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# An infinite-dimensional affine stochastic volatility model

Sonja Cox  | Sven Karbach  | Asma Khedher 

Korteweg-de Vries Institute for  
Mathematics, University of Amsterdam,  
Amsterdam, The Netherlands

## Correspondence

Sven Karbach, Korteweg-de Vries  
Institute for Mathematics, University of  
Amsterdam, Postbus 94248, NL-1090 GE  
Amsterdam, The Netherlands.  
Email: [sven@karbach.org](mailto:sven@karbach.org)

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## Abstract

We introduce a flexible and tractable infinite-dimensional stochastic volatility model. More specifically, we consider a Hilbert space valued Ornstein–Uhlenbeck-type process, whose instantaneous covariance is given by a pure-jump stochastic process taking values in the cone of positive self-adjoint Hilbert–Schmidt operators. The tractability of our model lies in the fact that the two processes involved are jointly *affine*, that is, we show that their characteristic function can be given explicitly in terms of the solutions to a set of generalized Riccati equations. The flexibility lies in the fact that we allow multiple modeling options for the instantaneous covariance process, including state-dependent jump intensity. Infinite dimensional volatility models arise, for example, when considering the dynamics of forward rate functions in the Heath–Jarrow–Morton–Musielà (HJMM) modeling framework using the Filipović space. In this setting, we discuss various examples: an infinite-dimensional version of the Barndorff–Nielsen–Shephard stochastic volatility model, as well as covariance processes with a state dependent intensity.

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**KEYWORDS**

forward price dynamics, Heath–Jarrow–Morton–Musielà framework, infinite-dimensional affine processes, Riccati equations, state-dependent jump intensity, stochastic volatility

**1 | INTRODUCTION**

In this paper, we propose a new class of *affine* stochastic volatility models  $(Y_t, X_t)_{t \geq 0}$ , where  $(Y_t)_{t \geq 0}$  takes values in a real separable Hilbert space  $(H, \langle \cdot, \cdot \rangle_H)$  and  $(X_t)_{t \geq 0}$  is a time-homogeneous *affine* Markov process taking values in  $\mathcal{H}^+ = \mathcal{L}_2^+(H)$ , the cone of positive self-adjoint Hilbert–Schmidt operators on  $H$ . The process  $X$  is taken from a class of affine processes introduced in Cox et al. (2020). The process  $(Y_t)_{t \geq 0}$  is modeled by the following stochastic differential equation:

$$dY_t = \mathcal{A}Y_t dt + X_t^{1/2} dW_t^Q, \quad t \geq 0, \quad Y_0 = y \in H, \tag{1}$$

where  $\mathcal{A} : \text{dom}(\mathcal{A}) \subseteq H \rightarrow H$  is a possibly unbounded operator with dense domain  $\text{dom}(\mathcal{A})$ , and  $(W_t^Q)_{t \geq 0}$  is a  $Q$ -Brownian motion independent of  $X$ , with  $Q$  a positive self-adjoint trace-class operator on  $H$ . Assuming that  $X$  is progressively measurable and using moment bounds on  $X$  established in Cox et al. (2020), the existence of a solution to Equation (1) is straightforward (see Lemma 2.8 below).

In Section 2.1, we show that under the assumption that the Markov process  $(X_t)_{t \geq 0}$  has càdlàg paths, it is a square-integrable semimartingale. This follows from the formulation of an associated martingale problem in terms of what we call a *weak generator* (see Definition 2.2) of the Markov process  $(X_t)_{t \geq 0}$  and yields the explicit representation of  $(X_t)_{t \geq 0}$  as

$$X_t = x + \int_0^t \left( b + B(X_s) + \int_{\mathcal{H}^+ \cap \{\|\xi\| > 1\}} \xi M(X_s, d\xi) \right) ds + J_t, \quad t \geq 0, \tag{2}$$

where  $x, b \in \mathcal{H}^+$ ,  $B \in \mathcal{L}(\mathcal{H})$  is a bounded linear operator, given  $y \in \mathcal{H}^+$  the measure  $M(y, \cdot) : \mathcal{B}(\mathcal{H}^+ \setminus \{0\}) \rightarrow \mathbb{R}$  is such that  $\nu^X(dt, d\xi) = M(X_t, d\xi)dt$  is the predictable compensator of the jump-measure of  $(X_t)_{t \geq 0}$ , and  $(J_t)_{t \geq 0}$  is a purely discontinuous  $\mathcal{H}^+$ -valued square integrable martingale. Moreover, by exploiting the results in Cox et al. (2020) and Metivier (1982), we adapt the proof of (Jacod & Shiryaev, 2003, Theorem II.2.42) to our infinite-dimensional setting to obtain the characteristic triplet (see Definition 1.1) of  $(X_t)_{t \geq 0}$  explicitly and show its affine form (see Proposition 2.6). The detailed parameter specifications are given in Assumption 2.1 below.

Our main motivation for studying Hilbert space-valued stochastic volatility models is the modeling of forward prices in commodity or fixed-income markets under the Heath–Jarrow–Morton–Musielà (HJMM) modeling paradigm (see for example, Benth and Krühner (2014, 2015), Filipović (2001), Carmona and Tehranchi (2006), Cont (2005)). In finite dimensions, multivariate stochastic volatility models with state-dependent volatility dynamics driven by Brownian noise and jumps are considered for example in Gourieroux and Sufana (2010), Caversaccio (2014), Leippold and Trojani (2008). The variance process  $X$  that we consider generalizes the Lévy-driven case considered in Benth et al. (2018) to a model allowing for state-dependent jump intensities, while

maintaining the desired *affine* property, which makes these models tractable. Stochastic volatilities with jumps describe the financial time series in energy and fixed-income markets well, as it is illustrated, for example, in Eydeland and Wolyniec (2002), Benth and Šaltytė Benth (2012), Cont (2001), Leippold and Trojani (2008). We refer in particular to Leippold and Trojani (2008) in which the authors discussed convincing empirical evidence for state-dependent jumps in the volatility.

Our *main contribution* lies in showing that our stochastic volatility model  $(Y, X)$  has the affine property, that is, we prove for all  $t \geq 0$  that the mixed Fourier–Laplace transform of  $(Y_t, X_t)$  is exponentially affine in the initial value  $(y, x) \in H \times \mathcal{H}^+$  and has a quasi-explicit formula in terms of a solution to generalized Riccati equations that are written in terms of the parameters of the model, see Theorem 3.3 below. For more on affine processes in various *finite dimensional* state spaces, see, for example, Cuchiero (2011), Duffie et al. (2003), Keller-Ressel et al. (2013), Keller-Ressel and Mayerhofer (2015), Spreij et al. (2011), Kallsen and Muhle-Karbe (2010), Cuchiero et al. (2011). In particular, Cuchiero et al. (2011) consider affine processes in the space of positive self-adjoint matrices, that is, they consider the finite-dimensional analog of our variance process  $X$ . Infinite-dimensional affine stochastic processes have been considered in, for example, Schmidt et al. (2020), Grafendorfer (2016), Cuchiero and Teichmann (2020), Benth et al. (2018), Benth and Simonsen (2018), Benth and Sgarra (2021). In particular, Benth et al. (2018), Benth and Simonsen (2018), Benth and Sgarra (2021) consider infinite-dimensional affine volatility models, however, they do not include state-dependent jump intensities.

The proof Theorem 3.3, that is, of the affine property of our stochastic volatility model  $(Y_t, X_t)_{t \geq 0}$ , is in Section 3. It involves considering an approximation  $(Y_t^{(n)}, X_t)_{t \geq 0}$  of  $(Y_t, X_t)_{t \geq 0}$  obtained by replacing  $\mathcal{A}$  in Equation (1) by its Yosida approximation. The use of the approximation allows us to exploit the semimartingale theory and standard techniques in order to show that the approximating process is affine. To show that the affine property holds for the limiting process  $(Y_t, X_t)_{t \geq 0}$ , we study the convergence of the generalized Riccati equations associated with  $(Y_t^{(n)}, X_t)_{t \geq 0}$  to those associated with  $(Y_t, X_t)_{t \geq 0}$ . We prove the existence of a unique solution to these generalized Riccati equations by exploiting infinite dimensional ODE results and using the quasi-monotonicity argument to show that the solution stays in the cone  $\mathcal{H}^+$ , see Deimling (1977) and Martin (1976). In order for the approach described above to succeed, we impose a commutativity-type condition on the covariance operator of the  $Q$ -Wiener process  $(W_t^Q)_{t \geq 0}$  and the stochastic volatility  $(X_t^{1/2})_{t \geq 0}$  (see Assumption 2.11 below). This condition is also imposed in Benth et al. (2018) and is rather limiting. However, we show that it can be avoided by considering a slightly different stochastic volatility model, see Remark 2.12 and the example in Section 4.4.

In Section 4, we consider a number of examples. For the process  $Y$ , we assume the setting proposed in Filipović (2001), Benth and Krühner (2014), which can be used to model arbitrage-free forward prices at time  $t \geq 0$  of a contract delivering an asset (commodity) or a stock at time  $t + x$ . In this case, the operator  $\mathcal{A}$  in Equation (1) is given by  $\mathcal{A} = \partial/\partial x$  and the space  $H$  is given by a Filipović space. For the process  $(X_t)_{t \geq 0}$ , we construct several examples in which we specify the drift and the jump parameters. We first show that the infinite dimensional lift of the multivariate Barndorff–Nielsen–Shephard model introduced in Benth et al. (2018) is a particular example of our model class. The stochastic variance process  $(X_t)_{t \geq 0}$  in this example is a stochastic differential equation driven by a *Lévy subordinator* in the space of self-adjoint Hilbert–Schmidt operators, as we show in Section 4.1.1. As mentioned above, this example does not involve state-dependent jump intensities. However, Sections 4.2–4.4 provide explicit parameter choices that *do* involve state-dependent jump intensities. In Section 4.2, we construct a variance process, which is essentially one-dimensional; evolving along a fixed vector  $z \in \mathcal{H}^+$ . In Section 4.3, we construct a truly

infinite-dimensional variance process  $X$ . In this example, both  $X_t, t \geq 0$ , and  $Q$  share a fixed orthonormal basis of eigenvectors. This is imposed to ensure that the commutativity condition given by Assumption 2.11 is satisfied. In Section 4.4, we avoid this commutativity condition by considering an example involving the alternative model discussed in Remark 2.12. In a subsequent article, we plan to compute option prices on forwards in commodity markets based on the models introduced here. In practice, these computations require the study of finite dimensional approximations of the variance process and its associated Riccati equations, which is being tackled in the working paper Karbach (2022).

### 1.1 | Layout of the article

In Section 2, we give an in-depth analysis of our stochastic volatility model and introduce sufficient parameter assumptions that ensure the well-posedness of our proposed model. Subsequently, in Section 3, we prove the affine-property of our joint model  $(Y_t, X_t)_{t \geq 0}$ . We split the proof into two parts, first in Section 3.1, we show the existence and uniqueness of solutions to the associated generalized Riccati equations under admissible parameter assumptions, thereafter in Section 3.2, we prove the affine transform formula. In Section 4, we give several examples of stochastic volatility models included in our model class by specifying various variance processes  $(X_t)_{t \geq 0}$ .

### 1.2 | Notation

For  $(X, \tau)$ , a topological vector space and  $S \subset X$  we let  $\mathcal{B}(S)$  denote the Borel- $\sigma$ -algebra generated by the relative topology on  $S$ . We denote by  $C^k([0, T]; S)$  the space of  $S$ -valued  $k$ -times continuously differentiable functions on  $[0, T]$ .

Throughout this article, we fix a separable, infinite-dimensional real Hilbert space  $(H, \langle \cdot, \cdot \rangle_H)$ . The space of bounded linear operators from  $H$  to  $H$  is denoted by  $\mathcal{L}(H)$ . The adjoint of an operator  $A \in \mathcal{L}(H)$  is denoted by  $A^*$ . We let  $\mathcal{L}_1(H) \subseteq \mathcal{L}(H)$  and  $\mathcal{L}_2(H) \subseteq \mathcal{L}(H)$  denote, respectively, the space of trace class operators and the space of Hilbert-Schmidt operators on  $H$ . Recall that  $\mathcal{L}_1(H)$  is a Banach space with the norm

$$\|A\|_{\mathcal{L}_1(H)} = \sum_{n=1}^{\infty} \langle (A^*A)^{1/2} e_n, e_n \rangle_H, \tag{3}$$

where  $(e_n)_{n \in \mathbb{N}}$  is an orthonormal basis for  $H$ . Moreover,  $\mathcal{L}_2(H)$  is a Hilbert space when endowed with the inner product

$$\langle A, B \rangle_{\mathcal{L}_2(H)} = \sum_{n=1}^{\infty} \langle Ae_n, Be_n \rangle_H. \tag{4}$$

Recall that for  $A \in \mathcal{L}(H)$  and  $B \in \mathcal{L}_2(H)$ , we have  $AB \in \mathcal{L}_2(H)$  and

$$\|AB\|_{\mathcal{L}_2(H)} \leq \|A\|_{\mathcal{L}(H)} \|B\|_{\mathcal{L}_2(H)}. \tag{5}$$

We define  $\mathcal{H}$  to be the space of all self-adjoint Hilbert–Schmidt operators on  $H$  and  $\mathcal{H}^+$  to be the cone of all positive operators in  $\mathcal{H}$ :

$$\mathcal{H} := \{A \in \mathcal{L}_2(H) : A = A^*\}, \quad \text{and} \quad \mathcal{H}^+ := \{A \in \mathcal{H} : \langle Ah, h \rangle_H \geq 0 \text{ for all } h \in H\}. \quad (6)$$

For notational brevity, we reserve  $\langle \cdot, \cdot \rangle$  to denote the inner product on  $\mathcal{L}_2(H)$ , and  $\|\cdot\|$  for the norm induced by  $\langle \cdot, \cdot \rangle$ . Note that  $\mathcal{H}$  is a closed subspace of  $\mathcal{L}_2(H)$ , and that  $\mathcal{H}^+$  is a self-dual cone in  $\mathcal{H}$ . For  $x, y \in \mathcal{H}$ , we write  $x \leq_{\mathcal{H}^+} y$  if  $y - x \in \mathcal{H}^+$  (and  $x \geq_{\mathcal{H}^+} y$  if  $x - y \in \mathcal{H}^+$ ). For  $a, b \in H$ , we let  $a \otimes b$  be the linear operator defined by  $a \otimes b(h) = \langle a, h \rangle_H b$  for every  $h \in H$ . Note that  $a \otimes a \in \mathcal{H}^+$  for every  $a \in H$ . When space is scarce, we shall write  $a^{\otimes 2} := a \otimes a$ .

