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Special issue on the occasion of Jaap Korevaar's 100-th birthday

Charting the q-Askey scheme. II. The q-Zhedanov scheme

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Dedicated to Jaap Korevaar on the occasion of his centennial birthday

Abstract

This is the second in a series of papers which intend to explore conceptual ways of distinguishing between families in the q-Askey scheme and uniform ways of parametrizing the families. For a system of polynomials $p_n(x)$ in the q-Askey scheme satisfying $Lp_n = h_n p_n$ with L a second order q-difference operator the q-Zhedanov algebra is the algebra generated by operators L and X (multiplication by x). It has two relations in which essentially five coefficients occur. Vanishing of one or more of the coefficients corresponds to a subfamily or limit family of the Askey–Wilson polynomials. An arrow from one family to another means that in the latter family one more coefficient vanishes. This yields the q-Zhedanov scheme given in this paper.

The q-hypergeometric expression of $p_n(x)$ can be interpreted as an expansion of $p_n(x)$ in terms of certain Newton polynomials. In our previous paper (Contemporary Math. 780) we used Verde-Star's clean parametrization of such expansions and we obtained a q-Verde-Star scheme, where vanishing of one or more of these parameters corresponds to a subfamily or limit family. The actions of the operators L and X on the Newton polynomials can be expressed in terms of the Verde-Star parameters, and thus the coefficients for the q-Zhedanov algebra can be expressed in terms of these parameters. There are interesting differences between the q-Verde-Star scheme and the q-Zhedanov scheme, which are discussed in the paper.

Keywords: q-Askey scheme; q-Zhedanov scheme; Askey–Wilson algebra

1. Introduction

Jaap Korevaar has worked on a wide range of topics in analysis, in particular complex and harmonic analysis and applications to number theory. As for special functions, he likes in

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particular the special functions of number theory, with the zeta function on top [24,25], but he does certainly not ignore the classical orthogonal polynomials. In [28] he used Hermite polynomials, and in work with his Ph.D. student Meyers [26,27] he used Legendre and ultraspherical polynomials. The paper [26] was a key reference in [29], a paper coauthored by Maryna Viazovska who was a 2022 Fields Medal winner.

After pioneering work of Hahn q-analogues of classical orthogonal polynomials got during 1975–1985 an increasing interest. This culminated in the introduction of the Askey–Wilson polynomials [8], a 4-parameter family of q-hypergeometric orthogonal polynomials which stand on top of the q-Askey scheme [3, Ch. 14]. In that scheme there are boxes representing families of q-hypergeometric orthogonal polynomials and arrows connecting them. An arrow stands for a limit transition or specialization and it diminishes the number of parameters by at least one.

This paper is the second in a series of papers which intend to explore conceptual ways of distinguishing between families in the scheme and uniform ways of parametrizing the families. First of all we do this for the q-Askey scheme, and tentatively later for its $q = 1$ limit (the original Askey scheme [3, Ch. 9]) and its $q = -1$ limit [15,22]. The present and the previous paper concentrate on the case of general q. The previous paper [1] was inspired by Verde-Star's paper [9] (see also Vinet and Zhedanov [16] and, for finite systems, Terwilliger [20], cf. [1, Remark 2.3]), which classifies the families in the q-Askey scheme in combination with a basis of Newton polynomials in terms of which they have nice expansions. There are uniform parameters describing this. The vanishing of one or more of the parameters corresponds to a subfamily.

The present paper classifies the q-Zhedanov algebras [2, §3] associated with families in the q-Askey scheme. Such an algebra is generated by the second order q-difference operator L for which the polynomials in the family are eigenfunctions, and by the operator of multiplication by x. Denote these two operators by K_1, K_2, respectively. Then the algebra is described by two relations which expand certain cubic forms in K_1, K_2 (repeated q-commutators) as linear combinations of K_1, K_2 and the identity operator. The vanishing of one or more of the coefficients in these linear combinations indicates a subfamily in the q-Askey scheme. The prototypical algebra of this type is the Askey–Wilson algebra, associated with the Askey–Wilson polynomials. See Zhedanov [13].

In Verde-Star's approach the action of L on the space of polynomials is conceptually described by its action on the corresponding Newton polynomials, while multiplication by x immediately yields an explicit action on the Newton polynomials. This makes it possible to express the coefficients of the q-Zhedanov algebra in terms of Verde-Star's parameters. From that it can be read off for which constraints on Verde-Star's parameters certain q-Zhedanov coefficients vanish. Since we already expressed in [1] Verde-Star's parameters in terms of the parameters of the corresponding family in the q-Askey scheme, we can associate vanishing properties of the q-Zhedanov parameters with families in the q-Askey scheme, and thus the q-Zhedanov scheme occurs. One can find these results in Sections 3–5, which form the heart of this paper. It should be observed that our work was anticipated, in the case of finite systems and by a different approach, by Terwilliger and Vidunas [19, Theorem 4.5 and 5.3] and Vidunas [21]. More limited q-Zhedanov schemes, not related to q-Verde-Star parameters, were earlier given by Mazzocco [23], see also [6].

Filling in this q-Zhedanov scheme gave some surprises. Several families in the q-Askey scheme which usually do not get much attention attain an independent status in the present scheme. We mention symmetric cases of Askey–Wilson, Al-Salam–Chihara and big q-Jacobi polynomials, and also q^2 versions of the continuous q-Jacobi polynomials and its subfamilies, which give interesting q-polynomials by quadratic transformation. In particular, continuous q^2-Hermite polynomials get after quadratic transformation a q-hypergeometric expression which allows to describe this family by Verde-Star's approach, something which was not possible in [9] for the continuous q-Hermite polynomials.

Very relevant for this paper is the fact that the Askey–Wilson algebra in its dependence on the Askey–Wilson parameters a, b, c, d is not only symmetric in these parameters, but also remains invariant if two parameters e, f from a, b, c, d are sent to q/e, q/f, respectively. Such a symmetry under $(a, b, c, d) \to (a, b, qd^{-1}, qc^{-1})$ for the Askey–Wilson DAHA [7,18], [6, §5.1] was communicated to me by Marta Mazzocco. She pointed for this to the DAHA symmetry $(a, b, c, d) \to (a, b, qc^{-1}, d)$ observed by Oblomkov [14, (2.11)]. Since Askey–Wilson polynomials are not invariant under these transformations (however, see Section 7), it means that a possible charting of the q-Zhedanov scheme cannot be a charting of the q-Askey scheme which completely distinguishes between individual systems of polynomials.

Duality is also an important theme in this paper, as it was also in the author's earlier paper [6] with Mazzocco. From the graphical q-Zhedanov scheme in Section 5 the dualities between the occurring families can be nicely seen.

It should be emphasized that, although the families from the q-Askey scheme are first of all known as orthogonal polynomials, we completely disregard orthogonality aspects in this paper. Only those algebraic aspects which give rise to a q-Verde-Star description and a nice associated q-Zhedanov algebra matter here for us.

The contents of this paper are as follows. In Section 2 we treat some (but not all) families from the q-Askey scheme. We concentrate on the families which play a particularly important role later in the paper. Duality properties are already discussed here. This section may be skipped on first reading. As already said, Section 3 on Verde Star's framework, Section 4 on using this framework in the context of the q-Zhedanov algebra, and Section 5 which presents the q-Zhedanov scheme in Fig. 1, form the heart of this paper. Some subsections of Section 4 consider examples of q-Zhedanov algebras in connection with q-Verde-Star parameters. Here the further symmetries just observed also enter. But these symmetries appear already in Section 2.3 in connection with duality, although not yet in connection with the Askey–Wilson algebra. In Section 6 the duals of continuous q^2-Hermite polynomials and of discrete q-Hermite I polynomials are discussed. Finally Section 7 gives some recapitulation, open questions and thoughts about an optimal q-Askey scheme.

Note. For definition and notation of q-shifted factorials and q-hypergeometric series we follow [4, §1.2]. We will only need terminating series:

$$ {}_r\phi_s\left(\begin{matrix} q^{-n}, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q, z\right) := \sum_{k=0}^{n} \frac{(q^{-n}; q)_k}{(q; q)_k} \frac{(a_2, \ldots, a_r; q)_k}{(b_1, \ldots, b_s; q)_k} \left((-1)^k q^{\frac{1}{2}k(k-1)}\right)^{s-r+1} z^k. $$

Here $(b_1, \ldots, b_s; q)_k := (b_1; q)_k \ldots (b_s; q)_k$ with $(b; q)_k := (1-b)(1-qb)\ldots(1-q^{k-1}b)$ the q-shifted factorial.

