.66; 5.61)

Table 2.7: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values when symmetric variance disturbances are present in Phase I for $n = 5$

		p				*ARL*			
k	$\hat{\sigma}$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$
30	$\tilde{S}$	0.40	0.011	0.021	0.21	2.91	147	156	8.78
						(6.13; 1.41)	(432; 39.2)	(10.2; 417)	(2.10; 36.4)
	$\bar{S}$	0.32	0.0068	0.033	0.27	3.55	200	77.4	4.61
						(6.77; 1.81)	(463; 55.6)	(8.73; 384)	(1.98; 12.3)
	$\bar{S}_{25}$	0.16	0.0026	0.089	0.43	7.70	449	20.9	2.58
						(17.6; 3.31)	(242; 197)	(3.93; 83.7)	(1.47; 5.03)
	$\overline{S_{20}}$	0.15	0.0025	0.10	0.46	7.92	456	15.8	2.35
						(16.7; 3.63)	(233; 207)	(3.88; 54.6)	(1.47; 4.18)
	$\bar{R}$	0.35	0.0084	0.025	0.23	3.24	175	118	6.19
						(6.52; 1.60)	(461; 41.8)	(9.69; 494)	(2.05; 20.4)
	$\overline{IQR}$	0.15	0.0025	0.10	0.46	7.79	454	15.9	2.35
						(16.5; 3.62)	(240; 205)	(3.93; 55.3)	(1.47; 4.19)
	$\bar{G}$	0.27	0.0050	0.045	0.32	4.08	246	43.2	3.52
						(7.38; 2.20)	(480; 84.8)	(7.69; 181)	(1.87; 7.45)
	$\overline{ADM}$	0.24	0.0042	0.054	0.35	4.56	284	31.2	3.09
						(8.11; 2.52)	(490; 110)	(7.03; 114)	(1.82; 5.85)
	$\overline{ADM'}$	0.16	0.0027	0.12	0.49	7.10	400	11.5	2.12
						(13.2; 3.72)	(236; 217)	(3.95; 31.7)	(1.48; 3.32)
	$\overline{MDM}$	0.15	0.0025	0.096	0.45	7.73	452	17.0	2.41
						(16.5; 3.52)	(249; 197)	(3.96; 59.8)	(1.48; 4.36)
	$\overline{MAD}$	0.18	0.0029	0.076	0.41	6.21	392	22.0	2.66
						(12.3; 3.07)	(392; 156)	(4.92; 80.3)	(1.59; 4.91)
	$D7$	0.15	0.0025	0.11	0.49	7.02	415	10.8	2.09
						(12.0; 4.03)	(294; 247)	(4.32; 25.9)	(1.53; 3.05)
75	$\tilde{S}$	0.41	0.010	0.016	0.20	2.61	121	123	6.10
						(4.23; 1.65)	(262; 44.7)	(18.4; 414)	(2.65; 15.2)
	$\bar{S}$	0.32	0.0063	0.030	0.28	3.30	180	49.4	3.85
						(5.00; 2.15)	(336; 82.0)	(13.1; 157)	(3.31; 7.00)
	$\bar{S}_{25}$	0.16	0.0025	0.11	0.48	6.72	414	11.7	2.15
						(11.4; 3.95)	(331; 241)	(4.62; 27.9)	(1.56; 3.12)
	$\overline{S_{20}}$	0.15	0.0024	0.12	0.50	7.06	426	9.90	2.04
						(11.3; 4.36)	(320; 279)	(4.56; 20.9)	(1.56; 2.79)
	$\bar{R}$	0.36	0.0079	0.022	0.24	2.96	149	77.3	4.71
						(4.60; 1.90)	(301; 62.6)	(15.6; 274)	(2.46; 9.68)
	$\overline{IQR}$	0.15	0.0025	0.12	0.50	6.90	417	10.0	2.04
						(11.0; 4.27)	(324; 271)	(4.59; 21.0)	(1.56; 2.80)
	$\bar{G}$	0.27	0.0048	0.044	0.34	3.83	229	29.3	3.11
						(5.59; 2.59)	(383; 118)	(10.5; 74.7)	(2.12; 4.86)
	$\overline{ADM}$	0.24	0.0041	0.055	0.37	4.25	266	22.1	2.80
						(6.10; 2.93)	(417; 146)	(9.23; 50.4)	(2.02; 4.07)
	$\overline{ADM'}$	0.16	0.0025	0.12	0.52	6.64	401	8.99	1.97
						(9.82; 4.44)	(331; 287)	(4.75; 16.8)	(1.58; 2.54)
	$\overline{MDM}$	0.15	0.0025	0.11	0.49	6.88	422	10.4	2.07
						(8.66; 3.61)	(443; 207)	(6.00; 30.7)	(1.71; 3.28)
	$\overline{MAD}$	0.19	0.0029	0.086	0.44	5.60	368	13.9	2.32
						(95.2; 33.2)	(138; 451)	(5.69; 39.3)	(2.13; 5.50)
	$D7$	0.16	0.0025	0.12	0.51	6.58	409	8.85	1.97
						(9.22; 4.65)	(370; 307)	(5.15; 15.2)	(1.66; 2.44)

Table 2.8: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values when symmetric variance disturbances are present in Phase I for $n = 9$

		p				ARL			
k	$\hat{\sigma}$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$
30	$\tilde{S}$	0.16	0.020	0.016	0.074	16.1	189	211	137
						(54.9; 1.46)	(444; 8.54)	(12.0; 34.0)	(3.08; 99.9)
	$\bar{S}$	0.063	0.0052	0.019	0.12	23.6	286	196	43.3
						(57.8; 5.15)	(420; 56.2)	(11.1; 262)	(2.95; 447)
	$\bar{S}_{25}$	0.023	0.0024	0.044	0.21	55.1	530	74.8	7.08
						(133; 20.5)	(140; 279)	(5.79; 421)	(2.13; 22.8)
	$\overline{S_{20}}$	0.025	0.0024	0.047	0.22	49.6	488	57.9	6.88
						(106; 19.0)	(173; 249)	(6.30; 354)	(2.24; 20.2)
	$\bar{R}$	0.061	0.0050	0.019	0.12	23.8	291	195	39.5
						(52.3; 5.51)	(420; 61.2)	(11.1; 284)	(2.95; 389)
	$\overline{IQR}$	0.025	0.0024	0.047	0.22	49.5	490	58.3	6.72
						(107; 19.1)	(176; 254)	(6.36; 339)	(2.24; 19.6)
	$\bar{G}$	0.053	0.0042	0.021	0.13	25.8	315	168	25.6
						(58.2; 6.94)	(407; 80.3)	(10.8; 375)	(2.91; 195)
	$\overline{ADM}$	0.047	0.0037	0.023	0.14	27.6	334	147	18.8
						(60.0; 8.23)	(403; 98.5)	(10.5; 458)	(2.88; 113)
	$\overline{ADM'}$	0.022	0.0025	0.069	0.28	51.1	448	21.0	3.86
						(90.7; 27.5)	(153; 378)	(5.94; 63.3)	(2.19; 7.12)
	$\overline{MDM}$	0.024	0.0023	0.045	0.22	48.8	492	56.0	6.05
						(104; 20.5)	(181; 275)	(6.48; 279)	(2.25; 16.8)
	$\overline{MAD}$	0.046	0.0037	0.022	0.13	29.4	352	176	23.9
						(68.9; 8.06)	(405; 96.4)	(9.89; 453)	(2.79; 166)
	$D7$	0.023	0.0024	0.062	0.27	47.5	452	22.6	4.00
						(81.3; 26.5)	(197; 363)	(6.71; 65.1)	(2.32; 7.24)
75	$\tilde{S}$	0.16	0.017	0.0079	0.038	10.5	130	263	163
						(32.5; 1.91)	(432; 13.9)	(31.1; 61.8)	(4.88; 185)
	$\bar{S}$	0.059	0.0046	0.013	0.11	20.2	264	186	19.7
						(39.5; 7.93)	(482; 93.3)	(19.1; 427)	(3.85; 95.0)
	$\bar{S}_{25}$	0.023	0.0023	0.057	0.26	48.5	473	26.7	4.31
						(86.8; 25.9)	(198; 343)	(6.71; 83.8)	(2.31; 8.27)
	$\overline{S_{20}}$	0.025	0.0023	0.056	0.26	44.0	453	26.2	4.27
						(72.8; 24.0)	(256; 328)	(7.64; 84.5)	(2.47; 8.15)
	$\bar{R}$	0.058	0.0045	0.013	0.11	20.4	268	177	17.6
						(39.4; 8.24)	(478; 99.1)	(19.2; 449)	(3.87; 78.1)
	$\overline{IQR}$	0.025	0.0023	0.056	0.26	44.0	453	25.7	4.27
						(72.7; 24.2)	(256; 332)	(7.66; 81.3)	(2.47; 8.02)
	$\bar{G}$	0.050	0.0038	0.016	0.13	22.7	299	135	12.1
						(41.2; 10.1)	(480; 124)	(17.5; 478)	(3.70; 43.3)
	$\overline{ADM}$	0.046	0.0035	0.019	0.14	24.5	320	107	9.66
						(42.9; 11.7)	(477; 147)	(16.2; 433)	(3.54; 29.6)
	$\overline{ADM'}$	0.021	0.0024	0.075	0.30	48.4	433	15.3	3.39
						(70.0; 32.9)	(231; 442)	(7.42; 30.2)	(2.42; 4.82)
	$\overline{MDM}$	0.025	0.0023	0.054	0.25	43.5	460	25.4	4.25
						(71.6; 25.5)	(268; 349)	(8.00; 72.2)	(2.52; 7.53)
	$\overline{MAD}$	0.044	0.0034	0.019	0.14	25.7	335	116	10.2
						(47.0; 11.7)	(492; 147)	(15.1; 472)	(3.42; 33.0)
	$D7$	0.022	0.0023	0.069	0.29	46.1	447	16.4	3.52
						(64.8; 32.2)	(278; 433)	(8.21; 31.9)	(2.55; 4.97)

Table 2.9: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values when asymmetric variance disturbances are present in Phase I for $n = 5$

		p				ARL			
k	$\hat{\sigma}$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$
30	$\tilde{S}$	0.69	0.11	0.025	0.093	1.90	68.9	176	122
						(5.92; 1.00)	(412; 1.63)	(10.9; 8.33)	(2.14; 44.3)
	$\bar{S}$	0.48	0.021	0.021	0.17	2.66	127	192	33.7
						(6.68; 1.09)	(460; 9.89)	(8.81; 123)	(1.97; 383)
	$\bar{S}_{25}$	0.15	0.0026	0.099	0.45	8.20	462	17.7	2.43
						(18.6; 3.52)	(208; 198)	(3.67; 66.8)	(1.44; 4.60)
	$\overline{S_{20}}$	0.14	0.0025	0.11	0.47	8.28	460	14.9	2.29
						(17.7; 3.63)	(204; 217)	(3.69; 51.7)	(1.44; 4.06)
	$\bar{R}$	0.50	0.024	0.017	0.15	2.52	116	214	43.3
						(6.45; 1.07)	(453; 8.73)	(9.72; 107)	(2.06; 469)
	$\overline{IQR}$	0.14	0.0025	0.11	0.47	8.18	459	15.0	2.29
						(17.4; 3.68)	(211; 268)	(3.71; 51.7)	(1.45; 4.05)
	$\bar{G}$	0.37	0.010	0.030	0.24	3.31	179	126	8.78
						(7.26; 1.37)	(478; 26.5)	(7.77; 390)	(1.89; 41.9)
	$\overline{ADM}$	0.31	0.0070	0.038	0.28	3.81	220	87.5	5.42
						(7.95; 1.63)	(491; 42.4)	(7.10; 490)	(1.82; 19.6)
	$\overline{ADM'}$	0.14	0.0026	0.14	0.53	7.95	413	9.00	1.95
						(14.4; 4.34)	(190; 277)	(3.67; 21.6)	(1.43; 2.83)
	$\overline{MDM}$	0.14	0.0025	0.10	0.47	8.16	461	15.3	2.31
						(17.2; 3.68)	(220; 216)	(3.77; 52.3)	(1.45; 4.08)
	$\overline{MAD}$	0.26	0.0052	0.049	0.32	4.79	291	74.2	4.73
						(10.8; 1.82)	(479; 56.1)	(5.68; 517)	(1.67; 16.2)
	$D7$	0.14	0.0025	0.13	0.52	7.69	426	9.07	1.96
						(12.9; 4.46)	(249; 291)	(4.02; 20.3)	(1.48; 2.74)
75	$\tilde{S}$	0.77	0.092	0.012	0.044	1.42	32.1	205	145
						(2.95; 1.00)	(150; 2.59)	(46.5; 18.5)	(3.97; 111)
	$\bar{S}$	0.49	0.017	0.012	0.15	2.27	94.3	204	14.3
						(4.26; 1.25)	(269; 19.3)	(18.3; 272)	(2.63; 71.1)
	$\bar{S}_{25}$	0.15	0.0025	0.12	0.50	7.18	424	10.4	2.06
						(12.1; 4.21)	(288; 266)	(4.31; 23.8)	(1.52; 2.92)
	$\overline{S_{20}}$	0.14	0.0024	0.13	0.52	7.36	429	9.33	1.99
						(12.0; 4.47)	(292; 293)	(4.33; 19.8)	(1.52; 2.72)
	$\bar{R}$	0.52	0.019	0.0093	0.13	2.11	82.0	233	18.6
						(3.93; 1.20)	(241; 16.9)	(21.7; 233)	(2.86; 102)
	$\overline{IQR}$	0.15	0.0025	0.12	0.51	7.20	420	9.28	1.99
						(11.6; 4.40)	(294; 284)	(4.36; 19.9)	(1.53; 2.73)
	$\bar{G}$	0.37	0.0086	0.024	0.24	2.97	151	92.8	5.25
						(5.03; 1.68)	(342; 46.9)	(12.7; 395)	(2.29; 14.1)
	$\overline{ADM}$	0.31	0.0064	0.034	0.29	3.41	191	54.7	3.95
						(5.58; 1.99)	(385; 69.3)	(10.7; 226)	(2.13; 8.53)
	$\overline{ADM'}$	0.14	0.0025	0.15	0.55	7.47	408	7.39	1.83
						(10.8; 5.11)	(277; 347)	(4.32; 12.7)	(1.52; 2.27)
	$\overline{MDM}$	0.15	0.0024	0.12	0.51	7.25	428	9.50	2.00
						(11.8; 4.47)	(297; 288)	(4.38; 20.1)	(1.53; 2.72)
	$\overline{MAD}$	0.26	0.0047	0.059	0.34	4.19	256	36.3	3.29
						(7.21; 2.29)	(470; 92.0)	(7.70; 143)	(1.88; 6.71)
	$D7$	0.14	0.0024	0.14	0.54	7.22	416	7.63	1.86
						(10.0; 5.17)	(321; 351)	(4.70; 12.4)	(1.56; 2.26)

Table 2.10: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values when asymmetric variance disturbances are present in Phase I for $n = 9$

		p				ARL			
k	$\hat{\sigma}$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$
30	$\tilde{S}$	0.10	0.0083	0.0038	0.035	12.1	153	370	92.9
						(26.3; 4.82)	(362; 51.7)	(63.6; 243)	(7.10; 476)
	$\bar{S}$	0.051	0.0038	0.011	0.10	21.3	286	172	13.1
						(36.8; 12.0)	(489; 152)	(27.0; 486)	(4.58; 34.6)
	$\bar{S}_{25}$	0.022	0.0024	0.047	0.22	57.2	534	66.1	6.47
						(136; 21.8)	(129; 291)	(5.54; 363)	(2.09; 19.6)
	$\overline{S_{20}}$	0.051	0.0039	0.010	0.086	24.3	317	285	27.4
						(54.9; 9.42)	(633; 115)	(18.4; 540)	(3.77; 133)
	$\bar{R}$	0.051	0.0038	0.011	0.10	21.4	286	176	12.2
						(37.3; 11.8)	(492; 150)	(27.0; 499)	(4.59; 36.4)
	$\overline{IQR}$	0.051	0.0039	0.010	0.086	24.3	316	286	28.1
						(55.0; 9.39)	(633; 115)	(18.4; 537)	(3.78; 134)
	$\bar{G}$	0.051	0.0038	0.011	0.10	21.4	286	174	13.1
						(37.1; 11.9)	(450; 151)	(27.1; 491)	(4.62; 35.2)
	$\overline{ADM}$	0.051	0.0038	0.010	0.099	21.4	287	178	13.5
						(37.1; 11.7)	(496; 149)	(26.4; 503)	(4.55; 37.2)
	$\overline{ADM'}$	0.020	0.0027	0.079	0.31	55.1	433	17.3	3.53
						(97.2; 30.3)	(130; 415)	(5.51; 48.1)	(2.11; 6.16)
	$\overline{MDM}$	0.051	0.0039	0.010	0.086	24.3	318	288	28.7
						(56.0; 9.25)	(635; 112)	(18.1; 532)	(3.75; 144)
	$\overline{MAD}$	0.051	0.0039	0.010	0.092	22.7	301	233	18.9
						(46.0; 10.3)	(573; 128)	(21.7; 563)	(4.09; 72.3)
	$D7$	0.025	0.0023	0.053	0.25	43.6	454	27.9	4.43
						(75.6; 24.2)	(240; 327)	(7.34; 85.2)	(2.44; 8.44)
75	$\tilde{S}$	0.080	0.0064	0.0041	0.050	13.5	175	330	32.9
						(22.9; 7.39)	(312; 87.4)	(73.0; 406)	(7.74; 117)
	$\bar{S}$	0.043	0.0032	0.017	0.14	24.1	326	77.6	7.77
						(34.2; 16.7)	(450; 222)	(26.7; 186)	(4.57; 13.3)
	$\bar{S}_{25}$	0.021	0.0024	0.065	0.28	51.7	467	22.5	3.99
						(92.7; 27.8)	(165; 384)	(6.21; 67.9)	(2.21; 7.39)
	$\overline{S_{20}}$	0.043	0.0032	0.016	0.13	25.3	337	112	9.48
						(43.1; 14.1)	(520; 183)	(19.4; 377)	(3.85; 23.0)
	$\bar{R}$	0.043	0.0032	0.017	0.14	24.0	323	78.1	7.81
						(33.9; 16.5)	(448; 218)	(26.1; 188)	(4.53; 13.6)
	$\overline{IQR}$	0.043	0.0032	0.016	0.13	0.043	337	112	9.51
						(43.0; 14.1)	(519; 183)	(19.3; 375)	(3.85; 23.1)
	$\bar{G}$	0.043	0.0032	0.017	0.14	23.9	323	77.6	7.82
						(33.7; 16.6)	(447; 220)	(26.5; 186)	(4.55; 13.5)
	$\overline{ADM}$	0.043	0.0032	0.017	0.14	23.9	323	78.9	7.84
						(34.3; 16.4)	(449; 217)	(26.2; 195)	(4.52; 13.7)
	$\overline{ADM'}$	0.020	0.0026	0.087	0.33	52.4	405	12.8	3.13
						(74.6; 36.5)	(194; 474)	(6.72; 23.7)	(2.31; 4.30)
	$\overline{MDM}$	0.043	0.0032	0.016	0.13	25.2	336	115	9.69
						(43.3; 14.0)	(522; 180)	(19.3; 392)	(3.85; 24.1)
	$\overline{MAD}$	0.043	0.0032	0.016	0.13	24.8	333	94.4	8.60
						(38.7; 15.3)	(494; 139)	(22.1; 288)	(4.13; 18.0)
	$D7$	0.023	0.0023	0.064	0.28	44.7	452	17.5	3.63
						(63.1; 31.1)	(295; 423)	(8.55; 34.2)	(2.60; 5.19)

Table 2.11: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values when localized variance disturbances are present in Phase I for $n = 5$

k	$\hat{\sigma}$	p $\lambda=0.5$	p $\lambda=1$	p $\lambda=1.5$	p $\lambda=2$	ARL $\lambda=0.5$	ARL $\lambda=1$	ARL $\lambda=1.5$	ARL $\lambda=2$
30	$\tilde{S}$	0.66	0.033	0.0037	0.054	1.60	43.1	344	49.3
						(2.60; 1.12)	(117; 12.0)	(93.7; 154)	(5.30; 263)
	$\bar{S}$	0.38	0.0088	0.015	0.20	2.75	132	111	5.63
						(4.22; 1.86)	(462; 59.3)	(21.8; 346)	(2.82; 11.4)
	$\bar{S}_{25}$	0.13	0.0025	0.12	0.49	9.49	481	12.8	2.17
						(21.5; 4.13)	(142; 255)	(3.24; 41.9)	(1.38; 3.74)
	$\overline{S_{20}}$	0.37	0.0089	0.015	0.18	3.09	164	201	8.68
						(6.29; 1.61)	(469; 42.3)	(14.2; 590)	(2.40; 30.0)
	$\bar{R}$	0.38	0.0087	0.015	0.20	2.80	136	120	5.86
						(4.41; 1.82)	(285; 57.5)	(20.9; 395)	(2.77; 12.7)
	$\overline{IQR}$	0.37	0.0089	0.014	0.18	3.09	162	202	8.59
						(6.25; 1.61)	(467; 42.3)	(14.5; 591)	(2.42; 29.2)
	$\bar{G}$	0.38	0.0088	0.015	0.20	2.75	133	113	5.68
						(4.23; 1.85)	(269; 59.4)	(21.7; 358)	(2.82; 11.7)
	$\overline{ADM}$	0.38	0.0087	0.015	0.20	2.79	135	117	5.79
						(4.37; 1.84)	(276; 58.0)	(21.1; 379)	(2.79; 12.2)
	$\overline{ADM'}$	0.12	0.0028	0.17	0.57	9.21	405	6.86	1.78
						(15.9; 5.33)	(147; 369)	(3.31; 14.0)	(1.39; 2.37)
	$\overline{MDM}$	0.37	0.0089	0.015	0.18	3.10	164	203	8.78
						(6.27; 1.61)	(475; 42.2)	(14.4; 587)	(2.39; 30.6)
	$\overline{MAD}$	0.37	0.0088	0.015	0.18	2.97	154	175	7.54
						(5.57; 1.66)	(406; 56.0)	(16.0; 567)	(2.50; 22.7)
	$D7$	0.16	0.0026	0.11	0.49	6.88	412	11.1	2.11
						(11.7; 3.99)	(302; 243)	(4.42; 26.3)	(1.53; 3.09)
75	$\tilde{S}$	0.58	0.021	0.0039	0.088	1.79	55.0	321	15.5
						(2.51; 1.33)	(110; 24.2)	(85.3; 346)	(5.11; 43.4)
	$\bar{S}$	0.32	0.0063	0.027	0.27	3.19	169	44.6	3.77
						(4.22; 2.43)	(264; 104)	(18.7; 97.0)	(2.66; 5.45)
	$\bar{S}_{25}$	0.13	0.0025	0.15	0.55	8.48	427	7.82	1.86
						(14.4; 4.97)	(195; 339)	(3.65; 16.3)	(1.44; 2.51)
	$\overline{S_{20}}$	0.31	0.0062	0.026	0.26	3.36	186	61.7	4.22
						(5.30; 2.17)	(371; 81.6)	(13.7; 205)	(2.36; 7.92)
	$\bar{R}$	0.32	0.0063	0.027	0.27	3.20	170	46.3	3.81
						(4.30; 2.39)	(273; 100)	(18.0; 106)	(2.62; 5.68)
	$\overline{IQR}$	0.32	0.0064	0.026	0.26	3.31	182	61.6	4.22
						(5.21; 2.14)	(361; 80.0)	(13.8; 199)	(2.37; 7.85)
	$\bar{G}$	0.32	0.0063	0.027	0.27	3.19	169	45.0	3.78
						(4.23; 2.43)	(266; 103)	(18.6; 97.9)	(2.66; 5.46)
	$\overline{ADM}$	0.32	0.0063	0.026	0.27	3.19	169	46.1	3.81
						(4.28; 2.40)	(270; 101)	(18.4; 104)	(2.64; 5.64)
	$\overline{ADM'}$	0.12	0.0027	0.18	0.59	8.68	384	5.90	1.69
						(12.2; 6.15)	(214; 421)	(3.83; 9.10)	(1.46; 1.99)
	$\overline{MDM}$	0.31	0.0062	0.026	0.26	3.37	187	62.0	4.22
						(5.28; 2.16)	(374; 82.0)	(13.8; 206)	(2.35; 7.96)
	$\overline{MAD}$	0.32	0.0063	0.026	0.26	3.28	179	56.5	4.09
						(4.91; 2.21)	(335; 85.8)	(14.9; 169)	(2.42; 7.15)
	$D7$	0.15	0.0024	0.13	0.53	6.91	414	8.16	1.91
						(9.61; 4.93)	(347; 332)	(4.92; 13.6)	(1.59; 2.44)

