Estimation methods for statistical process control

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Chapter 2

Standard Deviation Control Charts

2.1 Introduction

This chapter concerns the design and analysis of the standard deviation control chart with estimated limits. We consider an extensive range of statistics to estimate the in-control standard deviation (Phase I) and design the control chart for real-time process monitoring (Phase II) by determining the factors for the control limits. The Phase II performance of the design schemes is assessed when the Phase I data are uncontaminated and normally distributed as well as when the Phase I data are contaminated. We propose a robust estimation method based on the mean absolute deviation from the median supplemented with a simple screening method.

The chapter is structured as follows. The next section introduces various estimators of the standard deviation and assesses the MSE of the estimators. We then derive the Phase II control limits. Next, we describe the simulation procedure and simulation results. Furthermore, we discuss a real-world example implementing the various charts created. The chapter ends with some concluding remarks.

2.2 Proposed Phase I estimators

In practice the same statistic is generally used to estimate both the in-control standard deviation σ in Phase I and the standard deviation $\lambda\sigma$ in Phase II. Since the requirements for the estimators differ between the two phases, this is not always the best choice. In Phase I, an estimator should be efficient in uncontaminated situations and robust against disturbances, whereas in

Phase II the estimator should be sensitive to disturbances (cf. Jensen et al. (2006)). In the next two sections, we present the Phase I estimators considered in our study (Section 2.2.1) and evaluate the estimators by comparing their MSE (Section 2.2.2).

2.2.1 Standard deviation estimators

David (1998) gave a brief account of the history of standard deviation estimators. The traditional estimators are of course the pooled and the mean sample standard deviation and the mean sample range. Mahmoud et al. (2010) studied the relative efficiencies of these estimators for different sample sizes n and numbers of samples k. In deriving estimates of the in-control standard deviation, we will look at these as well as nine other estimators.

The first estimator of σ is based on the pooled sample standard deviation

$$\tilde{S} = \left(\frac{1}{k} \sum_{i=1}^{k} S_i^2 \right)^{1/2}, \tag{2.1}$$

where S_i is the i-th sample standard deviation defined by

$$S_i = \left(\frac{1}{n-1} \sum_{j=1}^{n} (X_{ij} - \bar{X}_i)^2 \right)^{1/2}.$$

An unbiased estimator is given by $\tilde{S}/c_4(k(n-1)+1)$, where $c_4(m)$ is defined by

$$c_4(m) = \left(\frac{2}{m-1} \right)^{1/2} \frac{\Gamma(m/2)}{\Gamma((m-1)/2)}.$$

The second estimator is based on the mean sample standard deviation

$$\bar{S} = \frac{1}{k} \sum_{i=1}^{k} S_i. \tag{2.2}$$

An unbiased estimator of σ is given by $\bar{S}/c_4(n)$.

Rocke (1989) proposed the trimmed mean of the sample ranges. In our study, we consider a variant of this estimator, namely the trimmed mean of the sample standard deviations because it is well known that the sample standard deviation is more robust than the sample range. The trimmed mean of the sample standard deviations is given by

$$\bar{S}_a = \frac{1}{k - \lceil ka \rceil} \times \left[\sum_{v=1}^{k-\lceil ka \rceil} \bar{S}_{(v)} \right], \tag{2.3}$$

where a denotes the percentage of samples to be trimmed, $\lceil z \rceil$ denotes the smallest integer not less than z and $\bar{S}_{(v)}$ denotes the v-th ordered value of the sample standard deviations. We consider the 25% trimmed mean of the sample standard deviations. To simplify the analysis, we trim an integer number of samples. For example, the 25% trimmed mean trims off the eight largest sample standard deviations when $k = 30$. To provide an unbiased estimate of σ for the normal case, the estimate must be divided by a normalizing constant. These constants are obtained from 100,000 simulation runs. For $n = 5$ and $k = 20, 30, 75$, the constants are 0.579, 0.585 and 0.568 respectively; for $n = 9$ and $k = 20, 30, 75$, the constants are 0.701, 0.705 and 0.693 respectively.

Because the above estimator trims off samples instead of individual observations, we expect the estimator to be robust against localized disturbances. We also consider a variant that is expected to be robust against diffuse disturbances, namely the mean sample standard deviation after trimming the observations in each sample

$$\overline{S_b} = \frac{1}{k} \sum_{i=1}^{k} S_i', \tag{2.4}$$

where S_i' is the standard deviation of sample i after trimming the observations, given by

$$S_i' = \left(\frac{1}{n - 2\lceil nb \rceil - 1} \sum_{v=\lceil nb \rceil + 1}^{n-\lceil nb \rceil} (X_{i(v)} - \bar{X}_i')^2 \right)^{1/2},$$

where

$$\bar{X}_i' = \frac{1}{n - 2\lceil nb \rceil} \sum_{v=\lceil nb \rceil + 1}^{n-\lceil nb \rceil} X_{i(v)},$$

with $X_{i(v)}$ the v-th ordered value in sample i and b the percentage of lowest and highest observations to be trimmed in each sample. In this study, we take 20% as our trimming percentage and, again, we trim an integer number of observations. The estimator trims off the smallest and largest observation for $n = 5$; it trims off the two smallest and the two largest observations for $n = 9$. The normalizing constant is 0.520 for $n = 5$ and 0.473 for $n = 9$.

The next estimator is based on the mean sample range

$$\bar{R} = \frac{1}{k}\sum_{i=1}^{k} R_i, \tag{2.5}$$

where R_i is the range of the i-th sample. An unbiased estimator of σ is $\bar{R}/d_2(n)$, where $d_2(n)$ is the expected range of a random $N(0,1)$ sample of size n. Values of $d_2(n)$ can be found in Duncan (1986), Table M.

The next estimator is based on the mean of the sample interquartile ranges

$$\overline{IQR} = \frac{1}{k}\sum_{i=1}^{k} IQR_i, \tag{2.6}$$

with IQR_i the interquartile range of sample i

$$IQR_i = X_{i(n-\lceil ne\rceil)} - X_{i(\lceil ne\rceil+1)}.$$

Thus, the same observations are trimmed off as in the calculation of S_i'. (Note that one would expect the IQR to correspond to $e = 0.25$. However, to simplify the analysis we only trim an integer number of observations.) The normalizing constant is 0.990 for $n = 5$ and 1.144 for $n = 9$.

We also consider an estimator based on the mean of the sample Gini's mean differences

$$\bar{G} = \frac{1}{k}\sum_{i=1}^{k} G_i, \tag{2.7}$$

where G_i is Gini's mean difference of sample i, defined by

$$G_i = \sum_{j=1}^{n-1}\sum_{l=j+1}^{n} |X_{ij} - X_{il}|/(n(n-1)/2),$$

representing the mean absolute difference between any two observations in the sample. This statistic was proposed by Gini (1912), although basically the same statistic had already been proposed by Jordan (1869). An unbiased estimator of σ is given by $\bar{G}/d_2(2)$. The appendix shows that the estimator based on Gini's mean difference can be rewritten as a linear function of order statistics and that Gini's mean difference is essentially the same as the so-called Downton estimator (Downton (1966)) and the probability-weighted moments estimator (Muhammad et al. (1993)). From David (1981, p. 191) it follows that the estimator derived from Gini's mean difference is highly efficient (98%) and is more robust to outliers than the estimators based on R or S.

An estimator of σ that is simpler and easier to interpret uses the mean of the sample average absolute deviation from the median, given by

$$\overline{ADM} = \frac{1}{k}\sum_{i=1}^{k} ADM_i, \tag{2.8}$$

where ADM_i is the average absolute deviation from the median of sample i, defined as

$$ADM_i = \frac{1}{n}\sum_{j=1}^{n} |X_{ij} - M_i|,$$

with M_i the median of sample i. An unbiased estimator of σ is given by $\overline{ADM}/t_2(n)$. Since it is difficult to obtain the constant $t_2(n)$ analytically, it is obtained by simulation. Extensive tables of $t_2(n)$ can be found in Riaz and Saghir (2009).

We also study the above estimator supplemented with a screening method based on control charting. Rocke (1989) proposed a two-stage procedure which first estimates σ by $\bar{R}$, then deletes any sample that exceeds the control limits and recomputes $\bar{R}$ using the remaining samples. Our approach follows a similar procedure. First, we estimate σ by $\overline{ADM}$ because $\overline{ADM}$ is expected to be more robust against outliers. For simplicity, our screening method uses the well known factors of the $S/c_4(n)$ control chart corresponding to the 3σ control limits in Phase I. Hence, the factors for the limits are 2.089 and 0 for $n = 5$ and 1.761 and 0.239 for $n = 9$ (cf. Table M in Duncan (1986)). We then chart $S/c_4(n)$, delete any sample that exceeds the control limits and recompute $\overline{ADM}$ using the remaining samples. We continue until all sample estimates fall within the limits. The normalizing constant is 0.996 for $n = 5$ and 0.998 for $n = 9$. The resulting estimator is denoted by $\overline{ADM}'$.

Next we study two other median statistics, namely the average of the sample medians of the absolute deviation from the median

$$\overline{MDM} = \frac{1}{k}\sum_{i=1}^{k} MDM_i, \tag{2.9}$$

with

$$MDM_i = median\{|X_{ij} - M_i|\},$$

and the mean of the sample medians of the average absolute deviation

$$\overline{MAD} = \frac{1}{k}\sum_{i=1}^{k} MAD_i, \tag{2.10}$$

with

$$MAD_i = median\{|X_{ij} - \overline{X}_i|\}.$$

The normalizing constant for $\overline{MDM}$ is 0.554 for $n = 5$ and 0.613 for $n = 9$. For $\overline{MAD}$, the normalizing constant is 0.627 for $n = 5$ and 0.658 for $n = 9$.

We also evaluate a robust estimator proposed by Tatum (1997). His method has proven to be robust to both diffuse and localized disturbances. The estimation method is constructed around a variant of the biweight A estimator. The method begins by calculating the residuals in each sample, which involves substracting the sample median from each value: $res_{ij} = X_{ij} - M_i$. If n is odd, then in each sample one of the residuals will be zero and is dropped. As a result, the total number of residuals is equal to $m' = nk$ when n is even and $m' = (n-1)k$ when n is odd. Tatum's estimator is given by

$$S_c^* = \frac{m'}{(m'-1)^{1/2}} \frac{(\sum_{i=1}^{k}\sum_{j:|u_{ij}|<1} res_{ij}^2(1-u_{ij}^2)^4)^{1/2}}{|\sum_{i=1}^{k}\sum_{j:|u_{ij}|<1}(1-u_{ij}^2)(1-5u_{ij}^2)|}, \tag{2.11}$$

where $u_{ij} = h_i res_{ij}/(cM^*)$, M^* is the median of the absolute values of all residuals,

$$h_i = \begin{cases} 1 & E_i \leq 4.5, \\ E_i - 3.5 & 4.5 < E_i \leq 7.5, \\ c & E_i > 7.5, \end{cases}$$

and $E_i = IQR_i/M^*$. The constant c is a tuning constant. Each value of c leads to a different estimator. Tatum showed that $c = 7$ gives an estimator that loses some efficiency when no disturbances are present, but gains efficiency when disturbances are present. We apply this value of c in our simulation study. Note that we have $h(i) = E_i - 3.5$ for $4.5 < E_i \leq 7.5$ in the equations instead of $h(i) = E_i - 4.5$ as presented by Tatum (Tatum 1997, p. 129). This was a typographical error in the formula, resulting in too much weight on localized disturbances and thus an overestimation of σ. An unbiased estimator of σ is given by $S_c^*/d^*(c, n, k)$, where $d^*(c, n, k)$ is the normalizing constant. During the implementation of the estimator we discovered that, for odd values of n, the values of $d^*(c, n, k)$ given by Table 1 in Tatum (1997) should be corrected. We use the corrected values, which are presented in Table 2.1 below. The resulting estimator is denoted by $D7$ as in Tatum (1997).

	$c = 7$				$c = 10$			
n	$k = 20$	$k = 30$	$k = 40$	$k = 75$	$k = 20$	$k = 30$	$k = 40$	$k = 75$
5	1.070	1.069	1.068	1.068	1.054	1.053	1.053	1.052
7	1.057	1.056	1.056	1.056	1.041	1.040	1.040	1.040
9	1.052	1.051	1.050	1.050	1.034	1.034	1.033	1.033
11	1.047	1.046	1.046	1.046	1.029	1.029	1.028	1.028
13	1.044	1.044	1.043	1.043	1.026	1.025	1.025	1.025
15	1.041	1.041	1.041	1.040	1.023	1.023	1.023	1.022

Table 2.1: Normalizing constants $d^*(c, n, k)$ for Tatum's estimator (S_c^*)

The estimators considered are summarized in Table 2.2.