2. Preliminaries about polynomials in the q-Askey scheme

2.1. Askey–Wilson polynomials and some subfamilies

Assume $a \neq 0$ and $ab, ac, ad, abcd \neq 1, q^{-1}, q^{-2}, \ldots$. Define *Askey–Wilson polynomials* by

$$ R_n(z; a, b, c, d \mid q) = R_n(z) := {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, az, az^{-1} \\ ab, ac, ad \end{matrix}; q, q\right). \tag{2.1} $$

Then $R_n(z)$ is a symmetric Laurent polynomial in z of degree n. In usual notation and normalization [3, (14.1.1)] the Askey–Wilson polynomials are written as

$$p_n\big(\tfrac{1}{2}(z+z^{-1}); a,b,c,d \mid q\big) := a^{-n}(ab,ac,ad;q)_n\, R_n(z;a,b,c,d \mid q).$$

In this form they are symmetric in a,b,c,d (see [8, p. 6]). In the normalization (2.1) the polynomials are still symmetric in b,c,d, but for $a \leftrightarrow b$ we have

$$R_n(z;a,b,c,d \mid q) = \left(\frac{a}{b}\right)^n \frac{(bc,bd;q)_n}{(ac,ad;q)_n}\, R_n(z;b,a,c,d \mid q), \tag{2.2}$$

where we assume in addition that $b \neq 0$ and $bc, bd \neq 1, q^{-1}, q^{-2}, \ldots$. For $c=-a$, $d=-b$ we have the *symmetric Askey–Wilson polynomials* [8, p. 7] with the symmetry

$$R_n(-z;a,b,-a,-b \mid q) = (-1)^n R_n(z;a,b,-a,-b \mid q). \tag{2.3}$$

The polynomials $R_n(z)$ given by (2.1) satisfy the eigenvalue equation [3, (14.1.7)]

$$(LR_n)(z) = (q^{-n} + abcdq^{n-1})R_n(z), \tag{2.4}$$

where

$$\begin{aligned}(Lf)(z) = \big(1+q^{-1}abcd\big)f(z) &+ \frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}\big(f(qz)-f(z)\big)\\ &+\frac{(a-z)(b-z)(c-z)(d-z)}{(1-z^2)(q-z^2)}\big(f(q^{-1}z)-f(z)\big).\end{aligned} \tag{2.5}$$

They also satisfy the three-term recurrence relation (for $n=1,2,\ldots$)

$$\begin{aligned}(z+z^{-1})R_n(z) =&(a+a^{-1})R_n(z)\\ &+\frac{(1-q^nab)(1-q^nac)(1-q^nad)(1-q^{n-1}abcd)}{a(1-q^{2n-1}abcd)(1-q^{2n}abcd)}(R_{n+1}(z)-R_n(z))\\ &+\frac{a(1-q^n)(1-q^{n-1}bc)(1-q^{n-1}bd)(1-q^{n-1}cd)}{(1-q^{2n-2}abcd)(1-q^{2n-1}abcd)}(R_{n-1}(z)-R_n(z)).\end{aligned} \tag{2.6}$$

The starting values $R_0(z)=1$ and $R_1(z)$ as given by (2.1) satisfy (2.6) in its reduced form

$$(z+z^{-1})R_0(z) = (a+a^{-1})R_0(z) + \frac{(1-ab)(1-ac)(1-ad)}{a(1-abcd)}(R_1(z)-R_0(z)).$$

Note that here the factors $(1-q^{-1}abcd)$ in numerator and denominator in (2.6) cancel, which even turns out to be correct if $abcd=q$.

In the following we mention a few special cases of Askey–Wilson which will play a remarkable role in the q-Zhedanov scheme. These special cases have their own, generally used, notations (see [3, Ch. 14]), but we will just denote them as a restricted $R_n(z)$.

Continuous dual q-Hahn. [3, (14.3.1)]

$$R_n(z;a,b,c,0 \mid q) = {}_3\phi_2\left(\begin{matrix}q^{-n}, az, az^{-1}\\ ab, ac\end{matrix}; q, q\right). \tag{2.7}$$

Al-Salam–Chihara. [3, (14.8.1)]

$$R_n(z;a,b,0,0 \mid q) = {}_3\phi_2\left(\begin{matrix}q^{-n}, az, az^{-1}\\ ab, 0\end{matrix}; q, q\right). \tag{2.8}$$

Symmetric Al-Salam–Chihara.

$$R_n(z;a,-a,0,0\mid q)={}_3\phi_2\left(\begin{matrix}q^{-n},az,az^{-1}\\-a^2,0\end{matrix};q,q\right). \tag{2.9}$$

For these polynomials the symmetry (2.3) holds with $b=0$.

Continuous q-Jacobi polynomials. [3, (14.10.1)]

$$R_n(z;a,-b,q^{\frac12}a,-q^{\frac12}b\mid q)={}_4\phi_3\left(\begin{matrix}q^{-n},q^na^2b^2,az,az^{-1}\\-ab,q^{\frac12}a^2,-q^{\frac12}ab\end{matrix};q,q\right). \tag{2.10}$$

Continuous q-Laguerre polynomials. [3, (14.18.1)],

$$R_n(z;a,q^{\frac12}a,0,0\mid q)={}_3\phi_2\left(\begin{matrix}q^{-n},az,az^{-1}\\q^{\frac12}a^2,0\end{matrix};q,q\right). \tag{2.11}$$

Continuous q-ultraspherical polynomials. [3, (14.10.17)]

$$R_n(z;a,-a,q^{\frac12}a,-q^{\frac12}a\mid q)={}_4\phi_3\left(\begin{matrix}q^{-n},q^na^4,az,az^{-1}\\-a^2,q^{\frac12}a^2,-q^{\frac12}a^2\end{matrix};q,q\right). \tag{2.12}$$

For these polynomials the symmetry (2.3) holds with $b=q^{\frac12}a$.

Continuous q-Hermite polynomials. [3, §14.26],

$$H_n\big(\tfrac12(z+z^{-1})\mid q\big):=\begin{cases}\lim\limits_{a\to0}a^{-n}R_n(z;a,q^{\frac12}a,0,0\mid q),\\ \lim\limits_{a\to0}a^{-n}R_n(z;a,-a,q^{\frac12}a,-q^{\frac12}a\mid q).\end{cases} \tag{2.13}$$

This is no longer a special case of Askey–Wilson, but a limit case. There is the symmetry $H_n(-x)=(-1)^nH_n(x)$.

$q\to q^2$ transformations. The transformation [4, (3.10.13)]

$${}_4\phi_3\left(\begin{matrix}a^2,b^2,c,d\\q^{\frac12}ab,-q^{\frac12}ab,-cd\end{matrix};q,q\right)={}_4\phi_3\left(\begin{matrix}a^2,b^2,c^2,d^2\\qa^2b^2,-cd,-qcd\end{matrix};q^2,q^2\right),$$

together with (2.3), allows to express the polynomials (2.10)–(2.13) for base q^2 in a different way:

$$R_n(z;a,-b,q^{\frac12},-q^{\frac12}\mid q)=R_n(z;a,-b,qa,-qb\mid q^2), \tag{2.14}$$

$$q^{-\frac12n}\frac{(-q;q)_n}{(-q^{\frac12}a;q)_n}R_n(z;q^{\frac12},-q^{\frac12},a,0\mid q)=a^{-n}R_n(z;a,qa,0,0\mid q^2), \tag{2.15}$$

$$q^{-\frac12n}\frac{(-q;q)_n}{(-a^2;q)_n}R_n(z;q^{\frac12},-q^{\frac12},a,-a\mid q)=a^{-n}R_n(z;a,-a,qa,-qa\mid q^2), \tag{2.16}$$

$$q^{-\frac12n}(-q;q)_nR_n(z;q^{\frac12},-q^{\frac12},0,0\mid q)=H_n\big(\tfrac12(z+z^{-1})\mid q^2\big). \tag{2.17}$$

We call the polynomials $R_n(z)$ on the left-hand sides the continuous q^2-Jacobi, continuous q^2-Laguerre, continuous q^2-ultraspherical and continuous q^2-Hermite polynomials, respectively. The continuous q^2-Jacobi polynomials as represented by the left-hand side of (2.14) were first observed by M. Rahman, see [3, p. 468].

2.2. Big q-Jacobi polynomials and some subfamilies

Assume $ab, a, c \neq q^{-1}, q^{-2}, \ldots$. Define *big q-Jacobi polynomials* [3, (14.5.1)] by

$$P_n(x;a,b,c;q) = P_n(x) := {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{n+1}ab, x \\ qa, qc \end{matrix}; q, q\right). \tag{2.18}$$

They are limit cases [3, (14.1.18)] of Askey–Wilson polynomials:

$$P_n(x;a,b,c;q) = \lim_{\lambda\to 0} R_n(\lambda^{-1}x; \lambda, \lambda bc^{-1}, q\lambda^{-1}a, q\lambda^{-1}c \mid q). \tag{2.19}$$

The S_4 symmetry of Askey–Wilson in its parameters reduces to an $S_2 \times S_2$ symmetry for big q-Jacobi. Here the S_3 symmetry which is obvious from (2.1) reduces to a symmetry obvious from (2.18):

$$\begin{aligned} &P_n(x;a,b,c;q) = P_n(x;c,abc^{-1},a;q) \quad \text{or} \\ &P_n(x;a,bc,c;q) = P_n(x;c,ba,a;q) \quad \text{(equivalently).} \end{aligned} \tag{2.20}$$

The other symmetry is a limit case of (2.20) under the limit (2.19):

$$\begin{aligned} &P_n(x;a,b,c;q) = \left(\frac{c}{b}\right)^n \frac{(qabc^{-1}, qb;q)_n}{(qa,qc;q)_n} P_n(bc^{-1}x; abc^{-1}, c, b; q) \quad \text{or} \\ &c^{-n}(qac,qc;q)_n P_n(cx;ac,b,c;q) = b^{-n}(qab,qb;q)_n P_n(bx;ab,c,b;q) \quad \text{(equivalenty),} \end{aligned} \tag{2.21}$$

where we assume that $b, c \neq 0$ and (in the first equality) $a, b, c, abc^{-1} \neq q^{-1}, q^{-2}, \ldots$. In particular, we have the *symmetric big q-Jacobi polynomials*

$$P_n(x;a,a,-a;q) = {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{n+1}a^2, x \\ qa, -qa \end{matrix}; q, q\right), \tag{2.22}$$

which satisfy

$$P_n(-x;a,a,-a;q) = (-1)^n P_n(x;a,a,-a;q)$$

by (2.21) and (2.20).