Table 2.12: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values when localized variance disturbances are present in Phase I for $n = 9$

		p				ARL			
k	$\hat{\sigma}$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$
30	$\tilde{S}$	0.055	0.0042	0.010	0.094	20.5	271	204	15.8
						(38.8; 10.5)	(501; 131)	(23.7; 518)	(4.27; 49.0)
	$\bar{S}$	0.046	0.0035	0.015	0.12	24.1	318	139	11.3
						(44.2; 12.4)	(519; 158)	(18.1; 477)	(3.76; 32.5)
	$\bar{S}_{25}$	0.027	0.0024	0.033	0.18	47.8	509	112	9.29
						(117; 17.4)	(213; 230)	(6.82; 617)	(2.32; 33.7)
	$\overline{S_{20}}$	0.030	0.0025	0.031	0.18	40.3	456	97.3	8.65
						(87.6; 15.9)	(291; 211)	(8.22; 509)	(2.53; 29.5)
	$\bar{R}$	0.047	0.0035	0.015	0.12	23.8	316	147	11.8
						(43.9; 12.1)	(522; 155)	(18.4; 486)	(3.78; 34.1)
	$\overline{IQR}$	0.030	0.0025	0.031	0.18	40.3	458	96.7	8.51
						(87.8; 16.0)	(293; 212)	(8.24; 504)	(2.53; 28.9)
	$\bar{G}$	0.044	0.0033	0.017	0.13	25.4	336	119	9.98
						(45.7; 13.3)	(519; 172)	(16.8; 442)	(3.61; 26.7)
	$\overline{ADM}$	0.041	0.0031	0.019	0.14	26.7	351	107	9.25
						(47.9; 14.0)	(514; 182)	(15.7; 393)	(3.51; 24.2)
	$\overline{ADM'}$	0.032	0.0028	0.041	0.20	38.2	403	67.7	6.84
						(84.0; 15.5)	(202; 204)	(6.73; 337)	(2.32; 21.0)
	$\overline{MDM}$	0.030	0.0025	0.030	0.17	39.7	459	100	8.69
						(87.0; 16.2)	(319; 213)	(8.52; 497)	(2.58; 28.0)
	$\overline{MAD}$	0.039	0.0029	0.019	0.14	29.5	380	128	10.5
						(57.8; 13.7)	(531; 178)	(13.4; 510)	(3.26; 32.7)
	$D7$	0.031	0.0025	0.038	0.21	36.8	422	52.1	5.98
						(69.0; 17.8)	(304; 237)	(8.45; 216)	(2.60; 14.8)
75	$\tilde{S}$	0.054	0.0042	0.0091	0.097	19.2	257	156	12.0
						(29.0; 12.7)	(393; 162)	(39.0; 383)	(5.56; 24.6)
	$\bar{S}$	0.046	0.0034	0.014	0.13	22.8	307	96.8	8.79
						(33.7; 15.1)	(447; 197)	(27.2; 255)	(4.59; 16.8)
	$\bar{S}_{25}$	0.026	0.0023	0.044	0.22	42.2	468	38.1	5.14
						(76.3; 22.3)	(283; 302)	(3.21; 133)	(2.53; 10.6)
	$\overline{S_{20}}$	0.030	0.0024	0.036	0.21	35.8	434	42.7	5.48
						(59.9; 20.1)	(394; 274)	(10.6; 139)	(2.86; 11.1)
	$\bar{R}$	0.047	0.0035	0.014	0.12	22.3	301	101	9.00
						(33.3; 14.7)	(442; 192)	(27.5; 270)	(4.68; 17.4)
	$\overline{IQR}$	0.030	0.0024	0.037	0.21	35.7	434	41.4	5.45
						(59.9; 20.2)	(393; 275)	(10.5; 136)	(2.86; 11.0)
	$\bar{G}$	0.044	0.0032	0.017	0.14	23.9	322	81.6	7.96
						(35.0; 36.1)	(457; 211)	(24.5; 213)	(4.39; 14.5)
	$\overline{ADM}$	0.041	0.0031	0.019	0.15	25.1	338	70.8	7.36
						(36.5; 16.9)	(468; 223)	(22.4; 182)	(4.14; 13.2)
	$\overline{ADM'}$	0.030	0.0025	0.043	0.22	36.1	416	36.3	5.05
						(60.6; 20.0)	(321; 267)	(9.03; 119)	(2.69; 10.1)
	$\overline{MDM}$	0.031	0.0024	0.035	0.20	35.3	434	43.9	5.56
						(58.5; 20.4)	(411; 274)	(10.9; 140)	(2.92; 10.9)
	$\overline{MAD}$	0.039	0.0029	0.021	0.15	27.3	364	71.1	7.30
						(42.4; 17.1)	(504; 225)	(18.4; 208)	(3.78; 14.3)
	$D7$	0.030	0.0024	0.040	0.22	35.4	433	32.1	4.85
						(53.2; 22.6)	(410; 305)	(11.3; 82.0)	(2.97; 8.17)

Table 2.13: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values when diffuse mean disturbances are present in Phase I for $n = 5$

k	$\hat{\sigma}$	p $\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	ARL $\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$
30	$\tilde{S}$	0.41	0.010	0.014	0.18	2.62	122	149	6.74
						(4.40; 1.67)	(282; 46.3)	(19.6; 461)	(2.70; 16.5)
	$\bar{S}$	0.36	0.0082	0.019	0.22	2.95	150	103	5.40
						(4.97; 1.82)	(342; 56.7)	(15.0; 373)	(2.44; 12.4)
	$\bar{S}_{25}$	0.18	0.0031	0.069	0.38	6.51	402	32.9	3.04
						(14.8; 2.75)	(343; 132)	(4.55; 154)	(1.55; 6.82)
	$\overline{S_{20}}$	0.18	0.0029	0.077	0.41	6.67	411	25.1	2.76
						(14.3; 2.97)	(322; 150)	(4.52; 105)	(1.54; 5.64)
	$\bar{R}$	0.40	0.0098	0.014	0.19	2.72	129	152	6.86
						(4.70; 1.66)	(310; 45.8)	(18.1; 486)	(2.61; 17.8)
	$\overline{IQR}$	0.18	0.0029	0.076	0.40	6.57	409	25.8	2.79
						(14.2; 2.95)	(346; 147)	(4.62; 108)	(1.55; 5.72)
	$\bar{G}$	0.32	0.0065	0.027	0.26	3.33	183	66.5	4.33
						(5.59; 2.04)	(391; 73.5)	(12.0; 246)	(2.23; 8.91)
	$\overline{ADM}$	0.29	0.0054	0.034	0.29	3.72	218	50.0	3.81
						(6.24; 2.25)	(444; 90.0)	(10.3; 176)	(2.10; 7.44)
	$\overline{ADM'}$	0.23	0.0041	0.069	0.48	5.16	345	30.0	2.97
						(11.3; 2.44)	(326; 104)	(4.56; 130)	(1.56; 6.19)
	$\overline{MDM}$	0.18	0.0029	0.075	0.40	6.60	412	25.1	2.78
						(18.9; 3.01)	(360; 154)	(4.59; 104)	(1.56; 5.55)
	$\overline{MAD}$	0.23	0.0038	0.053	0.34	5.03	322	38.1	3.30
						(9.93; 2.52)	(529; 110)	(6.23; 163)	(1.74; 7.02)
	$D7$	0.19	0.0031	0.081	0.42	5.69	356	18.3	2.52
						(10.3; 3.02)	(387; 155)	(5.11; 59.2)	(1.61; 4.35)
75	$\tilde{S}$	0.41	0.010	0.012	0.19	2.49	109	112	5.74
						(3.41; 2.86)	(189; 60.0)	(31.7; 284)	(3.32; 10.0)
	$\bar{S}$	0.36	0.0080	0.019	0.23	2.82	137	72.8	4.64
						(3.89; 2.07)	(234; 74.8)	(22.4; 188)	(2.86; 7.68)
	$\bar{S}_{25}$	0.19	0.0029	0.081	0.43	5.75	372	16.4	2.44
						(9.78; 3.37)	(428; 185)	(5.51; 44.2)	(1.66; 3.80)
	$\overline{S_{20}}$	0.18	0.0028	0.089	0.45	5.92	384	14.0	2.31
						(9.63; 3.61)	(423; 205)	(5.48; 34.2)	(1.66; 3.41)
	$\bar{R}$	0.40	0.0097	0.013	0.19	2.55	114	108	5.62
						(3.57; 1.87)	(203; 60.0)	(28.6; 290)	(3.17; 10.0)
	$\overline{IQR}$	0.18	0.0029	0.087	0.45	5.77	373	14.3	2.33
						(9.32; 3.52)	(425; 197)	(5.58; 34.7)	(1.67; 3.44)
	$\bar{G}$	0.32	0.0064	0.027	0.27	3.18	169	47.4	3.84
						(4.40; 2.33)	(281; 95.3)	(17.1; 114)	(2.57; 5.91)
	$\overline{ADM}$	0.29	0.0054	0.034	0.30	3.51	199	35.7	3.41
						(4.83; 2.56)	(323; 113)	(14.0; 82.0)	(2.37; 5.03)
	$\overline{ADM'}$	0.22	0.0037	0.070	0.40	4.80	301	18.9	2.59
						(7.98; 2.94)	(432; 147)	(6.24; 50.0)	(1.73; 4.03)
	$\overline{MDM}$	0.18	0.0028	0.087	0.45	5.87	383	14.2	2.32
						(9.50; 3.61)	(430; 208)	(5.55; 34.0)	(1.67; 3.40)
	$\overline{MAD}$	0.23	0.0038	0.058	0.38	4.57	292	22.2	2.78
						(7.04; 2.97)	(467; 149)	(8.05; 55.9)	(1.92; 4.24)
	$D7$	0.20	0.0030	0.085	0.44	5.30	346	13.7	2.30
						(7.77; 3.59)	(439; 204)	(6.43; 28.4)	(1.76; 3.16)

Table 2.14: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values when diffuse mean disturbances are present in Phase I for $n = 9$

2.5 Real data example

In this section we demonstrate the implementation of the control charts created above. Our dataset was supplied by Grant and Leavenworth (1988, p.9). The operation concerns thread grinding a fitting for an aircraft hydraulic system. Table 2.15 shows the pitch diameters of the threads for 20 randomly chosen samples. Each sample consists of 5 observations.

The control charting process starts with estimating the in-control standard deviation σ (Phase I). We construct control charts based on the different Phase I estimators proposed. The estimates derived from these estimators are shown in Table 2.16 and are used to determine the Phase II control limits. For example, the estimate of σ based on $\tilde{S}$ is equal to 2.972 and Table 2.3 shows that the respective factors for the upper and lower control limits are 2.352 and 0.171. Consequently the Phase II control limits are 6.990 and 0.508. Figure 2.5 compares the Phase II control limits for the proposed estimators.

In the case of $\overline{ADM}'$, we apply a simple subgroup screening method. The factors for the Phase I control limits are 2.089 and 0 for $n = 5$. We first determine the $\overline{ADM}$ from the 20 samples, which generates 2.594. The Phase I control limits are 5.419 and 0. Then we determine the standard deviation $S/c_4(5)$ of each sample and delete samples whose standard deviation falls outside the initial control limits. For the example discussed here, the standard deviations of samples 8, 9 and 13 fall outside the control limits. The same procedure is repeated in the second iteration: new values for the in-control σ (2.041) and the Phase I control limits (4.263 and 0) are generated from the remaining samples and any sample whose standard deviation falls outside the control limits is deleted. In the second iteration, it appears no longer necessary to delete further samples.

The highest values of $\widehat{UCL}$ and $\widehat{LCL}$ are given by $\tilde{S}$, while the lowest values are given by $D7$ and $\overline{ADM}'$. Note, however, that the question of which estimator produces the best estimate can not be resolved from such a limited sample.

Sample	Observations					$S/c_4(5)$
1	36	35	34	33	32	1.682
2	31	31	34	32	30	1.613
3	30	30	32	30	32	1.165
4	32	33	33	32	35	1.303
5	32	34	37	37	35	2.257
6	32	32	31	33	33	0.890
7	33	33	36	32	31	1.990
8	23	33	36	35	36	5.856
9	43	36	35	24	31	7.424
10	36	35	36	41	41	3.138
11	34	38	35	34	38	2.180
12	36	38	39	39	40	1.613
13	36	40	35	26	33	5.477
14	36	35	37	34	33	1.682
15	30	37	33	34	35	2.754
16	28	31	33	33	33	2.331
17	33	30	34	33	35	1.990
18	27	28	29	27	30	1.387
19	35	36	29	27	32	4.079
20	33	35	35	39	36	2.331

Table 2.15: Measurements of pitch diameter of threads on aircraft fittings

$\hat{\sigma}$		$\widehat{UCL}$	$\widehat{LCL}$
$\tilde{S}$	2.972	6.990	0.508
$\bar{S}$	2.657	6.263	0.112
$\bar{S}_{25}$	2.193	5.930	0.366
$\overline{S_{25}}$	2.456	6.238	0.415
$\bar{R}$	2.666	6.302	0.456
$\overline{IQR}$	2.424	6.159	0.410
$\bar{G}$	2.623	6.188	0.449
$\overline{ADM}$	2.594	6.137	0.444
$\overline{ADM}'$	2.041	4.849	0.349
$\overline{MDM}$	2.256	5.762	0.381
$\overline{MAD}$	2.408	5.892	0.409
$D7$	2.067	4.911	0.353

Table 2.16: Control chart limits for pitch diameter

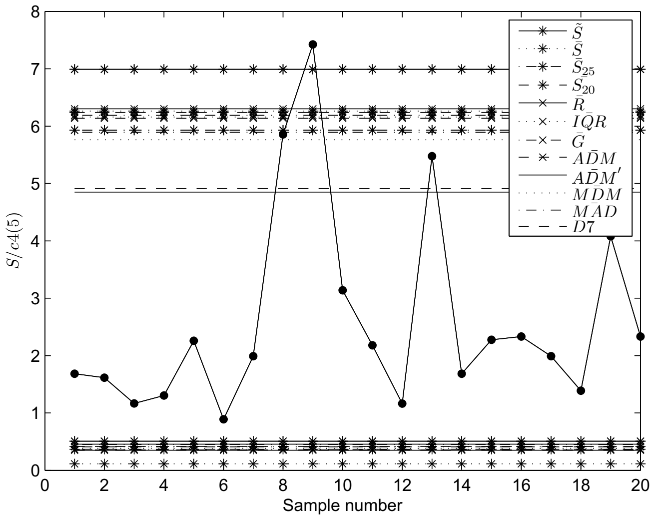


Figure 2.5: Control chart limits for pitch diameter

2.6 Concluding remarks

In this chapter we have compared 12 different estimators for designing standard deviation control charts and investigated their performance in Phase II. The added value of incorporating a simple screening procedure into an estimation method has proven to be substantial. This method performs better than estimators which remove samples ($\bar{S}_{25}$) or observations ($\overline{S_{20}}$ or $\overline{IQR}$) beforehand. The disadvantage of removing samples and/or observations beforehand is that too much information is lost in uncontaminated situations while, at the same time, the resulting estimates are biased in contaminated situations. The estimator $\overline{ADM}'$ retains far more information, deleting only extreme subgroups so that the final estimate is not affected substantially. Moreover, $\overline{ADM}'$ is intuitive and easy to implement. We recommend using $\overline{ADM}'$ when the dataset is likely to be contaminated by localized disturbances. On the other hand, we prefer $D7$ when the dataset is likely to be contaminated by mean diffuse disturbances because $D7$ is more robust against such disturbances. All in all, there appears to be no single best control-chart method that would cover every process and every company. ASTM 15D (1976, p.143) says it best: *"The final justification of a control chart criterion is its proven ability to detect assignable causes economically under practical conditions."*

2.7 Appendix

The literature proposes several estimators for the standard deviation of a normal distribution, including estimators based on Gini's mean difference, Downton's linear function of order statistics (Downton (1966)) and the probability-weighted moments estimator (Muhammad et al. (1993)).

Let $X_{i(1)}, X_{i(2)}, ..., X_{i(n)}$ denote the order statistics of sample i. According to David (1968), the sample statistic G_i can also be written as a function of order statistics

$$G_i = 2/(n(n-1)) \sum_{j=1}^{n} (2j - n - 1) X_{i(j)}. \tag{2.16}$$

Downton (1966) suggested as a possible unbiased estimator of σ the statistic

$$D_i = 1/\sqrt{\pi} \sum_{j=1}^{n} (2j - n - 1) X_{i(j)} / (n(n-1)), \tag{2.17}$$

and Muhammad et al. (1993) proposed the so-called probability weighted moments estimator of σ

$$S_{pw,i} = \sqrt{\pi}/n^2 \sum_{j=1}^{n} (2j - n - 1) X_{i(j)}. \tag{2.18}$$

It follows directly from (2.16), (2.17) and (2.18) that

$$G_i = 2/\sqrt{\pi} D_i = 2n/((n-1)\sqrt{\pi}) S_{pw,i}.$$

Chapter 3

A Robust Standard Deviation Control Chart

3.1 Introduction

In this chapter we investigate robust Phase I estimators for the subgroup standard deviation control chart. The estimators considered are the pooled standard deviation, the robust biweight A estimator of Tatum (1997) and several adaptive trimmers. Additionally, we look at an adaptive trimmer based on the mean deviation from the median, a statistic more resistant to diffuse outliers (see Chapter 2). For diffuse outliers, we think that an individuals chart would detect outliers more quickly. We therefore include an estimator based on the individuals chart. In order to measure the variability within and not between subgroups, we correct for differences in the location between subgroups. Moreover, we present an algorithm that combines the last two approaches. The performance of the estimators is evaluated by assessing their MSE under normality and in the presence of several types of contaminations. Finally, we derive factors for the Phase II limits of the standard deviation control chart and assess the performance of the control charts by means of a simulation study.

The chapter is structured as follows. The next section introduces the standard deviation estimators, demonstrates the implementation of the estimators by means of a real-world example and assesses their MSE. Next, we present the design schemes for the standard deviation control chart and derive the Phase II control limits. We then describe the simulation procedure, summarize simulation results and draw some final conclusions.

3.2 Proposed Phase I estimators

In this section we present six Phase I estimators, demonstrate the implementation of the estimators by means of a real data example and assess the efficiency of the estimators in terms of their MSE.

3.2.1 Standard deviation estimators

Recall that X_{ij}, $i = 1, 2, ..., k$ and $j = 1, 2, ..., n$, denote the Phase I data with n the subgroup size and k the number of subgroups.

The first estimator of σ is based on the pooled subgroup standard deviation $\tilde{S}$ (see (2.1)). This estimator provides a basis for comparison under normality when no contaminations are present. Mahmoud et al. (2010) showed that this estimator is more efficient than the mean of the subgroup standard deviations and the mean of the subgroup ranges when the data are normally distributed.

Second, we evaluate a robust estimator proposed by Tatum (1997) (see (2.11)). This estimator is denoted by $D7$ as in Tatum (1997).

We also include other procedures to obtain $\hat{\sigma}$. The first is a variant of Rocke (1989). Rocke's procedure first estimates σ by the mean subgroup range

$$\bar{R} = \frac{1}{k}\sum_{i=1}^{k} R_i,$$

where R_i is the range of the i-th subgroup. An unbiased estimator of σ under normality is $\bar{R}/d_2(n)$, where $d_2(n)$ is the expected range of a random $N(0, 1)$ subgroup of size n. Values of $d_2(n)$ can be found in Duncan (1986), Table M. Any subgroup that exceeds the Phase I control limits is deleted and $\bar{R}$ is recomputed from the remaining subgroups. Our approach is similar but continues until all subgroup ranges fall between the Phase I control limits. These are set at $\widehat{UCL} = U_n\bar{R}/d_2(n)$ and $\widehat{LCL} = L_n\bar{R}/d_2(n)$. For simplicity, we derive the factors U_n and L_n from the 0.99865 and 0.00135 quantiles of the distribution of $\bar{R}/d_2(n)$. Table 3.1 shows the factors for $n = 4, 5, 9$ as well as the constants added to obtain unbiased estimates once the data have been screened. Note that the factors and the constants are the same for $k = 20, 50, 100$. The resulting estimator is denoted by $\overline{R}^s$.

In addition, we evaluate an adaptive trimmer where the estimate of σ is obtained by the mean subgroup average deviation from the median instead

	$n=4$			$n=5$			$n=9$		
$\hat{\sigma}$	U_n	L_n	Constant	U_n	L_n	Constant	U_n	L_n	Constant
$\bar{R}^s$	2.321	0.170	1	2.305	0.172	1	1.950	0.330	1
$\overline{MD}^s$	2.321	0.170	0.998	2.305	0.172	1	1.950	0.330	1
$\overline{MD}^i$	3	-3	0.990	3	-3	0.975	3	-3	0.986
$\overline{MD}^{i,s}$	4.703	0.0018	1	3.225	0.035	1	2.485	0.142	1
	3	-3	0.988	3	-3	0.975	3	-3	0.986

Table 3.1: Factors for Phase I control limits for $k = 20, 50, 100$

of $\bar{R}$. The mean subgroup average deviation from the median is given by

$$\overline{MD} = \frac{1}{k}\sum_{i=1}^{k} MD_i,$$

where MD_i is the average absolute deviation from the median M_i of subgroup i defined by,

$$MD_i = \sum_{j=1}^{n} |X_{ij} - M_i|/n.$$

An unbiased estimator of σ is $\overline{MD}/t_2(n)$, where $t_2(n)$ equals $E(\overline{MD}/\sigma)$. Since it is difficult to obtain $E(\overline{MD})$ analytically, it is obtained by simulation. Extensive tables for $t_2(n)$ can be found in Riaz and Saghir (2009). The advantage of this estimator is that it is less sensitive to outliers than R (see Chapter 2). The resulting estimator is denoted by $\overline{MD}^s$. The values used for the Phase I control limits and the constants necessary to obtain unbiased estimates once the data have been screened are given in Table 3.1.

For subgroup control charts, only adaptive trimming methods based on the subgroup averages or subgroup standard deviations have been proposed in the literature so far. For diffuse outliers, however, an individuals chart should detect outliers more quickly. We therefore propose a screening method based on an individuals chart. The algorithm first calculates the residuals by subtracting the subgroup median from each observation in the corresponding subgroup. This ensures that the variability is measured within and not between subgroups. Next, an individuals chart of the residuals is constructed. The location of the chart μ is estimated by the mean of the subgroup medians, which is zero because the subgroup medians have been subtracted from the observations, and the standard deviation σ is estimated by $\overline{MD}$. To simplify, the factors for the individuals chart are 3 and -3 (see Table 3.1). The residuals that fall outside the control limits are excluded from the dataset. Then the procedure is repeated: the median values of the adjusted subgroups are determined, the residuals are calculated and the control limits of the individuals chart are computed. The residuals that now exceed the limits are removed. This continues until all residuals fall within the control limits. The resulting estimates of σ are slightly biased under normality. The constants necessary to obtain an unbiased estimate can be found in Table 3.1. The unbiased estimator is denoted by $\overline{MD}^i$.

The above procedure does not use the spread of the subgroups. Therefore, we finally propose an algorithm that combines the use of an individuals chart with subgroup screening. First, an initial estimate of σ is obtained via $\overline{MD}$. This estimate is then used to construct a standard deviation control chart so that the subgroups can be screened. Adopting R as a charting statistic will result in the exclusion of many subgroups, including many uncontaminated observations, when diffuse disturbances are present. For this reason, we employ IQR for screening purposes. The constants required to obtain an unbiased estimate of σ based on IQR are 0.594 for $n = 4$, 0.990 for $n = 5$ and 1.144 for $n = 9$. The values chosen for the Phase I control limits are presented in Table 3.1. The subgroup screening is continued until all IQR's fall within the limits. The resulting estimates of σ are unbiased and are used to screen observations with an individuals control chart (the procedure used to derive $\overline{MD}^i$). The final estimates of σ are slightly biased. The constants necessary to obtain an unbiased estimate can be found in Table 3.1. The unbiased estimator is denoted by $\overline{MD}^{i,s}$.

3.2.2 Real data example

In this section we demonstrate the estimation of σ in Phase I. Our dataset was supplied by Wadsworth et al. (2001, pp. 235-237). The operation concerns the melt index of a polyethylene compound. The data consist of 20 subgroups of size 4 (Table 3.2).

The factors used for the $n = 4, k = 20$ case are presented in Table 3.1. Note that $d_2(4) = 2.06$, $c_4(4) = 0.92$, $t_2(4) = 0.66$ and $d_{IQR}(4) = 0.59$. The estimates of σ obtained by $\tilde{S}$ and $D7$ are determined in one iteration and are 10.14 and 6.59 respectively. The values obtained by $\bar{R}^s$ and $\overline{MD}^s$ incorporate subgroup screening. The initial value of $\bar{R}^s$ is 8.96 and the respective upper and lower control limits are 20.80 and 1.52. The unbiased estimate of the range (i.e. $\bar{R}/d_2(4)$) of subgroup 3 falls above the control limit and so this subgroup is deleted. The second estimate of $\bar{R}^s$ equals

Sample	Observations				$R/d_2(4)$	$S/c_4(4)$	$MD/t_2(4)$	$IQR/d_{IQR}(4)$
1	218	224	220	231	6.31	6.23	6.46	6.73
2	238	236	247	234	6.31	6.23	5.70	3.37
3	280	228	228	221	28.65	29.70	22.42	0
4	210	249	241	246	18.94	19.51	16.72	8.42
5	243	240	230	230	6.31	7.33	8.74	16.84
6	225	250	258	244	16.03	15.26	14.82	10.10
7	240	238	240	243	2.43	2.24	1.90	0
8	244	248	265	234	15.06	14.02	13.30	6.73
9	238	233	252	243	9.23	8.80	9.12	8.42
10	228	238	220	230	8.74	8.03	7.60	3.37
11	218	232	230	226	6.80	6.72	6.84	6.73
12	226	231	236	242	7.77	7.43	7.98	8.42
13	224	221	230	222	4.37	4.38	4.18	3.37
14	230	220	227	226	4.86	4.55	4.18	1.68
15	224	228	226	240	7.77	7.80	6.84	3.37
16	232	240	241	232	4.37	5.35	6.46	13.47
17	243	250	248	250	3.40	3.59	3.42	3.37
18	247	238	244	230	8.26	8.14	8.74	10.10
19	224	228	228	246	10.68	10.69	8.36	0
20	236	230	230	232	2.91	3.07	3.04	3.37

Table 3.2: Melt index measurements

7.92 and the corresponding Phase I upper and lower control limits are 18.38 and 1.35. Now subgroup 4 does not meet the Phase I upper control limit and is removed. The third estimate of $\bar{R}^s$ is 7.31 and the control limits are 16.97 and 1.24. There are no further subgroups whose $R/d_2(4)$ exceeds the upper control limit. The resulting unbiased estimate of σ is 7.31. The $\overline{MD}^s$ procedure works in a similar way. In this case, subgroups 3 and 4 are again deleted. The final unbiased estimate is 7.03.

For the $\overline{MD}^i$ chart, we use a procedure based on the individuals control chart for the residuals. The residuals are calculated by substracting the subgroup median from each observation in the corresponding subgroup (see Table 3.3). The initial value of σ is 8.26 and the control limits of the individuals chart are 24.78 and -24.78. One residual in subgroup 3 and one residual in subgroup 4 fall outside the control limits. The corresponding observations are deleted from the dataset. The subgroup medians are determined from

Sample	Residuals			
1	-4.0	2.0	-2.0	9.0
2	1.0	-1.0	10.0	-3.0
3	52.0	0.0	0.0	-7.0
4	-33.5	5.5	-2.5	2.5
5	8.0	5.0	-5.0	-5.0
6	-22.0	3.0	11.0	-3.0
7	0.0	-2.0	0.0	3.0
8	-2.0	2.0	19.0	-12.0
9	-2.5	-7.5	11.5	2.5
10	-1.0	9.0	-9.0	1.0
11	-10.0	4.0	2.0	-2.0
12	-7.5	-2.5	2.5	8.5
13	1.0	-2.0	7.0	-1.0
14	3.5	-6.5	0.5	-0.5
15	-3.0	1.0	-1.0	13.0
16	-4.0	4.0	5.0	-4.0
17	-6.0	1.0	-1.0	1.0
18	6.0	-3.0	3.0	-11.0
19	-4.0	0.0	0.0	18.0
20	5.0	-1.0	-1.0	1.0

Table 3.3: Residuals of melt index measurements

the remaining observations and the residuals are recalculated. The second estimate of σ is 6.82 and the control limits are now 20.47 and -20.47. One residual in subgroup 6 falls below the lower control limit and so one observation is removed. Again, the medians are determined from the remaining observations and the residuals are recomputed. The third estimate of σ is 6.49 and the control chart has limits at 19.47 and -19.47. There are now no residuals that fall outside the control limits. The resulting unbiased estimate is 6.55.

For the $\overline{MD}^{i,s}$ chart, the first part of the procedure screens the subgroup IQR. The respective upper and lower control limits of the IQR chart are 38.86 and 0.015. The IQR of subgroups 3, 7 and 19 are 0 and so these subgroups are deleted. It is not necessary to delete any further subgroups. Next, individual observations are screened. The estimate of σ is 7.81 and the upper and lower control limits for the residuals are 23.45 and -23.45.

An outlier in subgroup 4 is deleted. The next estimate of σ is 7.18 with corresponding control limits 21.55 and -21.55. The outlier in subgroup 6 is removed from the dataset. Now σ is set at 6.79 with corresponding control limits 20.37 and -20.37. No further deletions are required. The unbiased estimate for the $\overline{MD}^{i,s}$ chart is 6.87.