Finally, throughout this article, we let  $\chi : \mathcal{H} \rightarrow \mathcal{H}$  denote the truncation function given by  $\chi(x) = x1_{\{\|x\| \leq 1\}}$ .

### 1.2.1 | Hilbert-valued semimartingales

We let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  be a filtered probability space and let  $(H, \langle \cdot, \cdot \rangle)$  be a separable Hilbert space. Let  $M = (M_t)_{t \geq 0}$  be an  $H$ -valued locally square-integrable martingale. Then we know from (Metivier, 1982, Theorem 21.6 and Section 23.3) that there exists a unique (up to a  $\mathbb{P}$ -null set) càdlàg predictable process  $\langle\langle M \rangle\rangle$  of finite variation taking values in the set of positive self-adjoint elements of  $\mathcal{L}_1(H)$  such that  $\langle\langle M \rangle\rangle_0 = 0$ , and  $M \otimes M - \langle\langle M \rangle\rangle$  is an  $\mathcal{L}_1(H)$ -valued local martingale.

Following (Metivier, 1982, Definition 23.7), an  $H$ -valued process  $X = (X_t)_{t \geq 0}$  is called a *semimartingale* if

$$X_t = X_0 + M_t + A_t, \quad t \geq 0, \quad (7)$$

where  $X_0$  is  $H$ -valued and  $\mathcal{F}_0$ -measurable,  $M$  is a  $H$ -valued locally square-integrable martingale with càdlàg paths such that  $M_0 = 0$  and  $A$  is an adapted  $H$ -valued càdlàg process of finite variation with  $A_0 = 0$ .

When the process  $A$  in Equation (7) is predictable, then  $X$  is said to be a *special semimartingale*. The decomposition (7) in this case is unique (see (Metivier, 1982, Theorem 23.6)) and is called the *canonical decomposition* of  $X$ . For a semimartingale  $X$ , we write  $\Delta X_t = X_t - X_{t-}$ , where  $X_{t-} = \lim_{s \rightarrow t-} X_s$ . Notice that when  $\|\Delta X\|$  is bounded, then  $X$  is a special semimartingale (see (Metivier, 1982, Chapter 4, Exercise 11)).

Two  $H$ -valued locally square-integrable martingales  $M$  and  $N$  are called *orthogonal* if the real-valued process  $(\langle M_t, N_t \rangle)_{t \geq 0}$  is a local martingale. Further, we call  $M$  a *purely discontinuous* local martingale if it is orthogonal to all continuous local martingales. An  $H$ -valued semimartingale can be written as (see (Metivier, 1982, Theorem 20.2))

$$X_t = X_0 + X_t^c + M_t^d + A_t, \quad t \geq 0, \quad (8)$$

where  $X_0$  is  $\mathcal{F}_0$ -measurable,  $X^c$  is a continuous local martingale with  $X_0^c = 0$ ,  $M^d$  is a locally square integrable martingale orthogonal to  $X^c$  with  $M_0^d = 0$ , and  $A$  is a càdlàg process of finite variation with  $A_0 = 0$ . The process  $X^c$  in Equation (8) is unique (up to a  $\mathbb{P}$  null set), see (Metivier, 1982, Chapter 4, Exercise 13).

We associate with the  $\mathcal{H}$ -valued semimartingale  $X$ , the integer-valued random measure  $\mu^X : \mathcal{B}([0, \infty) \times \mathcal{H}) \rightarrow \mathbb{N}$  given by

$$\mu^X(dt, d\xi) = \sum_{s \geq 0} \mathbf{1}_{\{\Delta X_s \neq 0\}} \delta_{(s, \Delta X_s)}(dt, d\xi), \tag{9}$$

where  $\delta_a$  denotes the Dirac measure at point  $a$ . Recall from (Jacod & Shiryaev, 2003, Theorem II.1.8), the existence and uniqueness (up to a  $\mathbb{P}$ -null set) of the predictable compensator  $\nu^X$  of  $\mu^X$ .

Given a semimartingale  $X$ , we define the “large jumps” process  $\check{X}$  by

$$\check{X} := \sum_{s \leq \cdot} \Delta X_s \mathbf{1}_{\{\|\Delta X_s\| > 1\}}, \tag{10}$$

and we define the “small jumps” process

$$\hat{X} = X - \check{X}. \tag{11}$$

Since  $\|\Delta \hat{X}\| \leq 1$ ,  $\hat{X}$  is a special semimartingale and hence it admits the unique decomposition

$$\hat{X}_t = X_0 + M_t^{\hat{X}} + A_t^{\hat{X}}, \quad t \geq 0, \tag{12}$$

where  $X_0$  is  $\mathcal{F}_0$ -measurable,  $M^{\hat{X}}$  is a local martingale with  $M_0^{\hat{X}} = 0$ , and  $A^{\hat{X}}$  is a predictable process of finite variation with  $A_0^{\hat{X}} = 0$ .

We are ready to introduce the characteristic triplet of an  $\mathcal{H}$ -valued semimartingale  $X$ .

**Definition 1.1.** Let  $X$  be an  $\mathcal{H}$ -valued semimartingale, let  $A^{\hat{X}}$  be the predictable process of finite variation from decomposition (12), let  $X^c$  be the continuous martingale part of  $X$  as provided by Equation (8), and let  $\nu^X$  be the predictable compensator of  $\mu^X$ , where  $\mu^X$  is defined by Equation (9). Then we call the triplet  $(A^{\hat{X}}, \langle X^c \rangle, \nu^X)$  the characteristic triplet of  $X$ . Note that the characteristic triplet consists of a predictable càdlàg  $\mathcal{H}$ -valued process of finite variation, a predictable càdlàg  $\mathcal{L}_1(\mathcal{H})$ -valued process of finite variation, and a predictable random measure on  $\mathcal{B}([0, \infty) \times \mathcal{H})$ .

## 2 | THE STOCHASTIC VOLATILITY MODEL

In this section, we specify our stochastic volatility model. First, in Section 2.1, we introduce the stochastic variance process  $X$ , which is an affine Markov process on the cone of positive self-adjoint Hilbert–Schmidt operators, the existence of which is established in Cox et al. (2020). We show that whenever the process  $X$  admits for a version with càdlàg paths, this version is actually a Markov semimartingale with characteristic triplet of an affine form and the representation (2) holds true. Subsequently, in Section 2.2, we show that given such a stochastic variance process  $X$  there exists a mild solution  $Y$  to Equation (1) with initial value  $y \in \mathcal{H}$ , which enables us to introduce our joint stochastic volatility model  $Z = (Y, X)$  (see Definition 2.9 below).



## 2.1 | The affine variance process

We model the stochastic variance process  $(X_t)_{t \geq 0}$  as a time-homogeneous *affine* Markov process on the state space  $\mathcal{H}^+$  in the sense of Cox et al. (2020). Recall that  $\chi : \mathcal{H} \rightarrow \mathcal{H}$ ,  $\chi(x) = x1_{\{\|x\| \leq 1\}}$  is our truncation function. Assume  $(b, B, m, \mu)$  to be an admissible parameter set in the following sense:

**Assumption 2.1.** An *admissible parameter set* consists of

- (i) a measure  $m : \mathcal{B}(\mathcal{H}^+ \setminus \{0\}) \rightarrow [0, \infty]$  such that
  - (a)  $\int_{\mathcal{H}^+ \setminus \{0\}} \|\xi\|^2 m(d\xi) < \infty$  and
  - (b)  $\int_{\mathcal{H}^+ \setminus \{0\}} |\langle \chi(\xi), h \rangle| m(d\xi) < \infty$  for all  $h \in \mathcal{H}$  and there exists an element  $I_m \in \mathcal{H}$  such that  $\langle I_m, h \rangle = \int_{\mathcal{H}^+ \setminus \{0\}} \langle \chi(\xi), h \rangle m(d\xi)$  for every  $h \in \mathcal{H}$ ;
- (ii) a vector  $b \in \mathcal{H}$  such that

$$\langle b, v \rangle - \int_{\mathcal{H}^+ \setminus \{0\}} \langle \chi(\xi), v \rangle m(d\xi) \geq 0 \quad \text{for all } v \in \mathcal{H}^+; \quad (13)$$

- (iii) a  $\mathcal{H}^+$ -valued measure  $\mu : \mathcal{B}(\mathcal{H}^+ \setminus \{0\}) \rightarrow \mathcal{H}^+$  such that

$$\int_{\mathcal{H}^+ \setminus \{0\}} \langle \chi(\xi), u \rangle \frac{\langle \mu(d\xi), x \rangle}{\|\xi\|^2} < \infty, \quad (14)$$

for all  $u, x \in \mathcal{H}^+$  satisfying  $\langle u, x \rangle = 0$ ,

- (iv) an operator  $B \in \mathcal{L}(\mathcal{H})$  with adjoint  $B^*$  satisfying

$$\langle B^*(u), x \rangle - \int_{\mathcal{H}^+ \setminus \{0\}} \langle \chi(\xi), u \rangle \frac{\langle \mu(d\xi), x \rangle}{\|\xi\|^2} \geq 0, \quad (15)$$

for all  $x, u \in \mathcal{H}^+$  satisfying  $\langle u, x \rangle = 0$ .

Given an admissible parameter set, the main result in (Cox et al., 2020, Theorem 2.8) ensures the existence of a square-integrable time-homogeneous  $\mathcal{H}^+$ -valued affine Markov process  $X$ . More specifically, (Cox et al., 2020, Theorem 2.8 and Proposition 4.17) imply Theorem 2.3 below, which we need in our derivations later. In order to state this result, we introduce our concept of a *weak generator*<sup>1</sup>, which is a minor modification of (Peszat & Zabczyk, 2007, Definition 9.36).

**Definition 2.2** (Weak generator). Let  $X$  be a square-integrable time-homogeneous  $\mathcal{H}^+$ -valued Markov process with transition semigroup  $(P_t)_{t \geq 0}$  acting on the space  $C_w(\mathcal{H}^+, \mathbb{R}) := \{f \in C(\mathcal{H}^+, \mathbb{R}) : \sup_{x \in \mathcal{H}^+} \frac{f(x)}{\|x\|^2+1} < \infty\}$ . Then the *weak generator*  $\mathcal{G} : \text{dom}(\mathcal{G}) \subseteq C_w(\mathcal{H}^+, \mathbb{R}) \rightarrow C_w(\mathcal{H}^+, \mathbb{R})$  of  $(P_t)_{t \geq 0}$  is defined as follows:  $f \in \text{dom}(\mathcal{G})$  if and only if there exists a  $g \in C_w(\mathcal{H}^+, \mathbb{R})$  such that

$$g(x) = \lim_{t \downarrow 0} \frac{P_t f(x) - f(x)}{t}, \quad (16)$$



and

$$P_t f(x) = f(x) + \int_0^t P_s g(x) ds, \tag{17}$$

for all  $x \in \mathcal{H}^+$ , and in this case we define  $\mathcal{G}f := g$ .

**Theorem 2.3.** *Let  $(b, B, m, \mu)$  be an admissible parameter set according to Assumption 2.1. Then there exist constants  $M, \omega \in [1, \infty)$  and a square-integrable time-homogeneous  $\mathcal{H}^+$ -valued Markov process  $X$  with transition semigroup  $(P_t)_{t \geq 0}$ , acting on functions  $f \in C_w(\mathcal{H}^+, \mathbb{R})$ , and weak generator  $(\mathcal{G}, \text{dom}(\mathcal{G}))$  such that the following holds:*

- (i)  $\mathbb{E}[\|X_t\|^2 | X_0 = x] \leq M e^{\omega t} (\|x\|^2 + 1)$  for all  $t \geq 0$ ,
- (ii)  $\text{lin}\{e^{-\langle \cdot, u \rangle} : u \in \mathcal{H}^+\} \cup \{\langle \cdot, u \rangle : u \in \mathcal{H}^+\} \subseteq \text{dom}(\mathcal{G})$ , and
- (iii) for every  $f \in \text{lin}\{e^{-\langle \cdot, u \rangle} : u \in \mathcal{H}^+\} \cup \{\langle \cdot, u \rangle : u \in \mathcal{H}^+\}$  we have:

$$\mathcal{G}f(x) = \langle b + B(x), f'(x) \rangle + \int_{\mathcal{H}^+ \setminus \{0\}} (f(x + \xi) - f(x) - \langle \chi(\xi), f'(x) \rangle) M(x, d\xi), \tag{18}$$

where  $M(x, d\xi) := m(i\xi) + \frac{\langle \mu(i\xi), x \rangle}{\|\xi\|^2}$ .

An additional assumption we want to impose on the affine variance processes under consideration is the requirement, that  $X$  must admit for a version with càdlàg paths.

**Assumption 2.4.** The time-homogeneous Markov process  $X$  associated with the parameters  $(b, B, m, \mu)$  of Assumption 2.1 has càdlàg paths.