2.2.2 Efficiency of proposed estimators

In order to compare the relative efficiency of the proposed Phase I estimators, we assess their MSE as was done in Tatum (1997). The MSE is estimated as

$$MSE = \frac{1}{N}\sum_{i=1}^{N}(\hat{\sigma}^i - \sigma)^2,$$

where $\hat{\sigma}^i$ is the unbiased estimate of the standard deviation in the i-th simulation run (note that $\hat{\sigma}^i$ differs from $\hat{\sigma}_i$, the latter denoting the Phase II estimate) and N is the number of simulation runs. We include the uncontaminated case, i.e. the situation where all the X_{ij}'s are from the $N(0, 1)$ distribution as well as four types of disturbances (cf. Tatum (1997)):

1. A model for diffuse symmetric variance disturbances in which each observation has a 95% probability of being drawn from the $N(0, 1)$ distri-

Estimator	Notation
Pooled sample standard deviation	$\tilde{S}$
Mean of sample standard deviations	$\bar{S}$
25% trimmed mean of sample standard deviations	$\bar{S}_{25}$
Mean of sample standard deviations after trimming sample observations	$\overline{S_{20}}$
Mean of sample ranges	$\bar{R}$
Mean of sample interquartile ranges	$\overline{IQR}$
Mean of sample Gini's mean differences	$\bar{G}$
Mean of sample averages of absolute deviation from median	$\overline{ADM}$
$\overline{AMD}$ after subgroup screening	$\overline{ADM}'$
Mean of sample medians of absolute deviation from median	$\overline{MDM}$
Mean of sample medians of absolute deviation from mean	$\overline{MAD}$
Tatum's robust estimator	$D7$

Table 2.2: Proposed estimators for the standard deviation

bution and a 5% probability of being drawn from the $N(0, a)$ distribution, with $a = 1.5, 2.0, ..., 5.5, 6.0$.

2. A model for diffuse asymmetric variance disturbances in which each observation is drawn from the $N(0, 1)$ distribution and has a 5% probability of having a multiple of a χ_1^2 variable added to it, with the multiplier equal to $0.5, 1.0, ..., 4.5, 5.0$.

3. A model for localized variance disturbances in which observations in 3 (when $k = 30$) or 6 (when $k = 75$) samples are drawn from the $N(0, a)$ distribution, with $a = 1.5, 2.0, ..., 5.5, 6.0$.

4. A model for diffuse mean disturbances in which each observation has a 95% probability of being drawn from the $N(0, 1)$ distribution and a 5% probability of being drawn from the $N(b, 1)$ distribution, with $b = 0.5, 1.0, ..., 9.0, 9.5$.

The MSE is obtained for $k = 30, 75$ subgroups of sizes $n = 5, 9$. The number of simulation runs N is equal to 50,000. (Note that Tatum (1997) used 10,000 simulation runs.)

The following results can be observed (see Figures 2.1-2.4). The y-intercepts show the MSE of the estimators when there are no contaminations. In this case, $\bar{S}_{25}$, $\overline{MDM}$, $\overline{S_{20}}$, $\overline{IQR}$ and $\overline{MAD}$ are less efficient than any of the other estimators because they use less information, while the other estimators are more or less equally efficient.

When symmetric diffuse variance disturbances are present (Figure 2.1), the best performing estimators are $D7$ and $\overline{ADM}'$. The fact that the performance of $\overline{ADM}'$ is similar to $D7$ is interesting because the former is more intuitive and the estimates are simpler to obtain. Tatum (1997) showed that the screening procedure based on the chart with σ estimated by $\bar{R}$ fails to match $D7$ in this situation. This is because R is more sensitive to outliers. Thus, using a robust statistic like $\overline{ADM}$, supplemented with subgroup screening by means of the control chart (resulting in $\overline{ADM}'$), works very well when symmetric diffuse outliers are present. The estimators $\bar{S}_{25}$, $\overline{S_{20}}$ $\overline{IQR}$ and $\overline{MDM}$ are more robust than the traditional estimators but less robust than $D7$ and $\overline{ADM}'$. Another result worth noting is that $\tilde{S}$ performs worst in this situation (comparable to the poor performance of $\bar{S}$ and $\bar{R}$). While others (e.g. Mahmoud et al. (2010)) recommend using this estimator because it is most efficient in the absence of contaminations, we see that its performance decreases most quickly when there are outliers. The estimators $\bar{G}$ and $\overline{ADM}$ are efficient when no contaminations are present and perform better than the traditional estimators ($\tilde{S}$, $\bar{S}$ and $\bar{R}$) in the case of occasional outliers. The effect is more pronounced for $n = 9$ than for $n = 5$.

When asymmetric diffuse variance disturbances are present (Figure 2.2), the same general results are found as for symmetric diffuse variance disturbances. Tatum (1997) showed that, when $n = 9$, $D7$ is superior to several other estimators, including the estimator resulting from subgroup screening based on $\bar{R}$. Our subgroup screening algorithm produces outcomes similar to Tatum's estimator. Note that, to estimate σ, we use an estimator that is less sensitive to outliers, namely $\overline{ADM}$ rather than $\bar{R}$.

In the case of localized variance disturbances (Figure 2.3), the estimator that performs best is $\overline{ADM}'$, followed by $D7$ and then by $\bar{S}_{25}$. It is interesting to see that $\overline{ADM}'$ performs substantially better than $D7$. In other words, screening based on the control charting procedure in Phase I seems more effective than using $D7$ when the data are contaminated by localized variance disturbances.

When diffuse mean disturbances are present in Phase I (Figure 2.4), $D7$ performs best, followed by $\overline{ADM}'$. The differences appear primarily for $n = 9$. When there is a possibility of this type of outlier in practice, we recommend using $D7$ or screening on the basis of an individuals chart. The latter is a subject for future research.

To summarize, the most efficient estimators are $D7$ and $\overline{ADM}'$ when there are diffuse variance disturbances; $\overline{ADM}'$ when there are localized variance disturbances; and $D7$ when there are mean shift disturbances.

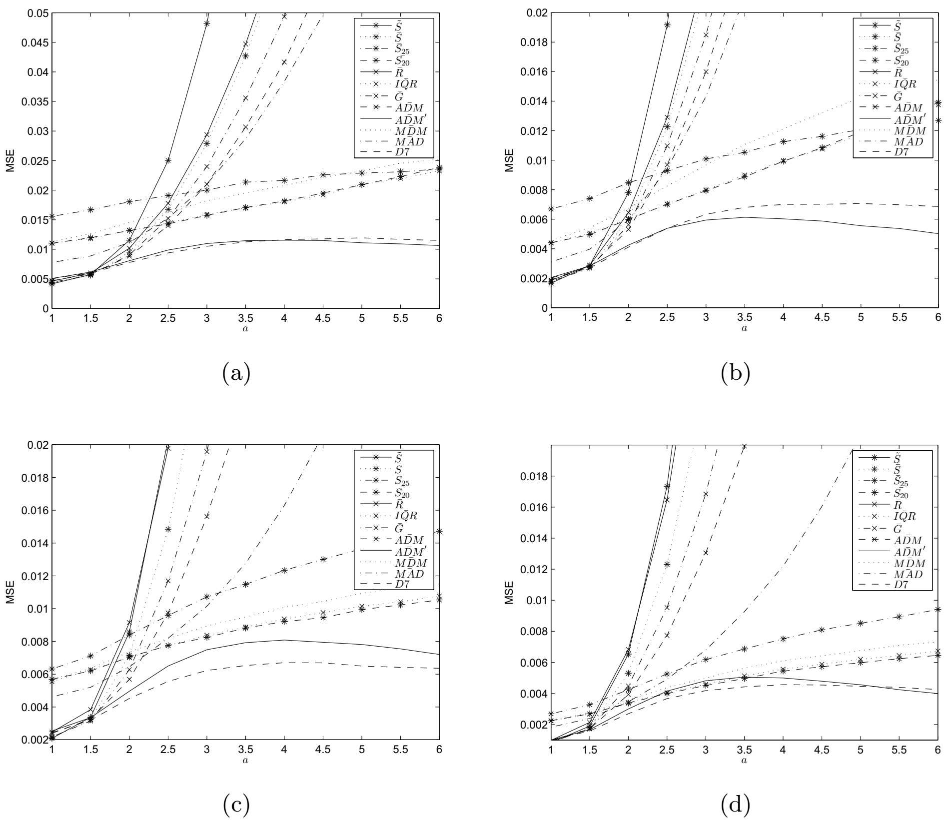


Figure 2.1: MSE of estimators when symmetric diffuse variance disturbances are present. (a) $n = 5, k = 30$ (b) $n = 5, k = 75$ (c) $n = 9, k = 30$ (d) $n = 9, k = 75$

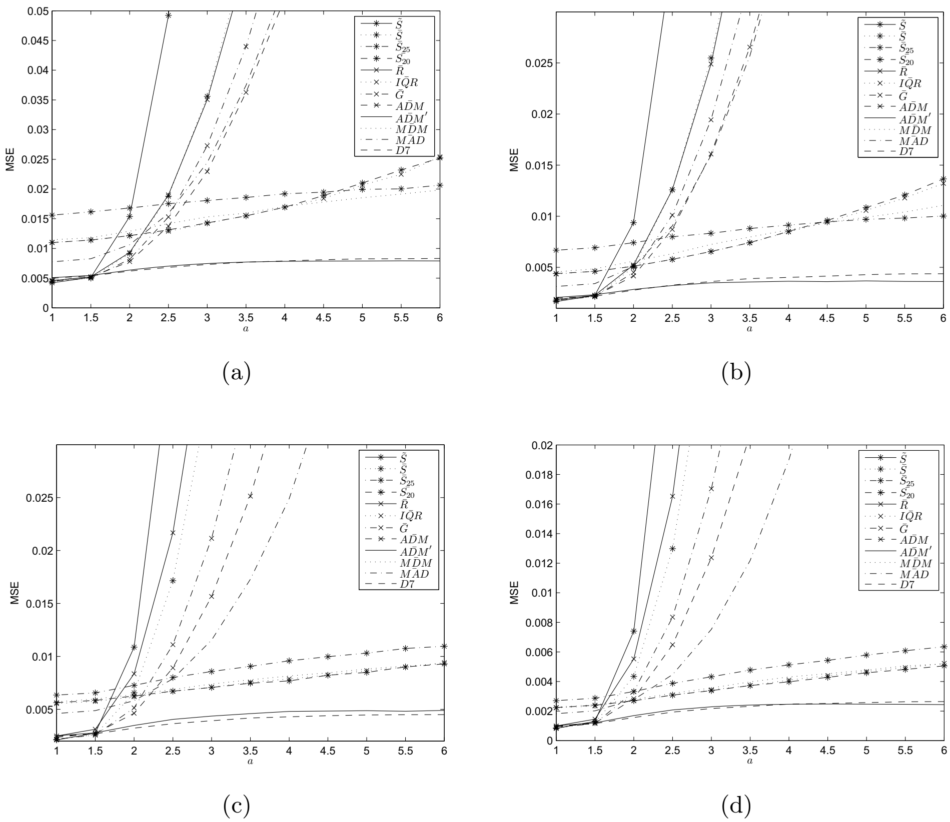
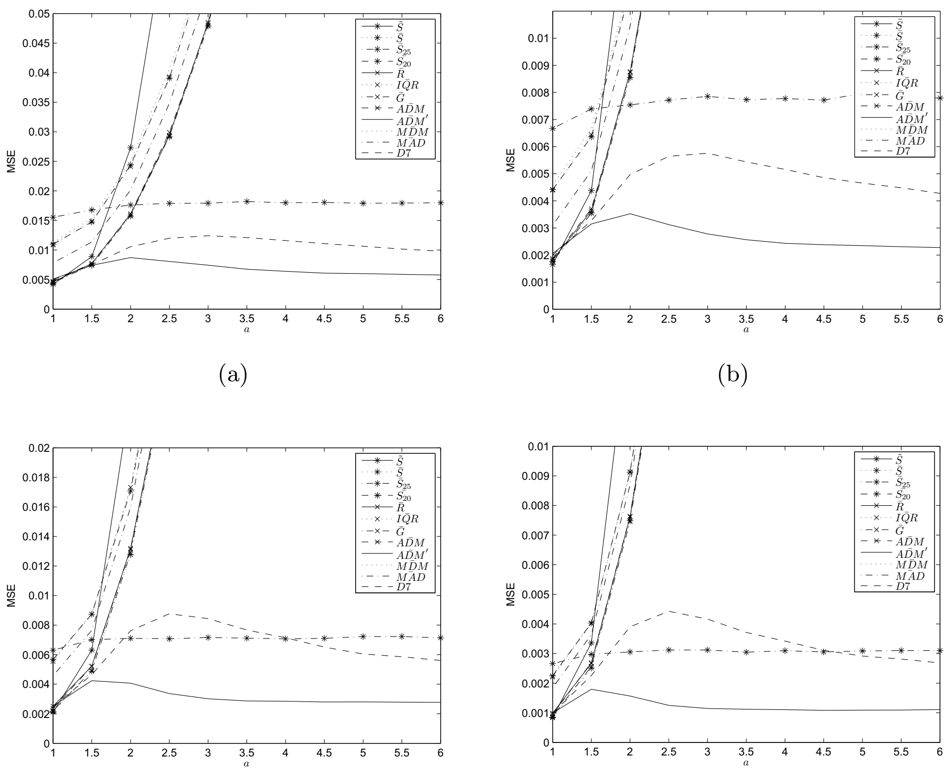


