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UNIVERSAL POINTS IN THE ASYMPTOTIC SPECTRUM OF TENSORS

MATTHIAS CHRISTANDL, PÉTER VRANA, AND JEROEN ZUIDDAM

1. INTRODUCTION

1.1. The asymptotic restriction problem. We study the asymptotic restriction problem, following the pioneering work of Volker Strassen [57–60]. The asymptotic restriction problem is a problem about multilinear maps $f : \mathbb{F}^{n_1} \times \cdots \times \mathbb{F}^{n_k} \rightarrow \mathbb{F}$ over an arbitrary field \mathbb{F} . Letting (e_1, \dots, e_{n_i}) be the standard basis of \mathbb{F}^{n_i} , one may equivalently think of f as the k -tensor $t \in \mathbb{F}^{n_1} \otimes \cdots \otimes \mathbb{F}^{n_k}$ defined by $t = \sum f(e_{a_1}, \dots, e_{a_k}) e_{a_1} \otimes \cdots \otimes e_{a_k}$ where a_i ranges over $\{1, \dots, n_i\}$. To state the asymptotic restriction problem we need the concepts of restriction and tensor product. Let $f : \mathbb{F}^{n_1} \times \cdots \times \mathbb{F}^{n_k} \rightarrow \mathbb{F}$ and $g : \mathbb{F}^{m_1} \times \cdots \times \mathbb{F}^{m_k} \rightarrow \mathbb{F}$ be multilinear maps. We say f *restricts to* g , and write $f \geq g$, if there are linear maps $A_i : \mathbb{F}^{m_i} \rightarrow \mathbb{F}^{n_i}$ such that $g = f \circ (A_1, \dots, A_k)$ where \circ denotes composition. We naturally define the *tensor product* $f \otimes g$ as the multilinear map $(\mathbb{F}^{n_1} \otimes \mathbb{F}^{m_1}) \times \cdots \times (\mathbb{F}^{n_k} \otimes \mathbb{F}^{m_k}) \rightarrow \mathbb{F}$ defined by $(v_1 \otimes w_1, \dots, v_k \otimes w_k) \mapsto f(v_1, \dots, v_k)g(w_1, \dots, w_k)$. We say f *restricts asymptotically to* g , written $f \gtrsim g$, if there is a sequence of natural numbers $a(n) \in o(n)$ such that

$$f^{\otimes(n+a(n))} \geq g^{\otimes n} \quad \text{when } n \rightarrow \infty.$$

The asymptotic restriction problem is: given f and g , decide whether $f \gtrsim g$.

Applications of the asymptotic restriction problem include computing the computational complexity of matrix multiplication in algebraic complexity theory [1, 9, 20, 26–28, 45, 47, 55, 56, 72] (see also [10, 18, 44, 46]), deciding the feasibility of an asymptotic transformation between pure quantum states via stochastic local operations and classical communication (SLOCC) in quantum information theory [6, 30, 38, 68], bounding the size of combinatorial structures like cap sets and tri-colored sum-free sets in additive combinatorics [2, 12, 29, 31, 32, 41, 63–65], and bounding the query complexity of certain properties in algebraic property testing [7, 8, 34, 36, 39, 52].

There are two central research directions in the asymptotic restriction problem, namely finding (1) *constructions*, which witness $f \gtrsim g$, and (2) *obstructions*, which prohibit $f \gtrsim g$. Matrix multiplication algorithms and efficient quantum protocols

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are examples of constructions. Lower bounds in the sense of computational complexity theory are examples of obstructions. Strassen introduced in 1986 the theory of asymptotic spectra of tensors to understand the asymptotic restriction problem [57, 59]. Deferring the details to the next subsection, this can be viewed as the theory of obstructions in the above sense. A remarkable result of this theory is that the asymptotic restriction problem for a family of tensors \mathcal{X} that is closed under direct sum and tensor product and contains the diagonal tensors $\langle n \rangle$ reduces to finding all maps $\mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$ that are

- (a) monotone under restriction \geq
- (b) multiplicative under tensor product \otimes
- (c) additive under direct sum \oplus
- (d) normalised to have value n at the unit tensor $\langle n \rangle$.