Due to the lack of local compactness of the underlying state space, standard Feller theory cannot be employed to establish Theorem 2.3. We overcame this problem by using *generalized* Feller semigroups, see (Cox et al., 2020, Section 4) and Cuchiero and Teichmann (2020). Unfortunately, the Markov processes associated to a generalized Feller semigroup need not have càdlàg paths (but see (Cuchiero & Teichmann, 2020, Theorem 2.13) for a positive result). Some (rather limiting) conditions that ensure that Assumption 2.4 is satisfied are provided in the lemma below. In ongoing work Karbach (2022), we hope to establish that in fact, Assumption 2.4 is always satisfied.

**Lemma 2.5.** *Assume that  $(b, B, m, \mu)$  is an admissible parameter set that fulfills either one of the following two cases:*

- (i) (the Lévy-driven case)  $\mu(d\xi) = 0$ ,
- (ii) (finite activity jumps)  $m(\mathcal{H}^+ \setminus \{0\}) < \infty$  and  $\int_{\mathcal{H}^+ \setminus \{0\}} \langle x, \frac{\mu(d\xi)}{\|\xi\|^2} \rangle < \infty$  for all  $x \in \mathcal{H}^+$ .

Then the affine Markov process  $(X_t)_{t \geq 0}$  associated to  $(b, B, m, \mu)$  admits for a version with càdlàg paths.

*Proof.* To prove (i), observe that the weak generator (18) associated to the admissible parameters  $(b, B, m, 0)$  is a weak generator of a Lévy driven SDE as described for example in (Peszat & Zabczyk,

2007, equation 9.37)) and hence the assertion follows from (Peszat & Zabczyk, 2007, Theorem 4.3). In case of (ii), the assertion follows from (Cox et al., 2020, Proposition 4.19).  $\square$

We show in the next proposition that the version of  $X$  with càdlàg paths is in fact a Markovian semimartingale.

**Proposition 2.6.** *Suppose that  $(b, B, m, \mu)$  is an admissible parameter set according to Assumption 2.1 and such that the associated affine Markov process  $X$  satisfies Assumption 2.4. Then there exists a version of  $(X_t)_{t \geq 0}$ , which is an  $\mathcal{H}^+$ -valued semimartingale with semimartingale characteristics  $(A, C, \nu^X)$  of the form*

$$A_t = \int_0^t b + B(X_s) ds \quad (19)$$

$$C_t = 0, \quad (20)$$

$$\nu^X(dt, d\xi) = M(X_t, d\xi) dt = \left( m(d\xi) + \langle X_t, \frac{\mu(d\xi)}{\|\xi\|^2} \rangle \right) dt. \quad (21)$$

Moreover, the following representation holds

$$X_t = X_0 + \int_0^t \left( b + B(X_s) + \int_{\mathcal{H}^+ \cap \{\|\xi\| > 1\}} \xi M(X_s, d\xi) \right) ds + J_t, \quad t \geq 0, \quad (22)$$

where  $J$  is a purely discontinuous square-integrable martingale.

In order to prove Proposition 2.6, we need the following result, which can be obtained by mimicking the proof of (Peszat & Zabczyk, 2007, Proposition 9.38).

**Proposition 2.7.** *Let  $X$  be a square-integrable time-homogeneous càdlàg Markov process on  $\mathcal{H}^+$  with transition semigroup  $(P_t)_{t \geq 0}$  acting on  $C_w(\mathcal{H}^+, \mathbb{R})$ , let  $\mathcal{G}$  be its weak generator and let  $f \in \text{dom}(\mathcal{G})$ . Define  $M_t = f(X_t) - f(X_0) - \int_0^t (\mathcal{G}f)(X_s) ds$ . Then  $(M_t)_{t \geq 0}$  is a real-valued martingale.*

*Proof of Proposition 2.6.* Let  $(e_n)_{n \in \mathbb{N}}$  be an orthonormal basis of  $\mathcal{H}$ , then for every  $n \in \mathbb{N}$ , we have  $e_n = e_n^+ - e_n^-$ , for  $e_n^+, e_n^- \in \mathcal{H}^+$ . By Theorem 2.3 and Proposition 2.7 applied to  $f = \langle \cdot, e_n \rangle$ , there exists a square-integrable martingale  $J^{(n)}$  such that

$$\begin{aligned} \langle X_t, e_n \rangle &= \langle X_0, e_n \rangle + \int_0^t \left( \langle b + B(X_s), e_n \rangle + \int_{\mathcal{H}^+ \cap \{\|\xi\| > 1\}} \langle \xi, e_n \rangle M(X_s, d\xi) \right) ds \\ &\quad + J_t^{(n)}, \quad t \geq 0. \end{aligned} \quad (23)$$

Noting that  $X = \sum_{n=1}^{\infty} \langle X, e_n \rangle e_n$ , we infer that  $X$  is an  $\mathcal{H}^+$ -valued semimartingale with the decomposition in Equation (22), where  $J = \sum_{n=1}^{\infty} J^{(n)} e_n$  is a square integrable  $\mathcal{H}$ -valued martingale.

We are left to show that  $J$  is purely discontinuous and to make the characteristic triplet of  $X$  explicit. These are known results in the finite-dimensional setting (see for instance (Jacod & Shiryaev, 2003, Theorem II.2.42)). Below, we adapt the proof of (Jacod & Shiryaev, 2003, Theorem II.2.42) to our setting. For that we decompose  $X = A^{\hat{X}} + N^{\hat{X}} + \check{X}$  as in Equations (11) and (12). Denote by  $(A^{\hat{X}}, C, \nu^{\hat{X}})$  the characteristic triplet of the semimartingale  $X$ . Let  $u \in \mathcal{H}^+$  be arbitrary and consider the function  $g_u = e^{-\langle \cdot, u \rangle}$ ,  $u \in \mathcal{H}^+$ . On the one hand, applying the Itô formula to  $g_u(X)$  (see for instance, (Metivier, 1982, Theorem 27.2)), yields that  $g_u(X)$  is a real-valued semimartingale and

$$\begin{aligned}
 e^{-\langle X_t, u \rangle} &= e^{-\langle X_0, u \rangle} - \int_0^t e^{-\langle X_{s-}, u \rangle} \langle u, dA_s^{\hat{X}} \rangle - \int_0^t e^{-\langle X_{s-}, u \rangle} \langle u, dN_s^{\hat{X}} \rangle \\
 &\quad + \frac{1}{2} \int_0^t e^{-\langle X_{s-}, u \rangle} \langle u \otimes u, dC_s \rangle_{\mathcal{L}_2(\mathcal{H})} + \int_0^t \int_{\mathcal{H}^+ \setminus \{0\}} e^{-\langle X_{s-}, u \rangle} K(\xi, u) \nu^{\hat{X}}(ds, d\xi) \\
 &\quad + \int_0^t \int_{\mathcal{H}^+ \setminus \{0\}} e^{-\langle X_{s-}, u \rangle} K(\xi, u) (\mu^{\hat{X}}(ds, d\xi) - \nu^{\hat{X}}(ds, d\xi)), \tag{24}
 \end{aligned}$$

where  $K(\xi, u) = e^{-\langle \xi, u \rangle} - 1 + \langle \chi(\xi), u \rangle$ . On the other hand, by Proposition 2.7 there exists a real-valued martingale  $I^u$  such that

$$\begin{aligned}
 e^{-\langle X_t, u \rangle} &= e^{-\langle X_0, u \rangle} + I_t^u - \int_0^t e^{-\langle X_{s-}, u \rangle} (\langle b + B(X_s), u \rangle) ds \\
 &\quad + \int_0^t \int_{\mathcal{H}^+ \setminus \{0\}} e^{-\langle X_{s-}, u \rangle} K(\xi, u) M(X_s, d\xi) ds, \quad t \geq 0. \tag{25}
 \end{aligned}$$

Note that for every  $t \geq 0$ , the integrals with respect to  $ds$  on the right-hand side of Equation (25) remain unchanged if we take the left-limits  $X_{s-}$  instead of  $X_s$ , as the number of jumps on  $[0, t]$  is at most countable. Moreover, as  $X$  takes values in  $\mathcal{H}^+$ , we have that  $g_u(X)$  is bounded and hence it is a special semimartingale and its canonical decomposition is unique. Therefore the finite variation part in formulas (24) and (25) must coincide, that is,

$$\begin{aligned}
 & - \int_0^t e^{-\langle X_{s-}, u \rangle} \left( \langle u, dA_s^{\hat{X}} \rangle + \frac{1}{2} \langle u \otimes u, dC_s \rangle_{\mathcal{L}_2(\mathcal{H})} + \int_{\mathcal{H}^+ \setminus \{0\}} K(\xi, u) \nu^{\hat{X}}(ds, d\xi) \right) \\
 &= - \int_0^t e^{-\langle X_{s-}, u \rangle} \left( \langle b + B(X_s), u \rangle + \int_{\mathcal{H}^+ \setminus \{0\}} K(\xi, u) M(X_s, d\xi) \right) ds, \tag{26}
 \end{aligned}$$

must hold for all  $t \geq 0$  almost surely. Now, by integrating  $e^{\langle X_{s-}, u \rangle}$  with respect to both sides of Equation (26) over  $[0, t]$ , we obtain

$$- \langle u, A_t^{\hat{X}} \rangle + \frac{1}{2} \langle u \otimes u, C_t \rangle_{\mathcal{L}_2(\mathcal{H})} + \int_{\mathcal{H}^+ \setminus \{0\}} K(\xi, u) \nu^{\hat{X}}([0, t], d\xi)$$

$$= - \langle u, \int_0^t b + B(X_s) ds \rangle + \int_0^t \int_{\mathcal{H}^+ \setminus \{0\}} K(\xi, u) M(X_s, d\xi) ds, \quad \forall t \geq 0 \text{ a.s.} \tag{27}$$

Now, following similar steps as in the proof of (Jacod & Shiryaev, 2003, Theorem II.2.42), we conclude that  $C_t = 0$ ,  $\nu^X([0, t], d\xi) = \int_0^t M(X_s, d\xi) ds$  and  $A_t^X = \int_0^t b + B(X_s) ds$ ,  $t \geq 0$ , and the statements of the proposition follow.  $\square$

## 2.2 | The joint stochastic volatility model

In this section, we present our joint model, see Definition 2.9 below, which involves taking the square root  $X^{1/2}$  of the process  $X$  from Theorem 2.3 as volatility for the  $H$ -valued process  $Y$  given by Equation (29) below.

Throughout this section, we consider the following setting: let  $(b, B, m, \mu)$  be a parameter set satisfying Assumption 2.1, let  $x \in \mathcal{H}^+$  and  $y \in H$ , and let  $Q \in \mathcal{L}_1(H)$  be self-adjoint and positive. Next, let  $X$  be the square-integrable time-homogeneous Markov process associated with the parameter set  $(b, B, m, \mu)$  the existence of which is guaranteed by Theorem 2.3; we denote the filtered probability space on which  $X$  is defined by  $(\Omega^1, \mathcal{F}^1, (\mathcal{F}_t^1)_{t \geq 0}, \mathbb{P}^1)$  and assume  $\mathbb{P}^1(X_0 = x) = 1$ . In addition, we let  $(\Omega^2, \mathcal{F}^2, (\mathcal{F}_t^2)_{t \geq 0}, \mathbb{P}^2)$  be another filtered probability space, which satisfies the usual conditions and allows for a  $Q$ -Wiener process  $W^Q : [0, \infty) \times \Omega \rightarrow H$ . Now set

$$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) := \left( \Omega^1 \times \Omega^2, (\mathcal{F}^1 \otimes \mathcal{F}^2), (\mathcal{F}_t^1 \otimes \mathcal{F}_t^2)_{t \geq 0}, \mathbb{P}^1 \otimes \mathbb{P}^2 \right), \tag{28}$$

and denote the expectation with respect to  $\mathbb{P}$  by  $\mathbb{E}$ . With slight abuse of notation, we consider  $X$  and  $W^Q$  to be processes on  $(\Omega, \mathcal{F}, \mathbb{F})$  (note that they are independent).

In addition, we assume  $(\mathcal{A}, \text{dom}(\mathcal{A}))$  to be the generator of a strongly continuous semigroup  $(S(t))_{t \geq 0}$  on  $H$ .

Now consider the following SDE, for which Lemma 2.8 below establishes the existence of a mild solution

$$\begin{cases} dY_t = \mathcal{A}Y_t dt + X_t^{1/2} dW_t^Q, & t \geq 0, \\ Y_0 = y. \end{cases} \tag{29}$$

**Lemma 2.8.** *Assume the setting described above, in particular, let  $(b, B, m, \mu)$  satisfy Assumption 2.1 and let  $X$  be the associated affine process. Moreover, let Assumption 2.4 hold. Then  $X$  is progressive,*

$$\mathbb{E} \left[ \int_0^t \|X_s^{1/2} Q^{1/2}\|^2 ds \right] < \infty, \tag{30}$$

and moreover

$$Y_t = S(t)y + \int_0^t S(t-s)X_s^{1/2} dW_s^Q, \quad t \geq 0, \tag{31}$$

is the unique mild solution to Equation (29).