Figure 2.2: MSE of estimators when asymmetric diffuse variance disturbances are present. (a) $n = 5, k = 30$ (b) $n = 5, k = 75$ (c) $n = 9, k = 30$ (d) $n = 9, k = 75$



(d)

Figure 2.3: MSE of estimators when localized variance disturbances are present. (a) $n = 5, k = 30$ (b) $n = 5, k = 75$ (c) $n = 9, k = 30$ (d) $n = 9, k = 75$

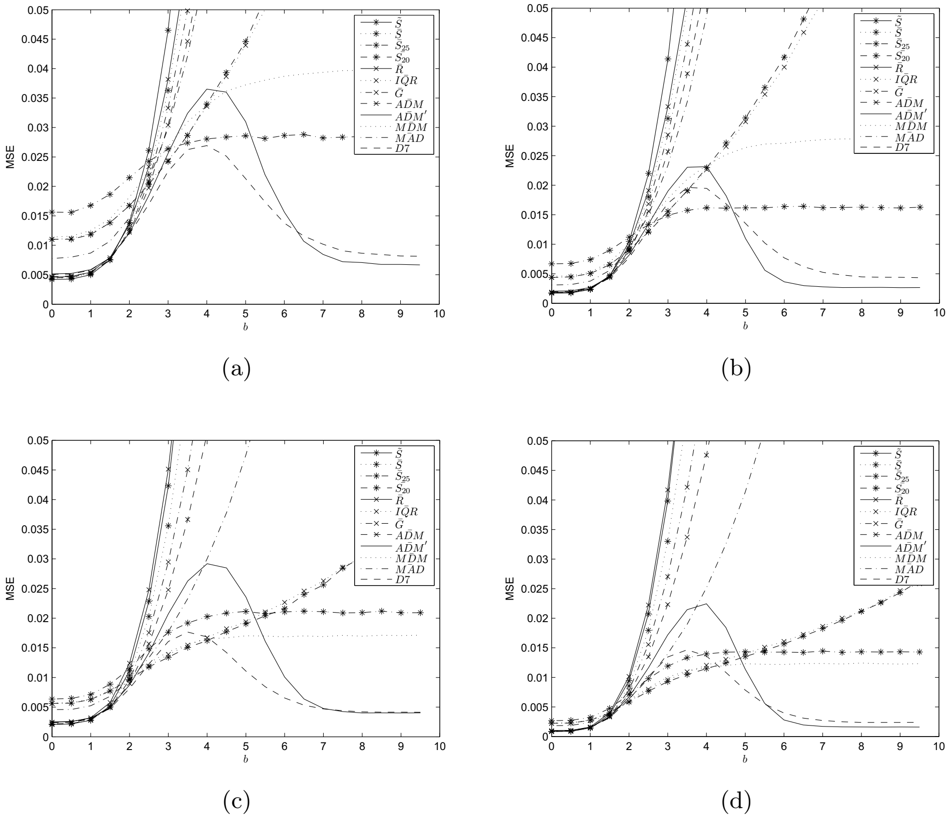


Figure 2.4: MSE of estimators when diffuse mean disturbances are present. (a) $n = 5, k = 30$ (b) $n = 5, k = 75$ (c) $n = 9, k = 30$ (d) $n = 9, k = 75$

2.3 Derivation of Phase II control limits

To control the unconditional in-control p, the design of the Phase II control charts requires a derivation of the factors U_n and L_n in (1.3). For the R chart, Hillier (1969) showed that when the limits are estimated, U_n and L_n derived for the σ-known case will not produce the desired signaling probability. To address this issue, he calculated the factors based on n, k and α in such a way that p equals α. Yang and Hillier (1970) derived correction factors for the S and $\tilde{S}$ charts. The solution suggested by Hillier (1969) is well known as a solution for short production runs. On the other hand, the ARL gives an indication of the expected run length and so is intuitively very appealing. The disadvantage of the ARL is, however, that it is determined by the occurrence of extremely long runs while in practice processes do not remain unchanged for a very long period (see also Does and Schriever (1992)). Nedumaran and Pignatiello (2001) developed an approach for constructing $\overline{X}$ control limits that attempt to match any percentile point of the run length distribution.

In this study we derive U_n and L_n so as to obtain the desired value for p. Later we will show that this issue is less important for the standard deviation control chart than for the $\overline{X}$ and X charts, because the estimation effect is less pronounced for the standard deviation control chart.

U_n and L_n depend on n, k and α. The Phase I estimators considered are the estimators presented in Table 2.2. We employ the same statistic, namely $S/c_4(n)$, as the Phase II charting statistic in each case so that any differences between the charts are entirely due to differences introduced by the Phase I estimators. We present the derivation of the factors for these charts below.

We start with the factors for the chart where $\hat{\sigma}$ is estimated by $\tilde{S}/c_4(k(n-1)+1)$. Exact results for this chart can be calculated and can also be found in Yang and Hillier (1970). We derive the factor for the upper control limit; the factor for the lower control limit can be obtained in a similar way. Note that S_i and $\tilde{S}$ are independent so the factors can be chosen as the upper and lower $\alpha/2$ quantiles of the distribution $S_i/\tilde{S}$. We can write $(S_i/\tilde{S})^2$ as $\frac{(n-1)S_i^2/\sigma^2}{k(n-1)\tilde{S}^2/\sigma^2}\cdot\frac{1/(n-1)}{1/k(n-1)}$, which is distributed as $\frac{\chi^2_{n-1}/(n-1)}{\chi^2_{k(n-1)}/(k(n-1))} = F_{n-1,k(n-1)}$, where χ^2_m denotes a chi-square distribution with m degrees of freedom and $F_{v,w}$ denotes an F distribution with v numerator degrees of freedom and w

denominator degrees of freedom. Hence

$$U_n = \frac{\sqrt{F_{n-1,k(n-1)}(1-\alpha/2)}c_4(k(n-1)+1)}{c_4(n)}. \tag{2.12}$$

For the charts based on the other Phase I estimators we use the result of Patnaik (1950). Patnaik approximated the distribution of $\bar{R}/\sigma$ by $a(n,k)\chi_{\nu(n,k)}/\sqrt{\nu_{(n,k)}}$, where $\chi_{\nu(n,k)}$ is the square root of a chi-square distribution with $\nu(n,k)$ degrees of freedom and $a(n,k)$ is a scale factor. The factors $a(n,k)$ and $\nu(n,k)$ are obtained by equating the first two moments of $\bar{R}/\sigma$ to the first two moments of $a(n,k)\chi_{\nu(n,k)}/\sqrt{\nu_{(n,k)}}$. Patnaik's approach can also be applied to approximate the distribution of $\hat{\sigma}/\sigma$, where $\hat{\sigma}$ is obtained via one of the unbiased estimators of the standard deviation in Phase I. Let $M_1 = E(\hat{\sigma}/\sigma) = 1$ and $M_2 = Var(\hat{\sigma}/\sigma)$. From Patnaik (1950) it follows that the values of $\nu(n,k)$ and $a(n,k)$ are

$$\nu(n,k) = 1/(-2+2\sqrt{1+2M_2+1/(16\nu(n,k))^3}), \tag{2.13}$$

$$a(n,k) = 1 + \frac{1}{4\nu(n,k)} + \frac{1}{32\nu^2(n,k)} - \frac{5}{128\nu^3(n,k)}. \tag{2.14}$$

Since $\frac{(S_i/\sigma)^2}{(c_4(n)\hat{\sigma}/\sigma)^2}$ is distributed as

$\frac{\chi^2_{n-1}/(n-1)}{c_4^2(n)a^2(n,k)\chi^2_{\nu(n,k)}/\nu(n,k)} = F_{n-1,\nu(n,k)}/(c_4^2(n)a^2(n,k))$, it follows that

$$U_n = \sqrt{F_{(n-1),\nu(n,k)}(1-\alpha/2)}/(c_4(n)a(n,k)). \tag{2.15}$$

Table 2.3 summarizes U_n and L_n for the control charts with $k = 20, 30, 75$ subgroups of sizes $n = 5, 9$ and $\alpha = 0.0027$. For other situations, values of M_2 can be derived by simulating $\hat{\sigma}/\sigma$. Then the constants $\nu(n,k)$ and $a(n,k)$ can be readily obtained from (2.13) and (2.14).

To judge the quality of the proposed corrections, we evaluate the unconditional probabilities of a false signal (p) in Phase II. The probabilities, presented in Table 2.4, are assessed using 50,000 simulation runs. This is enough to obtain a sufficiently small relative standard error.

2.4 Control chart performance

In this section we evaluate the performance of the design schemes presented above. The schemes are set up in the uncontaminated normal situation and several contaminated situations. We consider models similar to those used to assess the MSE with a, b and the multiplier equal to 4 to simulate the contaminated cases (see Section 2.2.2).

The performance of the design schemes is assessed in terms of the unconditional p and ARL as well as the conditional ARL associated with the 2.5% and 97.5% quantiles of the distribution of $\hat{\sigma}$. We consider different shifts in the standard deviation $\lambda\sigma$ in Phase II, namely λ equal to 0.5, 1, 1.5 and 2. The performance characteristics are obtained by simulation. The next section describes the simulation method, followed by the results for the control charts constructed in the uncontaminated situation and various contaminated situations.

2.4.1 Simulation procedure

The performance characteristics p and ARL for estimated control limits are determined by averaging the conditional characteristics, i.e. the characteristics for a given set of estimated control limits, over all possible values of the control limits. Recall the definitions of $p(F_i|\hat{\sigma})$ from (1.5), $E(RL|\hat{\sigma})$ from (1.6), $p = E(p(F_i|\hat{\sigma}))$ from (1.7) and $ARL = E(\frac{1}{p(F_i|\hat{\sigma})})$ from (1.8). These expectations are obtained by simulation: 50,000 datasets are generated and for each dataset $p(F_i|\hat{\sigma})$ and $E(RL|\hat{\sigma})$ are computed. By averaging these values we obtain the unconditional values.

2.4.2 Simulation results

First we consider the situation where the process follows a normal distribution and the Phase I data are not contaminated. We investigate the impact of the estimator used to estimate σ in Phase I. Tables 2.5 and 2.6 present the unconditional probability of one sample generating a signal (p), the unconditional average run length (ARL) and the upper and lower conditional ARL values corresponding to the upper and lower 0.025 quantiles of the distribution of $\hat{\sigma}$. When $\lambda = 1$, the process is in control, so we want p to be as low as possible and ARL to be as high as possible. When $\lambda \neq 1$, i.e. in the out-of-control situation, we want to achieve the opposite. The tables show that, when the limits are estimated, the in-control ARL is higher than the desired 370 (the control limits are chosen to provide an unconditional p of 0.0027), the value which is achieved when the limits are known. Note that the increase in the unconditional ARL due to the estimation process is not as large as for the $\overline{X}$ control chart. The reason is that for the $\overline{X}$ control chart the run length distribution is very right-skewed, generating a very large unconditional ARL. This seems to be less the case for standard deviation control charts.

We also study the conditional ARL values (or, equivalently, the conditional p values, since the conditional RL distribution is simply geometric with parameter equal to the conditional p). The first value in parentheses represents the ARL for the 2.5% quantile of the distribution of $\hat{\sigma}$, while the second value represents the ARL for the 97.5% quantile of the distribution of $\hat{\sigma}$. The results show that the conditional ARL values vary quite strongly, even when k equals 75. When λ equals 0.5, we see that a lower value of $\hat{\sigma}$ gives a higher ARL and vice versa. The reason is that a smaller value of $\hat{\sigma}$ in Phase I results in a lower value for the lower control limit and hence a lower probability of detecting a decrease in the standard deviation in Phase II. In the normal uncontaminated situation, we observe a nice pattern for all the estimators: the upper and lower conditional ARL values in the in-control situation are higher than in the out-of-control situation. However, this is not always the case when there are Phase I contaminations (Tables 2.7-2.14). Confining ourselves to the conditional ARL values in the contaminated case, we judge the upper and lower conditional ARL values to be satisfactory, provided that they do not change too much from the values observed in the uncontaminated normal case.

When we compare the estimators in the uncontaminated Phase I situation (Tables 2.5 and 2.6), $\tilde{S}$, $\overline{S}$, $\overline{R}$, $\overline{G}$, $\overline{ADM}$, $\overline{ADM}'$ and $D7$ produce very similar outcomes. The estimators $\overline{S}_{25}$, $\overline{S_{20}}$, $\overline{IQR}$, $\overline{MDM}$ and $\overline{MAD}$ are less powerful under normality.