The polynomials $P_n(x)$ given by (2.18) satisfy the eigenvalue equation [3, (14.5.5)]

$$(LP_n)(x) = (q^{-n} + abq^{n+1})P_n(x), \tag{2.23}$$

where

$$\begin{aligned} (Lf)(x) =&(1+qab)f(x) + qa(x-1)(bx-c)\big(f(qx)-f(x)\big) \\ &+ (x-qa)(x-qc)\big(f(q^{-1}x)-f(z)\big). \end{aligned} \tag{2.24}$$

They also satisfy the three-term recurrence relation (for $n = 1, 2, \ldots$)

$$\begin{aligned} xP_n(x) = P_n(x) &+ \frac{(1-q^{n+1}a)(1-q^{n+1}ab)(1-q^{n+1}c)}{(1-q^{2n+1}ab)(1-q^{2n+2}ab)} (P_{n+1}(x) - P_n(x)) \\ &- q^{n+1}ac\, \frac{(1-q^n)(1-q^nabc^{-1})(1-q^nb)}{(1-q^{2n}ab)(1-q^{2n+1}ab)} (P_{n-1}(x) - P_n(x)). \end{aligned} \tag{2.25}$$

The starting values $P_0(x) = 1$ and $P_1(x)$ as given by (2.18) satisfy (2.25) in its reduced form

$$xP_0(x) = P_0(x) + \frac{(1-qa)(1-qc)}{1-q^2ab} (P_1(x) - P_0(x)).$$

We mention some further subfamilies of the big q-Jacobi polynomials.

Little q-Jacobi. [3, (14.12.1) and p. 442, Remarks]

These can be seen as a special case of big q-Jacobi polynomials:

$$P_n(x;a,b,0;q) = {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{n+1}ab, x \\ qa, 0 \end{matrix}; q, q\right). \tag{2.26}$$

However, they are usually notated and defined as

$$p_n(x;a,b;q) := {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{n+1}ab \\ qa \end{matrix}; q, qx\right). \tag{2.27}$$

The expressions in (2.26) and (2.27) are related by

$$(-a)^n q^{\frac{1}{2}n(n+1)} \frac{(qb;q)_n}{(qa;q)_n}\, p_n((qa)^{-1}x;b,a;q) = P_n(x;a,b,0;q). \tag{2.28}$$

Big q-Laguerre. [3, 14.11.1]

$$P_n(x;a,0,c;q) = {}_3\phi_2\left(\begin{matrix} q^{-n}, x, 0 \\ qa, qc \end{matrix}; q, q\right). \tag{2.29}$$

2.3. Duality

In our charting of the q-Askey scheme we consider systems of polynomials $\{p_n(x)\}$ in the scheme up to dilation of x by a nonzero constant. In particular, taking -1 for this constant and noting from (2.1) that

$$R_n(-z;a,b,c,d \mid q) = R_n(z;-a,-b,-c,-d \mid q), \tag{2.30}$$

parameter values a,b,c,d and $-a,-b,-c,-d$ will be identified for charting purposes.

Define *dual parameters* $\tilde a, \tilde b, \tilde c, \tilde d$ in terms of the Askey–Wilson parameters a,b,c,d by

$$\tilde a = (q^{-1}abcd)^{\frac{1}{2}}, \quad \tilde b = ab/\tilde a, \quad \tilde c = ac/\tilde a, \quad \tilde d = ad/\tilde a, \tag{2.31}$$

see [7, §§ 5.7, 8.5] and [6, §2.3]. Because of the square root in the definition of $\tilde a$, Eq. (2.31) defines the values of the dual parameters up to possible common multiplication by -1. So for charting purposes this is harmless.

From (2.1) we have the duality relation

$$R_n(a^{-1}q^{-m};a,b,c,d \mid q) = R_m(\tilde a^{-1}q^{-n};\tilde a,\tilde b,\tilde c,\tilde d \mid q) \qquad (m,n \in \mathbb{Z}_{\ge 0}), \tag{2.32}$$

with both sides equal to

$${}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, q^{-m}, q^m a^2 \\ ab, ac, ad \end{matrix}; q, q\right). \tag{2.33}$$

Formula (2.32) gives a duality between the two Askey–Wilson polynomials $R_n(z;a,b,c,d \mid q)$ and $R_m(w;\tilde a,\tilde b,\tilde c,\tilde d \mid q)$.

Remark. The above concept of duality is rather weak. It certainly does not involve in general that orthogonality relations for the original system imply orthogonality for the dual system. See [10] for a treatment with many examples of dual systems in this stronger sense. Still our weak notion of duality, when occurring in the q-Askey scheme, usually makes it possible to relate the three-term recurrence relation and the second order q-difference eigenvalue equation for the one system in a formal way with the eigenvalue equation and the recurrence relation, respectively, for the dual system. For instance, the recurrence relation (2.6) for Askey–Wilson polynomials can be rewritten as

$$\begin{aligned} a(z+z^{-1})R_n(z) &= (1+q^{-1}\tilde a\tilde b\tilde c\tilde d)R_n(z) \\ &\quad+\frac{(1-q^n\tilde a^2)(1-q^n\tilde a\tilde b)(1-q^n\tilde a\tilde c)(1-q^n\tilde a\tilde d)}{(1-q^{2n}\tilde a^2)(1-q^{2n+1}\tilde a^2)}\,(R_{n+1}(z)-R_n(z)) \\ &\quad+\frac{(\tilde a-q^n\tilde a)(\tilde b-q^n\tilde a)(\tilde c-q^n\tilde a)(\tilde d-q^n\tilde a)}{(1-q^{2n}\tilde a^2)(q-q^{2n}\tilde a^2)}\,(R_{n-1}(z)-R_n(z)). \end{aligned} \tag{2.34}$$

Compare this with (2.5) with z replaced by $q^n\tilde a$ in the coefficients on the right-hand side.

Note that the definition of the dual parameters depends on a choice of one of the original parameters. In (2.31) this is a. The duals of the dual parameters $\tilde a,\tilde b,\tilde c,\tilde d$ with respect to $\tilde a$ are the original parameters a,b,c,d again, up to common multiplication by -1. (In the following this identification of parameters with their opposites will be silently assumed.) Now let a',b',c',d' be dual parameters of a,b,c,d with respect to b:

$$a'=\frac{ab}{(q^{-1}abcd)^{\frac12}},\quad b'=(q^{-1}abcd)^{\frac12},\quad c'=\frac{bc}{(q^{-1}abcd)^{\frac12}},\quad d'=\frac{bd}{(q^{-1}abcd)^{\frac12}}.$$

Then the dual parameters of a',b',c',d' with respect to a' are b,a,qd^{-1},qc^{-1}. This result can be alternatively stated by the commutative diagram

$$\begin{array}{ccc} a\;b\;c\;d & \stackrel{\sim}{\to} & \tilde a\;\tilde b\;\tilde c\;\tilde d \\ \downarrow & & \downarrow \\ a\;b\;\frac{q}{d}\;\frac{q}{c} & \stackrel{\sim}{\to} & \tilde b\;\tilde a\;\tilde c\;\tilde d \end{array}$$

From this we have:

Proposition 2.1. *Let $f(a,b,c,d)$ be a symmetric function which is also invariant under common multiplication of a,b,c,d by -1. Then f is symmetric in $\tilde a,\tilde b,\tilde c,\tilde d$ iff f is invariant under any mapping which sends an even number of its variables a,b,c,d to $q/a,q/b,q/c,q/d$, respectively.*

The Askey–Wilson polynomials, although invariant under permutations of a,b,c,d, are no longer invariant under the larger group which also involves mappings sending two parameters e,f out of a,b,c,d to qe^{-1},qf^{-1}, respectively, but such symmetries will play a role later in this paper, see Section 4.2. The larger group is isomorphic to the semidirect product $S_4\ltimes(\mathbb Z/2\mathbb Z)^3$ (where $(\mathbb Z/2\mathbb Z)^3$ is the group of an even number of sign changes of $1,2,3,4$), which is also the Weyl group for the root system D_4, see [17, §12.1].

Below we list some dualities obtained from (2.32), (2.33) by specialization or (possibly scaled) limit.

Continuous dual q-Hahn $\longleftrightarrow$ big q-Jacobi. [6, (44)]

$$\begin{aligned} &R_n(z;a,b,c,0\mid q)\;\longleftrightarrow\;P_m(y;q^{-1}ab,ab^{-1},q^{-1}ac;q) \\ &R_n(a^{-1}q^{-m};a,b,c,0\mid q)=P_m(q^{-n};q^{-1}ab,ab^{-1},q^{-1}ac;q) \\ &\qquad={}_3\phi_2\!\left(\begin{matrix}q^{-n},q^{-m},q^ma^2\\ ab,ac\end{matrix};q,q\right). \end{aligned} \tag{2.35}$$

Note that there is a one-to-one correspondence $\pm(a,b,c)\leftrightarrow(q^{-1}ab,ab^{-1},q^{-1}ac)$. By (2.2) there is a symmetry $(a,b,c)\leftrightarrow(b,a,c)$ for suitably normalized continuous dual q-Hahn. By the duality (2.35) this leads to a symmetry $(a,b,c)\leftrightarrow(a,b^{-1},b^{-1}c)$ for the big q-Jacobi parameters a,b,c, although it does not leave $P_n(x;a,b,c;q)$ invariant. Conversely, the symmetry (2.21) for big q-Jacobi polynomials leads to a symmetry $(a,b,c)\leftrightarrow(a,qc^{-1},qb^{-1})$ for the continuous dual Hahn parameters a,b,c, which, however, does not leave $R_n(z;a,b,c,0\mid q)$ invariant.