The final estimates for σ are presented in Table 3.4. The estimate based on $\tilde{S}$ is higher than the other estimates. This is because $\tilde{S}$ is more sensitive to outliers than the other estimators.

Chart	$\hat{\sigma}$
$\tilde{S}$	10.14
$D7$	6.59
$\bar{R}^s$	7.31
$\overline{MD}^s$	7.03
$\overline{MD}^i$	6.55
$\overline{MD}^{i,s}$	6.87

Table 3.4: Final estimates of σ

3.2.3 Efficiency of proposed estimators

In order to evaluate Phase I performance we now assess the MSE of the proposed Phase I estimators. Recall that the MSE is estimated as

$$MSE = \frac{1}{N}\sum_{i=1}^{N}(\hat{\sigma}^i - \sigma)^2,$$

where $\hat{\sigma}^i$ is the value of the unbiased estimate in the i-th simulation run and N is the number of simulation runs. Comparisons are made under normality and four types of disturbances (cf. Tatum (1997), see Section 2.2.2) but with an error rate of 6% in each case. The number of simulation runs N is equal to 50,000. (Note that Tatum (1997) used 10,000 simulation runs.)

Figure 3.1 shows the MSE values when diffuse symmetric variance disturbances are present. The y-intercepts show that the pooled standard deviation ($\tilde{S}$) has the lowest MSE when no disturbances are present. However, when the size of the disturbance (a) increases, the MSE increases quickly. The other estimators are more robust against outliers of this type. Those that use an individuals control chart to identify individual outliers, i.e. $\overline{MD}^i$ and $\overline{MD}^{i,s}$, coincide and perform best, followed by $D7$. The estimators based on only subgroup screening, namely $\bar{R}^s$ and $\overline{MD}^s$, turn out to perform less well in this situation. The reason is that they screen subgroup dispersion and ignore individual outliers. Note that $\bar{R}^s$ falls far short of $\overline{MD}^s$, because $\bar{R}^s$ uses $\bar{R}$ (rather than $\overline{MD}$) to estimate σ. As $\bar{R}$ is more sensitive to outliers, the Phase I limits are broader making it more difficult to detect outliers. This effect is particularly significant for $n = 9$, because a larger subgroup is more likely to be infected with an outlier.

When asymmetric diffuse disturbances are present (Figure 3.2), the results are comparable to the situation with diffuse symmetric disturbances: $\overline{MD}^i$ and $\overline{MD}^{i,s}$ coincide and perform best, followed by $D7$ and $\overline{MD}^s$. Note that in this situation $\tilde{S}$ and, for $n = 9$, $\bar{R}$ perform badly.

Figure 3.3 shows the results in situations with localized disturbances. The estimators incorporating subgroup screening ($\bar{R}^s$ and $\overline{MD}^s$) perform best. The estimator $\overline{MD}^{i,s}$ performs better than $D7$ in this situation. Finally, $\overline{MD}^i$ does not perform as well in this case because it does not take into account information on the subgroup spread.

The results for diffuse mean disturbance are shown in Figure 3.4. We can conclude that $\tilde{S}$ and $\bar{R}^s$ coincide for $n = 9$ and perform far worse than the other estimators. $\overline{MD}^s$ performs better but not as well as $D7$ and not as well as the estimators using an individuals chart to identify individual outliers. The reason is that $\overline{MD}^s$ is less capable of detecting such outliers. The estimators $\overline{MD}^{i,s}$ and $\overline{MD}^i$ coincide and perform best in this situation.

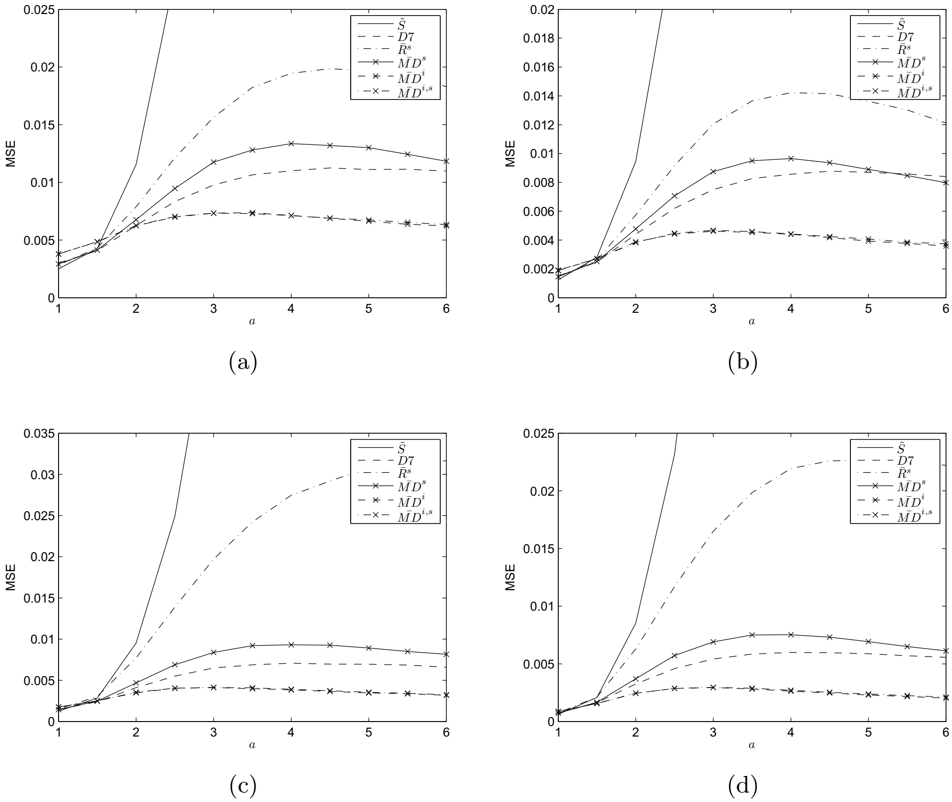


Figure 3.1: MSE of estimators when symmetric diffuse variance disturbances are present. (a) $n = 5, k = 50$ (b) $n = 5, k = 100$ (c) $n = 9, k = 50$ (d) $n = 9, k = 100$

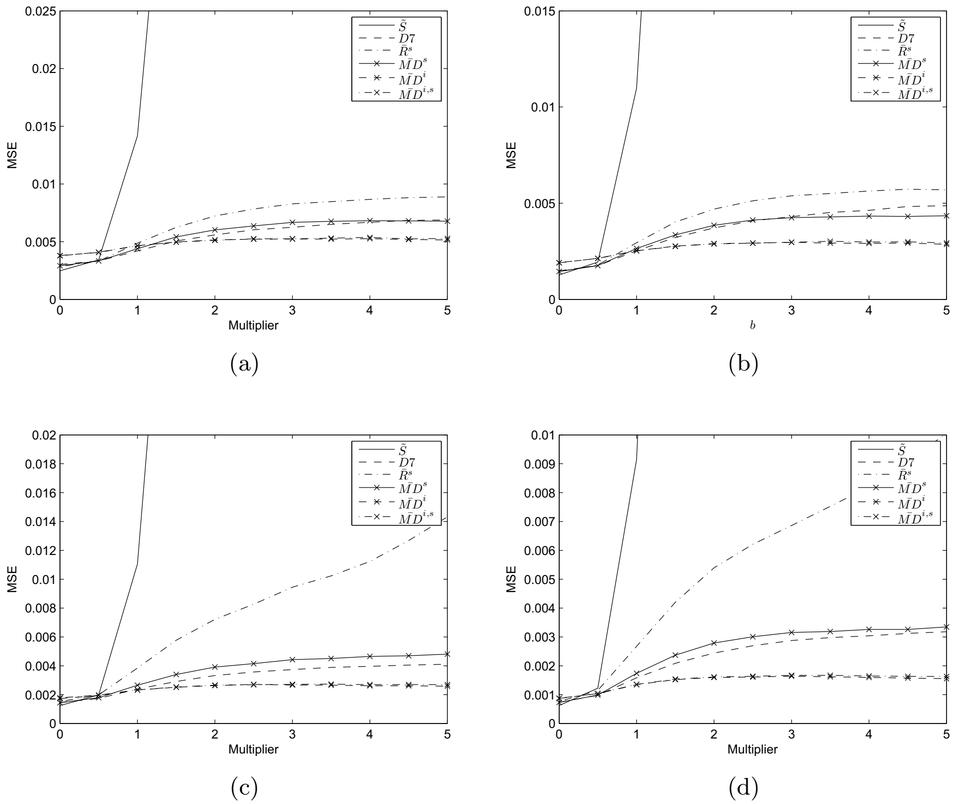
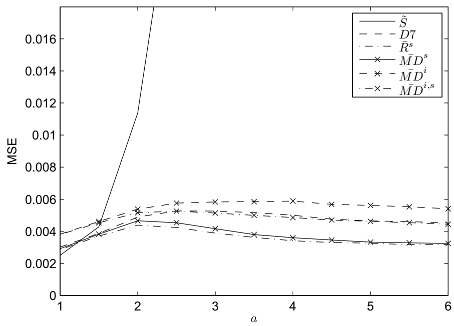
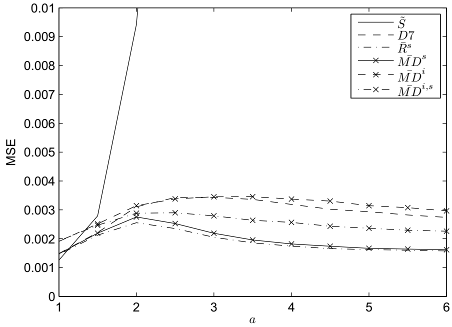


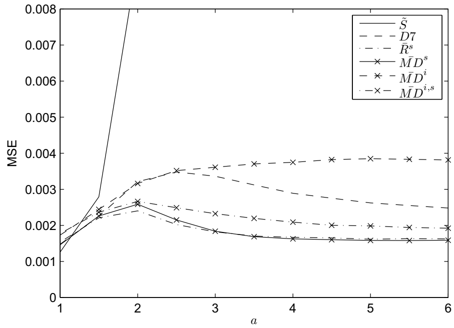
Figure 3.2: MSE of estimators when asymmetric diffuse variance disturbances are present. (a) $n = 5, k = 50$ (b) $n = 5, k = 100$ (c) $n = 9, k = 50$ (d) $n = 9, k = 100$



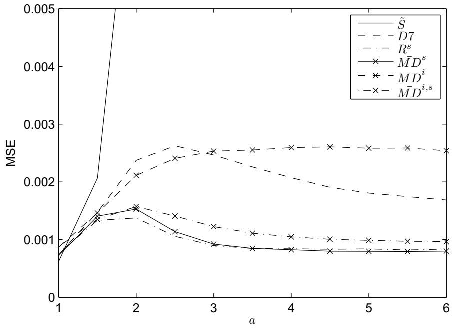
(a)



(b)



(c)



(d)

Figure 3.3: MSE of estimators when localized variance disturbances are present. (a) $n = 5, k = 50$ (b) $n = 5, k = 100$ (c) $n = 9, k = 50$ (d) $n = 9, k = 100$

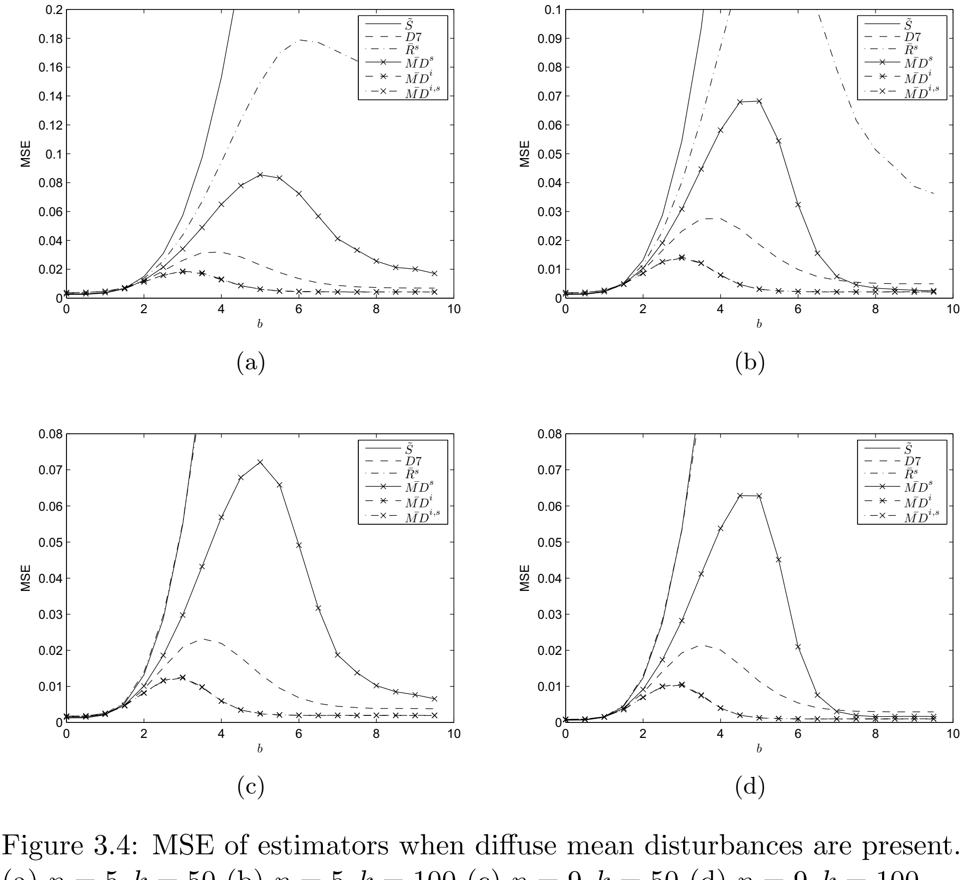


Figure 3.4: MSE of estimators when diffuse mean disturbances are present. (a) $n = 5, k = 50$ (b) $n = 5, k = 100$ (c) $n = 9, k = 50$ (d) $n = 9, k = 100$

3.3 Derivation of Phase II control limits

Equation (1.3) gives control limits for the standard deviation control chart with σ estimated in Phase I. We estimate $\lambda\sigma$ in Phase II by $S/c_4(n)$ for all charts. One of the criteria used to assess Phase II performance is the ARL. To allow comparison, U_n and L_n are chosen such that the unconditional ARL equals 370 and, for each chart, the ARL's for the upper and lower control limits are similar. U_n and L_n can not be obtained easily in analytic form and are obtained from 50,000 simulation runs. Table 3.5 presents U_n and L_n for $n = 5, 9$ and $k = 50, 100$.

		$k = 50$		$k = 100$	
n	$\hat{\sigma}$	U_n	L_n	U_n	L_n
5	$\tilde{S}$	2.230	0.163	2.236	0.169
	$D7$	2.225	0.162	2.236	0.167
	$\bar{R}^s$	2.226	0.163	2.236	0.168
	$\overline{MD}^s$	2.226	0.162	2.237	0.167
	$\overline{MD}^i$	2.217	0.160	2.333	0.166
	$\overline{MD}^{i,s}$	2.217	0.160	2.333	0.166
9	$\tilde{S}$	1.832	0.343	1.835	0.347
	$D7$	1.830	0.341	1.834	0.346
	$\bar{R}^s$	1.831	0.342	1.835	0.347
	$\overline{MD}^s$	1.830	0.342	1.835	0.346
	$\overline{MD}^i$	1.829	0.341	1.833	0.346
	$\overline{MD}^{i,s}$	1.829	0.341	1.833	0.346

Table 3.5: Factors U_n and L_n to determine Phase II control limits for $n = 5, 9$ and $k = 50, 100$

3.4 Control chart performance

We now evaluate the effect of the proposed estimators on the Phase II performance of the standard deviation control chart. We consider the same Phase I estimators as those used to asses the MSE with a, b and the multiplier equal to 4 to simulate the contaminated cases (see Section 2.2.2).

The performance of the control charts is assessed in terms of the unconditional ARL and SDRL. We compute these run length characteristics in an in-control and several out-of-control situations. We consider different shifts in the standard deviation $\lambda\sigma$, setting λ equal to 0.6, 1, 1.2 and 1.4. The performance characteristics are obtained by simulation. The next two sections describe the simulation method and the results for the control charts constructed in the uncontaminated and contaminated situations.

3.4.1 Simulation procedure

We use the same procedure as in Section 2.4.1. Enough replications of the procedure were performed to obtain sufficiently small relative estimated standard errors for the ARL. The relative standard error never exceeds 0.76%.

3.4.2 Simulation results

The ARL and SDRL are obtained in the in-control situation and in the out-of-control situation. When the process is in control, i.e. when $\lambda = 1$, we want the ARL and SDRL to be close to their intended values, namely 370. In the out-of-control situation, i.e. when $\lambda \neq 1$, we want to detect changes in the standard deviation as soon as possible, so the ARL should be as low as possible.

Table 3.6 shows the ARL and SDRL for the situation when the Phase I data are uncontaminated and normally distributed. The ARL is very similar across charts and the SDRL is slightly higher for the $\overline{MD}^i$ and $\overline{MD}^{i,s}$ charts.

Tables 3.7 and 3.8 show that, when there are disturbances in the Phase I data, the ARL values increase considerably. Thus, when the Phase I data are contaminated, changes in the process standard deviation are less likely to be detected. With diffuse disturbances (Table 3.7 and second half of Table 3.8), their impact is smallest for the charts based on $\overline{MD}^i$, $\overline{MD}^{i,s}$ and $D7$. When there are localized disturbances (second half of Table 3.8), the charts based on $\overline{R}^s$, $\overline{MD}^s$ and $\overline{MD}^{i,s}$ perform best, because these charts trim extreme subgroups. Note that in a number of cases the $\tilde{S}$, $\overline{R}^s$ and $\overline{MD}^s$ charts are ARL-biased: the in-control ARL is lower than the out-of-control ARL (cf. Jensen et al. (2006)).

Overall, the $\overline{MD}^{i,s}$ chart performs best. Under normality, this chart almost matches the standard chart based on $\tilde{S}$ and, in the presence of any contamination, the chart outperforms the alternatives.

			ARL				*SDRL*			
n	k	Chart	$\lambda = 0.6$	$\lambda = 1$	$\lambda = 1.2$	$\lambda = 1.4$	$\lambda = 0.6$	$\lambda = 1$	$\lambda = 1.2$	$\lambda = 1.4$
5	50	$\tilde{S}$	131	378	69.5	17.5	136	412	87.0	19.6
		$D7$	135	373	69.6	17.4	141	414	90.0	20.0
		$\bar{R}^s$	132	369	68.9	17.3	137	407	88.4	19.7
		$\overline{MD}^s$	135	375	69.7	17.4	141	414	90.0	20.0
		$\overline{MD}^i$	143	371	70.0	17.3	151	421	94.6	20.6
		$\overline{MD}^{i,s}$	143	371	69.9	17.4	151	421	95.0	20.7
	100	$\tilde{S}$	113	366	66.4	17.0	115	382	74.3	17.8
		$D7$	119	374	67.4	17.2	121	394	77.3	18.2
		$\bar{R}^s$	116	368	66.8	17.0	118	387	76.0	17.9
		$\overline{MD}^s$	119	375	67.3	17.3	121	394	76.8	18.2
		$\overline{MD}^i$	121	372	67.6	17.2	125	397	80.0	18.6
		$\overline{MD}^{i,s}$	122	371	67.8	17.2	125	396	80.2	18.5
9	50	$\tilde{S}$	28.4	371	43.6	9.02	29.3	392	51.6	9.30
		$D7$	29.5	373	43.9	9.06	30.8	397	53.5	9.50
		$\bar{R}^s$	29.2	392	43.9	9.05	30.5	392	53.9	9.52
		$\overline{MD}^s$	29.1	369	43.9	9.02	30.3	393	53.4	9.45
		$\overline{MD}^i$	29.8	368	44.7	9.11	31.4	395	56.0	9.71
		$\overline{MD}^{i,s}$	29.9	366	44.4	9.05	31.6	394	56.4	9.65
	100	$\tilde{S}$	26.0	373	42.2	8.90	26.2	382	45.7	8.77
		$D7$	26.4	375	42.8	8.98	26.7	386	47.1	8.92
		$\bar{R}^s$	26.3	369	42.3	8.93	26.6	380	46.8	8.89
		$\overline{MD}^s$	26.6	375	42.6	8.96	26.8	386	42.6	8.91
		$\overline{MD}^i$	26.7	369	42.2	8.92	27.1	382	47.3	8.94
		$\overline{MD}^{i,s}$	26.7	367	42.3	8.90	27.2	380	47.5	8.92

Table 3.6: ARL and SDRL under normality

				ARL				*SDRL*			
	n	k	Chart	$\lambda = 0.6$	$\lambda = 1$	$\lambda = 1.2$	$\lambda = 1.4$	$\lambda = 0.6$	$\lambda = 1$	$\lambda = 1.2$	$\lambda = 1.4$
$N(0,1)$	5	50	$\tilde{S}$	43.9	297	425	303	50.8	337	452	391
&			$D7$	102	484	159	34.7	108	498	215	47.2
$N(0,4)$			$\bar{R}^s$	92.2	464	206	50.1	99.9	479	286	87.2
(sym.)			$\overline{MD}^s$	101	474	171	38.2	108	492	171	58.6
			$\overline{MD}^i$	123	443	114	25.3	131	479	164	34.2
			$\overline{MD}^{i,s}$	122	446	114	25.6	131	481	164	34.6
		100	$\tilde{S}$	35.9	248	413	308	38.5	269	425	362
			$D7$	89.7	476	152	32.8	91.4	479	182	37.8
			$\bar{R}^s$	81.4	458	191	42.0	84.6	462	239	55.2
			$\overline{MD}^s$	89.0	470	159	34.4	89.0	470	159	42.0
			$\overline{MD}^i$	104	443	107	24.3	108	458	133	28.0
			$\overline{MD}^{i,s}$	104	444	108	28.0	108	458	134	28.0
	9	50	$\tilde{S}$	5.47	114	317	286	5.92	148	354	351
			$D7$	19.9	433	109	17.0	20.8	440	144	20.1
			$\bar{R}^s$	14.3	336	238	47.5	368	238	47.5	
			$\overline{MD}^s$	18.9	410	128	19.6	20.3	421	178	26.3
			$\overline{MD}^i$	23.8	420	75.2	12.9	25.3	433	102	15.0
			$\overline{MD}^{i,s}$	23.7	421	75.9	13.0	25.2	434	103	15.0
		100	$\tilde{S}$	4.86	95.6	305	302	4.75	110	331	345
			$D7$	17.9	420	104	16.5	18.0	423	122	17.6
			$\bar{R}^s$	12.9	323	225	36.0	12.9	343	276	53.4
			$\overline{MD}^s$	17.3	409	118	18.2	17.3	414	118	18.2
			$\overline{MD}^i$	21.3	420	70.0	12.4	21.7	424	82.2	13.1
			$\overline{MD}^{i,s}$	21.2	420	70.6	12.5	21.6	424	83.2	13.2
$N(0,1)$	5	50	$\tilde{S}$	23.2	149	231	266	38.4	243	325	355
&			$D7$	112	461	121	26.9	118	483	164	34.2
$N(0,4)$			$\bar{R}^s$	108	450	133	29.9	115	472	191	43.6
(asym.)			$\overline{MD}^s$	115	449	119	26.6	121	475	168	35.7
			$\overline{MD}^i$	130	419	92.7	21.7	138	461	131	27.5
			$\overline{MD}^{i,s}$	130	422	95.0	21.9	138	463	135	27.9
		100	$\tilde{S}$	14.9	98.0	183	262	21.3	147	253	262
			$D7$	98.0	458	116	25.9	100	466	138	28.8
			$\bar{R}^s$	95.2	447	122	27.3	97.9	455	151	31.9
			$\overline{MD}^s$	101	448	111	25.1	103	458	111	28.5
			$\overline{MD}^i$	111	419	88.9	21.1	114	438	109	23.5
			$\overline{MD}^{i,s}$	110	421	89.8	21.2	114	440	109	23.7
	9	50	$\tilde{S}$	2.34	33.3	93.6	165	3.27	79.5	186	263
			$D7$	22.7	435	80.0	13.5	23.7	443	104	15.2
			$\bar{R}^s$	18.8	399	140	22.6	20.6	415	207	39.7
			$\overline{MD}^s$	22.5	418	83.3	14.0	23.9	429	116	16.7
			$\overline{MD}^i$	26.0	407	61.1	11.2	27.5	425	80.5	12.6
			$\overline{MD}^{i,s}$	25.7	409	61.7	11.3	27.3	426	81.4	12.7
		100	$\tilde{S}$	1.79	18.8	59.5	140	1.72	34.5	112	220
			$D7$	20.4	429	75.9	13.2	20.5	431	87.2	13.7
			$\bar{R}^s$	17.1	396	123	18.9	17.6	403	159	22.9
			$\overline{MD}^s$	20.6	424	77.4	13.4	20.6	428	92.7	14.3
			$\overline{MD}^i$	23.2	408	57.4	10.9	23.6	415	66.3	11.2
			$\overline{MD}^{i,s}$	23.1	411	67.4	11.4	23.4	417	67.4	11.4

Table 3.7: ARL and SDRL when contaminations are present in Phase I

	n	k	Chart	*ARL* $\lambda = 0.6$	$\lambda = 1$	$\lambda = 1.2$	$\lambda = 1.4$	*SDRL* $\lambda = 0.6$	$\lambda = 1$	$\lambda = 1.2$	$\lambda = 1.4$
$N(0,1)$	5	50	$\tilde{S}$	42.8	293	436	308	48.3	330	459	390
&			$D7$	118	442	103	23.5	124	469	139	28.7
$N(0,4)$			$\bar{R}^s$	129	384	77.1	18.7	134	422	103	22.2
(loc.)			$\overline{MD}^s$	131	391	78.5	19.0	137	431	106	22.9
			$\overline{MD}^i$	127	428	100	22.9	136	468	143	29.5
			$\overline{MD}^{i,s}$	135	404	85.9	20.4	144	451	122	25.7
		100	$\tilde{S}$	35.6	245	419	312	37.7	263	429	361
			$D7$	103	441	99.1	23.0	105	451	117	25.1
			$\bar{R}^s$	113	382	73.1	18.3	113	401	85.1	19.7
			$\overline{MD}^s$	115	392	74.6	18.5	117	411	87.9	20.0
			$\overline{MD}^i$	109	429	95.0	22.1	112	446	117	25.0
			$\overline{MD}^{i,s}$	115	403	81.6	19.7	118	425	99.6	21.9
	9	50	$\tilde{S}$	5.34	110	324	295	5.55	136	357	254
			$D7$	24.3	427	67.9	12.1	25.4	438	86.9	13.3
			$\bar{R}^s$	28.8	372	46.0	9.31	30.2	396	57.8	9.93
			$\overline{MD}^s$	28.6	372	46.0	9.31	30.0	396	57.5	9.91
			$\overline{MD}^i$	23.9	420	74.4	12.8	25.5	433	100	14.7
			$\overline{MD}^{i,s}$	28.9	377	49.1	9.68	30.7	404	64.4	10.6
		100	$\tilde{S}$	4.80	93.8	306	309	4.59	105	329	347
			$D7$	21.8	425	65.3	11.9	22.0	428	74.0	12.2
			$\bar{R}^s$	25.9	372	43.9	9.12	26.2	384	49.1	9.14
			$\overline{MD}^s$	26.2	380	44.1	9.19	26.5	391	49.0	9.20
			$\overline{MD}^i$	21.4	420	69.4	12.4	21.8	424	81.6	13.0
			$\overline{MD}^{i,s}$	25.8	379	46.1	9.42	26.3	391	52.8	9.59
$N(0,1)$	5	50	$\tilde{S}$	40.2	280	470	335	42.8	300	480	395
&			$D7$	80.3	471	286	72.8	86.7	483	359	117
$N(4,1)$			$\bar{R}^s$	54.1	358	431	207	61.1	384	466	292
			$\overline{MD}^s$	65.0	409	394	144	73.0	431	450	220
			$\overline{MD}^i$	115	447	152	64.4	127	481	237	64.4
			$\overline{MD}^{i,s}$	115	449	152	63.5	127	483	234	63.5
		100	$\tilde{S}$	34.4	237	434	328	35.1	246	439	360
			$D7$	70.0	448	282	65.2	72.5	454	327	85.4
			$\bar{R}^s$	46.6	320	436	193	49.2	336	450	248
			$\overline{MD}^s$	55.9	378	399	129	58.9	392	429	173
			$\overline{MD}^i$	98.3	451	136	30.2	103	464	183	39.7
			$\overline{MD}^{i,s}$	98.0	451	136	30.2	103	464	182	39.4
	9	50	$\tilde{S}$	5.04	101	325	321	4.87	114	348	363
			$D7$	14.4	358	230	35.1	15.3	380	291	51.4
			$\bar{R}^s$	5.59	117	341	292	5.84	144	367	353
			$\overline{MD}^s$	9.53	231	370	99.0	10.4	266	404	151
			$\overline{MD}^i$	22.8	409	93.6	15.2	25.0	425	142	20.9
			$\overline{MD}^{i,s}$	22.8	410	92.1	15.2	25.0	425	140	21.0
		100	$\tilde{S}$	4.67	89.6	302	327	4.29	94.8	316	351
			$D7$	12.9	330	225	32.2	13.0	343	262	38.7
			$\bar{R}^s$	5.00	99.7	322	294	4.77	110	338	332
			$\overline{MD}^s$	8.59	207	387	88.6	8.64	226	403	117
			$\overline{MD}^i$	20.4	413	82.2	13.9	21.0	418	105	15.6
			$\overline{MD}^{i,s}$	20.4	414	81.7	13.8	21.0	418	104	15.4

Table 3.8: ARL and SDRL when contaminations are present in Phase I

3.5 Concluding remarks

In this chapter we consider several estimators of the standard deviation in Phase I of the control charting process. We have found that the performance of certain robust estimators is almost identical to the pooled subgroup standard deviation under normality, while the benefit of using such robust estimators can be substantial when there are disturbances. Following Rocke (1989, 1992), we have considered estimators that include a procedure for subgroup screening, but whereas Rocke used $\bar{R}$, we have used the average deviation from the median. This estimator performs better when there are localized disturbances and is much more robust against diffuse variance disturbances. However, when there are diffuse mean disturbances, the procedure loses efficiency.