*Proof.* The fact that  $X$  is progressive follows from the  $\mathbb{F}$ -adaptedness of  $X$  and Assumption 2.4. Moreover, it follows from Theorem 2.3 (i) and Hölder’s inequality that

$$\mathbb{E} \left\| X_t^{1/2} Q^{1/2} \right\|^2 \leq \|Q\|_{\mathcal{L}_1(H)} \mathbb{E} \left\| X_t^{1/2} \right\|_{\mathcal{L}(H)}^2 \leq \|Q\|_{\mathcal{L}_1(H)} \mathbb{E} \|X_t\| \tag{32}$$

$$\leq \sqrt{M} \|Q\|_{\mathcal{L}_1(H)} e^{\omega t/2} \sqrt{\mathbb{E} \|X_0\|^2 + 1}, \tag{33}$$

which implies Equation (30). Standard theory on infinite dimensional SDEs (see, for instance, (Gawarecki & Mandrekar, 2011, Section 3)) now yields the existence of a unique mild solution to Equation (29) given by Equation (31).  $\square$

**Definition 2.9.** Assume the setting described above, in particular, let  $(b, B, m, \mu)$  satisfy Assumption 2.1 and let  $X$  be the associated affine process. Moreover, let Assumption 2.4 hold and let  $Y$  be given by Equation (31). Then we refer to the  $H \times \mathcal{H}^+$ -valued process  $Z = (Y, X)$  as the *joint stochastic volatility model with affine pure-jump variance* (and with parameters  $(b, B, m, \mu, Q, \mathcal{A})$  and initial value  $(x, y)$ ). Note that the process  $(Z, (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}))$  is a (stochastically) weak solution to the following SDE in  $H \times \mathcal{H}$ :

$$\begin{cases} dZ_t = (\mathbf{b} + \mathbf{A}Z_t) dt + \Sigma(Z_t) d\mathbf{W}_t + d\mathbf{J}_t, & t \geq 0, \\ Z_0 = (y, x) \in H \times \mathcal{H}^+, \end{cases} \tag{34}$$

where  $\mathbf{b}, \mathbf{A}, \Sigma, \mathbf{B}$ , and  $\mathbf{J}$  are as follows:

$$\mathbf{b} := \begin{bmatrix} 0 \\ b + \int_{\mathcal{H}^+ \cap \{\|\xi\| > 1\}} \xi m(d\xi) \end{bmatrix}, \quad \mathbf{A} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} := \begin{bmatrix} \mathcal{A}z_1 \\ B(z_2) + \int_{\mathcal{H}^+ \cap \{\|\xi\| > 1\}} \xi \frac{\langle z_2, \mu(d\xi) \rangle}{\|\xi\|^2} \end{bmatrix}, \tag{35}$$

$$\Sigma(z) := \begin{bmatrix} (z_2)^{1/2} & 0 \\ 0 & 0 \end{bmatrix}, \quad d\mathbf{W} := \begin{bmatrix} dW^Q \\ 0 \end{bmatrix}, \quad \text{and} \quad d\mathbf{J} := \begin{bmatrix} 0 \\ dJ \end{bmatrix}, \tag{36}$$

where  $J$  is the purely discontinuous square-integrable martingale obtained from Proposition 2.6.

*Remark 2.10.* The assumption that  $W^Q$  is a  $Q$ -Wiener process can be weakened whilst maintaining all results presented in this article. Indeed, as  $X$  itself is already  $\mathcal{H}^+$  valued, it suffices to assume that  $Q \in \mathcal{L}_2(H)$  (instead of  $Q \in \mathcal{L}_1(H)$ ) (see also the proof of Lemma 2.8).

In order to show that our joint model is affine (see Theorem 3.3 below), we need one further assumption. This assumption is also imposed in Benth et al. (2018), see Proposition 3.2 of that article.

**Assumption 2.11.** There exists a positive and self-adjoint operator  $D \in \mathcal{L}(H)$  such that

$$X_t^{1/2} Q X_t^{1/2} = D^{1/2} X_t D^{1/2}, \quad \text{for all } t \geq 0. \tag{37}$$

To the best of our knowledge, all examples for which Assumption 2.11 holds are such that  $Q$  and  $X_t$  commute for all  $t \geq 0$ . In fact, as commuting self-adjoint and compact operators are jointly diagonalizable, this is difficult to ensure without assuming there exists a fixed orthonormal basis  $(e_n)_{n \in \mathbb{N}}$  of  $H$  that forms the eigenvectors of  $Q$  and of  $X_t, t \geq 0$ . Note that this essentially reduces the state space of  $X$  to the cone of positive, square-integrable sequences  $\ell_2^+$ , that is, we only model the eigenvalues of  $X$ , as the eigenvectors are fixed, see also Section 4.3. In conclusion, Assumption 2.11 is rather limiting. However, it can be circumvented if one considers a slightly different model, see Remarks 2.12 and 2.13 below.

*Remark 2.12.* Assumption 2.11 can be omitted if, instead of Equation (29), one assumes that the process  $Y$  in the joint model satisfies the following stochastic differential equation:

$$\begin{cases} dY_t = \mathcal{A}Y_t dt + D^{1/2}X_t^{1/2} dW_t, & t \geq 0, \\ Y_0 = y, \end{cases} \tag{38}$$

where  $W$  is an  $H$ -cylindrical Brownian motion (i.e.,  $dW_t$  is white noise) and  $D \in \mathcal{L}_1(H)$  is positive and self-adjoint (in fact,  $D \in \mathcal{H}^+$  suffices, see Remark 2.10). In this case, provided Assumptions 2.1 and 2.4 hold, we have

$$\mathbb{E} \left[ \int_0^t \|D^{1/2}X_s^{1/2}\|^2 ds \right] < \infty, \tag{39}$$

and

$$Y_t = S(t)y + \int_0^t S(t-s)D^{1/2}X_s^{1/2}dW_s, \quad t \geq 0, \tag{40}$$

is the unique mild solution to Equation (38), see also (Da Prato & Zabczyk, 1992, Chapter 4, Section 3). Moreover, Theorem 3.3 below remains valid: if  $Y$  is given by Equation (38) and Assumptions 2.1 and 2.4 hold, we obtain *exactly* the same expression for  $\mathbb{E}[e^{\langle Y_t, u_1 \rangle_H - \langle X_t, u_2 \rangle}]$ . In particular the joint model involving Equation (38) under Assumptions 2.1 and 2.4 coincides with the joint model involving Equation (29) under Assumptions 2.1–2.11, in the sense that for every fixed time  $t \geq 0$ , the distribution of  $(Y_t, X_t)$  is the same. We refer to Section 4.4 for an example of a joint model involving Equation (38).

*Remark 2.13.* If  $(\mathcal{A}, \text{dom}(\mathcal{A}))$  is the generator of an analytic semigroup and moreover  $\mathcal{A}^{-\alpha} \in \mathcal{L}_4(H)$  (equivalently,  $\mathcal{A}^{-2\alpha} \in \mathcal{H}$ ) for some  $\alpha \in [0, \frac{1}{2})$ , then a mild solution to Equation (29) exists even if  $W^Q$  is an  $H$ -cylindrical Brownian motion. These conditions are satisfied, for example, when  $\mathcal{A}$  is the Laplacian on  $\mathbb{R}^d$  for  $d \in \{1, 2, 3\}$ . We refer to Da Prato and Zabczyk (1992) for details.

Although this provides another way to circumvent Assumption 2.11 (as  $Q$  is the identity in this case), we will not investigate this setting any further: for the applications we have in mind  $(\mathcal{A}, \text{dom}(\mathcal{A}))$  fails to be the generator of an analytic semigroup. Note that to obtain the assertions of Theorem 3.3 in this setting, one would have to adapt its proof: one would not only have to approximate the operator  $\mathcal{A}$  but also the noise.

### 3 | THE JOINT STOCHASTIC VOLATILITY MODEL IS AFFINE

In this section, we present our main result, namely that the stochastic volatility model  $Z = (Y, X)$  in Definition 2.9 has the *affine property*, see Theorem 3.3. In particular, this means that we can express the mixed Fourier–Laplace transform  $\mathbb{E}[e^{i\langle Y_t, u \rangle_H - \langle X_t, v \rangle}]$  ( $u \in H, v \in \mathcal{H}^+$ ) in terms of the solution to *generalized Riccati equations* associated to the model parameters  $(b, B, m, \mu)$ ,  $\mathcal{A}$  and  $Q$  (respectively  $D$ ). In the upcoming subsection, we discuss the well-posedness of these generalized Riccati equations. Our main result, Theorem 3.3, is contained and proven in Section 3.2.

#### 3.1 | Analysis of the associated generalized Riccati equations

Let us fix an admissible parameter set  $(b, B, m, \mu)$  according to Assumptions 2.1 and a positive self-adjoint  $D \in \mathcal{L}(H)$ . Define  $F : \mathcal{H}^+ \rightarrow \mathbb{R}$  and  $R : iH \times \mathcal{H}^+ \rightarrow \mathcal{H}$ , respectively as

$$F(u) = \langle b, u \rangle - \int_{\mathcal{H}^+ \setminus \{0\}} (e^{-\langle \xi, u \rangle} - 1 + \langle \chi(\xi), u \rangle) m(d\xi), \tag{41}$$

$$R(h, u) = B^*(u) - \frac{1}{2} D^{1/2} h \otimes D^{1/2} h - \int_{\mathcal{H}^+ \setminus \{0\}} (e^{-\langle \xi, u \rangle} - 1 + \langle \chi(\xi), u \rangle) \frac{\mu(d\xi)}{\|\xi\|^2}. \tag{42}$$

Let  $(\mathcal{A}, \text{dom}(\mathcal{A}))$  be the generator of a strongly continuous semigroup  $(S(t))_{t \geq 0}$  and let  $(\mathcal{A}^*, \text{dom}(\mathcal{A}^*))$  be its adjoint. It is well known that  $(\mathcal{A}^*, \text{dom}(\mathcal{A}^*))$  generates the strongly continuous semigroup  $(S^*(t))_{t \geq 0}$  on  $H$ , see, for instance, (Goldstein, 1985, Theorem 4.3).

Let  $T \in \mathbb{R}^+, u_1 \in iH$  and  $u_2 \in \mathcal{H}^+$ . We consider the following system of differential equations, known as *generalized Riccati equations*:

$$\begin{cases} \frac{\partial \Phi}{\partial t}(t, u) = F(\psi_2(t, u)), & 0 < t \leq T, \quad \Phi(0, u) = 0, \\ \psi_1(t, u) = u_1 - i\mathcal{A}^* \left( \int_0^t \psi_1(s, u) ds \right), & 0 < t \leq T, \quad \psi_1(0, u) = u_1, \\ \frac{\partial \psi_2}{\partial t}(t, u) = R(\psi_1(t, u), \psi_2(t, u)), & 0 < t \leq T, \quad \psi_2(0, u) = u_2. \end{cases} \tag{43}$$

**Definition 3.1.** Let  $u = (u_1, u_2) \in iH \times \mathcal{H}^+$ . We say that  $(\Phi(\cdot, u), \Psi(\cdot, u)) := (\Phi(\cdot, u), (\psi_1(\cdot, u), \psi_2(\cdot, u))) : [0, T] \rightarrow \mathbb{R} \times iH \times \mathcal{H}$  is a *mild solution* to Equation (43) if  $\Phi(\cdot, u) \in C^1([0, T]; \mathbb{R}^+)$ ,  $\psi_1(\cdot, u) \in C([0, T]; iH)$ ,  $\psi_2(\cdot, u) \in C^1([0, T]; \mathcal{H}^+)$ , and the map  $(\Phi(\cdot, u), \Psi(\cdot, u))$  satisfies Equation (43).

In the following proposition, we show for every  $u = (u_1, u_2) \in iH \times \mathcal{H}^+$  the existence of a unique mild solution  $(\Phi(\cdot, u), \Psi(\cdot, u))$  to Equation (43).

**Proposition 3.2.** Let  $(b, B, m, \mu)$  be an admissible parameter set according to Assumption 2.1, let  $(\mathcal{A}, \text{dom}(\mathcal{A}))$  be the generator of a strongly continuous semigroup, and let  $D \in \mathcal{L}(H)$  be positive and self-adjoint. Then for every  $u \in iH \times \mathcal{H}^+$  and  $T \geq 0$  there exists a unique mild solution  $(\Phi(\cdot, u), \Psi(\cdot, u))$  to Equation (43) on  $[0, T]$ .



*Proof.* We set for  $k \in \mathbb{N}$ ,

$$m^{(k)}(d\xi) = \mathbf{1}_{\{\|\xi\| > 1/k\}} m(d\xi) \quad \text{and} \quad \mu^{(k)}(d\xi) = \mathbf{1}_{\{\|\xi\| > 1/k\}} \mu(d\xi). \tag{44}$$

Then for each  $k \in \mathbb{N}$ , we introduce  $F^{(k)} : \mathcal{H}^+ \rightarrow \mathbb{R}$  and  $R^{(k)} : i\mathcal{H} \times \mathcal{H}^+ \rightarrow \mathcal{H}$  defined, respectively, as

$$F^{(k)}(u) = \langle b, u \rangle - \int_{\mathcal{H}^+ \setminus \{0\}} (e^{-\langle \xi, u \rangle} - 1 + \langle \chi(\xi), u \rangle) m^{(k)}(d\xi), \tag{45}$$

$$R^{(k)}(h, u) = \tilde{R}^{(k)}(u) - \frac{1}{2} D^{1/2} h \otimes D^{1/2} h, \tag{46}$$

where  $\tilde{R}^{(k)}(u) = B^*(u) - \int_{\mathcal{H}^+ \setminus \{0\}} (e^{-\langle \xi, u \rangle} - 1 + \langle \chi(\xi), u \rangle) \frac{\mu^{(k)}(d\xi)}{\|\xi\|^2}, u \in \mathcal{H}^+$ . Consider for  $t \geq 0$ ,

$$\begin{cases} \frac{\partial \Phi^{(k)}}{\partial t}(t, u) = F^{(k)}(\psi_2^{(k)}(t, u)), & 0 < t \leq T, \quad \Phi^{(k)}(0, u) = 0, \\ \psi_1(t, u) = u_1 - i\mathcal{A}^* \left( i \int_0^t \psi_1(s, u) ds \right), & 0 < t \leq T, \quad \psi_1(0, u) = u_1, \\ \frac{\partial \psi_2^{(k)}}{\partial t}(t, u) = R^{(k)}(\psi_1(t, u), \psi_2^{(k)}(t, u)), & 0 < t \leq T, \quad \psi_2^{(k)}(0, u) = u_2. \end{cases} \tag{47}$$