The performance of the charts in the case of contaminated data is tabulated in Tables 2.7-2.14. The same general results are found as for the MSE comparisons. The most important points are as follows.

The chart based on $\tilde{S}$ is most powerful under normality; however its performance decreases most quickly when diffuse or localized disturbances occur. In light of this risk, we do not recommend using $\tilde{S}$.

The charts based on the estimators $\overline{S_{20}}$, $\overline{IQR}$ and $\overline{MDM}$ perform relatively well in response to diffuse disturbances but not very well when there are no contaminations.

Furthermore, the charts based on the estimators $\overline{G}$ and $\overline{ADM}$ are efficient under normality and are more efficient than the traditional charts based on $\tilde{S}$, $\overline{S}$ and $\overline{R}$ when diffuse outliers are present.

Finally, the charts based on the estimators $\overline{ADM}'$ and $D7$ perform equally well as the traditional charts in the uncontaminated case and substantially better than any of the other charts in contaminated situations. When mean diffuse disturbances are likely to occur in Phase I, we recommend using $D7$ because the control chart based on this estimator is more robust against such disturbances. When localized disturbances are likely to occur, we recommend using $\overline{ADM}'$. Advantages of the latter estimator are the ease of obtaining estimates and its intuitiveness: extreme samples, and hence the root cause of any disturbances, can be readily identified.

			Factors for control limits					
			$k = 20$		$k = 30$		$k = 75$	
n	$\hat{\sigma}$		U_n	L_n	U_n	L_n	U_n	L_n
5	$\tilde{S}$	Eq. (2.12)	2.352	0.171	2.315	0.172	2.272	0.173
	$\bar{S}$	Eq. (2.15)	2.357	0.171	2.318	0.172	2.274	0.173
	$\bar{S}_{25}$	Eq. (2.15)	2.704	0.167	2.527	0.169	2.359	0.171
	$\overline{S_{20}}$	Eq. (2.15)	2.540	0.169	2.438	0.170	2.319	0.172
	$\bar{R}$	Eq. (2.15)	2.364	0.171	2.322	0.172	2.275	0.173
	$\overline{IQR}$	Eq. (2.15)	2.541	0.169	2.439	0.170	2.318	0.172
	$\bar{G}$	Eq. (2.15)	2.359	0.171	2.320	0.172	2.275	0.173
	$\overline{ADM}$	Eq. (2.15)	2.366	0.171	2.324	0.172	2.275	0.173
	$\overline{ADM}'$	Eq. (2.15)	2.376	0.171	2.332	0.171	2.279	0.172
	$\overline{MDM}$	Eq. (2.15)	2.554	0.169	2.442	0.170	2.322	0.172
	$\overline{MAD}$	Eq. (2.15)	2.447	0.170	2.380	0.171	2.296	0.172
	$D7$	Eq. (2.15)	2.376	0.171	2.331	0.172	2.278	0.172
9	$\tilde{S}$	Eq. (2.12)	1.890	0.349	1.872	0.350	1.851	0.351
	$\bar{S}$	Eq. (2.15)	1.892	0.349	1.873	0.350	1.852	0.351
	$\bar{S}_{25}$	Eq. (2.15)	2.011	0.342	1.946	0.345	1.883	0.349
	$\overline{S_{20}}$	Eq. (2.15)	1.987	0.343	1.934	0.346	1.876	0.349
	$\bar{R}$	Eq. (2.15)	1.900	0.348	1.879	0.349	1.854	0.351
	$\overline{IQR}$	Eq. (2.15)	1.982	0.343	1.933	0.346	1.875	0.350
	$\bar{G}$	Eq. (2.15)	1.894	0.348	1.874	0.350	1.852	0.351
	$\overline{ADM}$	Eq. (2.15)	1.898	0.348	1.877	0.349	1.854	0.351
	$\overline{ADM}'$	Eq. (2.15)	1.901	0.348	1.879	0.349	1.854	0.351
	$\overline{MDM}$	Eq. (2.15)	1.987	0.343	1.936	0.346	1.876	0.349
	$\overline{MAD}$	Eq. (2.15)	1.956	0.345	1.915	0.347	1.868	0.350
	$D7$	Eq. (2.15)	1.901	0.348	1.879	0.349	1.854	0.351

Table 2.3: Factors U_n and L_n to determine Phase II control limits

			$p \times 10^2$					
			$k = 20$		$k = 30$		$k = 75$	
n	$\hat{\sigma}$		U_n	L_n	U_n	L_n	U_n	L_n
5	$\tilde{S}$	Eq. (2.12)	0.135	0.135	0.135	0.135	0.135	0.135
	$\bar{S}$	Eq. (2.15)	0.135	0.135	0.135	0.135	0.135	0.135
	$\bar{S}_{25}$	Eq. (2.15)	0.131	0.134	0.135	0.135	0.135	0.134
	$\overline{S_{20}}$	Eq. (2.15)	0.135	0.135	0.135	0.134	0.136	0.135
	$\bar{R}$	Eq. (2.15)	0.135	0.134	0.135	0.136	0.135	0.137
	$\overline{IQR}$	Eq. (2.15)	0.130	0.134	0.131	0.134	0.135	0.135
	$\bar{G}$	Eq. (2.15)	0.133	0.134	0.136	0.135	0.134	0.136
	$\overline{ADM}$	Eq. (2.15)	0.132	0.134	0.135	0.135	0.134	0.137
	$\overline{ADM}'$	Eq. (2.15)	0.136	0.135	0.137	0.133	0.134	0.134
	$\overline{MDM}$	Eq. (2.15)	0.130	0.136	0.133	0.134	0.134	0.136
	$\overline{MAD}$	Eq. (2.15)	0.131	0.135	0.130	0.135	0.134	0.135
	$D7$	Eq. (2.15)	0.135	0.135	0.134	0.136	0.136	0.133
9	$\tilde{S}$	Eq. (2.12)	0.134	0.135	0.134	0.135	0.134	0.135
	$\bar{S}$	Eq. (2.15)	0.136	0.134	0.136	0.136	0.134	0.135
	$\bar{S}_{25}$	Eq. (2.15)	0.141	0.135	0.137	0.134	0.134	0.135
	$\overline{S_{20}}$	Eq. (2.15)	0.131	0.134	0.135	0.135	0.135	0.133
	$\bar{R}$	Eq. (2.15)	0.135	0.135	0.134	0.134	0.134	0.136
	$\overline{IQR}$	Eq. (2.15)	0.134	0.134	0.133	0.135	0.133	0.137
	$\bar{G}$	Eq. (2.15)	0.133	0.134	0.134	0.136	0.134	0.136
	$\overline{ADM}$	Eq. (2.15)	0.134	0.135	0.134	0.134	0.133	0.136
	$\overline{ADM}'$	Eq. (2.15)	0.134	0.135	0.138	0.134	0.135	0.135
	$\overline{MDM}$	Eq. (2.15)	0.134	0.134	0.138	0.134	0.136	0.133
	$\overline{MAD}$	Eq. (2.15)	0.134	0.136	0.133	0.135	0.133	0.136
	$D7$	Eq. (2.15)	0.134	0.135	0.136	0.134	0.133	0.136

Table 2.4: In-control $p \times 10^2$ of control limits. The estimated relative standard error is never worse than 1%

		p				ARL			
k	$\hat{\sigma}$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$
30	$\tilde{S}$	0.019	0.0027	0.084	0.32	54.7	418	14.5	3.28
						(86.7; 33.7)	(151; 455)	(5.94; 33.0)	(2.18; 5.10)
	$\bar{S}$	0.019	0.0027	0.083	0.32	54.7	419	14.8	3.30
						(87.8; 33.4)	(150; 451)	(5.90; 34.2)	(2.18; 5.18)
	$\bar{S}_{25}$	0.019	0.0027	0.058	0.25	65.1	535	47.8	5.45
						(155; 24.7)	(91.9; 334)	(4.78; 248)	(1.95; 15.2)
	$\overline{S_{20}}$	0.019	0.0027	0.068	0.27	60.9	490	28.3	4.29
						(126; 27.1)	(104; 369)	(5.03; 113)	(2.01; 9.63)
	$\bar{R}$	0.019	0.0027	0.082	0.32	54.9	421	15.1	3.33
						(88.8; 32.9)	(147; 447)	(5.85; 35.9)	(2.17; 5.31)
	$\overline{IQR}$	0.019	0.0026	0.067	0.27	61.0	490	28.5	4.32
						(125; 27.0)	(106; 367)	(5.06; 114)	(2.01; 9.66)
	$\bar{G}$	0.019	0.0027	0.083	0.32	54.8	421	14.9	3.52
						(88.3; 33.2)	(148; 450)	(5.87; 35.1)	(2.17; 5.53)
	$\overline{ADM}$	0.020	0.0027	0.082	0.32	54.8	423	15.2	3.34
						(89.1; 32.8)	(148; 444)	(5.87; 36.6)	(2.16; 5.34)
	$\overline{ADM'}$	0.019	0.0027	0.081	0.31	56.5	434	15.7	3.39
						(95.2; 33.2)	(138; 451)	(5.69; 39.3)	(2.13; 5.50)
	$\overline{MDM}$	0.019	0.0027	0.067	0.27	60.9	490	29.2	4.37
						(129; 26.6)	(101; 365)	(5.02; 123)	(2.01; 10.0)
	$\overline{MAD}$	0.019	0.0026	0.074	0.29	57.7	457	20.4	3.77
						(107; 29.4)	(124; 404)	(5.40; 65.6)	(2.08; 7.20)
	$D7$	0.020	0.0027	0.081	0.31	55.1	427	15.7	3.38
						(92.0; 32.4)	(140; 442)	(5.72; 38.7)	(2.14; 5.49)
75	$\tilde{S}$	0.019	0.0027	0.090	0.34	52.7	391	11.9	3.02
						(70.8; 38.8)	(208; 479)	(6.98; 19.8)	(2.35; 3.92)
	$\bar{S}$	0.020	0.0027	0.090	0.34	52.7	392	12.0	3.03
						(71.2; 38.6)	(205; 478)	(6.94; 20.1)	(2.35; 3.94)
	$\bar{S}_{25}$	0.019	0.0027	0.077	0.30	57.1	446	18.2	3.60
						(101; 30.8)	(127; 423)	(5.24; 52.5)	(2.10; 6.44)
	$\overline{S_{20}}$	0.019	0.0027	0.083	0.32	54.8	420	14.8	3.30
						(88.2; 33.2)	(150; 451)	(5.90; 34.7)	(2.17; 5.20)
	$\bar{R}$	0.020	0.0027	0.090	0.34	52.3	391	12.1	3.05
						(71.1; 38.0)	(203; 475)	(6.88; 20.6)	(2.34; 4.00)
	$\overline{IQR}$	0.020	0.0027	0.083	0.32	54.7	419	14.8	3.30
						(88.0; 33.1)	(150; 450)	(5.90; 34.6)	(2.17; 5.20)
	$\bar{G}$	0.020	0.0027	0.090	0.34	52.2	391	12.0	3.04
						(70.8; 38.1)	(206; 476)	(6.93; 20.4)	(2.35; 3.98)
	$\overline{ADM}$	0.020	0.0027	0.090	0.34	52.2	391	12.1	3.04
						(71.4; 37.8)	(201; 474)	(6.87; 20.7)	(2.34; 4.00)
	$\overline{ADM'}$	0.019	0.0027	0.089	0.33	53.4	399	12.3	3.07
						(74.4; 38.2)	(194; 484)	(6.74; 21.6)	(2.32; 4.09)
	$\overline{MDM}$	0.020	0.0027	0.082	0.32	54.7	422	15.0	3.33
						(88.0; 32.8)	(150; 449)	(5.89; 36.1)	(2.17; 5.29)
	$\overline{MAD}$	0.019	0.0027	0.086	0.33	53.9	409	13.3	3.17
						(80.2; 35.4)	(174; 471)	(6.29; 27.2)	(2.25; 4.59)
	$D7$	0.019	0.0027	0.090	0.34	53.6	396	12.1	3.04
						(74.1; 38.5)	(197; 485)	(6.75; 21.1)	(2.32; 4.06)