To address this problem, we have proposed other algorithms, based on a procedure that also screens for individual outliers. The algorithms remove the variation between subgroups, so that only the variation within subgroups is measured. We have shown that these algorithms are very effective when there are diffuse disturbances. When there might also be localized disturbances, the method can be combined with subgroup screening based on the IQR. The latter procedure reveals a performance very similar to the robust estimator for the standard deviation control chart proposed by Tatum (1997). We think that this is a noteworthy outcome since the procedure is simple and intuitive. Moreover, it can be used to estimate σ in other practical applications.

Chapter 4

Location Estimators for $\overline{X}$ Control Charts

4.1 Introduction

This chapter studies estimation methods for the location parameter. We consider several robust location estimators as well as several estimation methods based on a Phase I analysis (recall that this is the use of a control chart to study a historical dataset retrospectively to identify disturbances). In addition, we propose a new type of Phase I analysis. The estimation methods are evaluated in terms of their MSE and their effect on the $\overline{X}$ control charts used for real-time process monitoring (Phase II). It turns out that the Phase I control chart based on the trimmed trimean far outperforms the existing estimation methods. This method has therefore proven to be very suitable for determining $\overline{X}$ Phase II control chart limits.

The remainder of the chapter is organized as follows. First, we present several Phase I sample statistics for the process location and compare their MSE. Then we describe some existing Phase I control charts and present a new algorithm for Phase I analysis. Following that, we present the design schemes for the $\overline{X}$ Phase II control chart and derive the control limits. Next, we describe the simulation procedure and present the effect of the proposed methods on Phase II performance. The final section offers some recommendations.

4.2 Proposed location estimators

To understand the behavior of the estimators it is again useful to distinguish diffuse and localized disturbances (cf. Tatum (1997)). As explained in Section 1.2, diffuse disturbances are outliers that are spread over all of the samples whereas localized disturbances affect all observations in one sample. We look at various types of estimators (both robust estimators and several estimation methods based on the principle of control charting) in Section 4.2.1 and compare their MSE in Section 4.2.2.

4.2.1 Location estimators

Recall that X_{ij}, $i = 1, 2, ..., k$ and $j = 1, 2, ..., n$, denote the Phase I data. The X_{ij}'s are assumed to be independent and largely $N(\mu, \sigma^2)$ distributed. We denote by $X_{i,(v)}$, $v = 1, 2, ..., n$, the v-th order statistic in sample i.

The first estimator that we consider is the mean of the sample means

$$\overline{\overline{X}} = \frac{1}{k}\sum_{i=1}^{k}\overline{X}_i = \frac{1}{k}\sum_{i=1}^{k}\left(\frac{1}{n}\sum_{j=1}^{n}X_{ij}\right).$$

This estimator is included to provide a basis for comparison, as it is the most efficient estimator for normally distributed data. However, it is well known that this estimator is not robust against outliers.

We also consider three robust estimators proposed earlier by Rocke (1989). They are the median of the sample means

$$M(\overline{X}) = median(\overline{X}_1, \overline{X}_2, ..., \overline{X}_k),$$

the mean of the sample medians

$$\overline{M} = \frac{1}{k}\sum_{i=1}^{k}M_i,$$

with M_i the median of sample i, and the trimmed mean of the sample means

$$\overline{\overline{X}}_\alpha = \frac{1}{k - 2\lceil k\alpha\rceil}\times\left[\sum_{v=\lceil k\alpha\rceil+1}^{k-\lceil k\alpha\rceil}\overline{X}_{(v)}\right],$$

where α denotes the percentage of samples to be trimmed, $\lceil z\rceil$ denotes the ceiling function, i.e. the smallest integer not less than z, and $\overline{X}_{(v)}$ denotes the v-th ordered value of the sample means. In our study, we consider the 20% trimmed mean, which trims the six smallest and the six largest sample means when $k = 30$. Of course other trimming percentages could have been used. In fact, we have also used 10% and 25% but the results with 20% are representative for this estimator.

Furthermore, our analysis includes the Hodges-Lehmann estimator (Hodges and Lehmann (1963)), an estimator based on the so-called Walsh averages. The h $(= n(n+1)/2)$ Walsh averages of sample i are

$$W_{i,k,l} = (X_{i,k} + X_{i,l})/2, k = 1, 2, ..., n, l = 1, 2, ..., n, k \leq l.$$

The Hodges-Lehmann estimate for sample i, denoted by HL_i, is defined as the median of the Walsh averages. Alloway and Raghavachari (1991) conducted a Monte Carlo simulation to determine whether the mean or the median of the sample Hodges-Lehmann estimates should be used to determine the final location estimate. They concluded that the mean of the sample values should be used

$$\overline{HL} = \frac{1}{k}\sum_{i=1}^{k} HL_i$$

and that the resulting estimate is unbiased.

We also include the trimean statistic. The trimean of sample i is the weighted average of the sample median and the two other quartiles

$$TM_i = (Q_{i,1} + 2Q_{i,2} + Q_{i,3})/4,$$

where $Q_{i,q}$ is the q-th quartile of sample i, $q = 1, 2, 3$ (cf. Tukey (1997) and Wang et al. (2007)). It also equals the average of the median and the midhinge $1/2\left(Q_{i,2} + \frac{Q_{i,1}+Q_{i,3}}{2}\right)$ (cf. Weisberg (1992)). We use the following definitions for the quartiles: $Q_{i,1} = X_{i,(a)}$ and $Q_{i,3} = X_{i,(b)}$ with $a = \lceil n/4 \rceil$ and $b = n - a + 1$. This means that $Q_{i,1}$ and $Q_{i,3}$ are defined as the second smallest and the second largest observations respectively for $5 \leq n \leq 8$, and as the third smallest and the third largest values respectively for $9 \leq n \leq 12$. Like the median and the midhinge, but unlike the sample mean, the trimean is a statistically resistant L-estimator (a linear combination of order statistics), with a breakdown point of 25% (see Wang et al. (2007)). According to Tukey (1977), using the trimean instead of the median gives a more useful assessment of location or centering. According to Weisberg

(1992), the “statistical resistance” benefit of the trimean as a measure of the center of a distribution is that it combines the median’s emphasis on center values with the midhinge’s attention to the extremes. The trimean is almost as resistant to extreme scores as the median and is less subject to sampling fluctuations than the arithmetic mean in extremely skewed distributions. Asymptotic distributional results of the trimean can be found in Wang et al. (2007). The location estimate analyzed below is the mean of the sample trimeans, i.e.

$$\overline{TM} = \frac{1}{k}\sum_{i=1}^{k} TM_i.$$

Finally, we consider a statistic that is expected to be robust against both diffuse and localized disturbances, namely the trimmed mean of the sample trimeans, defined by

$$\overline{TM}_\alpha = \frac{1}{k-2\lceil k\alpha\rceil} \times \left[\sum_{v=\lceil k\alpha\rceil+1}^{k-\lceil k\alpha\rceil} TM_{(v)}\right],$$

where $TM_{(v)}$ denotes the v-th ordered value of the sample trimeans. We consider the 20% trimmed trimean, which trims the six smallest and the six largest sample trimeans when $k = 30$.

The estimators outlined above are summarized in Table 4.1.

Estimator	Notation
Mean of sample means	$\overline{\overline{X}}$
Median of sample means	$M(\overline{X})$
Mean of sample medians	$\overline{M}$
20% trimmed mean of sample means	$\overline{\overline{X}}_{20}$
Mean of sample Hodges-Lehmann	$\overline{HL}$
Mean of sample trimeans	$\overline{TM}$
20% trimmed mean of sample trimeans	$\overline{TM}_{20}$

Table 4.1: Proposed location estimators

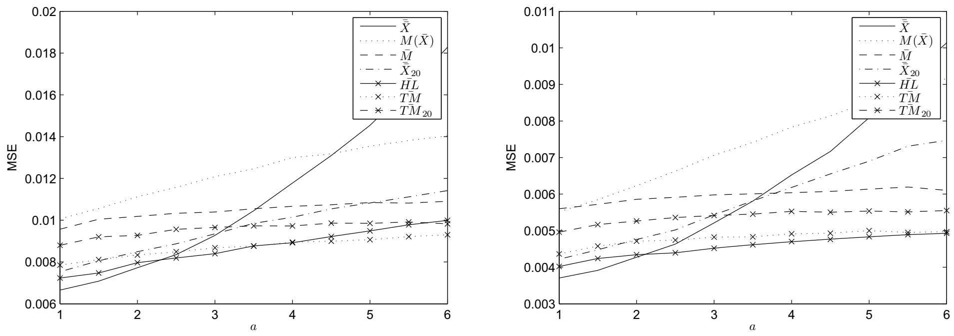
4.2.2 Efficiency of proposed estimators

As in Chapters 2 and 3, we follow Tatum (1997) and evaluate the estimators in terms of their MSE. In this case, the MSE is estimated as

$$MSE = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\hat{\mu}^i - \mu}{\sigma}\right)^2.$$

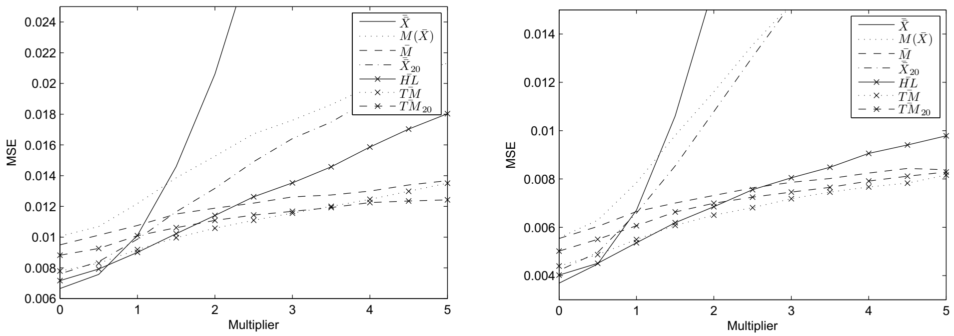
We include the uncontaminated case, i.e. the situation where all the X_{ij}'s are from the $N(0,1)$ distribution, as well as five types of disturbances. They are the four models described in Section 2.2.2 and

5. A model for localized mean disturbances in which observations in 3 out of 30 samples are drawn from the $N(a,1)$ distribution, with $a = 0.5, 1.0, ..., 5.5, 6.0$.



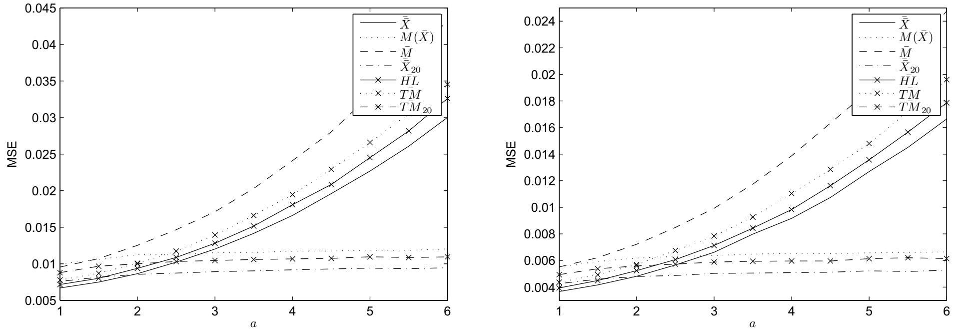
(a) (b)

Figure 4.1: MSE of estimators when symmetric diffuse variance disturbances are present for $k = 30$ (a) $n = 5$ (b) $n = 9$



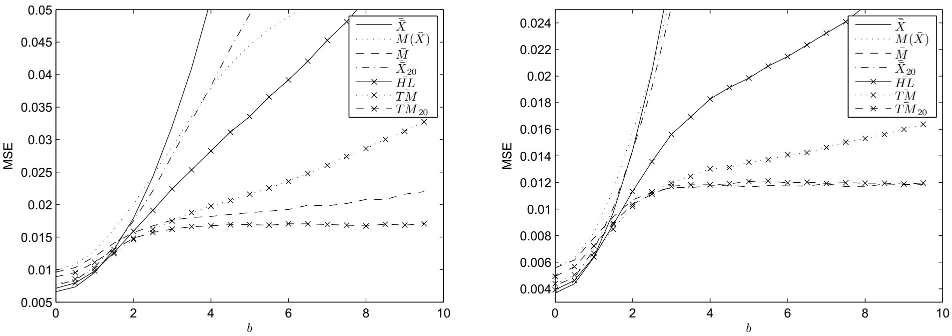
(a) (b)

Figure 4.2: MSE of estimators when asymmetric diffuse variance disturbances are present for $k = 30$ (a) $n = 5$ (b) $n = 9$



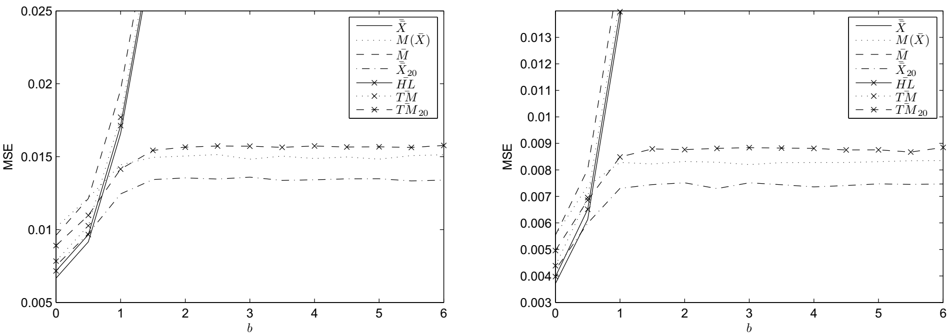
(a) (b)

Figure 4.3: MSE of estimators when localized variance disturbances are present for $k = 30$ (a) $n = 5$ (b) $n = 9$



(a) (b)

Figure 4.4: MSE of estimators when diffuse mean disturbances are present for $k = 30$ (a) $n = 5$ (b) $n = 9$



(a) (b)

Figure 4.5: MSE of estimators when localized mean disturbances are present for $k = 30$ (a) $n = 5$ (b) $n = 9$

The figures show that in the uncontaminated situation the most efficient estimator is $\overline{\overline{X}}$ as was to be expected. The estimators $\overline{HL}$, $\overline{\overline{X}}_{20}$ and $\overline{TM}$ are slightly less efficient followed by $\overline{TM}_{20}$, $\overline{M}$ and $M(\overline{X})$, the reason being that they use less information.

When diffuse symmetric variance disturbances are present (Figure 4.1), the best performing estimators are $\overline{HL}$ and $\overline{TM}$. The reason why $\overline{TM}$ performs well in this situation is that it filters out the extreme high and low values in each sample. $\overline{HL}$ also performs well because it obtains the sample statistic using the median of the Walsh averages, which is not sensitive to outliers. $\overline{M}$ and $\overline{TM}_{20}$ are as efficient in the contaminated situation as in the uncontaminated situation but they are outperformed by $\overline{HL}$ and $\overline{TM}$ because the latter estimators use more information. It is worth noting that the traditional estimator $\overline{\overline{X}}$ shows relatively bad results despite the symmetric character of the outliers. $M(\overline{X})$ and $\overline{\overline{X}}_{20}$ do not perform very well because these estimators focus on extreme samples whereas in the present situation the outliers are spread over all of the samples so that the non-trimmed samples are also infected.

When asymmetric variance disturbances are present (Figure 4.2), the most efficient estimators are $\overline{TM}$, $\overline{TM}_{20}$, $\overline{HL}$ and $\overline{M}$, performing particularly well relative to the other estimators for larger sample sizes. As for the symmetric diffuse case, the estimators that include a method to trim observations within a sample perform better than the methods that focus on sample trimming.

In the case of localized variance disturbances (Figure 4.3), the estimators based on the principle of trimming sample means rather than within-sample observations - $\overline{\overline{X}}_{20}$, $\overline{TM}_{20}$ and $M(\overline{X})$ - have the lowest MSE . The estimators $\overline{\overline{X}}$, $\overline{HL}$, $\overline{TM}$ and in particular $\overline{M}$ are less successful because these statistics only perform well if no more than a few observations in a sample are infected rather than all observations, as is the case here.

When diffuse mean disturbances are present (Figure 4.4), the results are comparable to the situation where there are diffuse asymmetric variance disturbances: $\overline{M}$, $\overline{TM}_{20}$ and $\overline{TM}$ perform best, followed by $\overline{HL}$. Note that in this situation $\overline{\overline{X}}$, $M(\overline{X})$ and $\overline{\overline{X}}_{20}$ perform badly.

When localized mean disturbances are present (Figure 4.5), the results are comparable to the situation where there are localized variance disturbances: the estimators based on the principle of trimming sample means, namely $\overline{\overline{X}}_{20}$, $M(\overline{X})$ and $\overline{TM}_{20}$, perform best.

To summarize, $\overline{M}$, $\overline{TM}$ and $\overline{TM}_{20}$ have the lowest MSE when there are diffuse disturbances. $\overline{M}$ and $\overline{TM}$ lose their efficiency advantage when contaminations take the form of localized mean or variance disturbances. In such situations, $M(\overline{X})$, $\overline{\overline{X}}_{20}$ and $\overline{TM}_{20}$, which involve trimming the sample means, perform relatively well. $\overline{TM}_{20}$ has the best performance overall because it is reasonably robust against all types of contaminations.

4.3 Proposed control chart location estimators

In-control process parameters can be obtained not only via robust statistics but also via Phase I control charting. In the latter case, control charts are used retrospectively to study a historical dataset and identify samples that are deemed out of control. The process parameters are then estimated from the in-control samples. In this section, we consider several Phase I analyses which apply the principle of control charting in order to generate robust estimates of process location. We study a Phase I control chart based on the commonly used estimator $\overline{\overline{X}}$ and a Phase I control chart based on the mean rank proposed by Jones-Farmer et al. (2009). Moreover, we propose two new types of Phase I analyses. The next section presents the various Phase I control charts and the following section shows the MSE of the proposed estimation methods.

4.3.1 Phase I control charts

The standard procedure in practice is to use the estimator $\overline{\overline{X}}$ for constructing the $\overline{X}$ Phase I control chart limits. The respective upper and lower control limits of the Phase I chart are given by $\widehat{UCL}_{\bar{\bar{X}}} = \overline{\overline{X}} + 3\hat{\sigma}/\sqrt{n}$ and $\widehat{LCL}_{\bar{\bar{X}}} = \overline{\overline{X}} - 3\hat{\sigma}/\sqrt{n}$, where we estimate σ by the robust standard deviation estimator proposed by Tatum (1997), using the corrected normalizing constants presented in Chapter 2. The samples whose $\overline{X}_i$ fall above $\widehat{UCL}_{\bar{\bar{X}}}$ or below $\widehat{LCL}_{\bar{\bar{X}}}$ are eliminated from the Phase I dataset. The final location estimate is the mean of the sample means of the remaining samples

$$\overline{\overline{X}}' = \frac{1}{k'} \sum_{i \epsilon K} \overline{X}_i \times I_{\widehat{LCL}_{\bar{\bar{X}}} \leq \bar{X}_i \leq \widehat{UCL}_{\bar{\bar{X}}}}(\overline{X}_i),$$

with K the set of samples which are not excluded, k' the number of non-excluded samples and I the indicator function. This adaptive trimmed mean estimator is denoted by $ATM_{\bar{\bar{X}}}$.

We also consider a Phase I analysis that is based on the mean rank proposed by Jones-Farmer et al. (2009). It is a nonparametric estimation method which treats the observations from the k mutually exclusive samples of size n as a single sample of $N = n \times k$ observations. Let $R_{ij} = 1, 2, ..., N$ denote the integer rank of observation X_{ij} in the pooled sample of size N. Let $\bar{R}_i = (\sum_{j=1}^{n} R_{ij})/n$ be the mean of the ranks in sample i. If the process is in control, the ranks should be distributed evenly throughout the samples. For an in-control process, the mean and variance of $\bar{R}_i$ are

$$E(\bar{R}_i) = \frac{N+1}{2}$$

and

$$Var(\bar{R}_i) = \frac{(N-n)(N+1)}{12n}.$$

According to the central limit theorem, the random variable

$$Z_i = \frac{\bar{R}_i - E(\bar{R}_i)}{\sqrt{var(\bar{R}_i)}}$$

follows approximately a standard normal distribution for large values of n. A control chart for these Z_i's can be constructed with center line equal to 0, upper control limit 3 and lower control limit -3. The samples with Z_i outside the Phase I control limits are considered to be out of control and are excluded from the dataset. The location estimate is obtained from the mean of the remaining sample means

$$\overline{\overline{X}}^* = \frac{1}{k^*} \sum_{i \epsilon K^*} \overline{X}_i \times I_{-3 \leq Z_i \leq 3}(Z_i),$$

with K^* the set of samples which are not excluded and k^* the number of non-excluded samples. This estimation method is denoted by ATM_{MR}.

We now present two new Phase I analyses based on the principle of control charting. For the first method, we build a Phase I control chart using a robust estimator. The advantage of a robust estimator over a sensitive estimator like $\overline{\overline{X}}$ is that the Phase I control limits are not affected by any disturbances so that the correct out-of-control observations are filtered out in Phase I. An estimator shown to be very robust by the MSE study in Section 4.2.2 is $\overline{TM}_{20}$. A disadvantage is that the estimator is not very efficient under normality. To address this, we use $\overline{TM}_{20}$ to construct the Phase I limits with which we screen $\overline{X}_i$ for disturbances, but then use the efficient estimator $\overline{\overline{X}}$ to obtain the location estimate from the remaining samples. The Phase I control limits are given by $\widehat{UCL}_{\bar{TM}_{20}} = \overline{TM}_{20} + 3\hat{\sigma}/\sqrt{n}$ and $\widehat{LCL}_{\bar{TM}_{20}} = \overline{TM}_{20} - 3\hat{\sigma}/\sqrt{n}$, where we estimate σ by Tatum's estimator. We then plot the $\overline{X}_i$'s on the Phase I control chart. The samples whose $\overline{X}_i$ falls outside the limits are regarded as out of control and removed from the dataset. The remaining samples are used to determine the grand sample mean

$$\overline{\overline{X}}^{\#} = \frac{1}{k^{\#}} \sum_{i \epsilon K^{\#}} \overline{X}_i \times I_{\widehat{LCL}_{\bar{TM}_{20}} \leq \bar{X}_i \leq \widehat{UCL}_{\bar{TM}_{20}}}(\overline{X}_i),$$

with $K^{\#}$ the set of samples which are not excluded and $k^{\#}$ the number of non-excluded samples. The resulting estimator is denoted by $ATM_{\bar{TM}_{20}}$.

The fourth type of Phase I control chart resembles the chart presented above, but employs a different method to screen for disturbances. The procedure consists of two steps.

In the first step we construct the control chart with limits as we did just before. Note that, for the sake of practical applicability, we use the same factor, namely 3, to derive the $\overline{X}$ and TM charts. We then plot the TM_i's of the Phase I samples on the control chart. Charting the TM_i's instead of the $\overline{X}_i$'s ensures that localized disturbances are identified and samples that contain only one single outlier are retained. A location estimator that is expected to be robust against localized mean disturbances is the mean of the sample trimeans of the samples that fall between the control limits

$$\overline{TM}' = \frac{1}{k^{\wedge}} \sum_{i \epsilon K^{\wedge}} TM_i \times I_{\widehat{LCL}_{\bar{TM}_{20}} \leq TM_i \leq \widehat{UCL}_{\bar{TM}_{20}}}(TM_i),$$

with $K^{\wedge}$ the set of samples which are not excluded and $k^{\wedge}$ the number of non-excluded samples.