Standard semigroup theory (see, e.g., (Engel & Nagel, 2000, Chapter II, Lemma 1.3)) ensures that the unique mild solution to Equation (47) is given by

$$\psi_1(t, (u_1, u_2)) = -iS^*(t)(iu_1), \quad t \in [0, T] \tag{48}$$

and  $\psi_1(\cdot, u) \in C([0, T]; i\mathcal{H})$ . Plugging  $\psi_1(t, u)$  into Equation (47), yields

$$\frac{\partial \psi_2^{(k)}}{\partial t}(t, u) = \tilde{R}^{(k)}(\psi_2^{(k)}(t, u)) + \frac{1}{2} D^{1/2} S^*(t)(iu_1) \otimes D^{1/2} S^*(t)(iu_1). \tag{49}$$

For  $k \in \mathbb{N}, u_1 \in i\mathcal{H}, t \in [0, T]$ , define  $\mathcal{R}_{u_1}^{(k)}(t, \cdot) : \mathcal{H}^+ \rightarrow \mathcal{H}$ , by

$$\mathcal{R}_{u_1}^{(k)}(t, h) = \tilde{R}^{(k)}(h) + \frac{1}{2} D^{1/2} S^*(t)(iu_1) \otimes D^{1/2} S^*(t)(iu_1). \tag{50}$$

By (Cox et al., 2020, Lemma 3.3) the function  $\tilde{R}^{(k)}$  is Lipschitz continuous on  $\mathcal{H}^+$  and since the term  $\frac{1}{2} D^{1/2} S^*(t)(iu_1) \otimes D^{1/2} S^*(t)(iu_1)$  does not depend on  $h$ , we conclude that for every  $t \in [0, T]$  and  $u_1 \in i\mathcal{H}$ , the function  $\mathcal{R}_{u_1}^{(k)}(t, \cdot)$  is Lipschitz continuous on  $\mathcal{H}^+$  as well, with the same Lipschitz constant as  $\tilde{R}^{(k)}$ . By (Cox et al., 2020, Lemma 3.2), for every  $k \in \mathbb{N}$ , the function  $\tilde{R}^{(k)}$  is quasi-monotone with respect to  $\mathcal{H}^+$  (see also (Cox et al., 2020, Definition 3.1) for the notion of quasi-monotonicity, and see (Deimling, 1977, Lemma 4.1 and Example 4.1) for relevant equivalent definitions). From this, we conclude that  $\mathcal{R}_{u_1}^{(k)}(t, \cdot)$  is also quasi-monotone for every  $t \in [0, T]$  and

$u_1 \in iH$ . Moreover, the growth condition

$$\|\mathcal{R}_{u_1}^{(k)}(t, u_2)\| \leq (\|B\|_{\mathcal{L}(H)} + 2k\|\mu(\mathcal{H}^+ \setminus \{0\})\|)\|u_2\| + \frac{1}{2}M^2 e^{2\omega t} \|D^{1/2}\|_{\mathcal{L}(H)}^2 \|u_1\|_H^2, \tag{51}$$

for every  $t \in [0, T]$ ,  $u_1 \in iH$  holds, where the constants  $M \geq 1$  and  $\omega \in \mathbb{R}$  are such that  $\|S^*(t)\|_{\mathcal{L}(H)} \leq M e^{\omega t}$ , for all  $t \geq 0$ , which exist for every strongly continuous semigroup, see (Engel & Nagel, 2000, Chapter I, Proposition 5.5). Thus the conditions of (Martin, 1976, Chapter 6, Theorem 3.1 and Proposition 3.2) are satisfied and we conclude from this the existence of a unique solution  $\psi_2^{(k)}(\cdot, u)$  on  $[0, T]$  to the equation

$$\frac{\partial \psi_2^{(k)}}{\partial t}(t, (u_1, u_2)) = \mathcal{R}_{u_1}^{(k)}\left(t, \psi_2^{(k)}(t, u)\right), \tag{52}$$

such that  $\psi_2^{(k)}(0, (u_1, u_2)) = u_2$ , hence  $\psi_2^{(k)}(\cdot, u)$  is the unique solution to Equation (47). By setting  $\Phi^{(k)}(t, u) = \int_0^t F^{(k)}(\psi_2^{(k)}(s, u))ds$  and the continuity of  $F^{(k)}$ , it follows that  $(\Phi^{(k)}(\cdot, u), \psi_1(\cdot, u), \psi_2^{(k)}(\cdot, u))$  is the unique mild solution to Equation (47) on  $[0, T]$ .

Now, let  $\mathcal{R}_{u_1} : [0, T] \times \mathcal{H}^+ \rightarrow \mathcal{H}$  be defined as the  $\mathcal{R}_{u_1}^{(k)}$  above, only with  $\tilde{R}^{(k)}$  replaced by  $\tilde{R}$ . By a similar reasoning as above and by (Cox et al., 2020, Lemma 3.2 and Remark 3.4), we conclude that  $\mathcal{R}_{u_1}(t, \cdot)$  is locally Lipschitz continuous on  $\mathcal{H}^+$  and quasi-monotone with respect to  $\mathcal{H}^+$  for every  $t \in [0, T]$  and  $u_1 \in iH$ . Thus by (Martin, 1976, Chapter 6, Theorem 3.1) for every  $t_0 \leq T$  and  $u_2 \in \mathcal{H}^+$ , there exists a  $t_0 < t_{\max} \leq T$  and a mapping  $\psi_{2,t_0}(\cdot, u) : [t_0, t_{\max}) \rightarrow \mathcal{H}^+$  such that

$$\frac{\partial \psi_{2,t_0}}{\partial t}(t, (u_1, u_2)) = \mathcal{R}_{u_1}(t, \psi_{2,t_0}(t, (u_1, u_2))), \quad \text{for } t \in [t_0, t_{\max}), \tag{53}$$

and  $\psi_{2,t_0}(t_0, (u_1, u_2)) = u_2$ . The function  $\mathcal{R}_{u_1}$  maps bounded sets of  $[0, \infty) \times \mathcal{H}^+$  into bounded sets of  $\mathcal{H}$ , thus by (Martin, 1976, Chapter 6, Proposition 1.1), it suffices to show that  $t \mapsto \psi_2(t, u) := \psi_{2,0}(t, u)$  is bounded throughout its lifetime, to conclude that  $t_{\max} = T$ . By arguing as in the proof of (Cox et al., 2020, Proposition 3.7), we conclude that for every  $t \geq 0$  and  $(u_1, u_2) \in iH \times \mathcal{H}^+$ , the sequence  $(\psi_2^{(k)}(t, u))_{k \in \mathbb{N}}$  is a nonincreasing sequence in  $\mathcal{H}^+$  converging to  $\psi_2(t, u) \geq 0$  for  $t \in [0, t_{\max})$ , hence

$$\|\psi_2(t, u)\| \leq \|\psi_2^{(k)}(t, u)\| \leq \|\psi_2^{(1)}(t, u)\|, \tag{54}$$

where the right-hand side is bounded on the whole  $[0, T]$ . Thus, we conclude that  $t_{\max} = T$  and  $\psi_2(\cdot, u)$  is the unique solution to Equation (43). Then again by inserting  $\psi_2(\cdot, u)$  into Equation (43) and the continuity of  $F$ , we conclude the existence of a unique solution  $\Phi(\cdot, u)$  of Equation (43) on  $[0, T]$ , and thus also of  $(\Phi(\cdot, u), \Psi(\cdot, u))$ , the unique mild solution to Equation (43) on  $[0, T]$ .  $\square$

### 3.2 | The affine property of our joint stochastic volatility model

Exploiting the existence of a solution to the generalized Riccati equation (43), we show in the following theorem that our joint stochastic volatility model  $Z = (X, Y)$  in Definition 2.9 has indeed the affine property.

**Theorem 3.3.** Let  $Z = (Y, X)$  be the stochastic volatility model in Definition 2.9 and let Assumption 2.11 hold. Moreover, let  $u = (u_1, u_2) \in iH \times \mathcal{H}$  and denote by  $(\Phi(\cdot, u), (\psi_1(\cdot, u), \psi_2(\cdot, u)))$  the mild solution to the generalized Riccati equation (43), the existence of which is guaranteed by Proposition 3.2. Then for all  $t \geq 0$ , it holds that

$$\mathbb{E} \left[ e^{\langle Y_t, u_1 \rangle_H - \langle X_t, u_2 \rangle} \right] = e^{-\Phi(t, u) + \langle y, \psi_1(t, u) \rangle_H - \langle x, \psi_2(t, u) \rangle}. \tag{55}$$

In applications, we are usually interested in distributional properties of the process  $(Y_t)_{t \geq 0}$ . Setting  $u_2 = 0$  in Equation (55), we obtain a quasi-explicit formula for the characteristic function of  $Y_t$  for  $t \geq 0$ . Due to its importance, we state it as a (trivial) corollary of Proposition 3.3.

**Corollary 3.4.** Let the assumption of Theorem 3.3 hold. Then the characteristic function of the process  $Y$  is exponential-affine in its initial value  $y \in H$  and the initial value  $x \in \mathcal{H}^+$  of the variance process  $X$ , more specifically, for all  $t \geq 0$  and  $u_1 \in iH$  we have

$$\mathbb{E} \left[ e^{\langle Y_t, u_1 \rangle_H} \right] = e^{-\Phi(t, (u_1, 0)) + \langle y, \psi_1(t, (u_1, 0)) \rangle_H - \langle x, \psi_2(t, (u_1, 0)) \rangle}. \tag{56}$$

In order to prove Theorem 3.3, we first consider the joint process  $(Y^{(n)}, X)$  obtained by replacing  $\mathcal{A}$  in Equation (29) by its Yosida approximation  $\mathcal{A}^{(n)} := n\mathcal{A}(nI - \mathcal{A})^{-1}$ . The use of the approximation will allow us to exploit the semimartingale theory and to apply the Itô formula and standard techniques in order to show that the approximating process  $(Y^{(n)}, X)$  is affine. Then we study the affine property for the limiting process (see Equation (58) below), when  $n$  goes to  $\infty$ .

Given the assumptions of Lemma 2.8, we know that inequality (30) holds. Therefore from standard theory on infinite dimensional SDEs (Da Prato & Zabczyk, 1992, Proposition 6.4), we know there exists a continuous adapted process  $Y^{(n)} : [0, \infty) \times \Omega \rightarrow H$  such that

$$Y_t^{(n)} = y + \int_0^t \mathcal{A}^{(n)} Y_s^{(n)} ds + \int_0^t X_s^{1/2} dW_s^Q, \quad t \geq 0. \tag{57}$$

Moreover, (Da Prato & Zabczyk, 1992, Proposition 7.5) ensures that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left\| Y_t^{(n)} - Y_t \right\|_H^2 \right] = 0. \tag{58}$$

See also (Cox & Hausenblas, 2013, Theorem 5.1, Definition 2.6) where convergence rates are obtained for Yosida approximations of SPDEs in the case the linear part of the drift is the generator of an analytic semigroup, for example, a Laplacian.

Regarding the corresponding Riccati equations, we have the following result:

**Proposition 3.5.** Let  $(b, B, m, \mu)$  satisfy Assumption 2.1, let  $(\mathcal{A}, \text{dom}(\mathcal{A}))$  be the generator of a strongly continuous semigroup, let  $D \in \mathcal{L}(H)$  be a positive self-adjoint operator, and let  $u \in iH \times \mathcal{H}^+$ . Moreover, let  $(\Phi(\cdot, u), (\psi_1(\cdot, u), \psi_2(\cdot, u)))$  be the mild solution to the generalized Riccati equation (43), and for  $n \in \mathbb{N}$ , let  $(\Phi^{(n)}(\cdot, u), (\psi_1^{(n)}(\cdot, u), \psi_2^{(n)}(\cdot, u)))$  be the solution to Equation (43) with  $\mathcal{A} = \mathcal{A}^{(n)}$ . Then

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} |\Phi^{(n)}(t, u) - \Phi(t, u)| = 0, \tag{59}$$

and

$$\lim_{n \rightarrow \infty} \left( \sup_{t \in [0, T]} \left\| \psi_1^{(n)}(t, u) - \psi_1(t, u) \right\|_H + \sup_{t \in [0, T]} \left\| \psi_2^{(n)}(t, u) - \psi_2(t, u) \right\| \right) = 0. \tag{60}$$

*Proof.* The uniform convergence of  $\psi_1^{(n)}(\cdot, u)$  to  $\psi_1(\cdot, u)$  on  $[0, T]$  is a well-known property of the Yosida approximation, see, for example, (Pazy, 1983, Proof of Theorem I.3.1).

Once this is established, the uniform convergence of  $\psi_2^{(n)}(\cdot, u)$  to  $\psi_2(\cdot, u)$  follows from (Martin, 1976, Chapter 6, Theorem 3.4). The uniform convergence of  $\Phi^{(n)}(\cdot, u)$  to  $\Phi(\cdot, u)$  follows from the uniform convergence of  $\psi_i^{(n)}(\cdot, u)$  to  $\psi_i(\cdot, u)$ ,  $i \in \{1, 2\}$ . Hence the statement of the proposition is proved.  $\square$

With Proposition 3.5 and classical stochastic calculus, we can now prove Theorem 3.3.