Table 2.5: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values under normality for $n = 5$

		p				*ARL*			
k	$\hat{\sigma}$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$
30	$\tilde{S}$	0.12	0.0027	0.17	0.58	9.05	402	6.48	1.74
						(14.4; 5.64)	(173; 394)	(3.52; 11.9)	(1.42; 2.22)
	$\bar{S}$	0.12	0.0027	0.17	0.58	9.02	400	6.52	1.75
						(14.4; 5.57)	(172; 390)	(3.50; 12.1)	(1.42; 2.23)
	$\bar{S}_{25}$	0.11	0.0027	0.14	0.52	10.6	479	10.4	2.02
						(24.2; 4.52)	(108; 296)	(2.97; 32.3)	(1.34; 3.35)
	$\overline{S_{20}}$	0.11	0.0027	0.15	0.53	10.3	462	9.60	1.96
						(22.3; 4.59)	(119; 306)	(3.05; 28.4)	(1.35; 3.15)
	$\bar{R}$	0.12	0.0027	0.17	0.58	9.23	410	6.74	1.77
						(15.1; 5.48)	(163; 383)	(3.44; 13.3)	(1.42; 2.31)
	$\overline{IQR}$	0.11	0.0027	0.15	0.54	10.3	463	9.52	1.96
						(22.0; 4.59)	(118; 306)	(3.06; 28.0)	(1.35; 3.13)
	$\bar{G}$	0.12	0.0027	0.17	0.58	9.02	401	6.55	1.75
						(14.5; 5.56)	(171; 387)	(3.50; 12.2)	(1.41; 2.25)
	$\overline{ADM}$	0.12	0.0027	0.17	0.58	9.17	409	6.69	1.76
						(15.1; 5.51)	(168; 387)	(3.46; 13.0)	(1.41; 2.29)
	$\overline{ADM'}$	0.12	0.0027	0.17	0.58	9.22	409	6.75	1.77
						(15.4; 5.49)	(160; 385)	(3.40; 13.1)	(1.40; 2.32)
	$\overline{MDM}$	0.11	0.0027	0.14	0.53	10.4	465	9.68	1.97
						(22.5; 4.59)	(115; 300)	(3.03; 29.0)	(1.35; 3.18)
	$\overline{MAD}$	0.12	0.0027	0.15	0.55	9.90	444	8.48	1.89
						(19.7; 4.82)	(130; 325)	(3.13; 22.0)	(1.37; 2.85)
	$D7$	0.12	0.0027	0.17	0.58	9.22	410	6.73	1.76
						(15.3; 5.50)	(165; 385)	(3.43; 13.1)	(1.40; 2.32)
75	$\tilde{S}$	0.12	0.0027	0.18	0.60	8.67	383	5.77	1.68
						(11.7; 6.43)	(235; 429)	(3.97; 8.39)	(1.48; 1.94)
	$\bar{S}$	0.12	0.0027	0.18	0.60	8.67	383	5.78	1.68
						(11.7; 6.39)	(229; 429)	(3.96; 8.45)	(1.48; 1.94)
	$\bar{S}_{25}$	0.12	0.0027	0.17	0.57	9.27	412	6.90	1.78
						(15.8; 5.42)	(156; 380)	(3.38; 13.9)	(1.40; 2.36)
	$\overline{S_{20}}$	0.12	0.0027	0.17	0.58	9.22	408	6.58	1.75
						(15.0; 5.58)	(169; 394)	(3.46; 12.4)	(1.41; 2.26)
	$\bar{R}$	0.12	0.0027	0.18	0.60	8.70	384	5.85	1.69
						(12.0; 6.26)	(225; 425)	(3.90; 8.86)	(1.47; 1.98)
	$\overline{IQR}$	0.12	0.0027	0.17	0.58	9.02	402	6.58	1.75
						(14.6; 5.52)	(196; 386)	(3.48; 12.4)	(1.41; 2.26)
	$\bar{G}$	0.12	0.0027	0.18	0.60	8.68	384	5.80	1.68
						(11.7; 6.38)	(231; 428)	(3.95; 8.53)	(1.48; 1.95)
	$\overline{ADM}$	0.12	0.0027	0.18	0.59	8.67	386	5.86	1.69
						(11.9; 6.28)	(225; 427)	(3.95; 8.78)	(1.47; 1.97)
	$\overline{ADM'}$	0.12	0.0027	0.18	0.60	8.70	384	5.86	1.69
						(12.0; 6.28)	(219; 425)	(3.87; 8.84)	(1.46; 1.98)
	$\overline{MDM}$	0.12	0.0027	0.17	0.58	9.23	407	6.58	1.75
						(95.2; 33.2)	(138; 451)	(5.69; 39.3)	(2.13; 5.50)
	$\overline{MAD}$	0.12	0.0027	0.17	0.58	8.96	398	6.35	1.73
						(13.9; 5.73)	(185; 401)	(3.60; 11.2)	(1.43; 2.18)
	$D7$	0.12	0.0027	0.18	0.60	8.69	385	5.84	1.70
						(12.0; 6.28)	(223; 426)	(3.89; 8.81)	(1.46; 1.98)

Table 2.6: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values under normality for $n = 9$

		p				*ARL*			
k	$\hat{\sigma}$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$
30	$\tilde{S}$	0.055	0.0043	0.016	0.11	23.0	293	195	22.9
						(52.0; 7.68)	(475; 92.0)	(13.2; 427)	(3.22; 131)
	$\bar{S}$	0.041	0.0031	0.024	0.15	28.7	359	104	9.45
						(55.8; 12.5)	(441; 158)	(11.5; 468)	(3.02; 30.8)
	$\bar{S}_{25}$	0.024	0.0024	0.040	0.20	52.7	526	84.4	7.60
						(128; 19.4)	(164; 262)	(6.00; 478)	(2.19; 24.7)
	$\overline{S_{20}}$	0.024	0.0023	0.045	0.22	48.5	493	54.9	5.97
						(103; 20.4)	(187; 275)	(6.50; 272)	(2.29; 16.6)
	$\bar{R}$	0.041	0.0032	0.023	0.15	28.4	356	110	9.82
						(55.4; 12.2)	(445; 157)	(11.8; 483)	(3.05; 32.7)
	$\overline{IQR}$	0.024	0.0023	0.044	0.22	48.3	493	55.0	5.99
						(103; 20.4)	(191; 276)	(6.56; 274)	(2.29; 16.2)
	$\bar{G}$	0.038	0.0029	0.026	0.16	30.2	379	86.0	8.10
						(56.6; 14.0)	(429; 183)	(11.3; 388)	(2.99; 23.3)
	$\overline{ADM}$	0.035	0.0027	0.029	0.17	31.8	393	73.2	7.28
						(58.9; 15.3)	(411; 200)	(10.8; 320)	(2.92; 19.3)
	$\overline{ADM'}$	0.024	0.0025	0.060	0.26	47.2	450	27.0	4.31
						(86.6; 24.0)	(178; 330)	(6.41; 94.1)	(2.25; 8.80)
	$\overline{MDM}$	0.026	0.0023	0.041	0.21	46.6	489	63.5	6.43
						(99.6; 19.3)	(213; 259)	(6.89; 323)	(2.35; 18.4)
	$\overline{MAD}$	0.033	0.0026	0.029	0.17	34.9	423	86.3	8.02
						(69.6; 15.6)	(382; 206)	(9.68; 412)	(2.78; 23.8)
	$D7$	0.025	0.0024	0.055	0.25	44.1	452	27.8	4.41
						(76.6; 23.9)	(230; 326)	(7.30; 90.8)	(2.40; 8.50)
75	$\tilde{S}$	0.054	0.0042	0.012	0.11	20.4	270	163	13.5
						(36.1; 10.3)	(467; 128)	(23.0; 478)	(4.23; 42.5)
	$\bar{S}$	0.040	0.0030	0.022	0.16	26.7	353	68.3	7.86
						(41.7; 16.0)	(482; 210)	(17.3; 217)	(3.67; 14.8)
	$\bar{S}_{25}$	0.024	0.0023	0.053	0.25	46.4	473	29.4	4.55
						(83.1; 24.7)	(219; 339)	(7.04; 97.3)	(2.38; 9.01)
	$\overline{S_{20}}$	0.025	0.0023	0.054	0.25	43.3	459	25.1	4.25
						(27.4; 25.6)	(276; 351)	(8.08; 68.7)	(2.52; 7.49)
	$\bar{R}$	0.041	0.0030	0.022	0.16	26.2	347	70.5	7.30
						(41.2; 15.7)	(478; 205)	(17.5; 226)	(3.69; 15.2)
	$\overline{IQR}$	0.025	0.0023	0.054	0.25	43.2	459	25.0	4.25
						(70.1; 25.7)	(276; 351)	(8.07; 96.0)	(2.52; 7.48)
	$\bar{G}$	0.037	0.0028	0.026	0.17	28.1	371	55.2	6.41
						(42.7; 17.6)	(477; 233)	(16.2; 161)	(3.53; 12.2)
	$\overline{ADM}$	0.035	0.0027	0.029	0.18	29.7	387	46.9	5.89
						(44.5; 18.9)	(470; 253)	(15.1; 128)	(3.41; 10.6)
	$\overline{ADM'}$	0.023	0.0023	0.065	0.28	44.7	445	18.2	3.69
						(65.9; 29.5)	(269; 403)	(8.00; 39.3)	(2.52; 5.56)
	$\overline{MDM}$	0.026	0.0023	0.049	0.24	41.5	458	28.4	4.51
						(67.8; 24.3)	(299; 332)	(8.57; 80.6)	(2.59; 8.12)
	$\overline{MAD}$	0.033	0.0025	0.031	0.19	32.2	413	45.8	5.78
						(50.7; 20.0)	(462; 264)	(13.0; 135)	(3.18; 10.9)
	$D7$	0.024	0.0022	0.059	0.27	42.7	454	19.5	3.82
						(61.1; 29.2)	(321; 399)	(9.05; 40.3)	(2.66; 5.61)

Table 2.7: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values when symmetric variance disturbances are present in Phase I for $n = 5$

k	$\hat{\sigma}$	p $\lambda=0.5$	p $\lambda=1$	p $\lambda=1.5$	p $\lambda=2$	ARL $\lambda=0.5$	ARL $\lambda=1$	ARL $\lambda=1.5$	ARL $\lambda=2$
30	$\tilde{S}$	0.40	0.011	0.021	0.21	2.91	147	156	8.78
						(6.13; 1.41)	(432; 39.2)	(10.2; 417)	(2.10; 36.4)
	$\bar{S}$	0.32	0.0068	0.033	0.27	3.55	200	77.4	4.61
						(6.77; 1.81)	(463; 55.6)	(8.73; 384)	(1.98; 12.3)
	$\bar{S}_{25}$	0.16	0.0026	0.089	0.43	7.70	449	20.9	2.58
						(17.6; 3.31)	(242; 197)	(3.93; 83.7)	(1.47; 5.03)
	$\overline{S_{20}}$	0.15	0.0025	0.10	0.46	7.92	456	15.8	2.35
						(16.7; 3.63)	(233; 207)	(3.88; 54.6)	(1.47; 4.18)
	$\bar{R}$	0.35	0.0084	0.025	0.23	3.24	175	118	6.19
						(6.52; 1.60)	(461; 41.8)	(9.69; 494)	(2.05; 20.4)
	$\overline{IQR}$	0.15	0.0025	0.10	0.46	7.79	454	15.9	2.35
						(16.5; 3.62)	(240; 205)	(3.93; 55.3)	(1.47; 4.19)
	$\bar{G}$	0.27	0.0050	0.045	0.32	4.08	246	43.2	3.52
						(7.38; 2.20)	(480; 84.8)	(7.69; 181)	(1.87; 7.45)
	$\overline{ADM}$	0.24	0.0042	0.054	0.35	4.56	284	31.2	3.09
						(8.11; 2.52)	(490; 110)	(7.03; 114)	(1.82; 5.85)
	$\overline{ADM'}$	0.16	0.0027	0.12	0.49	7.10	400	11.5	2.12
						(13.2; 3.72)	(236; 217)	(3.95; 31.7)	(1.48; 3.32)
	$\overline{MDM}$	0.15	0.0025	0.096	0.45	7.73	452	17.0	2.41
						(16.5; 3.52)	(249; 197)	(3.96; 59.8)	(1.48; 4.36)
	$\overline{MAD}$	0.18	0.0029	0.076	0.41	6.21	392	22.0	2.66
						(12.3; 3.07)	(392; 156)	(4.92; 80.3)	(1.59; 4.91)
	$D7$	0.15	0.0025	0.11	0.49	7.02	415	10.8	2.09
						(12.0; 4.03)	(294; 247)	(4.32; 25.9)	(1.53; 3.05)
75	$\tilde{S}$	0.41	0.010	0.016	0.20	2.61	121	123	6.10
						(4.23; 1.65)	(262; 44.7)	(18.4; 414)	(2.65; 15.2)
	$\bar{S}$	0.32	0.0063	0.030	0.28	3.30	180	49.4	3.85
						(5.00; 2.15)	(336; 82.0)	(13.1; 157)	(3.31; 7.00)
	$\bar{S}_{25}$	0.16	0.0025	0.11	0.48	6.72	414	11.7	2.15
						(11.4; 3.95)	(331; 241)	(4.62; 27.9)	(1.56; 3.12)
	$\overline{S_{20}}$	0.15	0.0024	0.12	0.50	7.06	426	9.90	2.04
						(11.3; 4.36)	(320; 279)	(4.56; 20.9)	(1.56; 2.79)
	$\bar{R}$	0.36	0.0079	0.022	0.24	2.96	149	77.3	4.71
						(4.60; 1.90)	(301; 62.6)	(15.6; 274)	(2.46; 9.68)
	$\overline{IQR}$	0.15	0.0025	0.12	0.50	6.90	417	10.0	2.04
						(11.0; 4.27)	(324; 271)	(4.59; 21.0)	(1.56; 2.80)
	$\bar{G}$	0.27	0.0048	0.044	0.34	3.83	229	29.3	3.11
						(5.59; 2.59)	(383; 118)	(10.5; 74.7)	(2.12; 4.86)
	$\overline{ADM}$	0.24	0.0041	0.055	0.37	4.25	266	22.1	2.80
						(6.10; 2.93)	(417; 146)	(9.23; 50.4)	(2.02; 4.07)
	$\overline{ADM'}$	0.16	0.0025	0.12	0.52	6.64	401	8.99	1.97
						(9.82; 4.44)	(331; 287)	(4.75; 16.8)	(1.58; 2.54)
	$\overline{MDM}$	0.15	0.0025	0.11	0.49	6.88	422	10.4	2.07
						(8.66; 3.61)	(443; 207)	(6.00; 30.7)	(1.71; 3.28)
	$\overline{MAD}$	0.19	0.0029	0.086	0.44	5.60	368	13.9	2.32
						(95.2; 33.2)	(138; 451)	(5.69; 39.3)	(2.13; 5.50)
	$D7$	0.16	0.0025	0.12	0.51	6.58	409	8.85	1.97
						(9.22; 4.65)	(370; 307)	(5.15; 15.2)	(1.66; 2.44)