Such maps are called *spectral points*. Here the *direct sum* $f \oplus g$ is defined naturally as the multilinear map $(\mathbb{F}^{n_1} \oplus \mathbb{F}^{m_1}) \times \cdots \times (\mathbb{F}^{n_k} \oplus \mathbb{F}^{m_k}) \rightarrow \mathbb{F}$ such that $(v_1 + w_1, \dots, v_k + w_k) \mapsto f(v_1, \dots, v_k) + g(w_1, \dots, w_k)$ where $v_i \in \mathbb{F}^{n_i}, w_i \in \mathbb{F}^{m_i}$, and the *unit tensor* $\langle n \rangle$ for $n \in \mathbb{N}$ is defined as the multilinear map $(\mathbb{F}^n)^{\times k} \rightarrow \mathbb{F}$ that maps $(e_{i_1}, \dots, e_{i_k})$ to 1 if $i_1 = \cdots = i_k$ and to 0 otherwise.

Properties (a) and (b) are natural properties to obtain an obstruction. Namely, suppose ξ satisfies (a) and (b), and let $f, g \in \mathcal{X}$. If $f \gtrsim g$, then by definition $f^{\otimes(n+o(n))} \geq g^{\otimes n}$, and (a) and (b) imply $\xi(f)^{n+o(n)} = \xi(f^{\otimes(n+o(n))}) \geq \xi(g^{\otimes n}) = \xi(g)^n$, which implies $\xi(f) \geq \xi(g)$. Turning this around, if $\xi(f) < \xi(g)$ then it is *not* the case that $f \gtrsim g$, so ξ yields an obstruction to $f \gtrsim g$.

Strassen [60] created a family of spectral points. Let $\mathcal{P}([k])$ be the set of all probability distributions on $[k] := \{1, 2, \dots, k\}$. Strassen defined a family of maps $\zeta^\theta : \{k\text{-tensors}\} \rightarrow \mathbb{R}_{\geq 0}$ parametrised by $\theta \in \mathcal{P}([k])$, named the support functionals, and he proved that the ζ^θ are spectral points for the family \mathcal{X} of oblique tensors, tensors whose support in some basis is an antichain, a strict (and nongeneric) subfamily of all tensors. In [62], for such tensors, ζ^θ has been given a formulation in terms of moment polytopes.

Our main result. *Universal spectral points* are spectral points for the family of *all* tensors.¹ The construction of non-trivial universal spectral points has been an open problem for more than thirty years. We introduce maps $F_\theta : \{\text{complex } k\text{-tensors}\} \rightarrow \mathbb{R}_{\geq 0}$ called the *quantum functionals* and we prove that they are universal spectral points. The quantum functionals are defined as follows (we will carefully define the quantum concepts later). For any $\theta \in \mathcal{P}([k])$, define

$$F_\theta(t) = 2^{E_\theta(t)},$$

$$E_\theta(t) = \sup_{g_i} \sum_{i=1}^k \theta(i) H(\text{Tr}_i(g_1, \dots, g_k) \cdot t).$$

¹Strassen [59, page 119] coined the term “universal spectral point” to refer to elements in the asymptotic spectrum of all tensors \mathcal{T} . We may indeed think of the asymptotic spectrum of all tensors \mathcal{T} as a *universal* object, because restricting all universal spectral points to any sub-semiring \mathcal{X} of tensors gives all the spectral points for that sub-semiring \mathcal{X} (this follows from [59, Cor. 2.7]; see also [74]). In other words, every element in the asymptotic spectrum of any given sub-semiring \mathcal{X} can be extended to an element of the spectrum of the full semiring of all k -tensors. In this way, knowing the universal spectral points allows us to exactly determine the asymptotic spectra for *all* sub-semirings of tensors.

where the supremum goes over invertible maps $g_i \in \text{GL}(\mathbb{C}^{n_i})$, H denotes the quantum entropy (that is, von Neumann entropy), and $\text{Tr}_i(g_1, \dots, g_k) \cdot t$ denotes a partial trace of $(g_1, \dots, g_k) \cdot t$ interpreted as a pure quantum state. To prove properties (a)–(d) we draw from invariant theory, quantum entropy, entanglement polytopes, Kronecker coefficients and Littlewood–Richardson coefficients, and in particular from work going back to Ness [49] and Brion [14] and more recently [70] and [51]. Let us briefly sketch the connection to moment polytopes, which in this context are called entanglement polytopes. Given a tensor s , let $r_1(s), \dots, r_k(s)$ be the single-system quantum marginal spectra of the tensor s viewed as a quantum state. Let G be the group $\text{GL}(\mathbb{C}^{n_1}) \times \dots \times \text{GL}(\mathbb{C}^{n_k})$. For a tensor t , define the set $\Delta_t = \{(r_1(s), \dots, r_k(s)) \mid s \in \overline{G \cdot t}\}$, where $\overline{G \cdot t}$ denotes the Euclidean closure (or equivalently Zariski closure) of the orbit $G \cdot t$. It is a non-trivial fact that Δ_t is a polytope, named the entanglement polytope. Then $E_\theta(t)$ equals the following optimisation over the entanglement polytope,