Al-Salam–Chihara $\longleftrightarrow$ *little* q*-Jacobi.* [6, (75)]

$$R_n(z;a,b,0,0\mid q)\;\longleftrightarrow\;P_m(y;q^{-1}ab,ab^{-1},0;q) \tag{2.36}$$
$$R_n(a^{-1}q^{-m};a,b,0,0\mid q)=P_m(q^{-n};q^{-1}ab,ab^{-1},0;q)={}_3\phi_2\left(\begin{matrix}q^{-n},q^{-m},q^ma^2\\ ab,0\end{matrix};q,q\right).$$

Continuous q^2*-Jacobi* $\longleftrightarrow$ *symmetric Askey–Wilson.*

$$R_n(z;a,-b,q^{\frac12},-q^{\frac12}\mid q)\;\longleftrightarrow\;R_m(w;(ab)^{\frac12},-(ab)^{\frac12},(qab^{-1})^{\frac12},-(qab^{-1})^{\frac12}\mid q), \tag{2.37}$$
$$\begin{aligned}R_n(a^{-1}q^{-m};a,-b,q^{\frac12},-q^{\frac12}\mid q)&=R_m((ab)^{-\frac12}q^{-n};(ab)^{\frac12},-(ab)^{\frac12},(qab^{-1})^{\frac12},-(qab^{-1})^{\frac12}\mid q)\\&={}_4\phi_3\left(\begin{matrix}q^{-n},q^nab,q^{-m},q^ma^2\\ -ab,q^{\frac12}a,-q^{\frac12}a\end{matrix};q,q\right).\end{aligned}$$

Continuous q^2*-Laguerre* $\longleftrightarrow$ *symmetric big* q*-Jacobi.*

$$R_n(z;a,q^{\frac12},-q^{\frac12},0\mid q)\;\longleftrightarrow\;P_m(y;q^{-\frac12}a,q^{-\frac12}a,-q^{-\frac12}a;q), \tag{2.38}$$
$$\begin{aligned}R_n(a^{-1}q^{-m};a,q^{\frac12},-q^{\frac12},0\mid q)&=P_m(a^{-1}q^{-n-\frac12};q^{-\frac12}a,q^{-\frac12}a,1,1;q)\\&={}_3\phi_2\left(\begin{matrix}q^{-n},q^{-m},q^ma^2\\ q^{\frac12}a,-q^{\frac12}a\end{matrix};q,q\right).\end{aligned}$$

Continuous q^2*-ultraspherical self-dual.*

$$R_n(z;a,-a,q^{\frac12},-q^{\frac12}\mid q)\;\longleftrightarrow\;R_m(w;a,-a,q^{\frac12},-q^{\frac12}\mid q), \tag{2.39}$$
$$\begin{aligned}R_n(a^{-1}q^{-m};a,-a,q^{\frac12},-q^{\frac12}\mid q)&=R_m(a^{-1}q^{-n};a,-a,q^{\frac12},-q^{\frac12}\mid q)\\&={}_4\phi_3\left(\begin{matrix}q^{-n},q^na^2,q^{-m},q^ma^2\\ -a^2,q^{\frac12}a,-q^{\frac12}a\end{matrix};q,q\right).\end{aligned}$$

3. Verde-Star's description of the q-Askey scheme

Monic polynomials u_n in the q-Askey scheme can be described by the data [9], [1, (3.1)–(3.4)]:

$$u_n(x)=\sum_{k=0}^n c_{n,k}\,v_k(x),\quad v_k(x)=\prod_{j=0}^{k-1}(x-x_j),\quad c_{n,k}=\prod_{j=k}^{n-1}\frac{g_{j+1}}{h_n-h_j}, \tag{3.1}$$
$$x_k=b_1q^k+b_2q^{-k},\quad h_k=a_1q^k+a_2q^{-k},\quad g_k=d_3q^{2k}+d_1q^k+d_0+d_2q^{-k}+d_4q^{-2k}, \tag{3.2}$$
$$\sum_{i=0}^4 d_i=0,\quad d_3=q^{-1}a_1b_1,\quad d_4=qa_2b_2, \tag{3.3}$$
$$a_2\ne a_1q^{\mathbb{Z}_{>0}},\quad \text{in particular,}\quad a_1\text{ or }a_2\ne 0,\qquad d_i\ne 0\text{ for some }i. \tag{3.4}$$

Define a linear operator L on the space of polynomials by

$$Lv_0 = 0, \qquad Lv_n = h_n v_n + g_n v_{n-1}, \quad n > 0, \tag{3.5}$$

or equivalently, in view of (3.1), by

$$Lu_n = h_n u_n, \quad n \geq 0. \tag{3.6}$$

For specific u_n in the q-Askey scheme the eigenvalue equation (3.5), (3.6) can be rewritten as a second order q-difference equation (the generalized Bochner property [5]).

In [1, (3.2)] x_k had an additional term b_0 and h_k had an additional term a_0, but without loss of generality these can be omitted, because the term b_0 only leads to a translation of the x-variable and the term a_0 only leads to adding a constant to the operator L in (3.5).

The polynomials u_n are determined by the 9 parameters $a_1, a_2, b_1, b_2, d_0, d_1, d_2, d_3, d_4$ under the 3 constraints (3.3). There are two other operations on these parameters which do not lead to essential changes. Simultaneous multiplication of a_1, a_2 and d_0, d_1, d_2, d_3, d_4 by $\mu \neq 0$ only leads to multiplication of L by μ. Simultaneous multiplication of b_1, b_2 and d_0, d_1, d_2, d_3, d_4 by $\rho \neq 0$ leads to replacing $u_n(x)$ by $\rho^n u_n(\rho^{-1}x)$. So essentially there are only 4 parameters, just as the Askey–Wilson polynomials (corresponding to the generic case of these parameters) have 4 parameters.

As observed in [1, §3], there are two further remarkable operations which can be performed on the 11 parameters:

$q \leftrightarrow q^{-1}$ *exchange*: $a_1 \leftrightarrow a_2,\ b_1 \leftrightarrow b_2,\ d_1 \leftrightarrow d_2,\ d_3 \leftrightarrow d_4$.

$x \leftrightarrow h$ *duality*: $a_1 \leftrightarrow b_1,\ a_2 \leftrightarrow b_2$; assume also that $b_2 \neq b_1 q^{\mathbb{Z}_{>0}}$, in particular, b_1 or $b_2 \neq 0$. This relates u_n given by (3.1) to its *dual* $\widetilde{u}_n$ given by

$$\widetilde{u}_n(x) = \sum_{k=0}^{n} \widetilde{c}_{n,k}\, \widetilde{v}_k(x), \quad \widetilde{v}_k(x) = \prod_{j=0}^{k-1}(x - h_j), \quad \widetilde{c}_{n,k} = \prod_{j=k}^{n-1} \frac{g_{j+1}}{x_n - x_j}. \tag{3.7}$$

If we put

$$U_n(x) := \prod_{j=0}^{n-1} \frac{h_n - h_j}{g_{j+1}} \times u_n(x) \ = \sum_{k=0}^{n} \prod_{j=0}^{k-1} \frac{(h_n - h_j)(x - x_j)}{g_{j+1}}, \tag{3.8}$$

$$\widetilde{U}_m(y) := \prod_{j=0}^{m-1} \frac{x_m - x_j}{g_{j+1}} \times \widetilde{u}_m(y) = \sum_{k=0}^{m} \prod_{j=0}^{k-1} \frac{(x_m - x_j)(y - h_j)}{g_{j+1}}, \tag{3.9}$$

then

$$U_n(x_m) = \widetilde{U}_m(h_n). \tag{3.10}$$

If U_n or $\tilde{U}_m$ are written in a special case with (3.2) substituted then they will appear as q-hypergeometric series.

4. q-Zhedanov algebra with q-Verde-Star parameters

For $\{u_n\}$ a system of polynomials in the q-Askey scheme consider operators K_1, K_2 acting on the space of polynomials in one variable x, K_1 being the second order q-difference operator which has the u_n as eigenfunctions and K_2 the operator of multiplication by x. Then

$$\begin{aligned} &(q + q^{-1})K_2K_1K_2 - K_2^2K_1 - K_1K_2^2 = C_1K_1 + DK_2 + G_1, \\ &(q + q^{-1})K_1K_2K_1 - K_1^2K_2 - K_2K_1^2 = C_2K_2 + DK_1 + G_2, \end{aligned} \tag{4.1}$$

for certain constants C_1, C_2, D, G_1, G_2. This is a rewriting of [2, (3.2)] by eliminating there K_3 and by substituting $R = 1-\frac{1}{2}(q+q^{-1})$. From [2, (3.2)] there would have been additional terms $A_2K_2^2$ and $A_1K_1^2$ on the right-hand sides of the two equalities in (4.1), respectively. But these can be removed by adding suitable constants to K_1 and K_2. There is also a Casimir operator Q, commuting with K_1 and K_2 and given by [2, (3.4)]:

$$\begin{aligned} Q = &-\frac{1}{2}(q+q^{-1})(K_1K_2^2K_1 + K_2K_1^2K_2) + K_1K_2K_1K_2 + K_2K_1K_2K_1 \\ &+ \frac{1}{2}(q+q^{-1})(C_1K_1^2 + C_2K_2^2) + D(K_1K_2 + K_2K_1) \\ &+ \frac{1}{2}(2+q+q^{-1})(G_1K_1 + G_2K_2), \end{aligned} \tag{4.2}$$

Then, for a certain constant ω,

$$Q = \omega. \tag{4.3}$$

Note the *duality* symmetry for (4.1), (4.2):

$$K_1 \leftrightarrow K_2, \quad C_1 \leftrightarrow C_2, \quad G_1 \leftrightarrow G_2. \tag{4.4}$$