Table 2.8: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values when symmetric variance disturbances are present in Phase I for $n = 9$

		p				ARL			
k	$\hat{\sigma}$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$
30	$\tilde{S}$	0.16	0.020	0.016	0.074	16.1	189	211	137
						(54.9; 1.46)	(444; 8.54)	(12.0; 34.0)	(3.08; 99.9)
	$\bar{S}$	0.063	0.0052	0.019	0.12	23.6	286	196	43.3
						(57.8; 5.15)	(420; 56.2)	(11.1; 262)	(2.95; 447)
	$\bar{S}_{25}$	0.023	0.0024	0.044	0.21	55.1	530	74.8	7.08
						(133; 20.5)	(140; 279)	(5.79; 421)	(2.13; 22.8)
	$\overline{S_{20}}$	0.025	0.0024	0.047	0.22	49.6	488	57.9	6.88
						(106; 19.0)	(173; 249)	(6.30; 354)	(2.24; 20.2)
	$\bar{R}$	0.061	0.0050	0.019	0.12	23.8	291	195	39.5
						(52.3; 5.51)	(420; 61.2)	(11.1; 284)	(2.95; 389)
	$\overline{IQR}$	0.025	0.0024	0.047	0.22	49.5	490	58.3	6.72
						(107; 19.1)	(176; 254)	(6.36; 339)	(2.24; 19.6)
	$\bar{G}$	0.053	0.0042	0.021	0.13	25.8	315	168	25.6
						(58.2; 6.94)	(407; 80.3)	(10.8; 375)	(2.91; 195)
	$\overline{ADM}$	0.047	0.0037	0.023	0.14	27.6	334	147	18.8
						(60.0; 8.23)	(403; 98.5)	(10.5; 458)	(2.88; 113)
	$\overline{ADM'}$	0.022	0.0025	0.069	0.28	51.1	448	21.0	3.86
						(90.7; 27.5)	(153; 378)	(5.94; 63.3)	(2.19; 7.12)
	$\overline{MDM}$	0.024	0.0023	0.045	0.22	48.8	492	56.0	6.05
						(104; 20.5)	(181; 275)	(6.48; 279)	(2.25; 16.8)
	$\overline{MAD}$	0.046	0.0037	0.022	0.13	29.4	352	176	23.9
						(68.9; 8.06)	(405; 96.4)	(9.89; 453)	(2.79; 166)
	$D7$	0.023	0.0024	0.062	0.27	47.5	452	22.6	4.00
						(81.3; 26.5)	(197; 363)	(6.71; 65.1)	(2.32; 7.24)
75	$\tilde{S}$	0.16	0.017	0.0079	0.038	10.5	130	263	163
						(32.5; 1.91)	(432; 13.9)	(31.1; 61.8)	(4.88; 185)
	$\bar{S}$	0.059	0.0046	0.013	0.11	20.2	264	186	19.7
						(39.5; 7.93)	(482; 93.3)	(19.1; 427)	(3.85; 95.0)
	$\bar{S}_{25}$	0.023	0.0023	0.057	0.26	48.5	473	26.7	4.31
						(86.8; 25.9)	(198; 343)	(6.71; 83.8)	(2.31; 8.27)
	$\overline{S_{20}}$	0.025	0.0023	0.056	0.26	44.0	453	26.2	4.27
						(72.8; 24.0)	(256; 328)	(7.64; 84.5)	(2.47; 8.15)
	$\bar{R}$	0.058	0.0045	0.013	0.11	20.4	268	177	17.6
						(39.4; 8.24)	(478; 99.1)	(19.2; 449)	(3.87; 78.1)
	$\overline{IQR}$	0.025	0.0023	0.056	0.26	44.0	453	25.7	4.27
						(72.7; 24.2)	(256; 332)	(7.66; 81.3)	(2.47; 8.02)
	$\bar{G}$	0.050	0.0038	0.016	0.13	22.7	299	135	12.1
						(41.2; 10.1)	(480; 124)	(17.5; 478)	(3.70; 43.3)
	$\overline{ADM}$	0.046	0.0035	0.019	0.14	24.5	320	107	9.66
						(42.9; 11.7)	(477; 147)	(16.2; 433)	(3.54; 29.6)
	$\overline{ADM'}$	0.021	0.0024	0.075	0.30	48.4	433	15.3	3.39
						(70.0; 32.9)	(231; 442)	(7.42; 30.2)	(2.42; 4.82)
	$\overline{MDM}$	0.025	0.0023	0.054	0.25	43.5	460	25.4	4.25
						(71.6; 25.5)	(268; 349)	(8.00; 72.2)	(2.52; 7.53)
	$\overline{MAD}$	0.044	0.0034	0.019	0.14	25.7	335	116	10.2
						(47.0; 11.7)	(492; 147)	(15.1; 472)	(3.42; 33.0)
	$D7$	0.022	0.0023	0.069	0.29	46.1	447	16.4	3.52
						(64.8; 32.2)	(278; 433)	(8.21; 31.9)	(2.55; 4.97)

Table 2.9: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values when asymmetric variance disturbances are present in Phase I for $n = 5$

		p				ARL			
k	$\hat{\sigma}$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$
30	$\tilde{S}$	0.69	0.11	0.025	0.093	1.90	68.9	176	122
						(5.92; 1.00)	(412; 1.63)	(10.9; 8.33)	(2.14; 44.3)
	$\bar{S}$	0.48	0.021	0.021	0.17	2.66	127	192	33.7
						(6.68; 1.09)	(460; 9.89)	(8.81; 123)	(1.97; 383)
	$\bar{S}_{25}$	0.15	0.0026	0.099	0.45	8.20	462	17.7	2.43
						(18.6; 3.52)	(208; 198)	(3.67; 66.8)	(1.44; 4.60)
	$\overline{S_{20}}$	0.14	0.0025	0.11	0.47	8.28	460	14.9	2.29
						(17.7; 3.63)	(204; 217)	(3.69; 51.7)	(1.44; 4.06)
	$\bar{R}$	0.50	0.024	0.017	0.15	2.52	116	214	43.3
						(6.45; 1.07)	(453; 8.73)	(9.72; 107)	(2.06; 469)
	$\overline{IQR}$	0.14	0.0025	0.11	0.47	8.18	459	15.0	2.29
						(17.4; 3.68)	(211; 268)	(3.71; 51.7)	(1.45; 4.05)
	$\bar{G}$	0.37	0.010	0.030	0.24	3.31	179	126	8.78
						(7.26; 1.37)	(478; 26.5)	(7.77; 390)	(1.89; 41.9)
	$\overline{ADM}$	0.31	0.0070	0.038	0.28	3.81	220	87.5	5.42
						(7.95; 1.63)	(491; 42.4)	(7.10; 490)	(1.82; 19.6)
	$\overline{ADM'}$	0.14	0.0026	0.14	0.53	7.95	413	9.00	1.95
						(14.4; 4.34)	(190; 277)	(3.67; 21.6)	(1.43; 2.83)
	$\overline{MDM}$	0.14	0.0025	0.10	0.47	8.16	461	15.3	2.31
						(17.2; 3.68)	(220; 216)	(3.77; 52.3)	(1.45; 4.08)
	$\overline{MAD}$	0.26	0.0052	0.049	0.32	4.79	291	74.2	4.73
						(10.8; 1.82)	(479; 56.1)	(5.68; 517)	(1.67; 16.2)
	$D7$	0.14	0.0025	0.13	0.52	7.69	426	9.07	1.96
						(12.9; 4.46)	(249; 291)	(4.02; 20.3)	(1.48; 2.74)
75	$\tilde{S}$	0.77	0.092	0.012	0.044	1.42	32.1	205	145
						(2.95; 1.00)	(150; 2.59)	(46.5; 18.5)	(3.97; 111)
	$\bar{S}$	0.49	0.017	0.012	0.15	2.27	94.3	204	14.3
						(4.26; 1.25)	(269; 19.3)	(18.3; 272)	(2.63; 71.1)
	$\bar{S}_{25}$	0.15	0.0025	0.12	0.50	7.18	424	10.4	2.06
						(12.1; 4.21)	(288; 266)	(4.31; 23.8)	(1.52; 2.92)
	$\overline{S_{20}}$	0.14	0.0024	0.13	0.52	7.36	429	9.33	1.99
						(12.0; 4.47)	(292; 293)	(4.33; 19.8)	(1.52; 2.72)
	$\bar{R}$	0.52	0.019	0.0093	0.13	2.11	82.0	233	18.6
						(3.93; 1.20)	(241; 16.9)	(21.7; 233)	(2.86; 102)
	$\overline{IQR}$	0.15	0.0025	0.12	0.51	7.20	420	9.28	1.99
						(11.6; 4.40)	(294; 284)	(4.36; 19.9)	(1.53; 2.73)
	$\bar{G}$	0.37	0.0086	0.024	0.24	2.97	151	92.8	5.25
						(5.03; 1.68)	(342; 46.9)	(12.7; 395)	(2.29; 14.1)
	$\overline{ADM}$	0.31	0.0064	0.034	0.29	3.41	191	54.7	3.95
						(5.58; 1.99)	(385; 69.3)	(10.7; 226)	(2.13; 8.53)
	$\overline{ADM'}$	0.14	0.0025	0.15	0.55	7.47	408	7.39	1.83
						(10.8; 5.11)	(277; 347)	(4.32; 12.7)	(1.52; 2.27)
	$\overline{MDM}$	0.15	0.0024	0.12	0.51	7.25	428	9.50	2.00
						(11.8; 4.47)	(297; 288)	(4.38; 20.1)	(1.53; 2.72)
	$\overline{MAD}$	0.26	0.0047	0.059	0.34	4.19	256	36.3	3.29
						(7.21; 2.29)	(470; 92.0)	(7.70; 143)	(1.88; 6.71)
	$D7$	0.14	0.0024	0.14	0.54	7.22	416	7.63	1.86
						(10.0; 5.17)	(321; 351)	(4.70; 12.4)	(1.56; 2.26)