$$E_\theta(t) = \sup \left\{ \sum_{i=1}^k \theta(i) H(\lambda^{(i)}) \mid (\lambda^{(1)}, \dots, \lambda^{(k)}) \in \Delta_t \right\},$$

which is a convex optimisation problem.

The definition of F_θ can in fact be extended to probability distributions θ on subsets of $\{1, 2, \dots, k\}$ in several ways. We explore these extensions and study the properties (a)–(d). One such extension, the lower quantum functional, satisfies (a) and (d) and is super-additive and super-multiplicative. Another extension, the upper quantum functional, satisfies (a) and (d) and is sub-additive and sub-multiplicative for θ that are what we call noncrossing. (The Strassen support functionals in fact similarly come in an upper and a lower version.)

Asymptotic rank and asymptotic subrank. In applications (e.g. the complexity of matrix multiplication) one is often interested in asymptotic restriction to or from a unit tensor. One commonly defines the *tensor rank*

$$R(f) = \min\{r \in \mathbb{N} \mid f \leq \langle r \rangle\},$$

and the *subrank*

$$Q(f) = \max\{s \in \mathbb{N} \mid \langle s \rangle \leq f\},$$

and asymptotically the *asymptotic rank* $\underline{R}(f) = \lim_{n \rightarrow \infty} R(f^{\otimes n})^{1/n}$ and the *asymptotic subrank* $\underline{Q}(f) = \lim_{n \rightarrow \infty} Q(f^{\otimes n})^{1/n}$. Clearly, maps that satisfy (a), (d) and super-multiplicativity are lower bounds on asymptotic rank, and maps that satisfy (a), (d) and sub-multiplicativity are upper bounds on asymptotic subrank. Spectral points (maps satisfying (a)–(d)) are thus between asymptotic subrank and asymptotic rank. The defining expression of asymptotic rank does not suggest any algorithm for approximating its value other than computing the rank for high tensor powers. (And in fact the asymptotic rank is not known to be computable.) In information theory, such an expression is called a multi-letter formula, and maps satisfying (a)–(d) are called single-letter formulas.

Cap set problem, slice rank and partition rank. In the study of the cap set problem and related problems in combinatorics, following the breakthrough result of Ellenberg and Gijswijt [32], the slice rank and partition rank of tensors were introduced [48, 64] and it was shown that they upper bound the aforementioned subrank. We show that the (upper) quantum functionals and Strassen’s (upper)

support functionals are an upper bound on slice rank and partition rank, asymptotically. As a consequence we prove that for the family of tight 3-tensors asymptotic subrank and asymptotic slice rank coincide. For complex tensors we exactly characterise the asymptotic slice rank in terms of the quantum functionals.

Although our universal spectral points, the quantum functionals, do not play a role for cap sets (we cannot work over \mathbb{C} for this problem), we think that a better understanding of our universal spectral points construction may lead to further progress on related combinatorial questions. In particular we remark in Example 4.14 how Strassen's support functionals recover the aforementioned cap set result.

1.2. Asymptotic spectra for tensors. To understand why we focus on maps satisfying the properties (a)–(d) we give a brief introduction to Strassen's theory of asymptotic spectra for tensors.