The operators K_1 and K_2 act on the Newton polynomials v_n (see (3.1) and (3.5)) by

$$K_1 v_n = h_n v_n + g_n v_{n-1}, \qquad K_2 v_n = v_{n+1} + x_n v_n.$$

So by the correspondence $f = \{f_n\}_{n=0}^\infty \longleftrightarrow \sum_n f_n v_n(x)$ these operators also act on sequences f by

$$(K_1 f)_n = h_n f_n + g_{n+1} f_{n+1}, \qquad (K_2 f)_n = x_n f_n + f_{n-1}. \tag{4.5}$$

Substitution of (4.5) on the left-hand sides of (4.1) leads to the right-hand sides with the constants explicitly expressed in terms of the parameters $a_1, a_2, b_1, b_2, d_0, d_1, d_2$. We get

$$\begin{aligned} &C_1 = (q-q^{-1})^2 b_1 b_2, \qquad C_2 = (q-q^{-1})^2 a_1 a_2, \\ &D \;= -(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2 \big((q^{-\frac{1}{2}}a_1 - q^{\frac{1}{2}}a_2)(q^{-\frac{1}{2}}b_1 - q^{\frac{1}{2}}b_2) + d_1 + d_2\big), \\ &G_1 = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})(q-q^{-1})(q^{-\frac{1}{2}}b_1 d_2 + q^{\frac{1}{2}} b_2 d_1), \\ &G_2 = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})(q-q^{-1})(q^{-\frac{1}{2}}a_1 d_2 + q^{\frac{1}{2}} a_2 d_1), \end{aligned} \tag{4.6}$$

We can also express ω in (4.3) in terms of these parameters:

$$\begin{aligned} \omega = (q+q^{-1}-2)\Big(&\big(q^{-1}a_1^2 + (q+q^{-1})a_1a_2 + qa_2^2\big)\big(q^{-1}b_1^2 + (q+q^{-1})b_1b_2 + qb_2^2\big) \\ &+ \big(d_1^2 + d_2^2 - (q+q^{-1})d_1d_2\big) + (q+q^{-1})(d_1+d_2)(a_1b_2 + a_2b_1) \\ &+ 2(d_1+d_2)(q^{-1}a_1b_1 + qa_2b_2)\Big). \end{aligned} \tag{4.7}$$

Since multiplication of K_1 by a nonzero constant μ and multiplication of K_2 by a nonzero constant ρ does not essentially change the algebra generated by K_1 and K_2, it follows from (4.1), (4.2) and (4.3) that this algebra is essentially determined by

$$\{\rho^2 C_1,\ \mu^2 C_2,\ \rho\mu D,\ \rho^2\mu\, G_1,\ \rho\mu^2\, G_2,\ \rho^2\mu^2\omega\} \tag{4.8}$$

with μ, ρ arbitrarily nonzero. By (4.6), (4.7) this precisely matches with the allowed multiplication by constants of the q-Verde-Star parameters in Section 3. So again there are essentially four parameters coming from the q-Zhedanov coefficients, just as the Askey–Wilson polynomials have four parameters.

The constants C_1, C_2 and D are invariant under $q \leftrightarrow q^{-1}$, but not so G_1, G_2 and ω. The property for G_1 or G_2 to be zero for all q is clearly invariant under $q \leftrightarrow q^{-1}$. If $q \leftrightarrow q^{-1}$ is combined with $a_1 \leftrightarrow a_2,\ b_1 \leftrightarrow b_2,\ d_1 \leftrightarrow d_2,\ d_3 \leftrightarrow d_4$, then all constants in (4.6), (4.7) remain unchanged.

By (4.6), (4.7) the duality symmetry $K_1 \leftrightarrow K_2$, $C_1 \leftrightarrow C_2$, $G_1 \leftrightarrow G_2$ in (4.4) is compatible with the $x \leftrightarrow h$ duality $a_1 \leftrightarrow b_1$, $a_2 \leftrightarrow b_2$ in Section 3.

In the following subsections we consider some special cases of the q-Zhedanov algebra in connection with the q-Verde-Star parameters: the general case associated with the Askey–Wilson polynomials, followed by a discussion of the $(c,d) \to (qd^{-1}, qc^{-1})$ symmetry, and finally the dual cases associated with the continuous dual q-Hahn and the big q-Jacobi polynomials.

4.1. The Askey–Wilson algebra

The generic case of the q-Zhedanov algebra is the algebra associated with the Askey–Wilson polynomials. Put $U_n(z+z^{-1}) = R_n(z; a, b, c, d \mid q)$. Then this can be written in the form (3.9) with

$$\begin{aligned} &x_k = aq^k + a^{-1}q^{-k}, \qquad h_k = q^{-k} + abcdq^{k-1},\\ &g_k = q^{-2k+1}a^{-1}(1-abq^{k-1})(1-acq^{k-1})(1-adq^{k-1})(1-q^k), \end{aligned} \tag{4.9}$$

so

$$\begin{aligned} &b_1 = a, \quad b_2 = a^{-1} \quad a_1 = q^{-1}abcd, \quad a_2 = 1,\\ &d_1 = -q^{-2}a(abcd + q(bc+bd+cd)), \quad d_2 = -(b+c+d+qa^{-1}). \end{aligned} \tag{4.10}$$

Let e_1, e_2, e_3, e_4 be the elementary symmetric polynomials in a, b, c, d:

$$\begin{aligned} &e_1 = a+b+c+d, \qquad e_2 = ab+ac+bc+ad+bd+cd,\\ &\qquad e_3 = abc+abd+acd+bcd, \qquad e_4 = abcd. \end{aligned} \tag{4.11}$$

Then, by (4.6) and (4.7),

$$\begin{aligned} &C_1 = (q-q^{-1})^2, \quad C_2 = q^{-1}(q-q^{-1})^2 e_4, \quad D = (1-q^{-1})^2(e_3+qe_1),\\ &G_1 = -q^{-3}(1-q)^2(1+q)(e_4+qe_2+q^2), \quad G_2 = -q^{-3}(1-q)^2(1+q)(e_1e_4+qe_3), \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} \omega = (q+q^{-1}-2)\Big(&e_1^2 + q^{-2}e_3^2 - (1+q^{-2})e_1e_3 - q^{-3}(1+q)^2e_2e_4\\ &+ q^{-3}(1-q^2)^2e_4 - q^{-1}(1+q)^2e_2\Big). \end{aligned} \tag{4.13}$$

In [6, (16)] the expressions (4.12) for C_1, C_2, D, G_1, G_2 were also obtained, but starting from the second order q-difference operator L given in (2.5), by which K_1 is represented in the polynomial representation [6, (17)] of the Askey–Wilson algebra. Note that indeed h_n as given by (4.9) equals the eigenvalue of the eigenfunction $R_n(z)$ in (2.4).

4.2. A further symmetry of the Askey–Wilson algebra

As observed in Section 2, nothing essentially changes for the q-Verde-Star parameters if L, h_n and g_n are simultaneously multiplied by a nonzero constant μ, or equivalently, if

a_1, a_2, g_1, g_2 are multiplied by μ. In the context of the q-Zhedanov algebra this means, by (4.8), that D, G_1 are multiplied by μ and C_2, G_2, Q, ω are multiplied by μ^2.

Now, in the case of the Askey–Wilson algebra, take $\mu = (q/(abcd))^{\frac{1}{2}} = \tilde{a}^{-1}$ (with $\tilde{a}$ given by (2.31). Then it follows for the q-Zhedanov algebra generated by K_1, K_2 with $K_1 = \tilde{a}^{-1}L$ that, by (4.12) and (4.13), $C_1, \tilde{a}^{-2}C_2, \tilde{a}^{-1}D, \tilde{a}^{-1}G_1, \tilde{a}^{-2}G_2$ and $\tilde{a}^{-2}\omega$ are not only invariant under permutations of a, b, c, d, but also invariant if two parameters e, f out of a, b, c, d are sent to qe^{-1}, qf^{-1}, respectively (the symmetries already discussed in Section 2.3). Such symmetries for the Askey–Wilson DAHA were pointed out to me by Marta Mazzocco. As was observed in Section 2.3, these symmetries generate a group which is isomorphic to the Weyl group of the root system D_4. For some version of the Askey–Wilson algebra a symmetry under the same group was earlier observed in [30, Section 4].

These further symmetries are relevant for the q-Zhedanov scheme (Fig. 1 in the next section). If in (4.12) some of the C_1, C_2, D, G_1, G_2 vanish under certain constraints on a, b, c, d then they will also vanish under transformed constraints by application of a symmetry. For instance, the constraints $c = -a$, $d = -b$ give symmetric Askey–Wilson (2.3) and cause D and G_2 to vanish. Then also the constraints $c = -qa^{-1}$, $d = -qb^{-1}$ make D and G_2 vanish. Furthermore, these symmetries induce symmetries on limit cases of the Askey–Wilson algebra.

The passage to the dual also becomes much nicer in this slightly adapted Askey–Wilson algebra. Indeed note that by (4.9), (4.10), (4.12)

$$\begin{aligned}
&b_1 = a, \quad b_2 = a^{-1}, \quad \tilde{a}^{-1}a_1 = \tilde{a}, \quad \tilde{a}^{-1}a_2 = \tilde{a}^{-1},\\
&x_k = aq^k + a^{-1}q^{-k}, \quad \tilde{a}^{-1}h_k = \tilde{a}q^k + \tilde{a}^{-1}q^{-k},\\
&C_1 = \tilde{a}^{-2}C_2 = (q-q^{-1})^2, \quad a^{-1}\tilde{D} = \tilde{a}^{-1}D, \quad a^{-1}\tilde{G}_1 = \tilde{a}^{-2}G_2, \quad a^{-2}\tilde{\omega} = \tilde{a}^{-2}\omega,
\end{aligned}$$

where $\tilde{D}$ is obtained from D in (4.12) by replacing a, b, c, d by $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$, respectively, and similarly for $\tilde{G}_1$ and $\tilde{\omega}$. Thus $\tilde{a}^{-1}D, \tilde{a}^{-1}G_1, \tilde{a}^{-2}G_2$ and $\tilde{a}^{-2}\omega$ are also invariant under permutations of $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$, which we already knew because of Proposition 2.1.