Although the remaining Phase I samples are expected to be free from localized mean disturbances, they could still contain diffuse disturbances. To eliminate such disturbances, the second step is to screen the individual observations using a Phase I individuals control chart with respective upper and lower control limits given by $\widehat{UCL}_{\bar{TM}'} = \overline{TM}' + 3\hat{\sigma}$ and $\widehat{LCL}_{\bar{TM}'} = \overline{TM}' - 3\hat{\sigma}$, where σ is estimated by Tatum's estimator. The observations X_{ij} that fall above $\widehat{UCL}_{\bar{TM}'}$ or below $\widehat{LCL}_{\bar{TM}'}$ are considered out of control and removed from the Phase I dataset. The final estimate is the mean of the sample means and is calculated from the observations deemed to be in control

$$\overline{\overline{X}}'' = \frac{1}{k''} \sum_{i \epsilon K''} \frac{1}{n_i'} \sum_{j \epsilon N_i'} X_{ij} \times I_{\widehat{LCL}_{\bar{TM}'} \leq X_{ij} \leq \widehat{UCL}_{\bar{TM}'}}(X_{ij}),$$

with K'' the set of samples which are not excluded, k'' the number of non-excluded samples, N_i' the set of non-excluded observations in sample i and n_i' the number of non-excluded observations in sample i. Note that we could also have used the double sum, divided by the sum of the n_i'. The advantage of our procedure is that, when a sample is infected by a localized disturbance, the disturbance will have a lower impact on the final location estimate when it is not detected. This estimation method is denoted by $ATM_{\bar{TM}'}$.

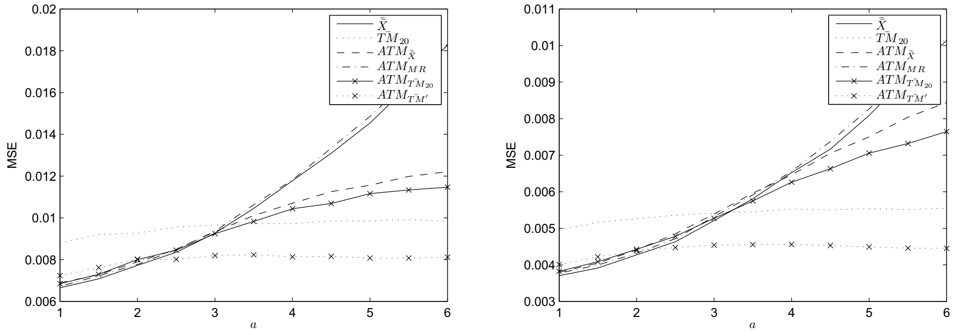
The proposed Phase I analyses are summarized in Table 4.2.

Phase I analysis	Notation
$\overline{\overline{X}}$ control chart with screening	$ATM_{\bar{\bar{X}}}$
Mean rank control chart with screening	ATM_{MR}
$\overline{TM}_{20}$ control chart with screening	$ATM_{\bar{TM}_{20}}$
$\overline{TM}'$ control chart with screening	$ATM_{\bar{TM}'}$

Table 4.2: Proposed Phase I analyses

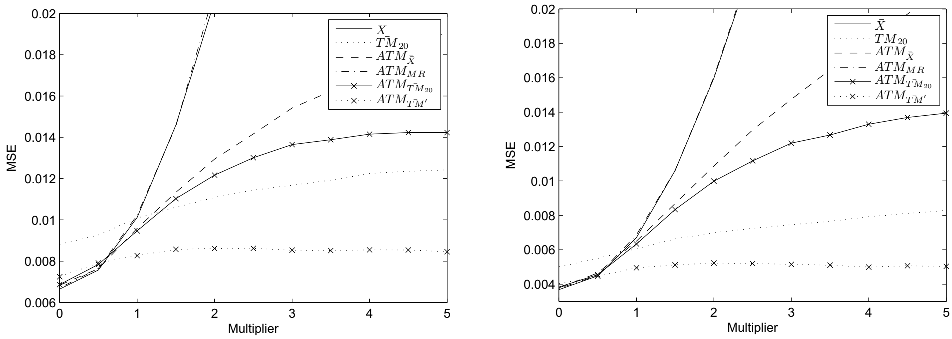
4.3.2 Efficiency of proposed Phase I control charts

To determine the efficiency of the proposed Phase I control charts, we consider the five types of contaminations defined in our MSE study of the statistics presented in Section 4.2.2. The MSE results for the Phase I control charts are given in Figures 4.6-4.10. To facilitate comparison, we have also included the MSE of the estimators $\overline{\overline{X}}$ and $\overline{TM}_{20}$.



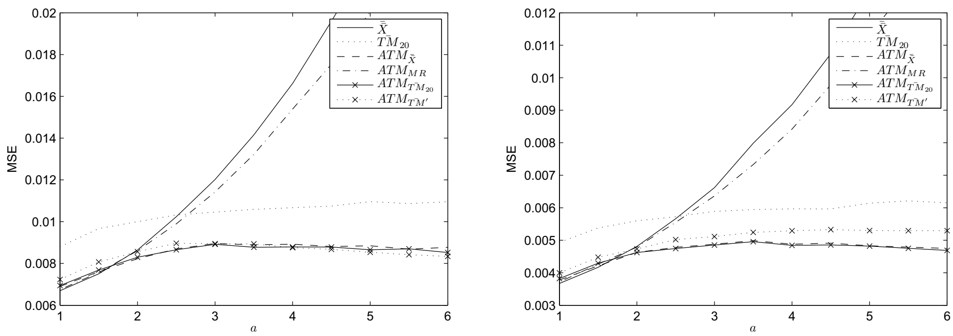
(a) (b)

Figure 4.6: MSE of estimators when symmetric diffuse variance disturbances are present for $k = 30$ (a) $n = 5$ (b) $n = 9$



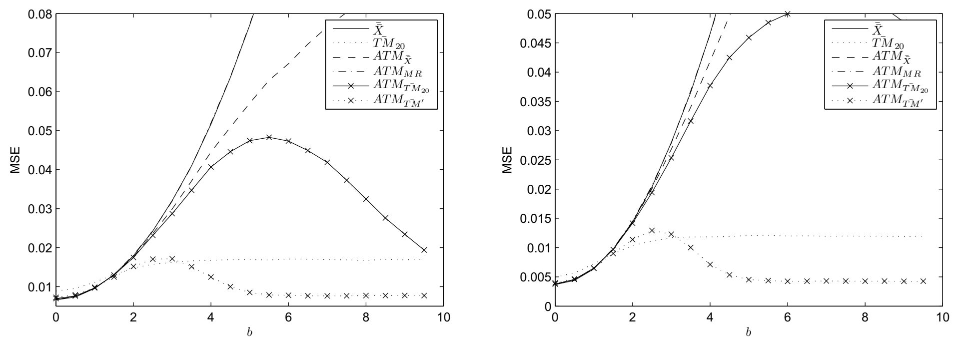
(a) (b)

Figure 4.7: MSE of estimators when asymmetric diffuse variance disturbances are present for $k = 30$ (a) $n = 5$ (b) $n = 9$



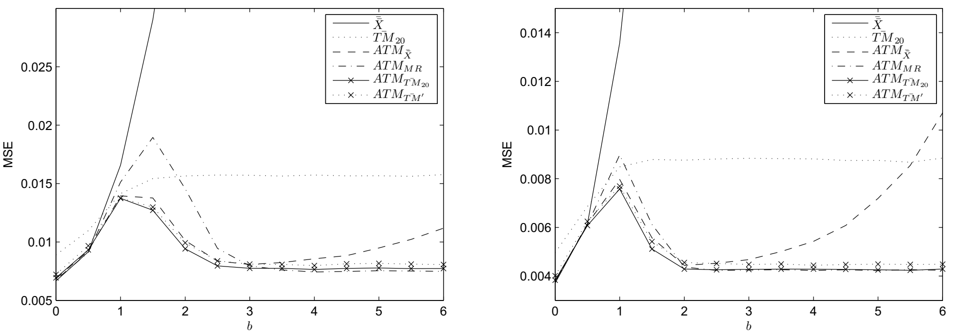
(a) (b)

Figure 4.8: MSE of estimators when localized variance disturbances are present for $k = 30$ (a) $n = 5$ (b) $n = 9$



(a) (b)

Figure 4.9: MSE of estimators when diffuse mean disturbances are present for $k = 30$ (a) $n = 5$ (b) $n = 9$



(a) (b)

Figure 4.10: MSE of estimators when localized mean disturbances are present for $k = 30$ (a) $n = 5$ (b) $n = 9$

The figures show that the standard Phase I analysis method, $ATM_{\bar{\bar{X}}}$, performs almost as well as $\overline{\overline{X}}$ under normality when no contaminations are present and seems to be robust against localized variance disturbances. However, the method loses efficiency in the other situations. Since $\overline{\overline{X}}$, the initial estimate of μ, is highly sensitive to disturbances, the Phase I limits are biased and fail to identify the correct out-of-control samples.

The mean rank method, denoted by ATM_{MR}, performs well under normality and when there are localized mean disturbances. The reason is that this estimator screens for samples with a mean rank significantly higher than that of the other samples. On the other hand, ATM_{MR} performs badly when diffuse outliers are present. The mean rank is not influenced by occasional outliers so that samples containing only one outlier are not filtered out and hence are included in the calculation of the grand sample mean.

The third method, $ATM_{\bar{TM}_{20}}$, which uses the robust estimator $\overline{TM}_{20}$ to construct a Phase I control chart, seems to be more efficient under normality than $\overline{TM}_{20}$ itself. The gain in efficiency can be explained by the use of an efficient estimator to obtain the final location estimate, once screening is complete. Thus, an efficient Phase I analysis does not require the use of an efficient estimator to construct the Phase I control chart.

The final method, $ATM_{\bar{TM}'}$, which first screens for localized disturbances and then for occasional outliers, far outperforms all estimation methods. The method is particularly powerful in the presence of diffuse disturbances, because its use of an individuals control chart in Phase I to identify single outliers increases the probability that such disturbances will be detected. For example, Figure 4.9 represents the situation where diffuse mean disturbances are present. The efficiency of the estimator improves for high b values because the disturbances are more likely to fall outside the control limits and are therefore more likely to be detected.

4.4 Derivation of Phase II control limits

We now turn to the effect of the proposed location estimators on the $\overline{X}$ control chart performance in Phase II. The formulae for the $\overline{X}$ control limits with estimated parameters are given by (1.4). For the Phase II control limits, we only estimate the in-control mean μ; we treat the in-control standard deviation σ as known because we want to isolate the effect of estimating the location parameter. The factor C_n that is used to obtain accurate control limits when the process parameters are estimated is derived such that the probability of a false signal equals the desired probability of a false signal. Except for the estimator $\overline{\overline{X}}$, C_n can not be obtained easily in analytic form and is therefore obtained by means of simulation. The factors are chosen such that p is equal to 0.0027 under normality. 50,000 simulation runs are used. For $k = 30$ $n = 5$ and $n = 9$, the resulting factors are equal to 3.05 for $\overline{\overline{X}}$, $ATM_{\bar{\bar{X}}}$, $ATM_{\bar{TM}_{20}}$ and $ATM_{\bar{TM}'}$; 3.06 for $\overline{\overline{X}}_{20}$ and $\overline{TM}$ and 3.07 for $M(\overline{X})$, $\overline{M}$, $\overline{HL}$, $\overline{TM}_{20}$ and $ATM_{\bar{M}R}$.

4.5 Control chart performance

In this section we evaluate the effect on $\overline{X}$ Phase II performance of the proposed location statistics and estimation methods based on Phase I control charting. We consider the same Phase I situations as those used to assess the MSE with a, b and the multiplier equal to 4 to simulate the contaminated cases (see Section 4.2.2).

We use the unconditional run length distribution to assess performance. Specifically, we look at several characteristics of that distribution, namely the average run length (ARL) and the standard deviation of the run length (SDRL). We also report the probability that one sample gives a signal (p). We compute these characteristics in an in-control and several out-of-control situations. We consider different shifts of size $\delta\sigma$ in the mean, setting δ equal to 0, 0.5, 1 and 2. The performance characteristics are obtained by simulation. Section 4.5.1 describes the simulation method and Section 4.5.2 gives simulation results identifying which control charts perform best in the uncontaminated and various contaminated situations.

4.5.1 Simulation procedure

The performance characteristics p and ARL for estimated control limits are determined by averaging the conditional characteristics, i.e. the characteristics for a given set of estimated control limits, over all possible values of the control limits. The corresponding definitions of $p(F_i|\hat{\mu})$, $E(RL|\hat{\mu})$, $p = E(p(F_i|\hat{\mu}))$ and $ARL = E(\frac{1}{p(F_i|\hat{\mu})})$ are obtained from (1.5)-(1.8), with all variables conditioned on $\hat{\mu}$ rather than $\hat{\sigma}$ and $\hat{\sigma} = \sigma$. These expectations are obtained by simulation: numerous datasets are generated and for each dataset $p(F_i|\hat{\mu})$ and $E(RL|\hat{\mu})$ are computed. By averaging these values we obtain the unconditional values. The unconditional standard deviation is determined by (1.9).

Enough replications of the above procedure were performed to obtain sufficiently small relative estimated standard errors for p and ARL. The relative standard error of the estimates is never higher than 0.60%.

4.5.2 Simulation results

First, we consider the situation where the process follows a normal distribution and the Phase I data are not contaminated. We investigate the impact of the estimator used to estimate μ in Phase I. Table 4.3 presents p and the ARL when the process mean equals $\mu + \delta\sigma$. When the process is in control ($\delta = 0$), we want p to be as low as possible and ARL to be as high as possible. In the out-of-control situation ($\delta \neq 0$), we want to achieve the opposite. We can see that in the absence of any contamination (Table 4.3), the efficiency of the estimators is very similar. We can therefore conclude that using a more robust location estimator does not have a substantial impact on control chart performance in the uncontaminated situation.

The Phase II control charts based on the estimation methods $M(\overline{X})$, $\overline{\overline{X}}_{20}$, $\overline{TM}_{20}$, $ATM_{\overline{\overline{X}}}$ and ATM_{MR} perform relatively well when localized disturbances are present, while the charts based on $\overline{M}$, $\overline{HL}$, $\overline{TM}$ and $\overline{TM}_{20}$ perform relatively well when diffuse disturbances are present (see Tables 4.4-4.8).

The Phase II chart based on $ATM_{\overline{TM}'}$ performs best: this chart is as efficient as $\overline{\overline{X}}$ in the uncontaminated normal situation and its performance does not change much when contaminations come into play. Moreover, the chart outperforms the other methods in all situations because it successfully filters out both diffuse and localized disturbances. In the presence of asymmetric disturbances, in particular, the added value of this estimation method is substantial.

When localized mean disturbances are present, we see a strange phenomenon for the $\overline{X}$, $\overline{M}$, $\overline{HL}$ and $\overline{TM}$ charts: the in-control ARL is lower than the out-of-control ARL for $\delta = 0.5$. In other words, these charts are more likely to give a signal in the in-control situation than in the out-of-control situation for $\delta = 0.5$ and hence, in the presence of disturbances, are highly ARL-biased (cf. Jensen et al. (2006)).

n	$\hat{\mu}$	p $\delta=0$	$\delta=0.5$	$\delta=1$	$\delta=2$	ARL and $SDRL$ $\delta=0$	$\delta=0.5$	$\delta=1$	$\delta=2$
5	$\overline{\overline{X}}$	0.0027	0.029	0.21	0.92	384 (392)	41.7 (49.4)	5.03 (4.90)	1.09 (0.32)
	$M(\overline{X})$	0.0027	0.028	0.21	0.91	390 (406)	46.2 (59.9)	5.31 (5.43)	1.10 (0.33)
	$\overline{M}$	0.0027	0.028	0.21	0.91	392 (407)	45.9 (59.0)	5.29 (5.37)	1.10 (0.33)
	$\overline{\overline{X}}_{20}$	0.0027	0.028	0.21	0.92	391 (401)	43.3 (52.4)	5.14 (5.08)	1.09 (0.32)
	$\overline{HL}$	0.0027	0.029	0.21	0.92	380 (389)	42.0 (50.4)	5.05 (4.94)	1.09 (0.32)
	$\overline{TM}$	0.0027	0.028	0.21	0.92	390 (400)	43.4 (53.0)	5.14 (5.09)	1.09 (0.32)
	$\overline{TM}_{20}$	0.0027	0.028	0.21	0.92	396 (410)	45.3 (56.9)	5.26 (5.29)	1.09 (0.33)
	$ATM_{\bar{\bar{X}}}$	0.0027	0.029	0.21	0.92	383 (392)	41.8 (49.6)	5.04 (4.92)	1.09 (0.32)
	ATM_{MR}	0.0027	0.029	0.21	0.92	383 (391)	41.5 (49.1)	5.04 (4.89)	1.09 (0.32)
	$ATM_{\bar{TM}_{20}}$	0.0027	0.029	0.21	0.92	382 (391)	41.8 (49.9)	5.04 (4.92)	1.09 (0.32)
	$ATM_{\bar{TM}'}$	0.0027	0.029	0.21	0.92	381 (390)	42.0 (50.3)	5.06 (4.96)	1.09 (0.32)
9	$\overline{\overline{X}}$	0.0027	0.064	0.48	1.00	384 (393)	17.9 (20.0)	2.13 (1.62)	1.00 (0.043)
	$M(\overline{X})$	0.0027	0.063	0.47	1.00	390 (405)	19.5 (23.5)	2.19 (1.74)	1.00 (0.046)
	$\overline{M}$	0.0027	0.063	0.47	1.00	390 (405)	19.5 (23.6)	2.19 (1.74)	1.00 (0.046)
	$\overline{\overline{X}}_{20}$	0.0027	0.063	0.48	1.00	391 (401)	18.5 (21.1)	2.15 (1.66)	1.00 (0.044)
	$\overline{HL}$	0.0027	0.064	0.48	1.00	380 (389)	18.0 (20.4)	2.13 (1.63)	1.00 (0.043)
	$\overline{TM}$	0.0027	0.063	0.48	1.00	390 (400)	18.6 (21.4)	2.16 (1.67)	1.00 (0.045)
	$\overline{TM}_{20}$	0.0027	0.062	0.47	1.00	395 (409)	19.3 (22.7)	2.18 (1.71)	1.00 (0.046)
	$ATM_{\bar{\bar{X}}}$	0.0027	0.064	0.48	1.00	382 (391)	17.9 (20.1)	2.13 (1.63)	1.00 (0.043)
	ATM_{MR}	0.0027	0.064	0.48	1.00	382 (391)	17.9 (20.1)	2.13 (1.63)	1.00 (0.043)
	$ATM_{\bar{TM}_{20}}$	0.0027	0.064	0.48	1.00	382 (391)	18.0 (20.2)	2.13 (1.63)	1.00 (0.043)
	$ATM_{\bar{TM}'}$	0.0027	0.064	0.48	1.00	380 (390)	18.0 (20.5)	2.13 (1.64)	1.00 (0.043)

Table 4.3: p, ARL and (in parentheses) SDRL of corrected limits under normality for $k = 30$

		p				ARL and $SDRL$			
n	$\hat{\mu}$	$\delta = 0$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 0$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$
5	$\overline{\overline{X}}$	0.0030	0.030	0.21	0.92	358 (375)	45.0 (60.9)	5.21 (5.44)	1.09 (0.33)
	$M(\overline{X})$	0.0029	0.029	0.21	0.91	375 (395)	48.4 (67.7)	5.41 (5.77)	1.10 (0.34)
	$\overline{M}$	0.0028	0.029	0.21	0.91	387 (403)	46.5 (61.5)	5.34 (5.52)	1.10 (0.33)
	$\overline{\overline{X}}_{20}$	0.0028	0.029	0.21	0.92	376 (392)	45.1 (58.8)	5.23 (5.35)	1.09 (0.33)
	$\overline{HL}$	0.0029	0.029	0.21	0.92	370 (383)	43.2 (54.5)	5.13 (5.15)	1.09 (0.32)
	$\overline{TM}$	0.0027	0.029	0.21	0.92	384 (396)	44.2 (55.6)	5.19 (5.21)	1.09 (0.32)
	$\overline{TM}_{20}$	0.0027	0.028	0.21	0.91	390 (405)	46.0 (59.2)	5.29 (5.38)	1.10 (0.33)
	$ATM_{\bar{\bar{X}}}$	0.0030	0.030	0.21	0.92	362 (378)	44.4 (58.8)	5.18 (5.33)	1.09 (0.32)
	ATM_{MR}	0.0030	0.030	0.21	0.92	357 (374)	45.1 (61.6)	5.21 (5.44)	1.09 (0.33)
	$ATM_{\bar{TM}_{20}}$	0.0029	0.030	0.21	0.92	364 (379)	44.3 (58.3)	5.18 (5.31)	1.09 (0.32)
	$ATM_{\bar{TM}'}$	0.0028	0.029	0.21	0.92	375 (386)	42.7 (52.8)	5.09 (5.06)	1.09 (0.32)
9	$\overline{\overline{X}}$	0.0030	0.066	0.48	1.00	358 (375)	19.1 (24.0)	2.17 (1.73)	1.00 (0.045)
	$M(\overline{X})$	0.0030	0.065	0.47	1.00	371 (393)	20.6 (27.5)	2.23 (1.83)	1.00 (0.048)
	$\overline{M}$	0.0028	0.063	0.47	1.00	386 (403)	19.8 (24.5)	2.20 (1.75)	1.00 (0.047)
	$\overline{\overline{X}}_{20}$	0.0029	0.064	0.48	1.00	373 (389)	19.4 (24.1)	2.18 (1.74)	1.00 (0.046)
	$\overline{HL}$	0.0028	0.064	0.48	1.00	373 (385)	18.4 (21.5)	2.14 (1.66)	1.00 (0.044)
	$\overline{TM}$	0.0027	0.063	0.48	1.00	384 (397)	18.8 (22.1)	2.16 (1.69)	1.00 (0.045)
	$\overline{TM}_{20}$	0.0027	0.063	0.47	1.00	391 (406)	19.4 (23.3)	2.19 (1.74)	1.00 (0.046)
	$ATM_{\bar{\bar{X}}}$	0.0030	0.066	0.48	1.00	358 (375)	19.2 (24.0)	2.17 (1.73)	1.00 (0.046)
	ATM_{MR}	0.0031	0.066	0.48	1.00	356 (374)	19.1 (24.1)	2.17 (1.73)	1.00 (0.046)
	$ATM_{\bar{TM}_{20}}$	0.0030	0.066	0.48	1.00	360 (376)	19.0 (23.5)	2.16 (1.71)	1.00 (0.046)
	$ATM_{\bar{TM}'}$	0.0028	0.064	0.48	1.00	375 (386)	18.3 (21.3)	2.14 (1.65)	1.00 (0.044)

Table 4.4: p, ARL and (in parentheses) SDRL of corrected limits when symmetric variance disturbances are present for $k = 30$

n	$\hat{\mu}$	p $\delta = 0$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	ARL and $SDRL$ $\delta = 0$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$
5	$\overline{\overline{X}}$	0.0076	0.012	0.12	0.82	233 (295)	143 (210)	12.8 (24.0)	1.23 (0.60)
	$M(\overline{X})$	0.0034	0.019	0.16	0.88	347 (378)	77.0 (110)	7.34 (8.26)	1.14 (0.41)
	$\overline{M}$	0.0029	0.022	0.18	0.90	374 (395)	61.6 (82.4)	6.32 (6.73)	1.12 (0.37)
	$\overline{\overline{X}}_{20}$	0.0034	0.018	0.16	0.88	337 (366)	75.9 (103)	7.22 (7.91)	1.14 (0.41)
	$\overline{HL}$	0.0033	0.020	0.17	0.89	340 (363)	67.3 (90.4)	6.73 (7.36)	1.13 (0.39)
	$\overline{TM}$	0.0030	0.021	0.17	0.89	365 (384)	61.2 (69.2)	6.33 (6.61)	1.12 (0.37)
	$\overline{TM}_{20}$	0.0029	0.022	0.18	0.90	379 (398)	60.4 (78.7)	6.26 (6.56)	1.12 (0.37)
	$ATM_{\overline{\overline{X}}}$	0.0034	0.020	0.17	0.89	334 (359)	70.0 (95.0)	6.86 (7.51)	1.13 (0.39)
	ATM_{MR}	0.0076	0.012	0.12	0.82	232 (294)	143 (209)	13.0 (26.7)	1.24 (0.61)
	$ATM_{\bar{TM}_{20}}$	0.0032	0.021	0.17	0.89	347 (367)	62.9 (82.9)	6.46 (6.84)	1.12 (0.38)
	$ATM_{\bar{TM}'}$	0.0028	0.025	0.20	0.91	373 (385)	48.9 (60.2)	5.53 (5.55)	1.10 (0.34)
9	$\overline{X}$	0.011	0.020	0.27	0.99	175 (239)	89.5 (148)	4.58 (6.42)	1.01 (0.12)
	$M(\overline{X})$	0.0044	0.035	0.36	0.99	299 (346)	42.1 (63.4)	3.04 (2.89)	1.01 (0.077)
	$\overline{M}$	0.0031	0.048	0.42	1.00	366 (389)	26.8 (34.1)	2.51 (2.12)	1.00 (0.058)
	$\overline{\overline{X}}_{20}$	0.0047	0.033	0.35	0.99	280 (324)	42.7 (61.0)	3.09 (2.89)	1.01 (0.078)
	$\overline{HL}$	0.0033	0.044	0.41	1.00	336 (358)	27.9 (33.9)	2.57 (2.15)	1.00 (0.059)
	$\overline{TM}$	0.0031	0.046	0.42	1.00	356 (378)	26.4 (31.9)	2.51 (2.08)	1.00 (0.057)
	$\overline{TM}_{20}$	0.0030	0.047	0.42	1.00	368 (391)	26.8 (33.2)	2.52 (2.11)	1.00 (0.058)
	$ATM_{\overline{\overline{X}}}$	0.0046	0.035	0.36	0.99	282 (322)	39.6 (57.4)	2.98 (2.76)	1.01 (0.074)
	ATM_{MR}	0.011	0.021	0.27	0.99	175 (240)	89.3 (148)	4.58 (6.41)	1.01 (0.12)
	$ATM_{\bar{TM}_{20}}$	0.0039	0.039	0.38	1.00	307 (339)	33.3 (44.1)	2.77 (2.44)	1.00 (0.066)
	$ATM_{\bar{TM}'}$	0.0028	0.055	0.45	1.00	370 (383)	21.4 (24.9)	2.30 (1.83)	1.00 (0.050)