We begin by putting an equivalence relation on tensors to get rid of trivialities. Restriction \geq and asymptotic restriction \gtrsim are both preorders (reflexive and transitive). We say f is isomorphic to g , and write $f \cong g$, if there are bijective linear maps $A_i : \mathbb{F}^{m_i} \rightarrow \mathbb{F}^{n_i}$ such that $g = f \circ (A_1, \dots, A_k)$. We say f and g are equivalent, and write $f \sim g$, if there are null maps $f_0 : \mathbb{F}^{a_1} \times \dots \times \mathbb{F}^{a_k} \rightarrow \mathbb{F} : x \mapsto 0$ and $g_0 : \mathbb{F}^{b_1} \times \dots \times \mathbb{F}^{b_k} \rightarrow \mathbb{F} : x \mapsto 0$ such that $f \oplus f_0 \cong g \oplus g_0$. The equivalence relation \sim is in fact the equivalence relation generated by the restriction preorder \geq . Let \mathcal{T} be the set of \sim -equivalence classes of \mathbb{F} -multilinear maps of order k . Direct sum and tensor product naturally carry over to \mathcal{T} , and \mathcal{T} becomes a semiring with additive unit $\langle 0 \rangle$ and multiplicative unit $\langle 1 \rangle$ (more precisely, the equivalence classes of those tensors). Restriction \geq induces a partial order on \mathcal{T} (reflexive, antisymmetric, transitive), and asymptotic restriction \gtrsim induces a preorder on \mathcal{T} . Both behave well with respect to the semiring operations, and naturally $n \geq m$ if and only if $\langle n \rangle \geq \langle m \rangle$.

The theory of asymptotic spectra revolves around Theorem 1.2 proved by Strassen [57, 59]. Given a topological space Δ we denote by $\mathcal{C}^+(\Delta)$ the semiring of continuous maps $\Delta \rightarrow \mathbb{R}_{\geq 0}$.

Definition 1.1. Given a semiring $\mathcal{X} \subseteq \mathcal{T}$, an *asymptotic spectrum for \mathcal{X}* consists of a compact space Δ and a semiring homomorphism $\phi : \mathcal{X} \rightarrow \mathcal{C}^+(\Delta)$ such that the following two conditions hold:

- (1) $\phi(\mathcal{X})$ separates the points of Δ ²
- (2) for all $a, b \in \mathcal{X}$, $a \lesssim b$ iff $\phi(a) \leq \phi(b)$ pointwise on Δ .

Theorem 1.2 (Spectral theorem). *For any semiring $\mathcal{X} \subseteq \mathcal{T}$, any asymptotic spectrum (Δ, ϕ) is isomorphic to the following,*

- (1) $\Delta = \{\geq\text{-monotone homomorphisms } \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}\}$
- (2) $\phi : \mathcal{X} \rightarrow \mathcal{C}^+(\Delta) : a \mapsto \hat{a}$ where $\hat{a} : \Delta \rightarrow \mathbb{R}_{\geq 0} : \xi \mapsto \xi(a)$.

We call this pair (Δ, ϕ) the *asymptotic spectrum of \mathcal{X}* and refer to it as $\Delta(\mathcal{X})$.

Theorem 1.2 is proved by a non-trivial reduction to the structure theory of Stone–Kadison–Dubois for a certain class of ordered rings (see [4]).

Remark 1.3. We note that $\Delta(\mathcal{X})$ may equivalently be defined with degeneration \triangleright instead of restriction \geq . Over \mathbb{C} , we say f degenerates to g , written,

²That is, for every two distinct points $\xi_1, \xi_2 \in \Delta$ there is a semiring element $a \in \mathcal{X}$ such that $\phi(a)(\xi_1) \neq \phi(a)(\xi_2)$.

$f \supseteq g$, if g is in the Euclidean closure (or equivalently Zariski closure) of the orbit $\mathrm{GL}(\mathbb{C}^{n_1}) \times \cdots \times \mathrm{GL}(\mathbb{C}^{n_k}) \cdot f$. It is a non-trivial fact from algebraic geometry (see [43, Lemma III.2.3.1] or [18, Theorem 20.24]) that there is a degeneration $f \supseteq g$ if and only if there are linear maps $A_i \in \mathrm{GL}(\mathbb{C}(\varepsilon)^{n_i})$ such that $(A_1, \dots, A_k) \cdot f = g + \varepsilon^1 g_1 + \cdots + \varepsilon^e g_e$ for some elements g_1, \dots, g_e , where $\mathbb{C}(\varepsilon)$ is the field \mathbb{C} extended with the formal variable ε . The latter definition of degeneration is valid when \mathbb{C} is replaced by an arbitrary field \mathbb{F} and that is how degeneration is defined for an arbitrary field. Degeneration is weaker than restriction: $f \geq g$ implies $f \supseteq g$. Asymptotically, however, the notions coincide: $f \gtrsim g$ if and only if $f^{\otimes(n+o(n))} \supseteq g^{\otimes n}$. We mention that, analogous to restriction, degeneration gives rise to *border rank* and *border subrank*, $\underline{\mathbf{R}}(f) = \min\{r \in \mathbb{N} \mid f \leq \langle r \rangle\}$, $\underline{\mathbf{Q}}(f) = \max\{s \in \mathbb{N} \mid \langle s \rangle \leq f\}$.