4.3. The dual algebras for continuous dual q-Hahn and big q-Jacobi

For obtaining the continuous dual q-Hahn algebra put $d = 0$ in (4.9)–(4.13). Then $e_4 = 0$ and e_1, e_2, e_3 can be considered as elementary symmetric polynomials in a, b, c. We obtain

$$x_k = aq^k + a^{-1}q^{-k}, \quad h_k = q^{-k}, \quad g_k = q^{-2k+1}a^{-1}(1-abq^{k-1})(1-acq^{k-1})(1-q^k), \tag{4.14}$$

$$b_1 = a, \quad b_2 = a^{-1} \quad a_1 = 0, \quad a_2 = 1, \quad d_1 = -q^{-1}abc, \quad d_2 = -(b+c+qa^{-1}), \tag{4.15}$$

$$\begin{aligned}
&C_1 = (q-q^{-1})^2, \quad C_2 = 0, \quad D = (1-q^{-1})^2(e_3+qe_1),\\
&G_1 = -(1-q^{-1})^2(1+q)(e_2+q), \quad G_2 = -(1-q^{-1})^2(1+q)e_3.
\end{aligned} \tag{4.16}$$

See also [6, (50)]. Note that $U_n(z+z^{-1}) = R_n(z; a, b, c, 0 \mid q)$ can be written in the form (3.9) with x_k, h_k, g_k given by (4.14), and that $h_n = q^{-n}$ is the eigenvalue of L for $d = 0$ in (2.4).

Just as for the Askey–Wilson algebra nothing essentially changes if we multiply h_n, g_n by μ, and a_1, a_2, g_1, g_2 by μ, and D, G_1 by μ, and C_2, G_2 by μ^2 (here we have omitted ω). If we now take $\mu = (q/(abc))^{\frac{1}{2}}$ then the resulting $C_1, \mu^2C_2, \mu D, \mu G_1, \mu^2G_2$ are not only invariant under permutations of a, b, c, but also invariant if two parameters e, f out of a, b, c, are sent to qe^{-1}, qf^{-1}, respectively.

With x_k, h_k, g_k given by (4.14) we get from (3.9) that $\tilde{U}_m(y) = P_m(y; q^{-1}ab, ab^{-1}, q^{-1}ac; q)$, a big q-Jacobi polynomial as we saw it already in the duality (2.35).

Below we give data associated with the big q-Jacobi polynomial $U_n(x) = P_n(x; a, b, c; q)$.

$$x_k = q^{-k}, \quad h_k = q^{-k} + abq^{k+1}, \quad g_k = q^{1-2k}(1-aq^k)(1-cq^k)(1-q^k), \tag{4.17}$$

$$b_1 = 0, \quad b_2 = 1, \quad a_1 = qab, \quad a_2 = 1, \quad d_1 = -qac, \quad d_2 = -q(a+c+1), \tag{4.18}$$

$$\begin{aligned} &C_1 = 0, \quad C_2 = q^{-1}ab(1-q^2)^2, \quad D = (1-q)^2(ab+ac+a+c), \\ &G_1 = -(1-q)^2(1+q)ac, \quad G_2 = -(1-q)^2(1+q)a(ab+bc+b+c). \end{aligned} \tag{4.19}$$

If in (4.17)–(4.19) we multiply $h_k, g_k, a_1, a_2, d_1, d_2, D, G_1$ by $(qab)^{-\frac{1}{2}}$ and C_2, G_2 by $(qab)^{-1}$, and next replace a, b, c by $q^{-1}ab, ab^{-1}, q^{-1}ac$ then we obtain the data which are dual to the data (4.14)–(4.16).

If in (4.17)–(4.19) we multiply $h_k, g_k, a_1, a_2, d_1, d_2, D, G_1$ by $(qab)^{-\frac{1}{2}}$ and C_2, G_2 by $(qab)^{-1}$, and if we multiply $x_k, g_k, b_1, b_2, d_1, d_2, D, G_2$ by $(ab^{-1}c^2)^{-\frac{1}{4}}$ and C_1, G_1 by $(ab^{-1}c^2)^{-\frac{1}{2}}$ then $U_n(x) = P_n\big((ab^{-1}c^2)^{\frac{1}{4}}x; a, b, c; q\big)$. Moreover the big q-Jacobi algebra then has the invariances corresponding to the ones we observed for the continuous dual Hahn algebra. We see that C_1, C_2, D, G_1, G_2 are invariant under the transformations $(a, b, c) \to (c, abc^{-1}, a)$, $(a, b, c) \to (abc^{-1}, c, b)$ and $(a, b, c) \to (a, b^{-1}, b^{-1}c)$. This nicely corresponds with equalities (2.20), (2.21) and with the symmetry observed in connection with (2.35).

5. The q-Zhedanov scheme

It will turn out that vanishing of some of the coefficients C_1, C_2, D, G_1, G_2 in the q-Zhedanov algebra with relations (4.1) is a characterizing property of corresponding polynomials in the q-Askey scheme. An arrow then will indicate that one more of the five constants becomes zero. We arrange the five coefficients in an array

$$\begin{array}{ccc} C_1 & & C_2 \\ & D & \\ G_1 & & G_2 \end{array} \tag{5.1}$$

By (4.4) we can pass to the dual q-Zhedanov algebra by reflection of the array with respect to the vertical axis.

We will replace coefficients in this array by symbols $\bullet$ or $\circ$, where $\bullet$ denotes any value (including zero) and $\circ$ denotes zero. From the explicit data for the q-Verde-Star parameters of the families in the q-Askey scheme as given in [1, Appendix A] we can obtain the corresponding vanishing pattern for the array (5.1) by means of (4.6). Then connect the arrays by arrows such that in the direction of the arrow at least one $\bullet$ is turned into $\circ$. The resulting graph is in Fig. 1.

Let us number the rows in the scheme from top to bottom by 1 to 6. In each row list the successive arrays from left to right by a, b, ...

Note the following dualities in Fig. 1:

1a, **3d**, **4b**, **5c**, **6a** are self-dual.

2a $\leftrightarrow$ **2b**, **3a** $\leftrightarrow$ **3b**, **3c** $\leftrightarrow$ **3e**, **4a** $\leftrightarrow$ **4e**, **4c** $\leftrightarrow$ **4g**, **4d** $\leftrightarrow$ **4f** are dual pairs.

The duals of **5a** and **5b** are not included in the scheme, but will be discussed in Section 6.

The arrays in Fig. 1 correspond with families in the q-Askey scheme as given in the list below. Usually we mention only one family for an array, but also subfamilies still corresponding to that array. There may be more families for the same array, by $q \leftrightarrow q^{-1}$ exchange, and there may be discrete families. If one wants to add these, one can use [3, Ch. 14]. We do not bother at all about orthogonality properties. The only things that matter are that the polynomials

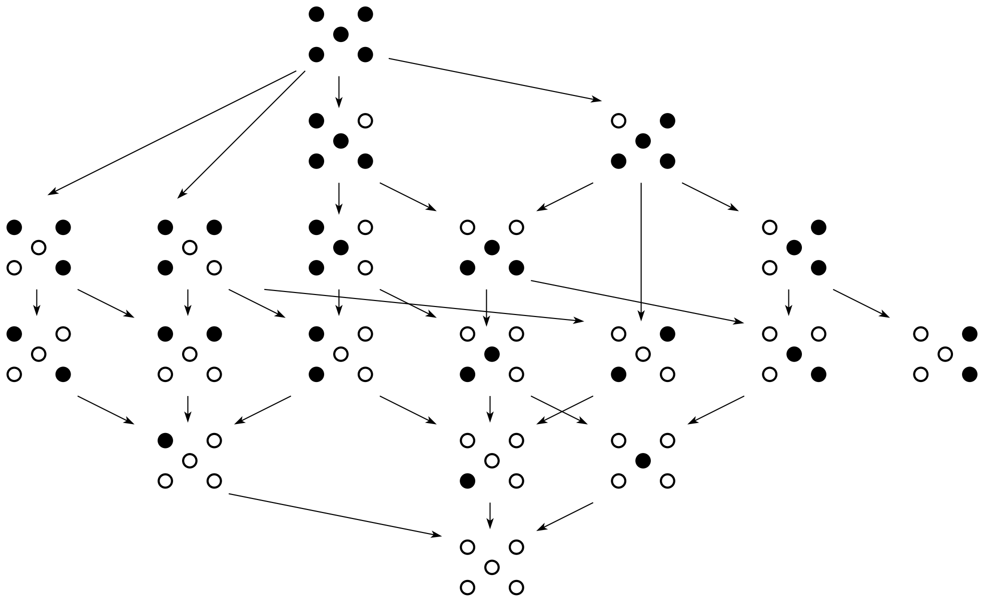


Fig. 1. The q-Zhedanov scheme. In referring to this rows are numbered top to bottom, and in each row the elements are labeled by the letters a, b, c, ... from left to right.

are eigenfunctions of a second order q-difference operator and satisfy a three-term recurrence relation.