Table 4.5: p, ARL and (in parentheses) SDRL of corrected limits when asymmetric variance disturbances are present for $k = 30$

		p				ARL and $SDRL$			
n	$\hat{\mu}$	$\delta=0$	$\delta=0.5$	$\delta=1$	$\delta=2$	$\delta=0$	$\delta=0.5$	$\delta=1$	$\delta=2$
5	$\overline{\overline{X}}$	0.0034	0.032	0.22	0.91	337 (361)	48.7 (73.0)	5.42 (6.09)	1.10 (0.34)
	$M(\overline{X})$	0.0028	0.029	0.21	0.91	382 (400)	47.4 (64.2)	5.40 (5.66)	1.10 (0.33)
	$\overline{M}$	0.0037	0.033	0.22	0.91	335 (372)	57.2 (98.5)	5.90 (7.45)	1.11 (0.36)
	$\overline{\overline{X}}_{20}$	0.0028	0.029	0.21	0.92	382 (395)	44.3 (55.8)	5.21 (5.25)	1.09 (0.32)
	$\overline{HL}$	0.0035	0.032	0.22	0.91	332 (358)	50.0 (76.8)	5.46 (6.22)	1.10 (0.34)
	$\overline{TM}$	0.0035	0.032	0.22	0.91	338 (368)	52.3 (83.4)	5.62 (6.63)	1.10 (0.35)
	$\overline{TM}_{20}$	0.0028	0.029	0.21	0.91	387 (403)	46.6 (61.7)	5.36 (5.55)	1.10 (0.33)
	$ATM_{\overline{\overline{X}}}$	0.0028	0.029	0.21	0.92	371 (382)	43.1 (53.7)	5.12 (5.13)	1.09 (0.32)
	ATM_{MR}	0.0033	0.031	0.22	0.92	342 (364)	47.7 (69.5)	5.34 (5.83)	1.10 (0.33)
	$ATM_{\overline{TM}_{20}}$	0.0028	0.029	0.21	0.92	371 (384)	42.9 (53.8)	5.12 (5.13)	1.09 (0.32)
	$ATM_{\overline{TM}'}$	0.0028	0.029	0.21	0.92	372 (384)	43.0 (53.7)	5.11 (5.10)	1.09 (0.32)
9	$\overline{\overline{X}}$	0.0034	0.068	0.48	1.00	337 (362)	20.4 (28.6)	2.21 (1.84)	1.00 (0.048)
	$M(\overline{X})$	0.0028	0.064	0.47	1.00	381 (400)	19.9 (25.0)	2.21 (1.78)	1.00 (0.047)
	$\overline{M}$	0.0038	0.069	0.47	1.00	332 (370)	24.0 (40.4)	2.32 (2.12)	1.00 (0.054)
	$\overline{\overline{X}}_{20}$	0.0028	0.064	0.48	1.00	382 (395)	18.9 (22.3)	2.17 (1.69)	1.00 (0.045)
	$\overline{HL}$	0.0035	0.069	0.48	1.00	333 (359)	20.8 (30.1)	2.22 (1.88)	1.00 (0.049)
	$\overline{TM}$	0.0035	0.068	0.48	1.00	338 (368)	21.8 (32.7)	2.25 (1.94)	1.00 (0.051)
	$\overline{TM}_{20}$	0.0028	0.063	0.47	1.00	385 (402)	19.7 (24.2)	2.20 (1.76)	1.00 (0.047)
	$ATM_{\overline{\overline{X}}}$	0.0028	0.065	0.48	1.00	372 (384)	18.4 (21.6)	2.15 (1.67)	1.00 (0.044)
	ATM_{MR}	0.0033	0.068	0.48	1.00	342 (364)	20.0 (27.2)	2.20 (1.81)	1.00 (0.048)
	$ATM_{\overline{TM}_{20}}$	0.0028	0.065	0.48	1.00	372 (384)	18.4 (21.7)	2.15 (1.67)	1.00 (0.044)
	$ATM_{\overline{TM}'}$	0.0029	0.065	0.48	1.00	368 (381)	18.6 (22.2)	2.15 (1.68)	1.00 (0.045)

Table 4.6: p, ARL and (in parentheses) SDRL of corrected limits when localized variance disturbances are present for $k = 30$

		p				ARL and $SDRL$			
n	$\hat{\mu}$	$\delta = 0$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	$\delta = 0$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$
5	$\overline{\overline{X}}$	0.0061	0.011	0.11	0.83	224 (271)	137 (182)	10.9 (12.9)	1.22 (0.53)
	$M(\overline{X})$	0.0048	0.014	0.13	0.85	289 (340)	115 (168)	9.57 (12.2)	1.19 (0.49)
	$\overline{M}$	0.0033	0.019	0.16	0.88	351 (380)	74.6 (103)	7.12 (7.89)	1.14 (0.41)
	$\overline{\overline{X}}_{20}$	0.0049	0.013	0.13	0.85	271 (316)	115 (158)	9.52 (11.2)	1.19 (0.49)
	$\overline{HL}$	0.0042	0.015	0.14	0.86	290 (326)	93.0 (126)	8.22 (9.18)	1.16 (0.45)
	$\overline{TM}$	0.0035	0.017	0.15	0.88	333 (361)	77.8 (103)	7.33 (7.95)	1.14 (0.41)
	$\overline{TM}_{20}$	0.0032	0.019	0.16	0.88	356 (383)	72.5 (97.0)	7.00 (7.57)	1.13 (0.40)
	$ATM_{\overline{\overline{X}}}$	0.0055	0.012	0.12	0.84	245 (392)	123 (167)	10.0 (12.0)	1.20 (0.51)
	ATM_{MR}	0.0061	0.011	0.11	0.83	224 (272)	136 (182)	10.9 (12.9)	1.21 (0.53)
	$ATM_{\bar{TM}_{20}}$	0.0052	0.013	0.12	0.85	257 (302)	116 (159)	9.58 (11.2)	1.19 (0.49)
	$ATM_{\bar{TM}'}$	0.0031	0.024	0.19	0.90	356 (374)	57.0 (78.1)	6.01 (6.39)	1.11 (0.36)
9	$\overline{\overline{X}}$	0.0089	0.018	0.26	0.99	161 (208)	77.3 (107)	4.17 (4.16)	1.01 (0.11)
	$M(\overline{X})$	0.0074	0.023	0.29	0.99	212 (274)	71.7 (111)	3.96 (4.17)	1.01 (0.10)
	$\overline{M}$	0.0035	0.041	0.39	1.00	339 (371)	32.4 (42.7)	2.72 (2.39)	1.00 (0.065)
	$\overline{\overline{X}}_{20}$	0.0075	0.021	0.28	0.99	193 (246)	68.9 (98.5)	3.91 (3.90)	1.01 (0.10)
	$\overline{HL}$	0.0046	0.033	0.35	0.99	272 (310)	39.6 (51.4)	3.00 (2.69)	1.01 (0.074)
	$\overline{TM}$	0.0037	0.038	0.38	1.00	317 (349)	33.6 (42.7)	2.79 (2.43)	1.00 (0.067)
	$\overline{TM}_{20}$	0.0035	0.039	0.38	1.00	336 (368)	32.9 (42.0)	2.75 (2.40)	1.00 (0.066)
	$ATM_{\overline{\overline{X}}}$	0.0081	0.020	0.28	0.99	177 (227)	71.0 (99.9)	3.98 (3.96)	1.01 (0.10)
	ATM_{MR}	0.0089	0.019	0.26	0.99	162 (209)	77.8 (108)	4.16 (4.15)	1.01 (0.11)
	$ATM_{\bar{TM}_{20}}$	0.0074	0.022	0.29	0.99	193 (242)	64.9 (91.2)	3.80 (3.73)	1.00 (0.098)
	$ATM_{\bar{TM}'}$	0.0031	0.051	0.43	1.00	352 (371)	24.5 (30.7)	2.42 (2.00)	1.00 (0.054)

Table 4.7: p, ARL and (in parentheses) SDRL of corrected limits when diffuse mean disturbances are present for $k = 30$

n	$\hat{\mu}$	p $\delta = 0$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$	ARL and $SDRL$ $\delta = 0$	$\delta = 0.5$	$\delta = 1$	$\delta = 2$
5	$\overline{X}$	0.017	0.0034	0.046	0.70	72.3 (87.5)	329 (351)	25.0 (28.8)	1.45 (0.83)
	$M(\overline{X})$	0.0031	0.021	0.17	0.89	366 (389)	66.0 (89.3)	6.62 (7.15)	1.13 (0.38)
	$\overline{M}$	0.017	0.0033	0.045	0.69	80.1 (105)	343 (372)	27.3 (33.6)	1.47 (0.86)
	$\overline{\overline{X}}_{20}$	0.0030	0.020	0.17	0.89	360 (379)	64.8 (81.8)	6.57 (6.81)	1.13 (0.38)
	$\overline{HL}$	0.017	0.0034	0.047	0.70	72.8 (89.3)	327 (350)	25.2 (29.2)	1.45 (0.83)
	$\overline{TM}$	0.017	0.0033	0.046	0.69	75.6 (94.4)	337 (362)	26.0 (30.7)	1.46 (0.84)
	$\overline{TM}_{20}$	0.0031	0.019	0.16	0.89	360 (385)	70.4 (92.0)	6.89 (7.32)	1.13 (0.40)
	$ATM_{\bar{\bar{X}}}$	0.0028	0.026	0.20	0.91	373 (385)	47.5 (49.1)	5.44 (5.47)	1.10 (0.33)
	ATM_{MR}	0.0028	0.029	0.21	0.92	378 (388)	42.1 (50.9)	5.08 (4.99)	1.09 (0.32)
	$ATM_{\bar{TM}_{20}}$	0.0028	0.029	0.21	0.92	378 (388)	42.8 (52.1)	5.12 (5.07)	1.09 (0.32)
	$ATM_{\bar{TM}'}$	0.0028	0.029	0.21	0.92	375 (386)	43.4 (53.4)	5.14 (5.11)	1.09 (0.32)
9	$\overline{X}$	0.035	0.0039	0.11	0.96	34.3 (30.7)	293 (321)	10.1 (10.7)	1.04 (0.22)
	$M(\overline{X})$	0.0031	0.048	0.42	1.00	366 (389)	26.8 (34.3)	2.51 (2.12)	1.00 (0.058)
	$\overline{M}$	0.034	0.0039	0.11	0.95	37.9 (48.5)	308 (346)	10.8 (12.3)	1.05 (0.23)
	$\overline{\overline{X}}_{20}$	0.0030	0.046	0.41	1.00	359 (379)	26.3 (31.5)	2.51 (2.07)	1.00 (0.057)
	$\overline{HL}$	0.035	0.0039	0.11	0.96	34.5 (40.8)	293 (322)	10.1 (10.9)	1.05 (0.22)
	$\overline{TM}$	0.034	0.0039	0.11	0.96	35.7 (43.1)	301 (333)	10.4 (11.4)	1.05 (0.22)
	$\overline{TM}_{20}$	0.0031	0.044	0.41	1.00	361 (385)	28.4 (35.4)	2.58 (2.18)	1.00 (0.060)
	$ATM_{\bar{\bar{X}}}$	0.0029	0.055	0.45	1.00	366 (380)	21.7 (25.9)	2.30 (1.85)	1.00 (0.050)
	ATM_{MR}	0.0028	0.064	0.48	1.00	377 (387)	18.2 (20.9)	2.14 (1.64)	1.00 (0.044)
	$ATM_{\bar{TM}_{20}}$	0.0028	0.063	0.48	1.00	378 (388)	18.4 (21.1)	2.15 (1.66)	1.00 (0.044)
	$ATM_{\bar{TM}'}$	0.0028	0.063	0.48	1.00	376 (386)	18.6 (21.5)	2.16 (1.67)	1.00 (0.044)

Table 4.8: p, ARL and (in parentheses) SDRL of corrected limits when localized mean disturbances are present for $k = 30$

4.6 Concluding remarks

In this chapter we have considered several Phase I estimators of the location parameter for use in establishing $\overline{\overline{X}}$ Phase II control chart limits. The collection includes robust estimators proposed in the existing literature as well as several Phase I analyses, which apply a control chart retrospectively to study a historical dataset. The estimators have been evaluated under various circumstances: the uncontaminated situation and various situations contaminated with diffuse symmetric and asymmetric variance disturbances, localized variance disturbances, diffuse mean disturbances and localized mean disturbances.

The standard methods suffer from a number of problems. Estimators that are based on the principle of trimming individual observations (e.g. $\overline{M}$ and $\overline{TM}$) perform reasonably well when there are diffuse disturbances but not when localized disturbances are present. In the latter situation, estimators that are based on the principle of trimming samples (e.g. $M(\overline{X})$ and $\overline{\overline{X}}_{20}$) are efficient. All of these methods are biased when there are asymmetric disturbances, as the trimming principle does not take into account the asymmetry of the disturbance.

A Phase I analysis, using a control chart to study a historical dataset retrospectively and trim the data adaptively, does take into account the distribution of the disturbance and is therefore very suitable for use during the estimation of μ. However, the standard method based on the $\overline{\overline{X}}$ Phase I control chart has certain limitations. First, the initial estimate, $\overline{\overline{X}}$, is very sensitive to outliers so that the Phase I limits are biased. As a consequence, the wrong data samples are often filtered out. Second, the sample mean is usually plotted on the Phase I control chart, which makes it difficult to detect outliers in individual observations. Moreover, deleting the entire sample instead of the single outlier reduces efficiency.

To address the problems encountered in the standard Phase I analysis, we have proposed a new type of Phase I analysis. The initial estimate of μ for the Phase I control chart is based on a trimmed version of the trimean, namely $\overline{TM}_{20}$, and a subsequent procedure for both sample screening and outlier screening (resulting in $ATM_{\bar{TM}'}$). The proposed method is efficient under normality and far outperforms the existing methods when disturbances are present. Consequently, $ATM_{\bar{TM}'}$ is a very effective method for estimating μ in the limits used to construct the $\overline{X}$ Phase II chart.

Chapter 5

A Robust $\overline{X}$ Control Chart

5.1 Introduction

This chapter studies alternative standard deviation estimators that serve as a basis to determine the $\overline{X}$ control chart limits used for real-time process monitoring (Phase II). Several existing (robust) estimation methods are considered. In addition, we propose a new estimation method based on a Phase I analysis, whereby a control chart is used to identify disturbances in a dataset retrospectively. The method constructs a Phase I control chart derived from the trimmed mean of the sample interquartile ranges in order to identify out-of-control data. An efficient estimator, namely the mean of the sample standard deviations, is used to obtain the final standard deviation estimate from the remaining data. The estimation methods are evaluated in terms of their MSE and their effects on the performance of the $\overline{X}$ Phase II control chart. It is shown that the newly proposed estimation method is efficient under normality and performs substantially better than standard methods when disturbances are present in Phase I.

The chapter is structured as follows. We first present the estimation methods for the standard deviation and assess the MSE of the estimators. In the following sections, we present the design schemes for the $\overline{X}$ Phase II control chart and derive the control limits. Next, we describe the simulation procedure and present the effect of the proposed methods on Phase II performance. The final section offers some recommendations.

5.2 Proposed Phase I estimators

We analyze various types of standard deviation estimators and compare them under various types of disturbances. The next two sections introduce the standard deviation and location estimators respectively, while the third section presents the MSE of the standard deviation estimators.

5.2.1 Standard deviation estimators

We again denote the Phase I data by X_{ij}, $i = 1, 2, ..., k$ and $j = 1, 2, ..., n$. The X_{ij}'s are assumed to be independent and $N(\mu, \sigma^2)$ distributed. We denote by $X_{i,(v)}$, $v = 1, 2, ..., n$, the v-th order statistic in sample i. We look at several robust estimators proposed in the existing literature and introduce a new method incorporating a Phase I control chart.

We consider the traditional estimators of the standard deviation, namely the mean of the sample standard deviations $\overline{S}$ (see (2.2)), the mean of the sample ranges $\overline{R}$ (see (2.5)) and the mean of the sample interquartile ranges (see (2.6)). We saw in Chapter 2 that $\overline{S}$ is slightly less efficient under normality than the pooled sample standard deviation $\tilde{S}$.

Rocke (1989) proposed the trimmed mean of the sample interquartile ranges

$$\overline{IQR}_{\alpha} = \frac{1}{k - 2\lceil k\alpha \rceil} \times \left[\sum_{v=\lceil k\alpha \rceil + 1}^{k - \lceil k\alpha \rceil} IQR_{(v)} \right],$$

where $IQR_{(v)}$ denotes the v-th ordered value of the sample interquartile ranges. We consider the 20% trimmed mean of the sample interquartile ranges, which trims the ten smallest and the ten largest sample interquartile ranges when $k = 50$ and the twenty smallest and the twenty largest sample interquartile ranges when $k = 100$. An unbiased estimator of σ is given by $\overline{IQR}_{20}/d_{I\bar{Q}R_{20}}$, where $d_{I\bar{Q}R_{20}}$ is a normalizing constant. The value of this constant is 0.925 for $n = 5$ and 1.108 for $n = 9$.

We also evaluate a robust estimator proposed by Tatum (1997). This estimator is defined in (2.11).

We now present a new estimation method based on the principle of Phase I control charting. We build a Phase I control chart using a robust estimator for the standard deviation, namely $\overline{IQR}_{20}$. A disadvantage of this estimator is that it is not very efficient under normality. To address this, we use $\overline{IQR}_{20}$ to construct the Phase I limits with which we screen the estimation data for disturbances, but then use the efficient estimator $\bar{S}$ to obtain a standard deviation estimate from the remaining data. The Phase I standard deviation control chart limits are given by $\widehat{UCL}_{I\bar{Q}R_{20}} = U_n \overline{IQR}_{20}/d_{I\bar{Q}R_{20}}$ and $\widehat{LCL}_{I\bar{Q}R_{20}} = L_n \overline{IQR}_{20}/d_{I\bar{Q}R_{20}}$. For simplicity, we derive U_n and L_n from the 0.99865 and 0.00135 quantiles of the distribution of IQR/d_{IQR}. These quantiles are obtained from 1,000,000 simulation runs. The respective values of U_n and L_n are 3.220 and 0.035 for $n = 5$ and 2.487 and 0.145 for $n = 9$. We then plot the IQR_i/d_{IQR}'s of the Phase I samples on the Phase I control chart. Charting the IQR instead of the sample standard deviation or the sample range ensures that localized variance disturbances are identified and samples that contain only one single outlier are retained. A standard deviation estimator that is expected to be robust against localized variance disturbances is based on the mean of the sample interquartile ranges of the samples that fall between the control limits

$$\overline{IQR}' = \frac{1}{k'} \sum_{i \epsilon K} IQR_i \times 1_{\widehat{LCL}_{I\bar{Q}R_{20}} \leq IQR_i/d_{IQR} \leq \widehat{UCL}_{I\bar{Q}R_{20}}}(IQR_i),$$

with K the set of samples which are not excluded and k' the number of non-excluded samples. The resulting estimate $\overline{IQR}'/d_{IQR}$ is unbiased.

Although the remaining Phase I samples are expected to be free from localized variance disturbances, they could still contain diffuse disturbances. To eliminate such disturbances, the next step is to screen the individual observations using a Phase I individuals control chart. To screen the individual observations, we determine the residuals in each sample by subtracting the trimean value from each observation in the corresponding sample: $resid_{ij} = X_{ij} - TM_i$ with

$$TM_i = (Q_{i,1} + 2Q_{i,2} + Q_{i,3})/4.$$

Note that $Q_{i,2}$ is the median of sample i. Substracting the sample trimeans ensures that the variability is measured within samples and not between samples. The control limits of the individuals chart are given by $\widehat{UCL}_{I\bar{Q}R'} = 3\overline{IQR}'/d_{IQR}$ and $\widehat{LCL}_{I\bar{Q}R'} = -3\overline{IQR}'/d_{IQR}$. The residuals $resid_{ij}$ that fall above $\widehat{UCL}_{I\bar{Q}R'}$ or below $\widehat{LCL}_{I\bar{Q}R'}$ are considered out of control and their corresponding observations are removed from the Phase I dataset. The final estimate is the mean of the sample standard deviations S_i and is calculated from the observations deemed to be in control

$$\bar{S}' = \frac{1}{k^{\wedge}} \sum_{i \epsilon K^{\wedge}} S_i(\{X_{ij} \times 1_{\widehat{LCL}_{I\bar{Q}R'} \leq resid_{ij} \leq \widehat{UCL}_{I\bar{Q}R'}}\})(X_{ij}),$$

with $K^{\wedge}$ the set of samples which are not excluded and $k^{\wedge}$ the number of non-excluded samples. The normalizing constant is 0.980 for $n = 5$ and 0.984 for $n = 9$. This adaptively trimmed standard deviation is denoted by ATS.

The proposed standard deviation estimators are summarized in Table 5.1.

Estimator	Notation
Mean of sample standard deviations	$\bar{S}$
Mean of sample ranges	$\bar{R}$
Mean of sample interquartile ranges	$\overline{IQR}$
20% trimmed mean of sample interquartile ranges	$\overline{IQR}_{20}$
Tatum's estimator	$D7$
$\overline{IQR}'$ control chart with screening	ATS

Table 5.1: Proposed standard deviation estimators

5.2.2 Location estimator

The above standard deviation estimators are used to construct the $\overline{X}$ Phase II control limits. To ensure a fair comparison, we use the same location estimator in each case. The location estimation method uses a procedure similar to ATS above. Chapter 4 showed that this procedure performs better than the standard procedures based on estimators such as the mean, median, trimmed mean and Hodges-Lehmann. Again, the procedure consists of two steps.

In the first step we determine a location estimator that is robust against both localized and diffuse mean disturbances, namely the 20% trimmed mean of the sample trimeans

$$\overline{TM}_{\alpha} = \frac{1}{k - 2\lceil k\alpha \rceil} \times \left[\sum_{v=\lceil k\alpha \rceil + 1}^{k - \lceil k\alpha \rceil} TM_{(v)} \right].$$

Note that we start with the entire dataset. The respective upper and lower control limits for the sample location are given by $\widehat{UCL}_{\bar{TM}_{20}} = \overline{TM}_{20} + 3\hat{\sigma}/\sqrt{n}$ and $\widehat{LCL}_{\bar{TM}_{20}} = \overline{TM}_{20} - 3\hat{\sigma}/\sqrt{n}$, where σ is estimated by the corresponding standard deviation estimator from Table 5.1. We then plot the TM_i's of the Phase I samples on the control chart. Charting the TM_i's instead of the $\overline{X}_i$'s ensures that localized disturbances are identified and samples that contain only one single outlier are retained. A location estimator that is expected to be robust against localized mean disturbances is the mean of the sample trimeans of the samples that fall between the control limits

$$\overline{TM}' = \frac{1}{k^*} \sum_{i \epsilon K^*} TM_i \times 1_{\widehat{LCL}_{\bar{TM}_{20}} \leq TM_i \leq \widehat{UCL}_{\bar{TM}_{20}}}(TM_i),$$

with K^* the set of samples which are not excluded and k^* the number of non-excluded samples.

Although the remaining Phase I samples are expected to be free from localized mean disturbances, they could still contain diffuse disturbances. To eliminate such disturbances, the next step is to screen the individual observations using a Phase I individuals control chart with respective upper and lower control limits given by $\widehat{UCL}_{\bar{TM}'} = \overline{TM}' + 3\hat{\sigma}$ and $\widehat{LCL}_{\bar{TM}'} = \overline{TM}' - 3\hat{\sigma}$, where σ is estimated by the corresponding standard deviation estimator from Table 5.1. The observations X_{ij} that fall above $\widehat{UCL}_{\bar{TM}'}$ or below $\widehat{LCL}_{\bar{TM}'}$ are considered out of control and removed from the Phase I dataset. The final estimate is based on the mean of the sample means and is calculated from the observations deemed to be in control

$$\overline{\overline{X}}' = \frac{1}{k''} \sum_{i \epsilon K''} \frac{1}{n_i'} \sum_{j \epsilon N_i'} X_{ij} \times 1_{\widehat{LCL}_{\bar{TM}'} \leq X_{ij} \leq \widehat{UCL}_{\bar{TM}'}}(X_{ij}),$$

with K'' the samples which are not excluded, k'' the number of non-excluded samples, N_i' the non-excluded observations in sample i and n_i' the number of non-excluded observations in sample i.

5.2.3 Efficiency of proposed standard deviation estimators

We again use the MSE of the proposed standard deviation estimators to evaluate their efficiency. The MSE is estimated as

$$MSE = \frac{1}{N} \sum_{i=1}^{N} \left(\hat{\sigma}^i - \sigma \right)^2,$$

where $\hat{\sigma}^i$ is the value of the unbiased estimate in the i-th simulation run and N is the number of simulation runs. We consider the uncontaminated case, i.e. the situation where all the X_{ij}'s are from the $N(0,1)$ distribution, as well as four types of disturbances (see Section 2.2.2).

The MSE is obtained for $k = 50, 100$ subgroups of sizes $n = 5, 9$. The number of simulation runs N is equal to 50,000.