*Proof of Theorem 3.3.* Let  $T \geq 0$  and  $u = (u_1, u_2) \in iH \times H^+$  be arbitrary. Moreover, let  $(\Phi^{(n)}(\cdot, u), \Psi^{(n)}(\cdot, u))$ ,  $n \in \mathbb{N}$ , be the solution to Equation (43) with  $\mathcal{A} = \mathcal{A}^{(n)}$  (the  $n$ th Yosida approximation). Note that as  $\mathcal{A}^{(n)}$  is bounded,  $\Psi^{(n)}(\cdot, u) = (\psi_1^{(n)}(\cdot, u), \psi_2^{(n)}(\cdot, u))$  is differentiable. Define the function  $f_u^{(n)}(t, y, x) : [0, T] \times H \times H^+ \rightarrow \mathbb{C}$  as follows:

$$f_u^{(n)}(t, y, x) = e^{-\Phi^{(n)}(T-t, u) + \langle y, \psi_1^{(n)}(T-t, u) \rangle_H - \langle x, \psi_2^{(n)}(T-t, u) \rangle}. \tag{61}$$

Observe that  $f_u^{(n)} \in C_b^{1,2,1}([0, T] \times H \times H^+)$  and it holds

$$\begin{aligned} \frac{\partial}{\partial t} f_u^{(n)}(t, y, x) &= \left( \frac{\partial \Phi^{(n)}}{\partial t}(T-t, u) - \langle y, \frac{\partial \psi_1^{(n)}}{\partial t}(T-t, u) \rangle_H \right. \\ &\quad \left. + \langle x, \frac{\partial \psi_2^{(n)}}{\partial t}(T-t, u) \rangle \right) f_u^{(n)}(t, y, x) \\ &= \left( F \left( \psi_2^{(n)}(T-t, u) \right) - \langle y, (\mathcal{A}^{(n)})^* \psi_1^{(n)}(T-t, u) \rangle_H \right. \\ &\quad \left. + \langle x, R \left( \psi_1^{(n)}(T-t, u), \psi_2^{(n)}(T-t, u) \right) \rangle \right) f_u^{(n)}(t, y, x). \end{aligned} \tag{62}$$

As before we write  $K : H \times H \rightarrow \mathbb{R}$  for the function  $K(u, v) = e^{-\langle u, v \rangle} - 1 + \langle \chi(u), v \rangle$ ,  $\tilde{K} : H \times H \rightarrow \mathbb{R}$  for  $\tilde{K}(u, v) = e^{-\langle u, v \rangle} - 1 + \langle u, v \rangle$  and  $\tilde{\mu}^X(ds, d\xi) = \mu^X(ds, d\xi) - M(X_s, d\xi)ds$ . Then applying the Itô formula to  $(f_u^{(n)}(t, Y_t^{(n)}, X_t))_{0 \leq t \leq T}$ , yields

$$\begin{aligned} f_u^{(n)}(t, Y_t^{(n)}, X_t) &= f_u^{(n)}(0, Y_0, X_0) + \int_0^t \frac{\partial}{\partial t} f_u^{(n)}(s, Y_s^{(n)}, X_{s-}) ds \\ &\quad - \int_0^t f_u^{(n)}(s, Y_s^{(n)}, X_{s-}) \langle b + B(X_{s-}), \psi_2^{(n)}(T-s, u) \rangle ds \\ &\quad + \int_0^t f_u^{(n)}(s, Y_s^{(n)}, X_{s-}) \langle \mathcal{A}^{(n)} Y_s^{(n)}, \psi_1^{(n)}(T-s, u) \rangle_H ds \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2} \int_0^t f_u^{(n)}(s, Y_s^{(n)}, X_{s-}) \langle X_{s-}^{1/2} Q X_{s-}^{1/2}, \psi_1^{(n)}(T-s, u) \otimes \psi_1^{(n)}(T-s, u) \rangle ds \\
 & + \int_0^t \int_{H^+ \setminus \{0\}} f_u^{(n)}(s, Y_s^{(n)}, X_{s-}) K(\xi, \psi_2^{(n)}(T-s, u)) M(X_s, d\xi) ds \\
 & + \int_0^t f_u^{(n)}(s, Y_s^{(n)}, X_{s-}) \langle \psi_1^{(n)}(T-s, u), X_{s-}^{1/2} dW_s^Q \rangle_H \\
 & + \int_0^t \int_{H^+ \setminus \{0\}} f_u^{(n)}(s, Y_s^{(n)}, X_{s-}) \tilde{K}(\xi, \psi_2^{(n)}(T-s, u)) \bar{\mu}^X(ds, d\xi) \\
 & - \int_0^t f_u^{(n)}(s, Y_s^{(n)}, X_{s-}) \langle \psi_2^{(n)}(T-s, u), dJ_s \rangle.
 \end{aligned} \tag{63}$$

From Equation (62), we infer

$$\begin{aligned}
 f_u^{(n)}(t, Y_t^{(n)}, X_t) & = \int_0^t f_u^{(n)}(s, Y_s^{(n)}, X_{s-}) \langle \psi_1^{(n)}(T-s, u), X_{s-}^{1/2} dW_s^Q \rangle_H \\
 & + \int_0^t \int_{H^+ \setminus \{0\}} f_u^{(n)}(s, Y_s^{(n)}, X_{s-}) \tilde{K}(\xi, \psi_2^{(n)}(T-s, u)) \bar{\mu}^X(ds, d\xi) \\
 & - \int_0^t f_u^{(n)}(s, Y_s^{(n)}, X_{s-}) \langle \psi_2^{(n)}(T-s, u), dJ_s \rangle.
 \end{aligned} \tag{64}$$

We hence conclude that the process  $f_u^{(n)}(t, Y_t^{(n)}, X_t)$ ,  $t \in [0, T]$  is a local martingale. Furthermore, since it is bounded on  $[0, T]$ , it is a martingale and it holds

$$\mathbb{E} \left[ e^{\langle Y_T^{(n)}, u_1 \rangle_H - \langle X_T, u_2 \rangle} \right] = \mathbb{E} \left[ e^{-\Phi^{(n)}(T, u) + \langle Y_0^{(n)}, \psi_1^{(n)}(T, u) \rangle_H - \langle X_0, \psi_2^{(n)}(T, u) \rangle} \right] \tag{65}$$

$$= e^{-\Phi^{(n)}(T, u) + \langle y, \psi_1^{(n)}(T, u) \rangle_H - \langle x, \psi_2^{(n)}(T, u) \rangle}. \tag{66}$$

Now taking limits for  $n \rightarrow \infty$ , evoking Equation (58) and Proposition 3.5 and since  $T \geq 0$  was arbitrary, we conclude the proof.  $\square$

## 4 | EXAMPLES

In this section, we discuss several examples that are included in our class of joint stochastic volatility models with affine pure-jump variance. In all the examples, we assume that the first component  $Y$  is modeled in the abstract setting of Definition 2.9, that means we do not specify  $Q$  or  $\mathcal{A}$  any further, however, we stress here that the HJMM modeling framework as described in Filipović (2001), Benth and Krühner (2014) serves as the main example. In this case, the Hilbert space  $H$  is the Filipović space of all absolutely continuous functions  $f : \mathbb{R}^+ \rightarrow \mathbb{R}$  such that  $|f(0)|^2 +$

$\int_0^\infty w(x)f'(x)^2 dx < \infty$ , where  $w : \mathbb{R}^+ \rightarrow [1, \infty)$  is an increasing continuous function such that  $w^{-1/3}$  is integrable, for example,  $w(x) = e^{\beta x}$  for some constant  $\beta > 0$ . Then  $H$  is indeed a separable Hilbert space when equipped with the inner product  $\langle f, g \rangle_w = f(0)g(0) + \int_0^\infty w(x)f'(x)g'(x) dx$ . In this setting, the operator  $\mathcal{A}$  is given as the first derivative in space, that is,  $\mathcal{A} = \partial/\partial x$ . Our focus here is on correct specifications of the parameter set  $(b, B, m, \mu)$  and the initial value  $X_0 = x \in \mathcal{H}^+$  such that Assumption 2.1 holds and the associated process  $(X_t)_{t \geq 0}$  satisfies Assumption 2.4 as well as the joint process  $(Y, X)$  satisfies Assumption 2.11.

In Section 4.1, we show that an Ornstein–Uhlenbeck process driven by a Lévy subordinator in  $\mathcal{H}^+$  is included in our model class for the variance process  $X$ , which is implied by the parameter choice  $\mu = 0$ . Consequently, in Section 4.1.1, we conclude that our class of stochastic volatility models extends the infinite-dimensional lift of the BNS stochastic volatility model introduced in Benth et al. (2018).

In the subsequent examples, we focus on variance processes admitting for state-dependent jump intensities. Comparable to the Lévy-driven case, these examples have the advantage to model *the volatility clustering* phenomenon. This was, for example, illustrated in Leippold and Trojani (2008) in a finite-dimensional setting where the state space is the cone of symmetric positive semi-definite matrices. In this latter paper, it was shown in a numerical example that for this type of models, the volatilities and jump intensities are time-varying leading to a clustering of jump events in phases of high jump intensities.

In Section 4.2, we construct a variance process  $X$ , which takes values in  $\{x + \lambda z : \lambda \geq 0\}$  for some fixed  $z \in \mathcal{H}^+$ . This is somewhat of a toy model: although the variance process is infinite-dimensional, its randomness is one-dimensional. In Section 4.3, we consider a *truly* infinite-dimensional stochastic variance process  $X$ . However, to ensure that Assumption 2.11 is satisfied, we assume that both  $Q$  and  $X_t, t \geq 0$  are diagonalizable with respect to the same fixed orthonormal basis. We close this section with Section 4.4 in which we show the benefits of the model discussed in Remark 2.12, which does not require Assumption 2.11 and thus allows for a more general variance process.

## 4.1 | The operator-valued BNS SV model

In Benth et al. (2018), the authors introduced an operator-valued volatility model that is an extension of the finite-dimensional model introduced in Barndorff-Nielsen and Stelzer (2007) (and thus they named it the operator-valued BNS SV model). In their model, it is assumed that the volatility process  $X$  is driven by a Lévy process  $(L_t)_{t \geq 0}$ . In order to ensure that  $X$  is positive, they assume that  $t \mapsto L_t$  is almost surely increasing with respect to  $\mathcal{H}^+$ , that is, that  $L$  is an  $\mathcal{H}^+$ -subordinator. This holds if and only if for any fixed  $t \geq 0$ , we have  $\mathbb{P}(L_t \in \mathcal{H}^+) = 1$ , (see also (Pérez-Abreu & Rocha-Arteaga, 2006, Proposition 9)). Roughly speaking, the model considered in Benth et al. (2018) amounts to taking  $\mu \equiv 0$  in our setting (i.e., to considering a stochastic volatility model  $Z = (Y, X)$  in Definition 2.9 with parameters  $(b, B, m, 0, Q, A)$ ). Indeed, in Section 4.1.1 below, we demonstrate that the model introduced in Benth et al. (2018) is fully contained in our setting.

First, however, we show for this stochastic volatility model that the characteristic function of  $Y_t, t \in [0, T]$  can be made explicit up to the Laplace exponent of the driving Lévy subordinator, see Proposition 4.1 below.

**Proposition 4.1.** *Let  $(b, B, m, 0)$  satisfy Assumption 2.1 and let  $X$  be the associated affine process with  $X_0 = x \in H^+$ . Moreover, let  $Q \in \mathcal{L}_1(H)$  be positive and self-adjoint such that Assumption 2.11 holds and  $\mathcal{A} : \text{dom}(\mathcal{A}) \subseteq H \rightarrow H$  be the generator of the strongly continuous semigroup  $(S(t))_{t \geq 0}$ . Then for every  $y \in H$ , the mild solution  $Y$  of Equation (29) exists and for all  $v_1 \in H$  and  $t \geq 0$  it holds that*

$$\begin{aligned} \mathbb{E} \left[ e^{i \langle Y_t, v_1 \rangle_H} \right] &= \exp \left( i \langle y, S^*(t)v_1 \rangle_H \right) \\ &\quad \times \exp \left( - \int_0^t \varphi_L \left( \frac{1}{2} \int_0^s e^{(s-\tau)B^*} (D^{1/2}S^*(\tau)v_1)^{\otimes 2} d\tau \right) ds \right) \\ &\quad \times \exp \left( - \frac{1}{2} \langle x, \int_0^t e^{\tau B^*} (D^{1/2}S^*(t-\tau)v_1)^{\otimes 2} d\tau \rangle \right), \end{aligned} \tag{67}$$

where  $\varphi_L : H \rightarrow \mathbb{C}$  denotes the Laplace exponent of the Lévy process  $L$  with characteristics  $(b, 0, m)$  and is given by

$$\varphi_L(u) = \langle b, u \rangle - \int_{H^+ \setminus \{0\}} \left( e^{-\langle \xi, u \rangle} - 1 + \langle \chi(\xi), u \rangle \right) m(d\xi), \quad u \in H^+. \tag{68}$$

*Proof.* The admissible parameter set  $(b, B, m, 0)$  corresponds to the solution  $X$  of a linear stochastic differential equation driven by a Lévy process  $(L_t)_{t \geq 0}$  with characteristics  $(b, 0, m)$ . It is easy to see that  $X$  has càdlàg paths and hence Assumption 2.4 is satisfied. Thus we are in the situation of Corollary 3.4 and conclude that the affine transform formula (56) holds with  $(\Phi(\cdot, v), (\psi_1(\cdot, v), \psi_2(\cdot, v)))$  being the mild solution to the generalized Riccati equations associated with  $(b, B, m, 0)$  and initial value  $v = (v_1, 0)$  for  $v_1 \in H$ . Hence, it is left to show that the solutions have the explicit form as indicated by formula (67). Indeed, observe that the unique mild solution to Equation (43) is given by  $\psi_1(t, (v_1, 0)) = iS^*(t)v_1$ . Then inserting  $\psi_1(\cdot, (v_1, 0))$  into Equation (43) and recalling that  $\mu = 0$  yields

$$\frac{\partial \psi_2}{\partial s}(s, (v_1, 0)) = B^*(\psi_2(s, (v_1, 0))) + \frac{1}{2} D^{1/2}S^*(s)v_1 \otimes D^{1/2}S^*(s)v_1. \tag{69}$$