As much as possible formulas for the occurring families in the list will be given in terms of Askey–Wilson $R_n(z; a, b, c, d \mid q)$, $x = z + z^{-1}$ (see (2.1)) or big q-Jacobi $P_n(x; a, b, c; q)$ (see (2.18)). Otherwise a formula is given as a limit case of R_n or P_n or as some ${}_r\phi_s$ q-hypergeometric function. Everything is up to constant factors.

1a. Askey–Wilson $R_n(z; a, b, c, d \mid q)$, see (2.1).

2a. continuous dual q-Hahn $R_n(z; a, b, c, 0 \mid q)$, see (2.7).

2b. big q-Jacobi $P_n(x; a, b, c; q)$, see (2.18).

3a. continuous q^2-Jacobi $R_n(z; a, b, q^{\frac{1}{2}}, -q^{\frac{1}{2}} \mid q)$, see (2.14).

3b. symmetric Askey–Wilson $R_n(z; a, b, -a, -b \mid q)$ (see (2.3)) or $R_n(z; a, b, -qa^{-1}, -qb^{-1} \mid q)$.

3c. Al-Salam–Chihara $R_n(z; a, b, 0, 0 \mid q)$ (see (2.8)) with subfamily continuous big q-Hermite $R_n(z; a, 0, 0, 0 \mid q)$, see [3, (14.8.1)].

3d. big q-Laguerre $P_n(x; a, 0, c; q)$, see (2.29).

3e. little q-Jacobi $P_n(x; a, b, 0; q)$ (see (2.26)) with limit family q-Bessel $\lim_{b\to 0} P_n(-ab^{-1}x; -q^{-1}ab^{-1}, b, 0; q) = {}_2\phi_1(q^{-n}, -q^n a; 0; q, qx)$ (see [3, (14.22.1)]).

For $a=q^{-N-1}$, $b=-p$ little q-Jacobi becomes q-Krawtchouk $P_n(x;q^{-N-1},-p,0;q)$ (see [3, (14.15.1)]).

4a. continuous q^2-Laguerre $R_n(z;q^{\frac12},-q^{\frac12},a,0\mid q)$, see (2.15).

4b. continuous q^2-ultraspherical $R_n(z;q^{\frac12},-q^{\frac12},a,-a\mid q)$ (see (2.16)) or $R_n(z;q^{\frac12},-q^{\frac12},a,-qa^{-1}\mid q)$.

4c. symmetric Al-Salam–Chihara $R_n(z;a,-a,0,0\mid q)$ (see (2.9)) with limit family continuous q-Hermite $\lim_{a\to 0}a^{-n}R_n(z;a,-a,0,0\mid q)=H_n\big(\frac12(z+z^{-1})\mid q\big)$ (see (2.13)).

4d. Al-Salam–Carlitz I
$\lim_{a\to 0}a^{-n}P_n(qax;a,0,ac;q)=(-c)^n q^{\frac12 n(n+1)}{}_2\phi_1(q^{-n},x^{-1};0;q,qc^{-1}x)$ (see [3, (14.24.1)]).

4e. symmetric big q-Jacobi $P_n(x;a,a,-a;q)$ (see (2.22)) or $P_n(x;-1,-1,c;q)$.

4f. q-Laguerre $\lim_{a\to-\infty}P_n(-x;a,b,0;q)=(qb;q)_n\,{}_1\phi_1(q^{-n};qb;q,-q^{n+1}bx)$ (see (2.27), (2.28), [3, (14.21.1), (14.12.13)]) with limit family Stieltjes–Wigert $\lim_{b\to 0}\lim_{a\to-\infty}P_n(-b^{-1}x;a,b,0;q)={}_1\phi_1(q^{-n};0;q,-q^{n+1}x)$ (see [3, (14.27.1)]). q-Charlier is q-Laguerre with x,b replaced by $-x,-b^{-1}$ (see [3, (14.23.1)]). This case also contains little q-Laguerre, which can be essentially identified with q^{-1}-Laguerre, see [3, p. 521].

4 g. special little q-Jacobi $P_n(a,-1,0;q)$. For $a=q^{-N-1}$ this is q-Krawtchouk with $p=q^{-N}$, see [3, (14.15.1)].

5a. continuous q^2-Hermite $R_n(z;q^{\frac12},-q^{\frac12},0,0\mid q)$, see (2.17).

5b. discrete q-Hermite I
$\lim_{a\to 0}a^{-n}P_n(qax;a,0,-a;q)=q^{\frac12 n(n+1)}{}_2\phi_1(q^{-n},x^{-1};0;q,-qx)$ (see [3, (14.28.1)]).

5c. $x^n(x^{-1};q)_n$

6a. x^n

Remarks. **1.** For obtaining this list we started with the q-Verde-Star data given in [1, Appendix A]. From these data, with a_0 and b_0 being put to zero, the q-Zhedanov coefficients C_1,C_2,D,G_1,G_2 can be computed by (4.6), and it can be read off which of these five coefficients vanish. Moreover, for a given family, it can be seen for which constraints on the parameters a further coefficient will vanish.

2. It is interesting to compare the scheme in Fig. 1 with the q-Verde-Star scheme [1, Figure 1]:

- First, the present scheme is more compact because it does not discern between q and q^{-1}, and also because it does not bother about the Newton polynomials v_k in which u_n is expanded, so each family occurs in at most one place in the scheme.
- Furthermore, some separate families in [1, Figure 1] here merge as subfamilies or limit families with other families (continuous big q-Hermite, q-Bessel and Stieltjes–Wigert).
- On the other hand, the present scheme is richer because many subfamilies here get an independent status which they did not have in the q-Verde-Star scheme. We mention symmetric Askey–Wilson, symmetric Al-Salam–Chihara, symmetric big q-Jacobi, continuous q^2-Jacobi, continuous q^2-Laguerre, continuous q^2-ultraspherical, discrete q-Hermite and q-Krawtchouk with $p=q^{-N}$.

- Continuous q-Hermite could not be included in the q-Verde-Star scheme, because it cannot be written in the form (3.1), but a q-Zhedanov algebra can be associated with it. Therefore, in the present scheme it occurs, but only as a limit family of symmetric Al-Salam–Chihara. However, continuous q^2-Hermite, obtained by the quadratic transformation (2.17), has its own place in our scheme. We also see from (2.17) that continuous q^2-Hermite will live in the q-Verde-Star scheme as a subfamily of Askey–Wilson, but not with a separate place in that scheme. To some extent this overcomes the defect that continuous q-Hermite could not be handled by Verde-Star [9].

3. The cases **5c** and **6** are degenerate. They are eigenfunctions of a first order q-difference operator and they satisfy a two-term recurrence relation.

4. It turns out that with each of the 32 possible arrays a family of polynomials can be associated, but we have not included all of these in the scheme, because some families did not seem to be of sufficient interest. The following cases were missing in the scheme.

$\begin{smallmatrix}\bullet&&\bullet\\&\circ&\\\bullet&&\bullet\end{smallmatrix}$ Askey–Wilson $R_n(z;a,b,c,d\mid q)$ with $abc+abd+acd+bcd+q(a+b+c+d)=0$.

$\begin{smallmatrix}\bullet&&\bullet\\&\bullet&\\\circ&&\bullet\end{smallmatrix}$ idem with $abcd+q(ab+ac+ad+bc+bd+cd)+q^2=0$.

$\begin{smallmatrix}\bullet&&\bullet\\&\bullet&\\\bullet&&\circ\end{smallmatrix}$ idem with $abcd(a+b+c+d)+q(abc+abd+acd+bcd)=0$.

$\begin{smallmatrix}\bullet&&\bullet\\&\bullet&\\\circ&&\circ\end{smallmatrix}$ idem with the previous two constraints.

$\begin{smallmatrix}\bullet&&\circ\\&\circ&\\\bullet&&\bullet\end{smallmatrix}$ continuous dual q-Hahn $R_n(z;a,b,c,0\mid q)$ with $abc+q(a+b+c)=0$.

$\begin{smallmatrix}\bullet&&\circ\\&\bullet&\\\circ&&\bullet\end{smallmatrix}$ idem with $ab+ac+bc+q=0$.

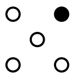
$\begin{smallmatrix}\circ&&\bullet\\&\circ&\\\bullet&&\bullet\end{smallmatrix}$ big q-Jacobi $P_n(x;a,b,c;q)$ with $a+c+a(b+c)=0$.

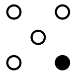
$\begin{smallmatrix}\circ&&\bullet\\&\bullet&\\\bullet&&\circ\end{smallmatrix}$ idem with $b+c+b(a+c)=0$.

$\begin{smallmatrix}\bullet&&\circ\\&\bullet&\\\circ&&\circ\end{smallmatrix}$ Al-Salam–Chihara $R_n(z;a,b,0,0\mid q)$ with $ab+q=0$.

$\begin{smallmatrix}\circ&&\bullet\\&\bullet&\\\circ&&\circ\end{smallmatrix}$ special little q-Jacobi $P_n(x;-1,b,0;q)$.

$\begin{smallmatrix}\circ&&\circ\\&\circ&\\\bullet&&\bullet\end{smallmatrix}$ big q-Laguerre $P_n(x;a,0,c;q)$ with $a+c+ac=0$,

 dual continuous q^2-Hermite ${}_3\phi_2\left({q^{-n},q^{n+1},x \atop -q,0};q,q\right)$ (see Section 6).

 special q-Charlier ${}_2\phi_1\left({q^{-n},-x \atop 0};q,-q^{n+1}\right)$ (see Section 6).

5. In Sections 4.2, 4.3 we already observed symmetries for q-Zhedanov algebras which are not satisfied by the corresponding Askey–Wilson, continuous dual q-Hahn and big q-Jacobi polynomials. Such symmetries will also be present in algebras associated with families in the third and fourth row of Fig. 1.