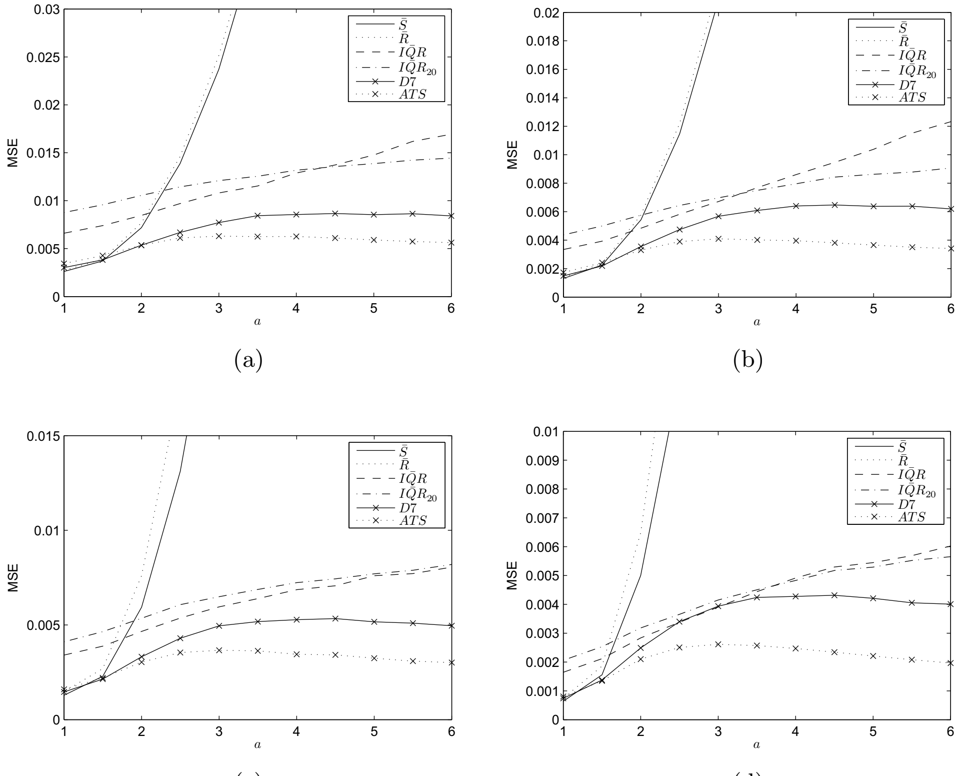


Figure 5.1: MSE of estimators when symmetric diffuse variance disturbances are present. (a) $n = 5, k = 50$ (b) $n = 5, k = 100$ (c) $n = 9, k = 50$ (d) $n = 9, k = 100$

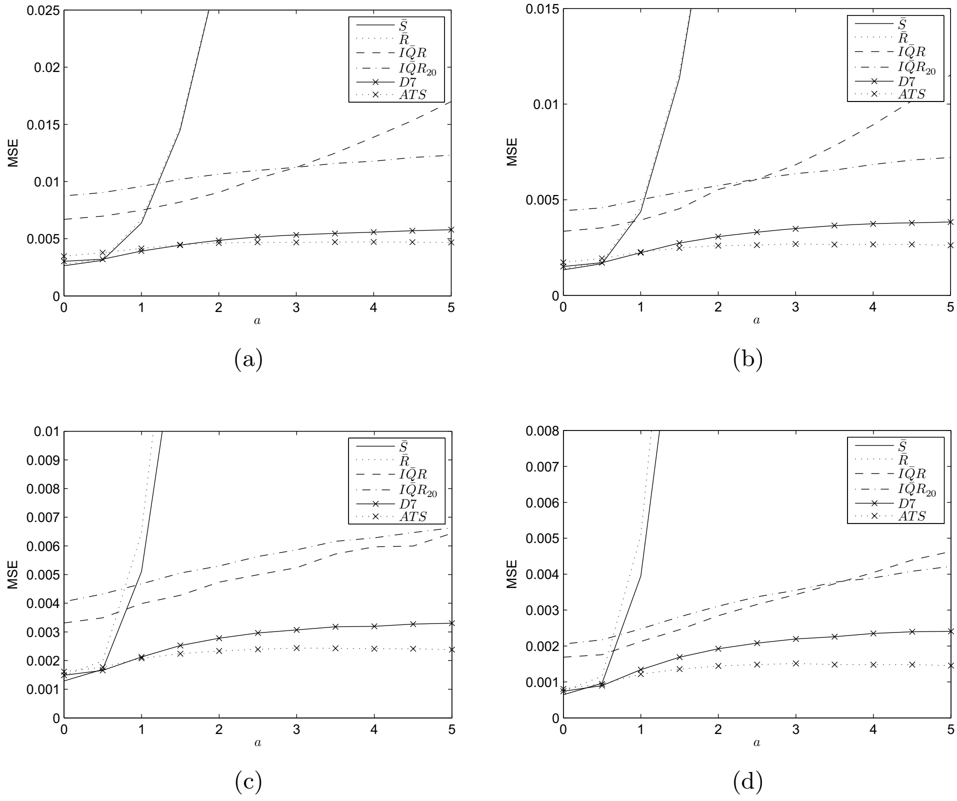


Figure 5.2: MSE of estimators when asymmetric diffuse variance disturbances are present. (a) $n = 5, k = 50$ (b) $n = 5, k = 100$ (c) $n = 9, k = 50$ (d) $n = 9, k = 100$

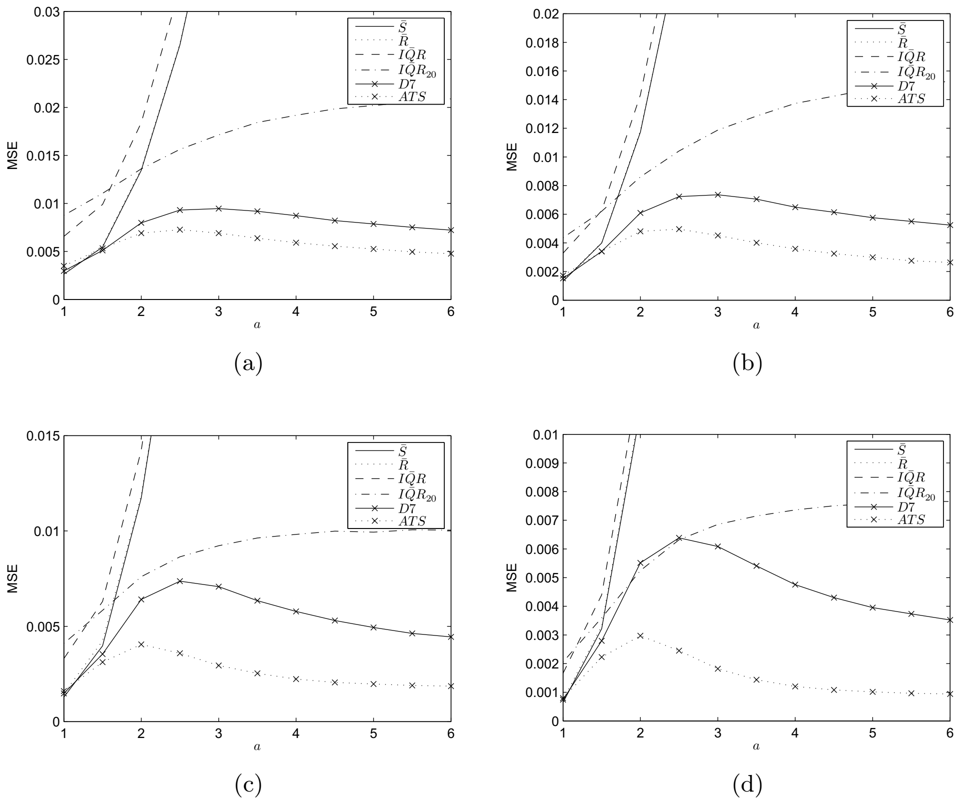


Figure 5.3: MSE of estimators when localized variance disturbances are present. (a) $n = 5, k = 50$ (b) $n = 5, k = 100$ (c) $n = 9, k = 50$ (d) $n = 9, k = 100$

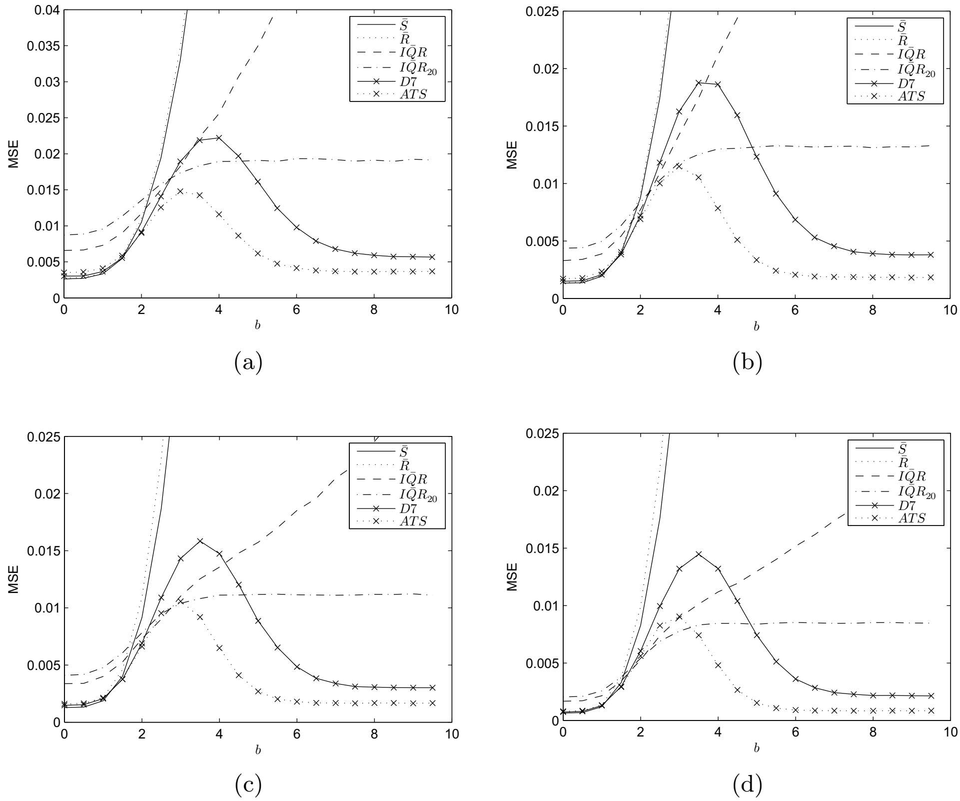


Figure 5.4: MSE of estimators when diffuse mean disturbances are present. (a) $n = 5, k = 50$ (b) $n = 5, k = 100$ (c) $n = 9, k = 50$ (d) $n = 9, k = 100$

Figures 5.1-5.4 show the MSE of the proposed estimators. The following results can be observed. The standard estimators $\bar{S}$ and $\bar{R}$ are not robust against either localized or diffuse disturbances. The $\overline{IQR}$ is less efficient under normality when there are no contaminations, but performs reasonably well when there are diffuse disturbances. The reason why $\overline{IQR}$ performs so well in these situations is that it trims the highest and lowest observations in each sample. However, this estimator remains biased when there are asymmetric diffuse disturbances because the trimming method does not take the distribution of the contaminations into account. Furthermore, this estimator is not efficient when there are localized variance disturbances as it trims only the observations within the sample instead of the sample interquartile ranges.

An estimator that combines within-sample and between-sample trimming, namely $\overline{IQR}_{20}$, performs reasonably well for all types of contaminations. However, its efficiency is relatively low under normality. $D7$ is efficient under normality as well as for contaminated data but relatively less so when the contamination consists of localized variance disturbances.

The estimator ATS is slightly less efficient under normality than the standard estimators, but much more robust than $\overline{IQR}$ and $\overline{IQR}_{20}$. Moreover, it shows outstanding performance when contaminations are present. We can therefore conclude that this estimator effectively filters out extreme observations.

5.3 Derivation of Phase II control limits

We now turn to the effect of the proposed estimators on the performance of the $\overline{X}$ Phase II control chart. The formulae for the $\overline{X}$ control limits with estimated parameters are given by (1.4). The factor C_n that is used to obtain accurate control limits when the process parameters are estimated is derived such that the probability of a false signal equals the chosen type I error probability p. The factors can not be obtained easily in analytic form. Therefore, they are obtained by means of simulation. The chosen type I error probability p is 0.0027. 50,000 simulation runs are used. The resulting factors are presented in Table 5.2.

	Factors for control limits			
	$n=5$		$n=9$	
$\hat{\sigma}$	$k=50$	$k=100$	$k=50$	$k=100$
$\bar{S}$	3.065	3.030	3.050	3.025
$\overline{R}$	3.070	3.035	3.055	3.025
$\overline{IQR}$	3.125	3.060	3.080	3.040
$\overline{IQR}_{20}$	3.155	3.080	3.090	3.045
$D7$	3.070	3.035	3.050	3.025
ATS	3.085	3.040	3.055	3.025

Table 5.2: Factors C_n to determine Phase II control limits

5.4 Control chart performance

In this section we evaluate the effect on $\overline{X}$ Phase II performance of the proposed standard deviation estimators. We consider the same Phase I situations as those used to assess the MSE with a, b and the multiplier equal to 4 to simulate the contaminated cases (see Section 5.2.3).

The performance of the Phase II control charts is assessed in terms of the unconditional p and ARL as well as the conditional ARL. The conditional ARL values express the ARL values for the control limits associated with the 2.5% and 95.7% quantiles of p in the in-control situation. We consider different shifts of size $\delta\sigma$ in the mean in Phase II, namely δ equal to 0, 0.25, 0.5 and 1. The performance characteristics are obtained by simulation. Sections 5.4.1 and 5.4.2 describe the simulation procedure and simulation results, enabling a comparison of control charts across the uncontaminated and contaminated situations.

5.4.1 Simulation procedure

The performance characteristics p and ARL for estimated control limits are determined by averaging the conditional characteristics, i.e. the characteristics for a given set of estimated control limits, over all possible values of the control limits. The corresponding definitions of $p(F_i|\hat{\mu},\hat{\sigma})$, $E(RL|\hat{\mu},\hat{\sigma})$, $p = E(p(F_i|\hat{\mu},\hat{\sigma}))$ and $ARL = E(\frac{1}{p(F_i|\hat{\mu},\hat{\sigma})})$ are obtained from (1.5)-(1.8), where the variables are conditioned on $\hat{\mu}$ and $\hat{\sigma}$. These expectations are obtained by simulation: numerous datasets are generated and for each dataset $p(F_i|\hat{\mu},\hat{\sigma})$ and $E(RL|\hat{\mu},\hat{\sigma})$ are computed. By averaging these values we obtain the unconditional values.

Enough replications of the above procedure were performed to obtain sufficiently small relative estimated standard errors for p and ARL. The relative standard error of the estimates is never higher than 0.80%.

5.4.2 Simulation results

The performance metrics are obtained in the in-control situation as well as in the out-of-control situation. When the process is in control ($\delta = 0$), we want p to be as low as possible and ARL to be as high as possible. In the out-of-control situation ($\delta \neq 0$), we want to achieve the opposite.

Tables 5.3 and 5.4 show the results for the $\overline{X}$ Phase II charts under normality. In this case we have estimated both the in-control μ and σ in Phase I. Unlike the $\overline{X}$ Phase II performance presented in Chapter 4, where only the mean was estimated to isolate the effect of location estimation, the ARL values are much higher than the desired 370. Thus, estimating the process standard deviation as well as the mean has substantially more impact on the $\overline{X}$ Phase II control chart than only estimating the process mean.

The conditional ARL values are presented in parentheses. The first value represents the ARL for the control limits associated with the 97.5% quantile of the simulated p in the in-control situation, while the second value represents the ARL for the control limits associated with the 2.5% quantile of the simulated p in the in-control situation. The results show that the conditional ARL values vary quite strongly, even when k equals 100.

In the absence of any contamination, the charts based on $\bar{S}$, $\bar{R}$, $D7$ and ATS show comparable performance. The charts based on $\overline{IQR}$ and $\overline{IQR}_{20}$ are less powerful under normality.

The analysis shows that, when there are disturbances in the Phase I data, the performance of all charts changes considerably: there is a sizeable decrease in p and increase in ARL. In other words, when the Phase I data are contaminated, shifts in the process mean are less quickly detected. When symmetric disturbances are present (Tables 5.5 and 5.6), their impact is the smallest for the charts based on $\overline{IQR}$, $\overline{IQR}_{20}$, $D7$ and ATS. These charts are also least affected when there are asymmetric disturbances (Tables 5.7 and 5.8). Both tables show that the chart based on ATS outperforms the others.

When there are localized disturbances (Tables 5.9 and 5.10), the charts based on the estimators $D7$ and ATS perform best, the reason being that these charts trim extreme samples. Finally, in the case of diffuse mean disturbances (Tables 5.11 and 5.12), the charts based on $\overline{IQR}_{20}$, $D7$ and ATS perform better than the other charts.

Overall, the ATS chart performs best. Under normality, the chart essentially matches the performance of the standard charts based on $\bar{S}$ and $\bar{R}$ and, in the presence of any contamination, the chart outperforms the alternatives.

5.5 Concluding remarks

We have analyzed several estimation methods for the standard deviation parameter and compared the MSE of the estimators under a range of circumstances: the uncontaminated situation and various situations contaminated with diffuse symmetric and asymmetric variance disturbances, localized variance disturbances and diffuse mean disturbances. We have also investigated the effect of estimating the standard deviation on $\overline{X}$ Phase II control chart performance when the methods are used to determine the Phase II limits.

One of the proposed methods has allowed us to address certain problems exhibited by the standard methods. Estimators that trim observations (e.g. $\overline{IQR}$) perform reasonably well when there are diffuse disturbances but not when there are localized disturbances. In the latter case, estimators that include a method to trim sample statistics (e.g. $\overline{IQR}_{20}$) are efficient. But all such methods are biased when disturbances are asymmetric, as the trimming does not take asymmetry into account.

A Phase I analysis - using a control chart to study a historical dataset retrospectively and trim the data adaptively - does consider the distribution of the disturbance, making it well-suited to the estimation of σ. In this chapter we have proposed a new type of Phase I analysis. The initial estimate of σ for the Phase I control chart is given by an estimator that is robust against both diffuse and localized disturbances, namely $\overline{IQR}_{20}$. We have shown that this estimator is not very efficient under normality. However, when $\overline{IQR}_{20}$ is only used to construct the Phase I control chart limits, and when the standard estimation method $\bar{S}$ is used to determine the final estimate of σ after screening, the resulting estimator (ATS) is efficient under normality. Moreover, ATS outperforms the other estimation methods when there are contaminations. It is therefore a suitable method for determining the value of σ in the $\overline{X}$ Phase II control chart limits.

Note that in Chapter 3 we evaluated the standard deviation estimator $\overline{MD}^{i,s}$, whose performance essentially matches that of ATS. The difference is that, in the case of ATS, the estimators used in Phase I and Phase II are not the same. The estimator used in Phase I is very robust so that the Phase I control limits better represent the in-control state of the process. As a result, only one iteration of the screening step is required. We think that, on the whole, ATS is of more practical use.

			p		
k	Chart	$\delta = 0$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$
50	$\bar{S}$	0.0027	0.0073	0.029	0.21
	$\bar{R}$	0.0027	0.0073	0.028	0.21
	$\overline{IQR}$	0.0027	0.0071	0.027	0.20
	$\overline{IQR}_{20}$	0.0027	0.0069	0.026	0.19
	$D7$	0.0027	0.0073	0.028	0.21
	ATS	0.0027	0.0072	0.028	0.20
100	$\bar{S}$	0.0027	0.0075	0.029	0.22
	$\bar{R}$	0.0027	0.0074	0.029	0.22
	$\overline{IQR}$	0.0027	0.0074	0.029	0.21
	$\overline{IQR}_{20}$	0.0027	0.0072	0.028	0.21
	$D7$	0.0027	0.0074	0.029	0.21
	ATS	0.0027	0.0073	0.029	0.21
			ARL		
k	Chart	$\delta = 0$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$
50	$\bar{S}$	489	193	44.9	5.24
		(155; 1256)	(64.1; 418)	(18.6; 85.7)	(3.21; 7.94)
	$\bar{R}$	500	196	46.2	5.30
		163; 1318)	(63.3; 489)	(18.4; 98.1)	(3.19; 8.58)
	$\overline{IQR}$	770	289	60.7	6.09
		(110; 3127)	(75.7; 1838)	(22.6; 307)	(3.48; 17.4)
	$\overline{IQR}_{20}$	1066	375	73.2	6.74
		(96.4; 5089)	(42.8; 1332)	(13.5; 227)	(2.69; 14.9)
	$D7$	507	201	46.2	5.30
		145; 1374)	(81.2; 493)	(23.1; 98.6)	(3.56; 8.63)
	ATS	543	211	48.4	5.45
		(135; 1536)	(57.8; 451)	(17.1; 91.2)	(3.06; 8.32)
100	$\bar{S}$	419	159	38.5	4.83
		(194; 818)	(77.1; 398)	(21.5; 83.7)	(3.48; 7.66)
	$\bar{R}$	428	161	38.9	4.85
		(193; 857)	(76.7; 282)	(21.4; 61.8)	(3.48; 6.50)
	$\overline{IQR}$	521	189	43.9	5.18
		(146; 1487)	(61.1; 522)	(17.9; 103)	(3.14; 8.89)
	$\overline{IQR}_{20}$	598	213	47.9	5.41
		(133; 1929)	(56.6; 553)	(16.8; 108)	(3.03; 9.26)
	$D7$	431	163	39.1	4.88
		(189; 877)	(81.6; 290)	(22.6; 63.2)	(3.57; 6.59)
	ATS	446	168	40.1	4.94
		(179; 940)	(80.0; 296)	(22.3; 64.3)	(3.53; 6.68)

Table 5.3: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values under normality for $n = 5$

			p		
k	Chart	$\delta = 0$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$
50	$\bar{S}$	0.0027	0.012	0.063	0.48
	$\bar{R}$	0.0027	0.012	0.063	0.48
	$\overline{IQR}$	0.0027	0.012	0.062	0.47
	$\overline{IQR}_{20}$	0.0027	0.012	0.061	0.46
	$D7$	0.0027	0.012	0.064	0.48
	ATS	0.0027	0.012	0.063	0.48
100	$\bar{S}$	0.0027	0.012	0.065	0.49
	$\bar{R}$	0.0027	0.012	0.063	0.48
	$\overline{IQR}$	0.0027	0.012	0.064	0.49
	$\overline{IQR}_{20}$	0.0027	0.012	0.064	0.48
	$D7$	0.0027	0.012	0.065	0.49
	ATS	0.0027	0.012	0.065	0.49

			ARL		
k	Chart	$\delta = 0$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$
50	$\bar{S}$	427	106	18.0	2.13
		189; 846)	(148; 157)	(23.8; 24.6)	(2.31; 2.48)
	$\bar{R}$	440	107	18.2	2.14
		(186; 923)	(53.1; 171)	(10.9; 26.2)	(1.75; 2.54)
	$\overline{IQR}$	537	124	20.1	2.22
		142; 1512)	(38.2; 257)	(8.61; 35.8)	(1.61; 2.94)
	$\overline{IQR}_{20}$	588	135	21.1	2.26
		(135; 1858)	(36.7; 388)	(8.37; 48.7)	(1.60; 3.36)
	$D7$	432	106	18.0	2.14
		(182; 885)	(65.8; 184)	(12.7; 27.6)	(1.85; 2.59)
	ATS	441	107	18.2	2.15
		(177; 931)	(127; 171)	(21.1; 26.3)	(2.21; 2.55)
100	$\bar{S}$	397	95.5	16.4	2.06
		(232; 643)	(56.4; 132)	(11.4; 21.6)	(1.79; 2.33)
	$\bar{R}$	398	92.8	16.4	2.06
		(223; 668)	(61.2; 135)	(12.1; 21.9)	(1.83; 2.35)
	$\overline{IQR}$	440	99.8	17.3	2.10
		(182; 941)	(47.2; 226)	(10.0; 32.1)	(1.71; 2.75)
	$\overline{IQR}_{20}$	462	103	17.6	2.12
		(174; 1066)	(46.6; 193)	(9.94; 28.7)	(1.70; 2.65)
	$D7$	403	93.1	16.4	2.07
		(225; 670)	(61.1; 130)	(12.1; 21.4)	(1.83; 2.33)
	ATS	401	92.9	16.4	2.07
		(219; 681)	(75.8; 143)	(14.1; 22.7)	(1.92; 2.38)

Table 5.4: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values under normality for $n = 9$

			p		
k	Chart	$\delta = 0$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$
50	$\bar{S}$	4.3×10^{-4}	0.0014	0.0070	0.081
	$\bar{R}$	4.1×10^{-4}	0.0013	0.0067	0.079
	$\overline{IQR}$	0.0015	0.0042	0.018	0.15
	$\overline{IQR}_{20}$	0.0016	0.0046	0.018	0.15
	$D7$	0.0015	0.0044	0.019	0.16
	ATS	0.0019	0.0053	0.022	0.17
100	$\bar{S}$	3.5×10^{-4}	0.0012	0.0065	0.082
	$\bar{R}$	3.3×10^{-4}	0.0012	0.0062	0.079
	$\overline{IQR}$	0.0014	0.0042	0.018	0.16
	$\overline{IQR}_{20}$	0.0016	0.0046	0.019	0.16
	$D7$	0.0015	0.0044	0.019	0.16
	ATS	0.0019	0.0053	0.022	0.18
			ARL		
k	Chart	$\delta = 0$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$
50	$\bar{S}$	4.3×10^{4}	3609	441	20.2
		(343; 1.4×10^{4})	(76.1; 1675)	(7.12; 58.7)	
	$\bar{R}$	1.4×10^{4}	364	475	21.1
		562; 7.8×10^{4})	(275; 1.5×10^{4})	(61.7; 1758)	(6.35; 61.1)
	$\overline{IQR}$	2041	657	120	9.08
		(166; 9948)	(109; 2979)	(30.1; 449)	(4.10; 33.3)
	$\overline{IQR}_{20}$	2288	746	128	9.28
		144; 1.2×10^{4})	(70.4; 2700)	(20.3; 413)	(3.34; 22.4)
	$D7$	1107	401	82.6	7.38
		(221; 3820)	(102; 1209)	(27.1; 209)	(3.94; 14.0)
	ATS	898	335	70.8	6.73
		(170; 3064)	(69.9; 860)	(19.9; 157)	(3.33; 11.7)
100	$\bar{S}$	5922	1599	255	15.2
		880; 2.4×10^{4})	(309; 5101)	(66.9; 713)	(6.80; 32.5)
	$\bar{R}$	6506	174	271	15.9
		(912; 2.7×10^{4})	(302; 6155)	(65.4; 837)	(6.73; 36.1)
	$\overline{IQR}$	1116	375	77.1	7.17
		240; 3632)	(99.8; 952)	(26.6; 171)	(3.91; 12.4)
	$\overline{IQR}_{20}$	1131	370	76.5	7.10
		204; 4062)	(83.7; 1046)	(23.0; 185)	(3.61; 13.1)
	$D7$	874	304	64.8	6.57
		(303; 2126)	(126; 785)	(32.1; 146)	(4.36; 11.0)
	ATS	693	247	55.1	5.96
		235; 1696)	(190; 507)	(49.3; 101)	(5.40; 8.83)

Table 5.5: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values when symmetric variance disturbances are present in Phase I for $n = 5$

				p	
k	Chart	$\delta = 0$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$
50	$\bar{S}$	2.9×10^{-4}	0.0019	0.015	0.23
	$\bar{R}$	2.0×10^{-4}	0.0014	0.011	0.20
	$\overline{IQR}$	0.0016	0.0078	0.045	0.40
	$\overline{IQR}_{20}$	0.0017	0.0079	0.046	0.41
	$D7$	0.0016	0.0078	0.046	0.41
	ATS	0.0019	0.0091	0.051	0.43
100	$\bar{S}$	2.5×10^{-4}	0.0017	0.014	0.23
	$\bar{R}$	1.7×10^{-4}	0.0012	0.011	0.20
	$\overline{IQR}$	0.0016	0.0079	0.047	0.42
	$\overline{IQR}_{20}$	0.0017	0.0081	0.048	0.42
	$D7$	0.0016	0.0077	0.046	0.42
	ATS	0.0019	0.0091	0.053	0.45
				ARL	
k	Chart	$\delta = 0$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$
50	$\bar{S}$	1.0×10^{4}	1445	127	5.14
		(934; 4.8×10^{4})	(185; 1.2×10^{4})	(67.7; 732)	(2.60; 14.0)
	$\bar{R}$	2.2×10^{4}	2702	206	6.42
		(1162; 1.2×10^{5})	(230; 2.2×10^{4})	(32.7; 1246)	(2.80; 19.3)
	$\overline{IQR}$	983	209	29.4	2.62
		(218; 3166)	(62.1; 459)	(12.2; 56.3)	(1.83; 3.66)
	$\overline{IQR}_{20}$	1047	219	30.3	2.64
		(196; 3690)	(109; 533)	(18.5; 63.2)	(2.11; 3.87)
	$D7$	799	180	26.7	2.52
		(271; 1945)	(85.7; 348)	(15.4; 45.0)	(2.00; 3.26)
	ATS	667	154	23.7	2.39
		(225; 1618)	(64.3; 274)	(12.5; 37.6)	(1.85; 3.00)
100	$\bar{S}$	6623	934	95.2	4.61
		(1446; 2.2×10^{4})	(244; 2308)	(34.4; 203)	(2.89; 7.15)
	$\bar{R}$	1.2×10^{4}	1492	135	5.49
		(1851; 4.6×10^{4})	(305; 5238)	(40.9; 390)	(3.13; 10.2)
	$\overline{IQR}$	785	160	24.6	2.44
		287; 1831)	(67.6; 416)	(13.1; 51.2)	(1.89; 3.42)
	$\overline{IQR}_{20}$	785	160	24.4	2.44
		(263; 1953)	(61.9; 317)	(12.2; 42.1)	(1.84; 3.17)
	$D7$	724	152	23.7	2.41
		(354; 1367)	(91.8; 240)	(16.3; 33.9)	(2.06; 2.86)
	ATS	591	129	20.9	2.28
		(287; 1106)	(71.4; 195)	(13.6; 29.0)	(1.92; 2.67)