By the variation of constant formula and recalling that  $\psi_2(0, (v_1, 0)) = 0$ , we conclude that the unique solution  $\psi_2(\cdot, (v_1, 0))$  is given by

$$\psi_2(t, (v_1, 0)) = \frac{1}{2} \int_0^t e^{(t-s)B^*} (D^{1/2}S^*(s)v_1 \otimes D^{1/2}S^*(s)v_1) ds \tag{70}$$

$$= \frac{1}{2} \int_0^t e^{\tau B^*} (D^{1/2}S^*(t-\tau)v_1 \otimes D^{1/2}S^*(t-\tau)v_1) d\tau. \tag{71}$$



Lastly, by inserting  $\psi_2(\cdot, (v_1, 0))$  into Equation (43) and since  $F$  is a continuous function, integrating Equation (43) with respect to  $t$  gives

$$\begin{aligned} \Phi(t, (v_1, 0)) &= \int_0^t \left( \langle b, \psi_2(s, (v_1, 0)) \rangle \right. \\ &\quad \left. - \int_{\mathcal{H}^+ \setminus \{0\}} (e^{-\langle \xi, \psi_2(s, (v_1, 0)) \rangle} - 1 + \langle \chi(\xi), \psi_2(s, (v_1, 0)) \rangle) m(d\xi) \right) ds \\ &= \int_0^t \varphi_L(\psi_2(s, (v_1, 0))) ds. \end{aligned} \tag{72}$$

Now, by inserting those formulas of  $\Phi(t, (v_1, 0))$ ,  $\psi_1(t, (v_1, 0))$  and  $\psi_2(t, (v_1, 0))$  into Equation (56), we obtain the desired formula. □

#### 4.1.1 | Comparison with the model introduced in Benth et al. (2018)

In Benth et al. (2018), the following infinite dimensional volatility model is considered for  $t \geq 0$ :

$$\begin{cases} dY_t = AY_t dt + \sqrt{X_t} dW_t^Q, \\ dX_t = B(X_t) dt + dL_t, \end{cases} \tag{73}$$

where  $(L_t)_{t \geq 0}$  is an  $\mathcal{L}_2(H)$ -valued Lévy process satisfying  $\mathbb{P}(L_t \in \mathcal{H}^+) = 1$  for every  $t \geq 0$ . Moreover, it is assumed that  $B : \mathcal{L}_2(H) \rightarrow \mathcal{L}_2(H)$  is of the form  $B(v) = cvc^*$  or  $B(v) = cv + vc^*$  for some  $c \in \mathcal{L}(H)$ . Finally,  $\mathcal{A} : \text{dom}(\mathcal{A}) \subseteq H \rightarrow H$  is assumed to be an unbounded operator generating a strongly continuous semigroup and  $(W_t)_{t \geq 0}$  is assumed to be an  $H$ -valued Brownian motion, which (at least, in the part of Benth et al. (2018) involving the affine property of  $(Y, X)$ ) is assumed to be independent of  $(L_t)_{t \geq 0}$  and with a covariance operator  $Q$  that satisfies Assumption 2.11.

In this section, we show that the joint volatility model (73) is a special case of our model in the case that  $\mu \equiv 0$ , more specifically, that (Benth et al., 2018, Proposition 3.2) is a special case of Proposition 4.1 above. To this end, we first remark that if  $\gamma \in \mathcal{L}_2(H)$ ,  $C \in \mathcal{L}_1(\mathcal{L}_2(H))$ , and  $\eta : \mathcal{B}(\mathcal{L}_2(H)) \rightarrow [0, \infty]$  are the characteristics of  $L$ , then  $C|_{\mathcal{H}} \equiv 0$  thanks to (Benth et al., 2018, Proposition 2.10). Moreover, in view of Lemma A.2, we have that  $\gamma \in \mathcal{H}$ ,  $C = 0$ , and  $\text{supp}(\eta) \subseteq \mathcal{H}$  (this answers an open question in Benth et al. (2018): see the discussion prior to Proposition 2.11 in that article). Finally, it is easily verified that  $B(\mathcal{H}) \subset \mathcal{H}$  in both cases described above, so although the ‘ambient’ space for  $X$  is  $\mathcal{L}_2(H)$  in Benth et al. (2018), one can, without loss of generality, take  $\mathcal{H}$  as ambient space for  $X$ .

Next, note that the process  $X$  in (73) has càdlàg paths by construction (see also Lemma 2.5), so Assumption 2.4 is satisfied. It remains to verify that Assumption 2.1 is met. Note that Assumption 2.1 (iii) is immediately satisfied as  $\mu \equiv 0$ . To verify that the two choices for  $B$  described above satisfy Assumption 2.1 (iv), we recall from (Benth et al., 2018, Lemma 2.2) that in these cases one has  $e^{tB}(\mathcal{H}^+) \subseteq \mathcal{H}^+$  for all  $t \geq 0$ , which, by (Lemmert & Volkman, 1998, Theorem 1), implies that  $B$  is quasi-monotone. Finally, Assumptions 2.1 (ii) and (i) hold due to the following result from Pérez-Abreu and Rocha-Arteaga (2006):

**Theorem 4.2.** Let  $(L_t)_{t \geq 0}$  be an  $\mathcal{H}$ -valued Lévy process with characteristic triplet  $(\gamma, C, \eta)$ . Then the following two statements are equivalent:

- i) for all  $t \geq 0$  we have  $\mathbb{P}(L_t \in \mathcal{H}^+) = 1$ ;
- ii)  $C = 0$ ,  $\text{supp}(\eta) \subseteq \mathcal{H}^+$  and there exists an  $I_\eta \in \mathcal{H}$  such that  $\xi \mapsto |\langle \chi(\xi), h \rangle|$  is  $\eta$ -integrable and  $\int_{\mathcal{H}^+ \setminus \{0\}} \langle \chi(\xi), h \rangle \eta(d\xi) = \langle I_\eta, h \rangle$  for all  $h \in \mathcal{H}$ , and such that  $\gamma - I_\eta \in \mathcal{H}^+$ .

*Proof.* First, note that  $\mathcal{H}^+$  is regular (see, e.g., (Karlin, 1959, Theorem 1)), i.e., any sequence  $(A_n)_{n \in \mathbb{N}}$  in  $\mathcal{H}$  satisfying  $A_1 \leq_{\mathcal{H}^+} A_2 \leq_{\mathcal{H}^+} \dots \leq_{\mathcal{H}^+} A$  for some  $A \in \mathcal{H}$  is convergent in  $\mathcal{H}$ . The cone is also normal: its dual  $\mathcal{H}^+$  is generating for  $\mathcal{H}$ . Thus  $\mathcal{H}^+$  is a regular normal proper cone in the terminology of Pérez-Abreu and Rocha-Arteaga (2006). Now, note that the implication “i)  $\Rightarrow$  ii)” follows from (Pérez-Abreu & Rocha-Arteaga, 2006, Theorem 18), and reverse implication follows from (Pérez-Abreu & Rocha-Arteaga, 2006, Theorem 10). □

### 4.2 | An essentially one-dimensional variance process

We now present a simple example of a pure-jump affine process  $(X_t)_{t \geq 0}$  on  $\mathcal{H}^+$  with state-dependent jump intensity. Starting from its initial value  $X_0 = x \in \mathcal{H}^+$  this process moves along a single vector  $z \in \mathcal{H}^+ \setminus \{0\}$  and is thus essentially one-dimensional. For this case we specify an admissible parameter set  $(b, B, m, \mu)$  such that the associated affine process  $X$  has càdlàg paths and is driven by a pure-jump process  $(J_t)_{t \geq 0}$  with jumps of size  $\xi \in (0, \infty)$  in the single direction  $z \in \mathcal{H}^+$  with  $\|z\| = 1$  and such that the jump-intensity depends on the current state of the process  $X$ . For the sake of simplicity, we let the constant parameters  $b$  and  $m$  be zero. Moreover, we shall fix the dependency structure by means of a fixed vector  $g \in \mathcal{H}^+ \setminus \{0\}$ . We then take a measure  $\eta : \mathcal{B}((0, \infty)) \rightarrow [0, \infty)$  such that  $\int_0^\infty \lambda^{-2} \eta(d\lambda) < \infty$  and define the vector valued measure  $\mu : \mathcal{B}(\mathcal{H}^+ \setminus \{0\}) \rightarrow \mathcal{H}^+$  by

$$\mu(A) := g\eta(\{\lambda \in \mathbb{R}^+ : \lambda z \in A\}). \tag{74}$$

From the assumption that  $\int_0^\infty \lambda^{-2} \eta(d\lambda) < \infty$  it follows that for every  $x \in \mathcal{H}^+$  the measure  $M(x, d\xi)$  on  $\mathcal{B}(\mathcal{H}^+ \setminus \{0\})$  defined by

$$M(x, d\xi) := \frac{\langle x, \mu(d\xi) \rangle}{\|\xi\|^2} \tag{75}$$

is finite and thus also

$$\int_{\mathcal{H}^+ \setminus \{0\}} \langle \chi(\xi), u \rangle \frac{\langle x, \mu(d\xi) \rangle}{\|\xi\|^2} = \int_0^1 \lambda^{-1} \eta(d\lambda) \langle z, u \rangle \langle g, x \rangle < \infty, \quad \forall u, x \in \mathcal{H}^+. \tag{76}$$

We now must find a linear operator  $B : \mathcal{H} \rightarrow \mathcal{H}$  such that

$$\langle B^*(u), x \rangle - \int_{\mathcal{H}^+ \setminus \{0\}} \langle \chi(\xi), u \rangle \frac{\langle x, \mu(d\xi) \rangle}{\|\xi\|^2} \geq 0, \tag{77}$$

whenever  $\langle x, u \rangle = 0$  for  $x, u \in \mathcal{H}^+$ . The simplest example is obtained by taking

$$B(u) := \int_{\mathcal{H}^+ \setminus \{0\}} \chi(\xi) \frac{\langle u, \mu(d\xi) \rangle}{\|\xi\|^2}, \quad u \in \mathcal{H}. \tag{78}$$

From this we see that  $B$  and  $\mu$  indeed satisfy condition (77) and conclude that the parameter set  $(0, B, 0, \mu)$  is an admissible parameter set according to Definition 2.1. Thus the existence of an associated affine process  $X$  on  $\mathcal{H}^+$  is guaranteed by Theorem 2.3. Since  $\int_{\mathcal{H}^+ \setminus \{0\}} \|\xi\|^{-2} \langle x, \mu(d\xi) \rangle < \infty$  for all  $x \in \mathcal{H}^+$ , it follows from Proposition 2.5 that Assumption 2.4 is satisfied as well. It remains to ensure that Assumption 2.11 is satisfied. For this purpose it suffices to assume that  $x$  and  $z$  commute with  $Q$ . Indeed, note that for  $u \in \{x + \lambda z : \lambda \in [0, \infty)\}$  we have  $B(u) \in \{\lambda z : \lambda \in [0, \infty)\}$ . Thus from the semimartingale representation (22), we see that  $X_t \in \{x + \lambda z : \lambda \in [0, \infty)\}$  for all  $t \geq 0$ , that means  $X_t$  commutes with  $Q$  for all  $t \geq 0$  and therefore Assumption 2.11 is satisfied.

### 4.3 | A state-dependent stochastic volatility model on a fixed ONB

In this example we specify an admissible parameter set  $(b, B, m, \mu)$  giving more general affine dynamics of the associated variance process  $X$  on  $\mathcal{H}^+$ . In the previous Section 4.2 we imposed additional commutativity assumptions on the initial value  $X_0 = x \in \mathcal{H}^+$ , the jump direction  $z$  and the covariance operator  $Q$ . In this example we allow for a more general jump behavior, while maintaining Assumption 2.11. To do so, we pick up the discussion preceding Remark 2.12 and note here that Assumption 2.11 is satisfied, whenever  $Q$  and  $X_t$  commute for all  $t \geq 0$ . Recall that  $Q$  and  $(X_t)_{t \geq 0}$  commute if and only if they are jointly diagonalizable. This motivates the consideration of a variance process  $X$  that is diagonalizable with respect to a fixed ONB.

More concretely, let  $(e_n)_{n \in \mathbb{N}}$  be an ONB of eigenvectors of the operator  $Q$ . We model  $X$  such that  $X_t$  ( $t \geq 0$ ) is diagonalizable with respect to the ONB  $(e_n)_{n \in \mathbb{N}}$ , i.e.

$$X_t = \sum_{i \in \mathbb{N}} \lambda_i(t) e_n \otimes e_n, \quad t \geq 0, \tag{79}$$

for the sequence of eigenvalues  $(\lambda_i(t))_{i \in \mathbb{N}}$  of  $X_t$  in  $\ell_2^+$ . Concerning the modeling of the dynamics of  $(X_t)_{t \geq 0}$ , this essentially means that we model the dynamics of the sequence of eigenvalues  $(\lambda_i(t))_{i \in \mathbb{N}}$  in  $\ell_2^+$  only.