Table 2.10: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values when asymmetric variance disturbances are present in Phase I for $n = 9$

		p				ARL			
k	$\hat{\sigma}$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$
30	$\tilde{S}$	0.10	0.0083	0.0038	0.035	12.1	153	370	92.9
						(26.3; 4.82)	(362; 51.7)	(63.6; 243)	(7.10; 476)
	$\bar{S}$	0.051	0.0038	0.011	0.10	21.3	286	172	13.1
						(36.8; 12.0)	(489; 152)	(27.0; 486)	(4.58; 34.6)
	$\bar{S}_{25}$	0.022	0.0024	0.047	0.22	57.2	534	66.1	6.47
						(136; 21.8)	(129; 291)	(5.54; 363)	(2.09; 19.6)
	$\overline{S_{20}}$	0.051	0.0039	0.010	0.086	24.3	317	285	27.4
						(54.9; 9.42)	(633; 115)	(18.4; 540)	(3.77; 133)
	$\bar{R}$	0.051	0.0038	0.011	0.10	21.4	286	176	12.2
						(37.3; 11.8)	(492; 150)	(27.0; 499)	(4.59; 36.4)
	$\overline{IQR}$	0.051	0.0039	0.010	0.086	24.3	316	286	28.1
						(55.0; 9.39)	(633; 115)	(18.4; 537)	(3.78; 134)
	$\bar{G}$	0.051	0.0038	0.011	0.10	21.4	286	174	13.1
						(37.1; 11.9)	(450; 151)	(27.1; 491)	(4.62; 35.2)
	$\overline{ADM}$	0.051	0.0038	0.010	0.099	21.4	287	178	13.5
						(37.1; 11.7)	(496; 149)	(26.4; 503)	(4.55; 37.2)
	$\overline{ADM'}$	0.020	0.0027	0.079	0.31	55.1	433	17.3	3.53
						(97.2; 30.3)	(130; 415)	(5.51; 48.1)	(2.11; 6.16)
	$\overline{MDM}$	0.051	0.0039	0.010	0.086	24.3	318	288	28.7
						(56.0; 9.25)	(635; 112)	(18.1; 532)	(3.75; 144)
	$\overline{MAD}$	0.051	0.0039	0.010	0.092	22.7	301	233	18.9
						(46.0; 10.3)	(573; 128)	(21.7; 563)	(4.09; 72.3)
	$D7$	0.025	0.0023	0.053	0.25	43.6	454	27.9	4.43
						(75.6; 24.2)	(240; 327)	(7.34; 85.2)	(2.44; 8.44)
75	$\tilde{S}$	0.080	0.0064	0.0041	0.050	13.5	175	330	32.9
						(22.9; 7.39)	(312; 87.4)	(73.0; 406)	(7.74; 117)
	$\bar{S}$	0.043	0.0032	0.017	0.14	24.1	326	77.6	7.77
						(34.2; 16.7)	(450; 222)	(26.7; 186)	(4.57; 13.3)
	$\bar{S}_{25}$	0.021	0.0024	0.065	0.28	51.7	467	22.5	3.99
						(92.7; 27.8)	(165; 384)	(6.21; 67.9)	(2.21; 7.39)
	$\overline{S_{20}}$	0.043	0.0032	0.016	0.13	25.3	337	112	9.48
						(43.1; 14.1)	(520; 183)	(19.4; 377)	(3.85; 23.0)
	$\bar{R}$	0.043	0.0032	0.017	0.14	24.0	323	78.1	7.81
						(33.9; 16.5)	(448; 218)	(26.1; 188)	(4.53; 13.6)
	$\overline{IQR}$	0.043	0.0032	0.016	0.13	0.043	337	112	9.51
						(43.0; 14.1)	(519; 183)	(19.3; 375)	(3.85; 23.1)
	$\bar{G}$	0.043	0.0032	0.017	0.14	23.9	323	77.6	7.82
						(33.7; 16.6)	(447; 220)	(26.5; 186)	(4.55; 13.5)
	$\overline{ADM}$	0.043	0.0032	0.017	0.14	23.9	323	78.9	7.84
						(34.3; 16.4)	(449; 217)	(26.2; 195)	(4.52; 13.7)
	$\overline{ADM'}$	0.020	0.0026	0.087	0.33	52.4	405	12.8	3.13
						(74.6; 36.5)	(194; 474)	(6.72; 23.7)	(2.31; 4.30)
	$\overline{MDM}$	0.043	0.0032	0.016	0.13	25.2	336	115	9.69
						(43.3; 14.0)	(522; 180)	(19.3; 392)	(3.85; 24.1)
	$\overline{MAD}$	0.043	0.0032	0.016	0.13	24.8	333	94.4	8.60
						(38.7; 15.3)	(494; 139)	(22.1; 288)	(4.13; 18.0)
	$D7$	0.023	0.0023	0.064	0.28	44.7	452	17.5	3.63
						(63.1; 31.1)	(295; 423)	(8.55; 34.2)	(2.60; 5.19)

Table 2.11: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values when localized variance disturbances are present in Phase I for $n = 5$

		p				ARL			
k	$\hat{\sigma}$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$
30	$\tilde{S}$	0.66	0.033	0.0037	0.054	1.60	43.1	344	49.3
						(2.60; 1.12)	(117; 12.0)	(93.7; 154)	(5.30; 263)
	$\bar{S}$	0.38	0.0088	0.015	0.20	2.75	132	111	5.63
						(4.22; 1.86)	(462; 59.3)	(21.8; 346)	(2.82; 11.4)
	$\bar{S}_{25}$	0.13	0.0025	0.12	0.49	9.49	481	12.8	2.17
						(21.5; 4.13)	(142; 255)	(3.24; 41.9)	(1.38; 3.74)
	$\overline{S_{20}}$	0.37	0.0089	0.015	0.18	3.09	164	201	8.68
						(6.29; 1.61)	(469; 42.3)	(14.2; 590)	(2.40; 30.0)
	$\bar{R}$	0.38	0.0087	0.015	0.20	2.80	136	120	5.86
						(4.41; 1.82)	(285; 57.5)	(20.9; 395)	(2.77; 12.7)
	$\overline{IQR}$	0.37	0.0089	0.014	0.18	3.09	162	202	8.59
						(6.25; 1.61)	(467; 42.3)	(14.5; 591)	(2.42; 29.2)
	$\bar{G}$	0.38	0.0088	0.015	0.20	2.75	133	113	5.68
						(4.23; 1.85)	(269; 59.4)	(21.7; 358)	(2.82; 11.7)
	$\overline{ADM}$	0.38	0.0087	0.015	0.20	2.79	135	117	5.79
						(4.37; 1.84)	(276; 58.0)	(21.1; 379)	(2.79; 12.2)
	$\overline{ADM'}$	0.12	0.0028	0.17	0.57	9.21	405	6.86	1.78
						(15.9; 5.33)	(147; 369)	(3.31; 14.0)	(1.39; 2.37)
	$\overline{MDM}$	0.37	0.0089	0.015	0.18	3.10	164	203	8.78
						(6.27; 1.61)	(475; 42.2)	(14.4; 587)	(2.39; 30.6)
	$\overline{MAD}$	0.37	0.0088	0.015	0.18	2.97	154	175	7.54
						(5.57; 1.66)	(406; 56.0)	(16.0; 567)	(2.50; 22.7)
	$D7$	0.16	0.0026	0.11	0.49	6.88	412	11.1	2.11
						(11.7; 3.99)	(302; 243)	(4.42; 26.3)	(1.53; 3.09)
75	$\tilde{S}$	0.58	0.021	0.0039	0.088	1.79	55.0	321	15.5
						(2.51; 1.33)	(110; 24.2)	(85.3; 346)	(5.11; 43.4)
	$\bar{S}$	0.32	0.0063	0.027	0.27	3.19	169	44.6	3.77
						(4.22; 2.43)	(264; 104)	(18.7; 97.0)	(2.66; 5.45)
	$\bar{S}_{25}$	0.13	0.0025	0.15	0.55	8.48	427	7.82	1.86
						(14.4; 4.97)	(195; 339)	(3.65; 16.3)	(1.44; 2.51)
	$\overline{S_{20}}$	0.31	0.0062	0.026	0.26	3.36	186	61.7	4.22
						(5.30; 2.17)	(371; 81.6)	(13.7; 205)	(2.36; 7.92)
	$\bar{R}$	0.32	0.0063	0.027	0.27	3.20	170	46.3	3.81
						(4.30; 2.39)	(273; 100)	(18.0; 106)	(2.62; 5.68)
	$\overline{IQR}$	0.32	0.0064	0.026	0.26	3.31	182	61.6	4.22
						(5.21; 2.14)	(361; 80.0)	(13.8; 199)	(2.37; 7.85)
	$\bar{G}$	0.32	0.0063	0.027	0.27	3.19	169	45.0	3.78
						(4.23; 2.43)	(266; 103)	(18.6; 97.9)	(2.66; 5.46)
	$\overline{ADM}$	0.32	0.0063	0.026	0.27	3.19	169	46.1	3.81
						(4.28; 2.40)	(270; 101)	(18.4; 104)	(2.64; 5.64)
	$\overline{ADM'}$	0.12	0.0027	0.18	0.59	8.68	384	5.90	1.69
						(12.2; 6.15)	(214; 421)	(3.83; 9.10)	(1.46; 1.99)
	$\overline{MDM}$	0.31	0.0062	0.026	0.26	3.37	187	62.0	4.22
						(5.28; 2.16)	(374; 82.0)	(13.8; 206)	(2.35; 7.96)
	$\overline{MAD}$	0.32	0.0063	0.026	0.26	3.28	179	56.5	4.09
						(4.91; 2.21)	(335; 85.8)	(14.9; 169)	(2.42; 7.15)
	$D7$	0.15	0.0024	0.13	0.53	6.91	414	8.16	1.91
						(9.61; 4.93)	(347; 332)	(4.92; 13.6)	(1.59; 2.44)

Table 2.12: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values when localized variance disturbances are present in Phase I for $n = 9$

		p				ARL			
k	$\hat{\sigma}$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$
30	$\tilde{S}$	0.055	0.0042	0.010	0.094	20.5	271	204	15.8
						(38.8; 10.5)	(501; 131)	(23.7; 518)	(4.27; 49.0)
	$\bar{S}$	0.046	0.0035	0.015	0.12	24.1	318	139	11.3
						(44.2; 12.4)	(519; 158)	(18.1; 477)	(3.76; 32.5)
	$\bar{S}_{25}$	0.027	0.0024	0.033	0.18	47.8	509	112	9.29
						(117; 17.4)	(213; 230)	(6.82; 617)	(2.32; 33.7)
	$\overline{S_{20}}$	0.030	0.0025	0.031	0.18	40.3	456	97.3	8.65
						(87.6; 15.9)	(291; 211)	(8.22; 509)	(2.53; 29.5)
	$\bar{R}$	0.047	0.0035	0.015	0.12	23.8	316	147	11.8
						(43.9; 12.1)	(522; 155)	(18.4; 486)	(3.78; 34.1)
	$\overline{IQR}$	0.030	0.0025	0.031	0.18	40.3	458	96.7	8.51
						(87.8; 16.0)	(293; 212)	(8.24; 504)	(2.53; 28.9)
	$\bar{G}$	0.044	0.0033	0.017	0.13	25.4	336	119	9.98
						(45.7; 13.3)	(519; 172)	(16.8; 442)	(3.61; 26.7)
	$\overline{ADM}$	0.041	0.0031	0.019	0.14	26.7	351	107	9.25
						(47.9; 14.0)	(514; 182)	(15.7; 393)	(3.51; 24.2)
	$\overline{ADM'}$	0.032	0.0028	0.041	0.20	38.2	403	67.7	6.84
						(84.0; 15.5)	(202; 204)	(6.73; 337)	(2.32; 21.0)
	$\overline{MDM}$	0.030	0.0025	0.030	0.17	39.7	459	100	8.69
						(87.0; 16.2)	(319; 213)	(8.52; 497)	(2.58; 28.0)
	$\overline{MAD}$	0.039	0.0029	0.019	0.14	29.5	380	128	10.5
						(57.8; 13.7)	(531; 178)	(13.4; 510)	(3.26; 32.7)
	$D7$	0.031	0.0025	0.038	0.21	36.8	422	52.1	5.98
						(69.0; 17.8)	(304; 237)	(8.45; 216)	(2.60; 14.8)
75	$\tilde{S}$	0.054	0.0042	0.0091	0.097	19.2	257	156	12.0
						(29.0; 12.7)	(393; 162)	(39.0; 383)	(5.56; 24.6)
	$\bar{S}$	0.046	0.0034	0.014	0.13	22.8	307	96.8	8.79
						(33.7; 15.1)	(447; 197)	(27.2; 255)	(4.59; 16.8)
	$\bar{S}_{25}$	0.026	0.0023	0.044	0.22	42.2	468	38.1	5.14
						(76.3; 22.3)	(283; 302)	(3.21; 133)	(2.53; 10.6)
	$\overline{S_{20}}$	0.030	0.0024	0.036	0.21	35.8	434	42.7	5.48
						(59.9; 20.1)	(394; 274)	(10.6; 139)	(2.86; 11.1)
	$\bar{R}$	0.047	0.0035	0.014	0.12	22.3	301	101	9.00
						(33.3; 14.7)	(442; 192)	(27.5; 270)	(4.68; 17.4)
	$\overline{IQR}$	0.030	0.0024	0.037	0.21	35.7	434	41.4	5.45
						(59.9; 20.2)	(393; 275)	(10.5; 136)	(2.86; 11.0)
	$\bar{G}$	0.044	0.0032	0.017	0.14	23.9	322	81.6	7.96
						(35.0; 36.1)	(457; 211)	(24.5; 213)	(4.39; 14.5)
	$\overline{ADM}$	0.041	0.0031	0.019	0.15	25.1	338	70.8	7.36
						(36.5; 16.9)	(468; 223)	(22.4; 182)	(4.14; 13.2)
	$\overline{ADM'}$	0.030	0.0025	0.043	0.22	36.1	416	36.3	5.05
						(60.6; 20.0)	(321; 267)	(9.03; 119)	(2.69; 10.1)
	$\overline{MDM}$	0.031	0.0024	0.035	0.20	35.3	434	43.9	5.56
						(58.5; 20.4)	(411; 274)	(10.9; 140)	(2.92; 10.9)
	$\overline{MAD}$	0.039	0.0029	0.021	0.15	27.3	364	71.1	7.30
						(42.4; 17.1)	(504; 225)	(18.4; 208)	(3.78; 14.3)
	$D7$	0.030	0.0024	0.040	0.22	35.4	433	32.1	4.85
						(53.2; 22.6)	(410; 305)	(11.3; 82.0)	(2.97; 8.17)