Table 5.6: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values when symmetric variance disturbances are present in Phase I for $n = 9$

		p			
k	Chart	$\delta = 0$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$
50	$\bar{S}$	2.9 $\times 10^{-4}$	9.2 $\times 10^{-4}$	0.0045	0.054
	$\bar{R}$	2.8 $\times 10^{-4}$	9.2 $\times 10^{-4}$	0.0046	0.055
	$\overline{IQR}$	0.0016	0.0045	0.018	0.15
	$\overline{IQR}_{20}$	0.0018	0.0050	0.020	0.16
	$D7$	0.0019	0.0053	0.022	0.17
	ATS	0.0022	0.0061	0.024	0.19
100	$\bar{S}$	1.7 $\times 10^{-4}$	6.07 $\times 10^{-4}$	0.0033	0.048
	$\bar{R}$	1.7 $\times 10^{-4}$	6.3 $\times 10^{-4}$	0.0034	0.049
	$\overline{IQR}$	0.0016	0.0045	0.019	0.16
	$\overline{IQR}_{20}$	0.0018	0.0051	0.021	0.17
	$D7$	0.0018	0.0053	0.022	0.18
	ATS	0.0022	0.0061	0.025	0.19
		ARL			
k	Chart	$\delta = 0$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$
50	$\bar{S}$	2.4 $\times 10^{10}$	4.1 $\times 10^{14}$	1.5 $\times 10^{8}$	6.2 $\times 10^{4}$
		549; 2.2 $\times 10^{7}$)	(277; 2.1 $\times 10^{7}$)	(62.3; 1.1 $\times 10^{6}$)	(6.38; 7196)
	$\bar{R}$	4.2 $\times 10^{9}$	2.9 $\times 10^{7}$	2.5 $\times 10^{7}$	3532
		(575; 2.2 $\times 10^{7}$)	(194; 1.4 $\times 10^{6}$)	(95.4; 9.6 $\times 10^{4}$)	(5.40; 1172)
	$\overline{IQR}$	4219	1376	158	9.88
		(158; 1324)	(72.3; 2951)	(20.6; 446)	(3.38; 23.6)
	$\overline{IQR}_{20}$	1944	670	118	8.74
		(132; 1.0 $\times 10^{4}$)	(62.8; 8338)	(18.4; 1152)	(3.17; 42.0)
	$D7$	820	318	67.7	6.62
		(189; 2565)	(76.4; 702)	(21.4; 132)	(3.47; 10.5)
	ATS	723	283	60.7	6.23
		(155; 2292)	(77.7; 647)	(22.0; 123)	(3.49; 10.1)
100	$\bar{S}$ 8.7 $\times 10^{5}$	6.7 $\times 10^{4}$	1.6 $\times 10^{4}$	123	
		(1107; 1.5 $\times 10^{6}$)	(467; 2.9 $\times 10^{5}$)	(94.7; 2.4 $\times 10^{4}$)	(8.34; 402)
	$\bar{R}$	6.8 $\times 10^{5}$	7.0 $\times 10^{6}$	5843	69.0
		1097; 1.0 $\times 10^{6}$)	(402; 1.4 $\times 10^{5}$)	(83.1; 1.3 $\times 10^{4}$)	(7.75; 260)
	$\overline{IQR}$	1204	400	78.8	7.23
		(223; 4523)	(109; 1148)	(29.0; 200)	(4.07; 13.8)
	$\overline{IQR}_{20}$	988	341	69.9	6.75
		188; 3510)	(80.2; 1006)	(22.3; 179)	(3.54; 12.7)
	$D7$	669	246	55.1	5.94
		(254; 1525)	(99.4; 450)	(26.4; 91.1)	(3.91; 8.30)
	ATS	571	213	49.0	5.54
		(209; 1312)	(83.8; 397)	(23.0; 82.0)	(3.62; 7.77)

Table 5.7: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values when asymmetric variance disturbances are present in Phase I for $n = 5$

			p		
k	Chart	$\delta = 0$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$
50	$\bar{S}$	1.3×10^{-4}	8.5×10^{-4}	0.0071	0.14
	$\bar{R}$	1.0×10^{-4}	6.7×10^{-4}	0.0058	0.12
	$\overline{IQR}$	0.0018	0.0084	0.048	0.42
	$\overline{IQR}_{20}$	0.0019	0.0087	0.049	0.42
	$D7$	0.0020	0.0092	0.052	0.44
	ATS	0.0023	0.010	0.056	0.45
100	$\bar{S}$	7.4×10^{-5}	5.7×10^{-4}	0.0054	0.13
	$\bar{R}$	5.5×10^{-5}	4.4×10^{-4}	0.0043	0.11
	$\overline{IQR}$	0.0018	0.0086	0.050	0.43
	$\overline{IQR}_{20}$	0.0019	0.0089	0.051	0.43
	$D7$	0.0019	0.0092	0.053	0.45
	ATS	0.0022	0.010	0.058	0.46
			ARL		
k	Chart	$\delta = 0$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$
50	$\bar{S}$	1.4×10^{8}	1.1×10^{7}	1.9×10^{5}	190
		1173; 1.7×10^{7})	(208; 1.1×10^{6})	(30.5; 3.5×10^{4})	(2.72; 166)
	$\bar{R}$	2.1×10^{9}	2.1×10^{8}	9.0×10^{10}	322
		(1408; 4.1×10^{7})	(260; 3.1×10^{6})	(36.0; 9.2×10^{4})	(2.92; 314)
	$\overline{IQR}$	880	200	28.5	2.56
		(194; 2921)	(49.7; 444)	(10.4; 54.7)	(1.73; 4.60)
	$\overline{IQR}_{20}$	905	203	28.9	2.57
		(177; 3084)	(61.5; 768)	(12.1; 82.3)	(1.81; 4.29)
	$D7$	628	154	23.6	2.38
		(232; 1423)	(131; 243)	(21.3; 34.4)	(2.23; 2.88)
	ATS	551	137	21.8	2.30
		(198; 1248)	(63.0; 216)	(12.3; 31.3)	(1.83; 2.76)
100	$\bar{S}$	4.8×10^{5}	3.6×10^{4}	1405	15.1
		(2429; 1.9×10^{6})	(577; 1.0×10^{5})	(66.0; 4594)	(3.86; 45.8)
	$\bar{R}$	1.2×10^{6}	1.0×10^{5}	2.5×10^{3}	19.9
		(3124; 3.6×10^{6})	(532; 2.9×10^{5})	(62.6; 1.1×10^{4})	(3.82; 77.3)
	$\overline{IQR}$	699	152	23.6	2.40
		(254; 1592)	(60.7; 443)	(12.1; 52.6)	(1.83; 3.44)
	$\overline{IQR}_{20}$	685	150	23.4	2.38
		(235; 1674)	(58.4; 354)	(11.7; 45.4)	(1.81; 3.24)
	$D7$	572	131	21.2	2.29
		(293; 1026)	(123; 192)	(20.1; 28.6)	(2.21; 2.64)
	ATS	496	116	19.4	2.20
		(251; 894)	(81.4; 178)	(14.9; 27.0)	(1.97; 2.57)

Table 5.8: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values when asymmetric variance disturbances are present in Phase I for $n = 9$

			p		
k	Chart	$\delta = 0$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$
50	$\bar{S}$	1.3×10^{-4}	5.2×10^{-4}	0.0030	0.047
	$\bar{R}$	1.3×10^{-4}	5.1×10^{-4}	0.0030	0.046
	$\overline{IQR}$	1.8×10^{-4}	6.3×10^{-4}	0.0033	0.047
	$\overline{IQR}_{20}$	0.0011	0.0033	0.014	0.13
	$D7$	0.0015	0.0044	0.018	0.16
	ATS	0.0021	0.0057	0.023	0.18
100	$\bar{S}$	1.2×10^{-4}	4.7×10^{-4}	0.0029	0.048
	$\bar{R}$	1.1×10^{-4}	4.6×10^{-4}	0.0029	0.047
	$\overline{IQR}$	1.4×10^{-4}	5.3×10^{-4}	0.0031	0.048
	$\overline{IQR}_{20}$	0.0011	0.0033	0.015	0.14
	$D7$	0.0014	0.0043	0.019	0.16
	ATS	0.0020	0.0056	0.023	0.19
			ARL		
k	Chart	$\delta = 0$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$
50	$\bar{S}$	2.3×10^{4}	6410	828	32.8
		(1843; 1.1×10^{5})	(1054; 2.1×10^{4})	(192; 2442)	(12.8; 76.9)
	$\bar{R}$	2.5×10^{4}	6946	868	33.7
		(1805; 1.3×10^{5})	(767; 2.3×10^{4})	(144; 2606)	(10.8; 80.7)
	$\overline{IQR}$	1.5×10^{5}	3.0×10^{4}	2.7×10^{3}	60.7
		(918; 7.2×10^{5})	(743; 1.2×10^{5})	(150; 1.1×10^{5})	(10.8; 231)
	$\overline{IQR}_{20}$	3703	1171	186	11.8
		(193; 2.1×10^{4})	(76.8; 4467)	(41.4; 636)	(3.48; 30.0)
	$D7$	1112	408	83.5	7.48
		(224; 3717)	(90.4; 1536)	(24.5; 257)	(3.74; 15.8)
	ATS	843	321	68.0	6.58
		(155; 2899)	(216; 780)	(70.2; 144)	(6.50; 11.2)
100	$\bar{S}$	1.5×10^{4}	3766	534	25.4
		(2865; 5.0×10^{4})	(780; 9525)	(144; 1222)	(11.1; 47.3)
	$\bar{R}$	1.6×10^{4}	3904	549	26.1
		(2870; 5.3×10^{4})	(1034; 1.2×10^{4})	(184; 1456)	(12.8; 53.0)
	$\overline{IQR}$	2.9×10^{4}	6682	827	32.4
		(1630; 1.6×10^{5})	(264; 2.7×10^{4})	(144; 3022)	(10.8; 90.0)
	$\overline{IQR}_{20}$	1775	554	105	8.68
		(285; 6672)	(107; 1654)	(28.1; 272)	(4.06; 16.9)
	$D7$	886	309	66.0	6.65
		(309; 2132)	(114; 629)	(29.6; 120)	(4.18; 9.89)
	ATS	652	236	52.9	5.80
		218; 1603)	(89.9; 468)	(24.4; 94.4)	(3.73; 8.47)

Table 5.9: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values when localized variance disturbances are present in Phase I for $n = 5$

				p	
k	Chart	$\delta = 0$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$
50	$\bar{S}$	1.2×10^{-4}	8.9×10^{-4}	0.0083	0.17
	$\bar{R}$	1.2×10^{-4}	9.0×10^{-4}	0.0083	0.17
	$\overline{IQR}$	1.4×10^{-4}	9.9×10^{-4}	0.0086	0.17
	$\overline{IQR}_{20}$	0.0014	0.0067	0.040	0.38
	$D7$	0.0016	0.0076	0.045	0.41
	ATS	0.0025	0.011	0.060	0.46
100	$\bar{S}$	1.1×10^{-4}	8.6×10^{-4}	0.0082	0.18
	$\bar{R}$	1.1×10^{-4}	8.6×10^{-4}	0.0083	0.18
	$\overline{IQR}$	1.2×10^{-4}	9.0×10^{-4}	0.0084	0.18
	$\overline{IQR}_{20}$	0.0014	0.0069	0.042	0.40
	$D7$	0.0015	0.0076	0.045	0.42
	ATS	0.0025	0.011	0.061	0.47
				ARL	
k	Chart	$\delta = 0$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$
50	$\bar{S}$	1.5×10^{4}	2.3×10^{3}	193	6.54
		(2724; 5.1×10^{4})	(505; 6959)	(59.9; 487)	(3.72; 11.4)
	$\bar{R}$	1.6×10^{4}	2432	201	6.69
		(2608; 6.0×10^{4})	(773; 1.6×10^{4})	(82.4; 930)	(4.26; 16.0)
	$\overline{IQR}$	3.0×10^{4}	4082	292	7.66
		(1671; 1.7×10^{5})	(277; 1.4×10^{4})	(38.0; 899)	(3.02; 16.7)
	$\overline{IQR}_{20}$	1281	273	35.7	2.82
		(229; 4581)	(87.2; 621)	(15.6; 71.4)	(2.00; 4.12)
	$D7$	823	190	27.6	2.55
		(273; 2013)	(118; 348)	(19.6; 45.1)	(2.18; 3.26)
	ATS	509	126	20.4	2.24
		(176; 1211)	(96.9; 225)	(17.0; 32.2)	(2.04; 2.79)
100	$\bar{S}$	1.2×10^{4}	1650	152	5.95
		(3975; 3.0×10^{4})	(690; 2965)	(56.6; 249)	(4.21; 8.00)
	$\bar{R}$	1.3×10^{4}	1691	154	5.98
		(3776; 3.3×10^{4})	(1534; 3841)	(142; 303)	(5.53; 8.84)
	$\overline{IQR}$	1686	2130	180	6.38
		(2599; 6.5×10^{4})	(404; 6258)	(50.8; 453)	(3.47; 11.2)
	$\overline{IQR}_{20}$	971	194	28.3	2.60
		(310; 2447)	(86.6l 387)	(45.6; 49.1)	(2.02; 3.41)
	$D7$	748	158	24.5	2.44
		(359; 1422)	(79.6; 249)	(14.8; 34.9)	(1.99; 2.90)
	ATS	455	105	18.0	2.14
		(226; 842)	(57.4; 184)	(11.6; 27.5)	(1.80; 2.58)

Table 5.10: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values when localized variance disturbances are present in Phase I for $n = 9$

			p		
k	Chart	$\delta = 0$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$
50	$\bar{S}$	2.5×10^{-4}	8.8×10^{-4}	0.0046	0.062
	$\bar{R}$	2.3×10^{-4}	8.3×10^{-4}	0.0045	0.060
	$\overline{IQR}$	9.4×10^{-4}	0.0028	0.012	0.12
	$\overline{IQR}_{20}$	0.0013	0.0036	0.015	0.13
	$D7$	9.6×10^{-4}	0.0029	0.013	0.12
	ATS	0.0017	0.0047	0.019	0.16
100	$\bar{S}$	2.1×10^{-4}	8.1×10^{-4}	0.0045	0.063
	$\bar{R}$	2.0×10^{-4}	7.6×10^{-4}	0.0042	0.061
	$\overline{IQR}$	8.7×10^{-4}	0.0027	0.012	0.12
	$\overline{IQR}_{20}$	0.0012	0.0036	0.016	0.14
	$D7$	8.8×10^{-4}	0.0028	0.013	0.13
	ATS	0.0016	0.0045	0.019	0.16
			ARL		
k	Chart	$\delta = 0$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$
50	$\bar{S}$	1.2×10^{4}	6444	976	32.3
		(991; 5.4×10^{4})	(341; 1.5×10^{4})	(72.5; 1754)	(7.15; 60.0)
	$\bar{R}$	1.3×10^{4}	7379	1117	35.3
		(1030; 6.1×10^{4})	(1068; 4.4×10^{4})	(216; 4737)	(13.4; 115)
	$\overline{IQR}$	4141	2120	306	15.4
		(245; 2.2×10^{4})	(108; 6419)	(28.4; 866)	(4.05; 36.4)
	$\overline{IQR}_{20}$	3177	1530	264	12.4
		(176; 1.7×10^{4})	(83.2; 9626)	(23.1; 1253)	(3.60; 45.4)
	$D7$	2269	1006	182	11.5
		(299; 9368)	(146; 3666)	(36.7; 359)	(4.67; 25.9)
	ATS	1343	598	115	8.57
		(175; 5726)	(108; 1457)	(29.6; 244)	(4.07; 15.7)
100	$\bar{S}$	7771	3188	484	22.4
		1554; 2.5×10^{4})	(823; 2.2×10^{3})	(155; 2638)	(11.2; 75.0)
	$\bar{R}$	8521	3520	521	23.6
		(1651; 2.8×10^{4})	(951; 1.5×10^{4})	(176; 1756)	(12.1; 57.6)
	$\overline{IQR}$	2173	790	143	10.2
		(366; 8109)	(142; 1924)	(35.4; 310)	(4.63; 18.4)
	$\overline{IQR}_{20}$	1564	568	106	8.65
		(262; 5767)	(99.9; 1570)	(26.5; 260)	(3.93; 16.2)
	$D7$	1662	618	117	9.27
		(433; 4800)	(172; 2467)	(421.3; 389)	(5.07; 20.5)
	ATS	934	350	73.2	6.94
		(247; 2762)	(94.9; 1218)	(25.4; 213)	(3.83; 13.9)

Table 5.11: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values when diffuse mean disturbances are present in Phase I for $n = 5$

		p			
k	Chart	$\delta = 0$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$
50	$\bar{S}$	2.2×10^{-4}	0.0014	0.011	0.20
	$\bar{R}$	1.4×10^{-4}	0.0010	0.0089	0.17
	$\overline{IQR}$	0.0012	0.0060	0.037	0.36
	$\overline{IQR}_{20}$	0.0014	0.0066	0.040	0.38
	$D7$	0.0011	0.0055	0.035	0.36
	ATS	0.0017	0.0081	0.047	0.41
100	$\bar{S}$	1.8×10^{-4}	0.0013	0.011	0.20
	$\bar{R}$	1.3×10^{-4}	9.7×10^{-4}	0.0087	0.17
	$\overline{IQR}$	0.0011	0.0059	0.037	0.38
	$\overline{IQR}_{20}$	0.0013	0.0068	0.041	0.397
	$D7$	0.0010	0.0054	0.035	0.37
	ATS	0.0017	0.0082	0.048	0.42

		ARL			
k	Chart	$\delta = 0$	$\delta = 0.25$	$\delta = 0.5$	$\delta = 1$
50	$\bar{S}$	9178	3394	272	7.00
		(1508; 3.2×10^4)	(2351; 5964)	(216; 425)	(6.50; 10.4)
	$\bar{R}$	1.5×10^4	6628	462	9.05
		(1923; 5.8×10^4)	(512; 7.8×10^4)	(60.0; 3614)	(3.67; 33.9)
	$\overline{IQR}$	1479	373	44.8	3.10
		(278; 5152)	(64.8; 693)	(12.7; 77.7)	(1.87; 4.29)
	$\overline{IQR}_{20}$	1305	332	41.0	2.95
		(235; 4557)	(391; 964)	(56.5; 98.8)	(3.29; 4.74)
	$D7$	1336	350	43.5	3.06
		(361; 3751)	(100; 533)	(17.4; 63.2)	(2.11; 3.87)
	ATS	819	215	30.0	2.59
		(228; 2339)	(159; 496)	(24.8; 58.8)	(2.37; 3.66)
100	$\bar{S}$	7369	2127	178	6.07
		(2201; 1.9×10^4)	(450; 1.5×10^4)	(54.6; 893)	(3.55; 14.9)
	$\bar{R}$	1.1×10^4	3640	273	7.51
		2834; 3.1×10^4)	(999; 7126)	(101; 488)	(4.67; 11.1)
	$\overline{IQR}$	1137	259	34.2	2.82
		(376; 2822)	(82.2; 592)	(15.1; 67.5)	(2.01; 3.92)
	$\overline{IQR}_{20}$	979	220	30.8	2.67
		(316; 2457)	(71.6; 373)	(13.6; 47.9)	(1.92; 3.38)
	$D7$	1166	265	35.8	2.87
		(484; 2483)	(101; 521)	(17.6; 61.1)	(2.14; 3.74)
	ATS	702	163	24.7	2.44
		303; 1479)	(93.3; 259)	(16.4; 36.0)	(2.05; 2.94)

Table 5.12: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values when diffuse mean disturbances are present in Phase I for $n = 9$

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Samenvatting

Schattingsmethoden voor statistische procesbeheersing

Processen zijn onderhevig aan variatie. In hoeverre een proces normaal functioneert kan worden bepaald met behulp van een regelkaart. Een regelkaart onderscheidt variatie veroorzaakt door speciale oorzaken en variatie veroorzaakt door gewone oorzaken. Variatie door gewone oorzaken is in ieder proces aanwezig: het is inherent aan het ontwerp van het proces. Als in een proces alleen variatie door gewone oorzaken aanwezig is, wordt het proces beheerst genoemd: het proces fluctueert binnen een voorspelbare bandbreedte. Speciale oorzaken van variatie zijn onderwerp voor verbetering. Speciale oorzaken zijn bijvoorbeeld onverwachte gebeurtenissen of een nieuwe leverancier voor inkomend materiaal.

De regelkaart is in 1924 ontwikkeld door dr. Walter A. Shewhart. Het is een grafiek van metingen van een proceskarakteristiek op de verticale as, uitgezet tegen de tijd op de horizontale as. Daarnaast worden in de grafiek twee regelgrenzen weergegeven: de Upper Control Limit (UCL) en de Lower Control Limit (LCL). Wanneer een meting boven de UCL of onder de LCL komt, geeft de regelkaart een signaal: het proces is niet langer beheerst. Indien de proceskarakteristiek een numerieke variabele is, is het gebruikelijk om zowel de locatie als de spreiding van de karakteristiek te monitoren. De spreiding wordt eerst onderzocht, gevolgd door de locatie. Veelal wordt als maat voor de locatie het gemiddelde genomen en voor de spreiding de standaarddeviatie. De regelgrenzen van de regelkaart voor de standaarddeviatie worden bepaald op basis van de standaarddeviatie van het proces; de regelgrenzen van de kaart voor het gemiddelde worden gebaseerd op zowel de standaarddeviatie als het gemiddelde van het proces. In de praktijk zijn het gemiddelde en de standaarddeviatie niet bekend en deze worden daarom geschat op basis van gemiddeld 30-100 steekproeven, ieder ter grootte van 5-9 metingen. Deze fase wordt ook wel Fase I genoemd. Het gebruik van de regelkaart om het proces te monitoren wordt aangeduid met Fase II. In de literatuur gebruikt men vaak de standaardschatters op basis van het steekproefgemiddelde en de steekproefstandaarddeviatie gebruikt uit Fase I. Het nadeel van deze methoden is dat ze gevoelig zijn voor uitschieters, verschuivingen en andere afwijkingen in de data. Dit proefschrift onderzoekt alternatieve schattingsmethoden voor de bepaling van de regelgrenzen in Fase II.

In hoofdstuk 2 worden twaalf schattingsmethoden voor de standaarddeviatie vergeleken. In de studie zijn de standaard schatters gebaseerd op de steekproefstandaarddeviatie en de steekproefrange opgenomen en verscheidene alternatieve schattingsmethoden voor de standaarddeviatie. Daarnaast wordt een analyse in Fase I onderzocht. Hierin wordt een regelkaart opgesteld, die wordt gebruikt om steekproeven met extreme spreiding te identificeren. De andere steekproeven worden vervolgens gebruikt om de standaarddeviatie te bepalen. De methoden worden beoordeeld door de schattingsfout te bepalen onder normaliteit en onder aanwezigheid van verschillende typen verstoringen. Daarnaast wordt onderzocht wat het effect van de methoden is op de kwaliteit van de regelkaart in Fase II.

Uit hoofdstuk 2 blijkt dat de meeste schatters voor de standaarddeviatie robuust zijn tegen ofwel diffuse verstoringen (uitschieters verspreid over de steekproeven), ofwel gelokaliseerde verstoringen (verstoringen die een gehele steekproef beïnvloeden), maar niet tegen beide. De robuuste schatter van Tatum blijkt goed te werken wanneer diffuse verstoringen aanwezig zijn in Fase I, terwijl de analyse in Fase I goed functioneert wanneer de data gelokaliseerde verstoringen bevatten.

In hoofdstuk 3 wordt geprobeerd een schattingsmethode voor de standaarddeviatie te ontwikkelen die goed functioneert voor beide typen verstoringen. Dit gebeurt door een verbetering aan te brengen in de analyse in Fase I (gebruikt in hoofdstuk 2), die steekproeven met extreme spreiding identificeert met behulp van een regelkaart. Deze analyse wordt aangevuld met een controle op diffuse verstoringen met behulp van een regelkaart voor individuele waarnemingen. De methode wordt vergeleken met een aantal andere robuuste methoden. De evaluatie vindt weer plaats op basis van de schattingsfout en het effect op de kwaliteit van de regelkaart voor de standaarddeviatie in Fase II. De nieuwe methode blijkt relatief robuust te zijn tegen de verschillende typen verstoringen.

De hoofdstukken 4 en 5 gaan over het opstellen van regelkaarten voor het gemiddelde. Daarvoor dienen het gemiddelde en de standaarddeviatie van het proces te worden geschat.

Hoofdstuk 4 onderzoekt verschillende schattingsmethoden voor het gemiddelde van het proces. In de studie wordt de standaard schattingsmethode (het gemiddelde van de steekproefgemiddelden) onderzocht evenals een aantal alternatieve locatieschatters en Fase I analyses. Daarnaast wordt een analyse voor Fase I voorgesteld, waarmee op een nog betere manier dan in hoofdstuk 2 vooraf uitschieters worden verwijderd. Dit kan door nog zorgvuldiger met verstoringen in de data om te gaan. De resultaten wijzen uit dat de regelkaart voor het gemiddelde op basis van de nieuwe analyse in Fase I, beter presteert dan de andere regelkaarten wanneer verstoringen aanwezig zijn in Fase I.

De methode uit hoofdstuk 4 wordt in hoofdstuk 5 ook toegepast op een schatter voor de standaarddeviatie. De nieuwe methoden uit hoofdstukken 4 en 5 zijn bovendien eenvoudig te combineren tot een regelkaart voor het gemiddelde. Uit de resultaten volgt dat bij gebruik van deze methoden zowel diffuse als gelokaliseerde verstoringen in Fase I minder impact hebben op de regelgrenzen, waardoor de regelkaart beter in staat is het proces te monitoren in Fase II.

Curriculum Vitae

Marit Schoonhoven werd geboren op 2 maart 1981 te Alphen aan den Rijn. Na de middelbare school in dezelfde plaats doorlopen te hebben studeerde zij van 1999 tot en met 2004 Bedrijfswiskunde en Informatica aan de Vrije Universiteit in Amsterdam. Na haar afstuderen in 2004 (cum laude) werkte Marit voor de ING Bank, waar ze afdelingen van ING adviseerde bij de ontwikkeling van kredietrisicomodellen. In 2007 besloot zij in dienst te treden van het Instituut voor Bedrijfs en Industriële Statistiek van de Universiteit van Amsterdam (IBIS UvA) om advisering te combineren met wetenschappelijk onderzoek.

IBIS UvA adviseert bedrijven, financiële instellingen en ziekenhuizen bij de implementatie van Lean Six Sigma, een programma voor kwaliteits- en efficiëntieverbetering. Het instituut traint medewerkers uit de lijn in het uitvoeren van verbeterprojecten binnen de organisatie. Voorbeelden van deze verbeterprojecten zijn het effiënter inrichten van het hypotheekofferte-traject bij een bank of het verhogen van de capaciteit op de afdeling cardiologie van een ziekenhuis. Marit heeft projecten begeleid bij AEGON, het Deventer Ziekenhuis, de Reinier de Graaf Groep en het Universitair Medisch Centrum Utrecht.

Naast advisering levert IBIS UvA een bijdrage aan de wetenschappelijke ontwikkeling van Lean Six Sigma en de industriële statistiek. Marit heeft van 2007-2011 onderzoek verricht naar methoden voor procesbeheersing, waarvan dit proefschrift het resultaat is.