6. Duals of continuous q^2-Hermite and discrete q-Hermite I

In this section we will phrase the various dualities in the notation of (3.8)–(3.10).

Continuous q^2-Laguerre $\longleftrightarrow$ special big q-Jacobi. For obtaining the dual of continuous q-Hermite we first consider a suitable dual of continuous q^2-Laguerre and then specialize the parameter.

Put $x_k = q^{k+\frac12} + q^{-k-\frac12}$, $h_k = q^{-k}$, $g_k = (q^{\frac12} - aq^k)(q^{-2k} - 1)$. Then by (4.6) we have $C_1 = (q-q^{-1})^2$, $C_2 = D = G_1 = 0$, $G_2 = (1+q^{-1})(1-q)^2 a$. Also (3.8)–(3.10) take the form

$$\begin{aligned} &U_n(z+z^{-1}) = {}_3\phi_2\left({q^{-n},\, q^{\frac12}z,\, q^{\frac12}z^{-1} \atop q^{\frac12}a,\, -q};q,q\right),\\ &\tilde U_m(y) = {}_3\phi_2\left({y,\, q^{-m},\, q^{m+1} \atop q^{\frac12}a,\, -q};q,q\right), \qquad U_n(q^{m+\frac12}+q^{-m-\frac12}) = \tilde U_m(q^{-n}). \end{aligned} \tag{6.1}$$

By (2.1), (2.15) and (2.18) we can identify the dual polynomials $U_n(z+z^{-1})$ and $\tilde U_m(y)$ as continuous q^2-Laguerre and special big q-Jacobi:

$$U_n(z+z^{-1}) = R_n(z; q^{\frac12}, -q^{\frac12}, a, 0 \mid q), \quad \tilde U_m(y) = P_m(y; -1, -1, q^{-\frac12}a; q). \tag{6.2}$$

It is precisely this dual pair which allows the specialization $a = 0$ which we need. The $\tilde U_m$ belong to the same array **4e** in Fig. 1 as symmetric big q-Jacobi. In fact, with the choice $x_k = aq^k + a^{-1}q^{-k}$, $h_k = q^{-k}$, $g_k = q^{1-2k}a^{-1}(1-q^k)(1-a^2q^{2k-1})$ we would have obtained the same values for C_1, C_2, D, G_1, G_2, but $U_n(z+z^{-1}) = R_n(z; a, q^{\frac12}, -q^{\frac12}, 0 \mid q)$ and for $\tilde U_m(y)$ the symmetric big q-Jacobi polynomial $P_m(y; q^{-\frac12}a, q^{-\frac12}a, -q^{-\frac12}a; q)$.

It is also remarkable that for both specializations of big q-Jacobi belonging to array **4e** the specialization of the three-term recurrence relation (2.25) gives (for $n = 1, 2, \ldots$) $xP_n(x)$ as a linear combination of $P_{n+1}(x)$ and $P_{n-1}(x)$ without a term for $P_n(x)$. Still only for symmetric big q-Jacobi $P_n(x; a, a, -a; q)$ we have $P_n(-x) = (-1)^n P_n(x)$, but for $P_n(x; -1, -1, c; q)$ the starting value $P_1(x)$ already lacks this symmetry.

Continuous q^2-Hermite $\longleftrightarrow$ very special big q-Jacobi. Just put $a = 0$ in the above case.

Note that the “dual continuous q^2-Hermite polynomials” $P_n(x) = P_n(x; -1, -1, 0; q)$ cannot be orthogonal with respect to a positive orthogonality measure. This is seen from the signs of the coefficients in the recurrence relation (a specialization of (2.25))

$$x\,P_n(x) = \frac{1}{1-q^{2n+1}}\,P_{n+1}(x) - \frac{q^{2n+1}}{1-q^{2n+1}}\,P_{n-1}(x), \quad n = 1,2,\ldots\,. \tag{6.3}$$

Al-Salam–Carlitz I $\longleftrightarrow$ q-Charlier. For obtaining the dual of discrete q-Hermite I we first consider the dual families of Al-Salam–Carlitz I and q-Charlier and then specialize the parameter.

Put $x_k = q^k$, $h_k = q^{-k}$, $g_k = a(q^{-k}-1)$. Then $C_1 = C_2 = 0$, $D = q^{-1}(1-q)^2(1-a)$, $G_1 = (1-q^{-1})^2(1+q)a$, $G_2 = 0$. Also (3.8)–(3.10) take the form

$$\begin{aligned} U_n(x) &= {}_2\phi_1\left(\begin{matrix} q^{-n}, x^{-1} \\ 0 \end{matrix}; q, -qa^{-1}x\right), \\ \widetilde{U}_m(y) &= {}_2\phi_1\left(\begin{matrix} y, q^{-m} \\ 0 \end{matrix}; q, -q^{m+1}a^{-1}\right), \qquad U_n(q^m) = \widetilde{U}_m(q^{-n}). \end{aligned} \tag{6.4}$$

Furthermore, by [3, (14.24.1), (14.23.1)], the expressions in terms of Al-Salam–Carlitz I and q-Charlier are:

$$U_n(x) = a^n q^{-\frac12 n(n-1)} U_n^{(-a)}(x;q), \qquad \widetilde{U}_m(y) = C_m(y;a;q). \tag{6.5}$$

Discrete q-Hermite I $\longleftrightarrow$ special q-Charlier. Just put $a = 1$ in the above case. Then D will also vanish and, by [3, (14.28.1), (14.23.1)], the expressions in terms of discrete q-Hermite I and q-Charlier are:

$$U_n(x) = q^{-\frac12 n(n-1)} h_n(x;q), \qquad \widetilde{U}_m(y) = C_m(y;1;q). \tag{6.6}$$

7. Charting the q-Askey scheme: conclusions and questions

From Sections 4 and 5 we can conclude that a system $\{p_n\}$, belonging to a family in the q-Askey scheme for some values of the parameters for that family, and considered up to dilation of the independent variable, gives rise to values of $C_1, C_2, D, G_1, G_2, \omega$ up to multiplication by certain powers of nonzero constants μ and ρ as in (4.8). In view of Sections 4.2, 4.3 and Remark 5 in Section 5 the system $\{p_n\}$ will not be uniquely determined by the equivalence class of $C_1, C_2, D, G_1, G_2, \omega$. If the p_n are Askey–Wilson polynomials then we may conjecture from Section 4.2 that its parameters $\pm(a,b,c,d)$ are determined by the equivalence class of $C_1, C_2, D, G_1, G_2, \omega$ up to permutations and up to sending any e, f from a,b,c,d to q/e, q/f, but even that is not yet clear.

Still it is of interest to let the equivalence classes (4.8) form a complex manifold. Here we impose that ω and at least one of the other five coordinates are nonzero. Then we can put ω and another nonvanishing coordinate equal to 1. This will fix μ and ρ, possibly up to sign. Then use the other four coordinates as local coordinates. There will still be point pairs which have to be identified.

Another question, more a matter of taste and of convenience than of mathematical rigor, is whether the usual version [3, p. 414] of the q-Askey scheme should be adapted in view of the q-Verde-Star scheme [1] and the q-Zhedanov scheme Fig. 1. I would say that an adapted scheme should not distinguish between q and q^{-1} variants of the same family. Neither should it distinguish between two cases of the same family with expansions in terms of different Newton polynomials. Interesting as these features of the q-Verde-Star scheme may be, they make the graphical display too large and too detailed. For the rest the adapted scheme should be the union of the q-Verde-Star and the q-Zhedanov scheme, since each of the schemes highlights some families which deserve special attention but are not in the other scheme. It would be convenient to have one scheme which starts with Askey–Wilson and another scheme which starts with q-Racah. In the lower rows these schemes would have several families in common.

Transcendental functions might also be associated with the q-Zhedanov scheme, as we will argue now. The q-Zhedanov algebra yielding the coefficients C_1, C_2, D, G_1, G_2 by (4.1) is generated by a second order q-difference operator K_1 and the operator K_2 which is multiplication by x. Polynomials associated with the algebra occur as eigenfunctions of K_1 for special eigenvalues. However, the algebra and its coefficients are independent of the choice of the eigenvalues. Other eigenfunctions of K_1 on another domain may yield generalized orthogonal systems leading to integral transforms. Notably, in addition to Askey–Wilson polynomials, there are Askey–Wilson functions, see Koelink and Stokman [12]. The same authors have in [11, Figure 1.2] a scheme, somewhat analogous to the q-Askey scheme, with limit cases of the Askey–Wilson functions. It would be interesting to match our scheme with a possibly extended version of the scheme in [11]. Passage to the transcendental case will also be needed for realizing some symmetries of the q-Zhedanov algebra. For instance, the symmetry $(a, b, c, d) \to (a, b, qd^{-1}, qc^{-1})$ of the Askey–Wilson algebra (see Section 4.2) is, by [11, (3.2)], visible for Askey–Wilson functions, while this symmetry is not present if we restrict ourselves to the polynomial case.

Schemes for nonsymmetric Askey–Wilson polynomials and their subfamilies, and for the corresponding DAHAs might be another follow-up topic. A Verde-Star type approach for nonsymmetric polynomials may be promising. However, in view of the cases already considered in [6], there does not seem to be a uniform approach for the DAHA as we do have by (4.1) for the q-Zhedanov algebras.

Acknowledgments

I thank Marta Mazzocco for communicating to me about the symmetry $(a, b, c, d) \to (a, b, qd^{-1}, qc^{-1})$ of the Askey–Wilson algebra. I also thank Raimundas Vidunas for pointing me to [19,21], and an anonymous referee for helpful comments.

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