We now come to a specification of the parameters  $(b, B, m, \mu)$  such that Assumption 2.1 is satisfied and moreover such that  $X_t$  is indeed diagonalizable with respect to  $(e_n)_{n \in \mathbb{N}}$  for all  $t \geq 0$ . Let the measure  $m : \mathcal{B}(\mathcal{H}^+ \setminus \{0\}) \rightarrow [0, \infty)$  be such that for  $A \in \mathcal{B}(\mathcal{H}^+ \setminus \{0\})$  we have

$$m(A) := \sum_{n \in \mathbb{N}} m_n(\{\lambda \in (0, \infty) : \lambda(e_n \otimes e_n) \in A\}), \tag{80}$$

for a sequence  $(m_n)_{n \in \mathbb{N}}$  of finite measures on  $\mathcal{B}((0, \infty))$  such that

$$\sum_{n \in \mathbb{N}} m_n((0, \infty)) < \infty \quad \text{and} \quad \sum_{n \in \mathbb{N}} \int_1^\infty \lambda^2 m_n(d\lambda) < \infty. \tag{81}$$

Then let  $\tilde{b} \in \mathcal{H}^+$  be diagonalizable with respect to  $(e_n)_{n \in \mathbb{N}}$  and set

$$b := \tilde{b} + \int_{\mathcal{H}^+ \setminus \{0\}} \chi(\xi) m(d\xi) = \tilde{b} + \sum_{n \in \mathbb{N}} \int_0^1 \lambda m_n(d\lambda) e_n \otimes e_n. \tag{82}$$

We see that  $b$  and  $m$  satisfy their respective conditions in Assumption 2.1. Now, let  $(g_n)_{n \in \mathbb{N}} \subseteq \mathcal{H}^+$  and define  $\mu(d\xi) : \mathcal{B}(\mathcal{H}^+ \setminus \{0\}) \rightarrow \mathcal{H}^+$  by

$$\mu(A) := \sum_{n \in \mathbb{N}} g_n \mu_n(\{\lambda \in (0, \infty) : \lambda(e_n \otimes e_n) \in A\}), \tag{83}$$

for a sequence of finite measures  $(\mu_n)_{n \in \mathbb{N}}$  on  $\mathcal{B}((0, \infty))$  such that

$$\sum_{n \in \mathbb{N}} g_n \mu_n((0, \infty)) \in \mathcal{H}^+ \quad \text{and} \quad \sum_{n \in \mathbb{N}} \int_0^1 \lambda^{-2} \mu_n(d\lambda) \langle g_n, x \rangle < \infty, \quad \forall x \in \mathcal{H}^+. \tag{84}$$

Moreover, let  $G \in \mathcal{H}$  be diagonalizable with respect to  $(e_n)_{n \in \mathbb{N}}$ , note that this implies that for any  $x \in \mathcal{H}^+$  that is diagonalizable with respect to  $(e_n)_{n \in \mathbb{N}}$ , we have that  $Gx + xG^*$  is diagonalizable with respect to  $(e_n)_{n \in \mathbb{N}}$  as well. We thus define the linear operator  $B : \mathcal{H} \rightarrow \mathcal{H}$  by

$$B(u) = Gu + uG^* + \int_{\mathcal{H}^+ \setminus \{0\}} \chi(\xi) \frac{\langle \mu(d\xi), u \rangle}{\|\xi\|^2}, \quad u \in \mathcal{H}. \tag{85}$$

Now, one can check that  $B$  and  $\mu$  indeed satisfy their respective conditions in Assumption 2.1. Due to the first condition on  $m$  in (81) and the second on  $\mu$  in (84), it follows from Proposition 2.5 that Assumption 2.4 is satisfied.

Again from the semimartingale representation (22) we conclude that for all  $t \geq 0$  the operator  $X_t$  is diagonalizable with respect to  $(e_n)_{n \in \mathbb{N}}$  and thus Assumption 2.11 is satisfied as well.

#### 4.4 | A general state-dependent stochastic volatility model

In this example we show that modeling under the alternative formulation of the model  $(Y, X)$  provided by Remark 2.12 gives considerably more freedom in the model parameter specification. We write  $\hat{b} = b + \int_{\mathcal{H}^+ \cap \{\|\xi\| > 1\}} \xi m(d\xi)$  and for every  $u \in \mathcal{H}$  we set  $\hat{B}(u) = B(u) + \int_{\mathcal{H}^+ \cap \{\|\xi\| > 1\}} \xi \frac{\langle u, \mu(d\xi) \rangle}{\|\xi\|^2}$ . We then see that for the stochastic volatility model  $(Y, X)$  given by the SDE

$$d(Y_t, X_t) = \begin{bmatrix} 0 \\ \hat{b} \end{bmatrix} + \begin{bmatrix} AY_t \\ \hat{B}(X_t) \end{bmatrix} dt + \begin{bmatrix} D^{1/2} X_t^{1/2} & 0 \\ 0 & 0 \end{bmatrix} d \begin{bmatrix} W_t \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ J_t \end{bmatrix}, \quad t \geq 0, \tag{86}$$

with  $(Y_0, X_0) = (y, x) \in H \times \mathcal{H}^+$ , and  $W = (W_t)_{t \geq 0}$  a cylindrical Brownian motion, the Assumption 2.11 can be dropped. Therefore, every admissible parameter set  $(b, B, m, \mu)$ , such that the associated affine process  $X$  satisfies Assumption 2.4 is a valid parameter choice. To emphasize the gained flexibility, we compare it with the example in Section 4.3. For simplicity, we let  $(e_n)_{n \in \mathbb{N}}$  be some ONB of  $H$  and specify  $m$  and  $\mu$  as in (80) and (83), respectively, with respect to this ONB.

This means that the noise in the variance process  $X$  again occurs on the diagonal only. However,  $Q$  need not be diagonalizable with respect to  $(e_n)_{n \in \mathbb{N}}$  and instead of taking  $b$  to be diagonalizable with respect to the ONB  $(e_n)_{n \in \mathbb{N}}$  and  $B$  of the particular form (85), we allow for a general drift  $\hat{b} \in \mathcal{H}$  such that  $\hat{b} - \int_{\mathcal{H}^+ \setminus \{0\}} \chi(\xi) m(d\xi) \geq 0$ . Moreover, let  $C$  be a bounded linear operator on  $H$  and define  $B \in \mathcal{L}(H)$  by

$$B(u) = Cu + uC^* + \Gamma(u), \tag{87}$$

for some  $\Gamma \in \mathcal{L}(H)$  with  $\Gamma(H^+) \subseteq H^+$  and such that

$$\langle \Gamma(x), u \rangle - \int_{\mathcal{H}^+ \setminus \{0\}} \langle \chi(\xi), u \rangle \frac{\langle x, \mu(d\xi) \rangle}{\|\xi\|^2} \geq 0. \tag{88}$$

We can again check, that  $(b, B, m, \mu)$  satisfies the Assumption 2.1 and the associated affine process  $X$  Assumption 2.4. Then according to Equation (22), the variance process  $X$  has the representation

$$X_t = X_0 + \int_0^t \left( b + CX_s + X_s C^* + \Gamma(X_s) + \int_{\mathcal{H}^+ \cap \{\|\xi\| \geq 1\}} \xi M(X_s, d\xi) \right) ds + J_t, \quad t \geq 0, \tag{89}$$

where  $M$  is as in Equation (21) and  $J$  is a purely discontinuous square integrable martingale with a compensator  $\nu^X$ . The dynamics (89) has a similar structure as the affine dynamics of covariance processes in finite dimensions presented in (Cuchiero et al., 2011, equation 1.2) in the pure-jump case. Indeed, both models have an affine drift and are driven by a pure-jump process whose compensator is an affine function of  $X$ . As mentioned above, this model will also demonstrate clustering behavior.

## 5 | CONCLUSION AND OUTLOOK

In Section 2, we introduce an infinite dimensional stochastic volatility model. More specifically, we consider a process  $Y$  that solves a linear SDE in a Hilbert space  $H$  with additive noise, where the variance of the noise is dictated by a process  $X$  taking values in the space of self-adjoint Hilbert-Schmidt operators on  $H$ . The process  $X$  is assumed to be an affine pure-jump process that allows for state-dependent jump intensities; its existence has been established in the previous work Cox et al. (2020) under certain admissibility conditions on the parameters involved (see Assumption 2.1).

In the derivation of the affine transform formula, we make use of Hilbert-valued semimartingale calculus, for this reason, we must assume that  $X$  has càdlàg paths (see Assumption 2.4). Currently, we establish existence of càdlàg paths under limited conditions (see Proposition 2.5). Relaxing these conditions is one of the aims of the working paper Karbach (2022) where the author considers finite-dimensional approximations (in particular, Galerkin approximations of the associated generalized Riccati equations are considered) and studies convergence of the variance process in the Skorohod topology.

Having introduced the joint model, we prove that it is affine (see Theorem 3.3). To this end, we need an additional “commutativity”-type assumption, see Assumption 2.11. This assumption is avoided by considering a slightly different model, see Remark 2.12 and Section 4.4.

Our model extends the model introduced in Benth et al. (2018), where the authors assume that  $X$  is driven by a suitably chosen Lévy process (see Section 4.1.1). In Section 4, we also discuss other concrete examples of our model.

Another way to avoid Assumption 2.11 would be to construct a variance process  $X$  that takes values in the space of self-adjoint *trace class* operators. Indeed, in this case, we can assume that the process  $W^Q$  driving  $Y$  is a cylindrical Brownian motion (i.e.,  $Q$  is the identity). However, taking the trace class operators as a state space is not trivial as this is a nonreflexive Banach space. We aim to pursue this direction of research in a forthcoming work.

Finally, in a subsequent work, we plan to consider the dynamics of forward rates in commodity markets modeled by our proposed stochastic volatility dynamics. Then study the problem of computing option prices on these forwards. In practice, these computations require finite-rank approximations of the associated generalized Riccati equations as being considered in Karbach (2022).

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## DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

## ORCID

Sonja Cox  <https://orcid.org/0000-0002-9417-1542>

Sven Karbach  <https://orcid.org/0000-0002-9262-9871>

Asma Khedher  <https://orcid.org/0000-0003-1528-3923>

## ENDNOTE

<sup>1</sup>Alternatively, we could work in the framework of generalized Feller semigroups and their generators, as we did in Cox et al. (2020), but this would require us to introduce more concepts.

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**APPENDIX A: AUXILIARY RESULTS**

**Lemma A.1.** *Let  $(H, \|\cdot\|, \langle \cdot, \cdot \rangle)$  be a separable real Hilbert space, let  $K \subseteq H$  be a cone such that  $H = K - K$  and let  $\mu_1, \mu_2 : B(H) \rightarrow \mathbb{R}$  be measures such that  $\text{supp}(\mu_1), \text{supp}(\mu_2) \subseteq K$ , and  $\int_H e^{-\langle x, y \rangle} \mu_1(dx) = \int_H e^{-\langle x, y \rangle} \mu_2(dx)$  for all  $y \in K$ . Then  $\mu_1 = \mu_2$ .*

*Proof.* Let  $(e_n)_{n \in \mathbb{N}}$  be an orthonormal basis for  $H$  and let  $e_n^+, e_n^- \in K$  be such that  $e_n = e_n^+ - e_n^-$ ,  $n \in \mathbb{N}$ . By Dynkin’s lemma, it suffices to prove that  $\mu_1$  and  $\mu_2$  coincide on sets of the type  $\cap_{i=1}^n \{ \langle \cdot, e_i^+ \rangle \in B_i^+, \langle \cdot, e_i^- \rangle \in B_i^- \}$ ,  $B_1^+, B_1^-, \dots, B_n^+, B_n^- \in B(\mathbb{R})$ , and  $n \in \mathbb{N}$ . This implies that it suffices to prove the lemma for the case  $H = \mathbb{R}^n$  and  $K = [0, \infty)^n \subseteq \mathbb{R}^n$ ,  $n \in \mathbb{N}$ . In this case, the result follows by a standard Stone–Weierstrass argument, see, for example, (Hytönen et al., 2017, Theorem E.1.14). □

**Lemma A.2.** *Let  $(H, \langle \cdot, \cdot \rangle)$  be a Hilbert space, let  $U \subseteq H$  be a closed linear subspace and let  $P_U : H \rightarrow U$  be the orthogonal projection of  $H$  onto  $U$ . Moreover, let  $(L_t)_{t \geq 0}$  be a  $H$ -valued Lévy process satisfying  $\mathbb{P}(L_1 \in U) = 1$  and let  $\gamma \in H$ ,  $C \in \mathcal{L}_1(H)$ , and  $\eta : B(H \setminus \{0\}) \rightarrow [0, \infty]$  be its characteristics. In addition, let  $\gamma_s \in U$ ,  $C_s \in \mathcal{L}_1(U)$ , and  $\eta_s : B(U \setminus \{0\}) \rightarrow [0, \infty]$  be the characteristics of  $L$  when interpreted as a  $(U, \langle \cdot, \cdot \rangle)$ -valued process. Then  $\gamma = \gamma_s$ ,  $C = C_s P_U$ , and  $\eta(A) = \eta_s(A \cap U)$  for all  $A \in B(H \setminus \{0\})$ . In particular,  $Ch = 0$  whenever  $h \in U^\perp$  and  $\text{supp}(\eta) \subseteq U$ .*

*Proof.* Define  $\tilde{\eta} : B(H \setminus \{0\}) \rightarrow [0, \infty]$  by  $\tilde{\eta}(A) = \eta_s(A \cap U)$ ,  $A \in B(H \setminus \{0\})$ . Then for all  $h \in H$  and  $t \geq 0$  we have, using that  $L_t \in U$  a.s.

$$\begin{aligned} \mathbb{E}(e^{i\langle L_t, h \rangle}) &= \mathbb{E}(e^{i\langle L_t, P_U h \rangle}) = \exp(t(i\langle \gamma_s, P_U h \rangle - \langle C_s P_U h, P_U h \rangle)) \\ &\quad \times \exp\left(t \int_{U \setminus \{0\}} (e^{i\langle \xi, P_U h \rangle} - 1 + i\langle \xi, P_U h \rangle 1_{\{\|\xi\| < 1\}}) \eta_s(d\xi)\right) \\ &= \exp\left(t\left(i\langle \gamma_s, h \rangle - \langle C_s P_U h, h \rangle + \int_{H \setminus \{0\}} (e^{i\langle \xi, h \rangle} - 1 + i\langle \xi, h \rangle 1_{\{\|\xi\| < 1\}}) \tilde{\eta}(d\xi)\right)\right) \end{aligned} \tag{A.1}$$

The result now follows from the uniqueness of the characteristic triplet. □