Table 2.13: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values when diffuse mean disturbances are present in Phase I for $n = 5$

		p				ARL			
k	$\hat{\sigma}$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = 2$
30	$\tilde{S}$	0.41	0.010	0.014	0.18	2.62	122	149	6.74
						(4.40; 1.67)	(282; 46.3)	(19.6; 461)	(2.70; 16.5)
	$\bar{S}$	0.36	0.0082	0.019	0.22	2.95	150	103	5.40
						(4.97; 1.82)	(342; 56.7)	(15.0; 373)	(2.44; 12.4)
	$\bar{S}_{25}$	0.18	0.0031	0.069	0.38	6.51	402	32.9	3.04
						(14.8; 2.75)	(343; 132)	(4.55; 154)	(1.55; 6.82)
	$\overline{S_{20}}$	0.18	0.0029	0.077	0.41	6.67	411	25.1	2.76
						(14.3; 2.97)	(322; 150)	(4.52; 105)	(1.54; 5.64)
	$\bar{R}$	0.40	0.0098	0.014	0.19	2.72	129	152	6.86
						(4.70; 1.66)	(310; 45.8)	(18.1; 486)	(2.61; 17.8)
	$\overline{IQR}$	0.18	0.0029	0.076	0.40	6.57	409	25.8	2.79
						(14.2; 2.95)	(346; 147)	(4.62; 108)	(1.55; 5.72)
	$\bar{G}$	0.32	0.0065	0.027	0.26	3.33	183	66.5	4.33
						(5.59; 2.04)	(391; 73.5)	(12.0; 246)	(2.23; 8.91)
	$\overline{ADM}$	0.29	0.0054	0.034	0.29	3.72	218	50.0	3.81
						(6.24; 2.25)	(444; 90.0)	(10.3; 176)	(2.10; 7.44)
	$\overline{ADM'}$	0.23	0.0041	0.069	0.48	5.16	345	30.0	2.97
						(11.3; 2.44)	(326; 104)	(4.56; 130)	(1.56; 6.19)
	$\overline{MDM}$	0.18	0.0029	0.075	0.40	6.60	412	25.1	2.78
						(18.9; 3.01)	(360; 154)	(4.59; 104)	(1.56; 5.55)
	$\overline{MAD}$	0.23	0.0038	0.053	0.34	5.03	322	38.1	3.30
						(9.93; 2.52)	(529; 110)	(6.23; 163)	(1.74; 7.02)
	$D7$	0.19	0.0031	0.081	0.42	5.69	356	18.3	2.52
						(10.3; 3.02)	(387; 155)	(5.11; 59.2)	(1.61; 4.35)
75	$\tilde{S}$	0.41	0.010	0.012	0.19	2.49	109	112	5.74
						(3.41; 2.86)	(189; 60.0)	(31.7; 284)	(3.32; 10.0)
	$\bar{S}$	0.36	0.0080	0.019	0.23	2.82	137	72.8	4.64
						(3.89; 2.07)	(234; 74.8)	(22.4; 188)	(2.86; 7.68)
	$\bar{S}_{25}$	0.19	0.0029	0.081	0.43	5.75	372	16.4	2.44
						(9.78; 3.37)	(428; 185)	(5.51; 44.2)	(1.66; 3.80)
	$\overline{S_{20}}$	0.18	0.0028	0.089	0.45	5.92	384	14.0	2.31
						(9.63; 3.61)	(423; 205)	(5.48; 34.2)	(1.66; 3.41)
	$\bar{R}$	0.40	0.0097	0.013	0.19	2.55	114	108	5.62
						(3.57; 1.87)	(203; 60.0)	(28.6; 290)	(3.17; 10.0)
	$\overline{IQR}$	0.18	0.0029	0.087	0.45	5.77	373	14.3	2.33
						(9.32; 3.52)	(425; 197)	(5.58; 34.7)	(1.67; 3.44)
	$\bar{G}$	0.32	0.0064	0.027	0.27	3.18	169	47.4	3.84
						(4.40; 2.33)	(281; 95.3)	(17.1; 114)	(2.57; 5.91)
	$\overline{ADM}$	0.29	0.0054	0.034	0.30	3.51	199	35.7	3.41
						(4.83; 2.56)	(323; 113)	(14.0; 82.0)	(2.37; 5.03)
	$\overline{ADM'}$	0.22	0.0037	0.070	0.40	4.80	301	18.9	2.59
						(7.98; 2.94)	(432; 147)	(6.24; 50.0)	(1.73; 4.03)
	$\overline{MDM}$	0.18	0.0028	0.087	0.45	5.87	383	14.2	2.32
						(9.50; 3.61)	(430; 208)	(5.55; 34.0)	(1.67; 3.40)
	$\overline{MAD}$	0.23	0.0038	0.058	0.38	4.57	292	22.2	2.78
						(7.04; 2.97)	(467; 149)	(8.05; 55.9)	(1.92; 4.24)
	$D7$	0.20	0.0030	0.085	0.44	5.30	346	13.7	2.30
						(7.77; 3.59)	(439; 204)	(6.43; 28.4)	(1.76; 3.16)

Table 2.14: Unconditional p and ARL and (in parentheses) the upper and lower conditional ARL values when diffuse mean disturbances are present in Phase I for $n = 9$

2.5 Real data example

In this section we demonstrate the implementation of the control charts created above. Our dataset was supplied by Grant and Leavenworth (1988, p.9). The operation concerns thread grinding a fitting for an aircraft hydraulic system. Table 2.15 shows the pitch diameters of the threads for 20 randomly chosen samples. Each sample consists of 5 observations.

The control charting process starts with estimating the in-control standard deviation σ (Phase I). We construct control charts based on the different Phase I estimators proposed. The estimates derived from these estimators are shown in Table 2.16 and are used to determine the Phase II control limits. For example, the estimate of σ based on $\tilde{S}$ is equal to 2.972 and Table 2.3 shows that the respective factors for the upper and lower control limits are 2.352 and 0.171. Consequently the Phase II control limits are 6.990 and 0.508. Figure 2.5 compares the Phase II control limits for the proposed estimators.

In the case of $\overline{ADM}'$, we apply a simple subgroup screening method. The factors for the Phase I control limits are 2.089 and 0 for $n = 5$. We first determine the $\overline{ADM}$ from the 20 samples, which generates 2.594. The Phase I control limits are 5.419 and 0. Then we determine the standard deviation $S/c_4(5)$ of each sample and delete samples whose standard deviation falls outside the initial control limits. For the example discussed here, the standard deviations of samples 8, 9 and 13 fall outside the control limits. The same procedure is repeated in the second iteration: new values for the in-control σ (2.041) and the Phase I control limits (4.263 and 0) are generated from the remaining samples and any sample whose standard deviation falls outside the control limits is deleted. In the second iteration, it appears no longer necessary to delete further samples.

The highest values of $\widehat{UCL}$ and $\widehat{LCL}$ are given by $\tilde{S}$, while the lowest values are given by $D7$ and $\overline{ADM}'$. Note, however, that the question of which estimator produces the best estimate can not be resolved from such a limited sample.

Sample	Observations					$S/c_4(5)$
1	36	35	34	33	32	1.682
2	31	31	34	32	30	1.613
3	30	30	32	30	32	1.165
4	32	33	33	32	35	1.303
5	32	34	37	37	35	2.257
6	32	32	31	33	33	0.890
7	33	33	36	32	31	1.990
8	23	33	36	35	36	5.856
9	43	36	35	24	31	7.424
10	36	35	36	41	41	3.138
11	34	38	35	34	38	2.180
12	36	38	39	39	40	1.613
13	36	40	35	26	33	5.477
14	36	35	37	34	33	1.682
15	30	37	33	34	35	2.754
16	28	31	33	33	33	2.331
17	33	30	34	33	35	1.990
18	27	28	29	27	30	1.387
19	35	36	29	27	32	4.079
20	33	35	35	39	36	2.331

Table 2.15: Measurements of pitch diameter of threads on aircraft fittings

$\hat{\sigma}$		$\widehat{UCL}$	$\widehat{LCL}$
$\tilde{S}$	2.972	6.990	0.508
$\bar{S}$	2.657	6.263	0.112
$\bar{S}_{25}$	2.193	5.930	0.366
$\overline{S_{25}}$	2.456	6.238	0.415
$\bar{R}$	2.666	6.302	0.456
$\overline{IQR}$	2.424	6.159	0.410
$\bar{G}$	2.623	6.188	0.449
$\overline{ADM}$	2.594	6.137	0.444
$\overline{ADM}'$	2.041	4.849	0.349
$\overline{MDM}$	2.256	5.762	0.381
$\overline{MAD}$	2.408	5.892	0.409
$D7$	2.067	4.911	0.353

Table 2.16: Control chart limits for pitch diameter

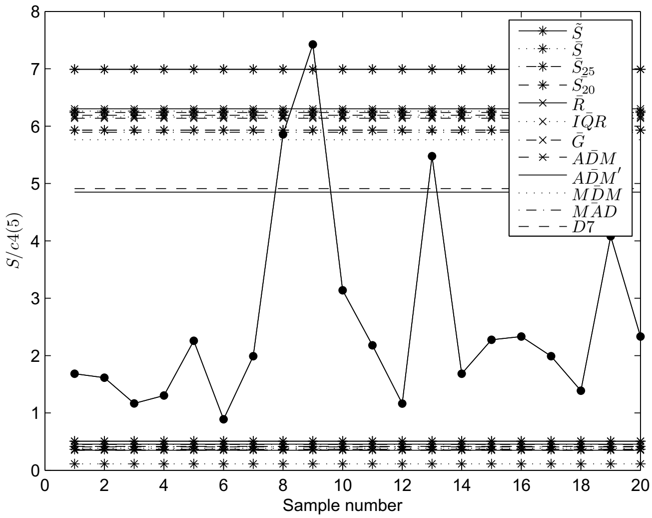


Figure 2.5: Control chart limits for pitch diameter

2.6 Concluding remarks

In this chapter we have compared 12 different estimators for designing standard deviation control charts and investigated their performance in Phase II. The added value of incorporating a simple screening procedure into an estimation method has proven to be substantial. This method performs better than estimators which remove samples ($\bar{S}_{25}$) or observations ($\overline{S_{20}}$ or $\overline{IQR}$) beforehand. The disadvantage of removing samples and/or observations beforehand is that too much information is lost in uncontaminated situations while, at the same time, the resulting estimates are biased in contaminated situations. The estimator $\overline{ADM}'$ retains far more information, deleting only extreme subgroups so that the final estimate is not affected substantially. Moreover, $\overline{ADM}'$ is intuitive and easy to implement. We recommend using $\overline{ADM}'$ when the dataset is likely to be contaminated by localized disturbances. On the other hand, we prefer $D7$ when the dataset is likely to be contaminated by mean diffuse disturbances because $D7$ is more robust against such disturbances. All in all, there appears to be no single best control-chart method that would cover every process and every company. ASTM 15D (1976, p.143) says it best: *"The final justification of a control chart criterion is its proven ability to detect assignable causes economically under practical conditions."*

2.7 Appendix

The literature proposes several estimators for the standard deviation of a normal distribution, including estimators based on Gini's mean difference, Downton's linear function of order statistics (Downton (1966)) and the probability-weighted moments estimator (Muhammad et al. (1993)).

Let $X_{i(1)}, X_{i(2)}, ..., X_{i(n)}$ denote the order statistics of sample i. According to David (1968), the sample statistic G_i can also be written as a function of order statistics

$$G_i = 2/(n(n-1)) \sum_{j=1}^{n} (2j - n - 1) X_{i(j)}. \tag{2.16}$$

Downton (1966) suggested as a possible unbiased estimator of σ the statistic

$$D_i = 1/\sqrt{\pi} \sum_{j=1}^{n} (2j - n - 1) X_{i(j)} / (n(n-1)), \tag{2.17}$$

and Muhammad et al. (1993) proposed the so-called probability weighted moments estimator of σ

$$S_{pw,i} = \sqrt{\pi}/n^2 \sum_{j=1}^{n} (2j - n - 1) X_{i(j)}. \tag{2.18}$$

It follows directly from (2.16), (2.17) and (2.18) that

$$G_i = 2/\sqrt{\pi} D_i = 2n/((n-1)\sqrt{\pi}) S_{pw,i}.$$