Remark 1.4. Let \mathcal{X} be a family of tensors over \mathbb{F} . Let $\mathbb{F} \subseteq \mathbb{K}$ be a field extension. We may view \mathcal{X} as a family of tensors $\mathcal{X}^{\mathbb{K}}$ over \mathbb{K} . The asymptotic spectra of \mathcal{X} and $\mathcal{X}^{\mathbb{K}}$ are equal [59, Theorem 3.10]. In particular, if $t \in \{0, 1\}^{n_1} \times \cdots \times \{0, 1\}^{n_k}$, then t can be viewed as a tensor $t^{\mathbb{F}}$ in $\mathbb{F}^{n_1} \otimes \cdots \otimes \mathbb{F}^{n_k}$ over any field \mathbb{F} , and by the above, if $\mathbb{F} \subseteq \mathbb{K}$ is the prime subfield of \mathbb{K} , then $\Delta(t^{\mathbb{F}}) = \Delta(t^{\mathbb{K}})$, and e.g. $\underline{\mathbf{R}}(t^{\mathbb{F}}) = \underline{\mathbf{R}}(t^{\mathbb{K}})$. A well-known example is that the exponent $\omega_{\mathbb{F}}$ of matrix multiplication over \mathbb{F} depends only on the characteristic of \mathbb{F} .

We reduced the asymptotic restriction problem $a \lesssim b$ to the problem of computing the asymptotic spectrum of \mathcal{T} or a sub-semiring $\mathcal{X} \subseteq \mathcal{T}$. An element $\xi \in \Delta(\mathcal{X})$ is called a *spectral point*. It is clear that constructing explicitly all elements in the asymptotic spectrum $\Delta(\mathcal{T})$ of all tensors \mathcal{T} is an important problem in the development of the theory of asymptotic spectra of tensors. Elements of $\Delta(\mathcal{T})$ are called *universal spectral points*, and they correspond precisely to maps $\xi : \{k\text{-tensors over } \mathbb{F}\} \rightarrow \mathbb{R}_{\geq 0}$ satisfying $\xi(s \oplus t) = \xi(s) + \xi(t)$, $\xi(s \otimes t) = \xi(s)\xi(t)$, $\xi(\langle 1 \rangle) = 1$, and $\xi(s) \leq \xi(t)$ whenever $s \leq t$ (i.e. properties (a)–(d)).

Remark 1.5. As remarked earlier, it is known that for any sub-semiring $\mathcal{X} \subseteq \mathcal{T}$ the restriction map $\Delta(\mathcal{T}) \rightarrow \Delta(\mathcal{X})$ (which maps any $\phi \in \Delta(\mathcal{T})$ to ϕ restricted to \mathcal{X}) is surjective [59]. In other words, knowing all universal spectral points tells us precisely what the spectral points are for *any* sub-semiring $\mathcal{X} \subseteq \mathcal{T}$.

Example 1.6 (Flattening ranks or gauge points, [59, Equation 3.10]). The flattening ranks are a finite collection of $2^{k-1} - 1$ universal spectral points (where as usual k is the order of our tensors) that have a simple definition in terms of matrix rank. Namely, let $S \sqcup \bar{S} = [k]$ be any bipartition of $[k]$ such that S and \bar{S} are not empty. (There are indeed $2^{k-1} - 1$ such bipartitions.) Then, given any multilinear map f , thought of as a tensor $f \in V_1 \otimes \cdots \otimes V_k$, we let $\zeta_S(f)$ be the rank of the matrix obtained from f under the natural grouping map $V_1 \otimes \cdots \otimes V_k \rightarrow (\otimes_{i \in S} V_i) \otimes (\otimes_{i \in \bar{S}} V_i)$. These maps $\zeta_S : \mathcal{T} \rightarrow \mathbb{R}_{\geq 0}$ are called the *flattening ranks* or the *gauge points* (following the $k = 3$ terminology of [59, Equation 3.10]) and it is easy to verify that the ζ_S are universal spectral points. Prior to this paper the gauge points were the only known universal spectral points.

It can be useful to think of the asymptotic spectrum as a compact subspace of \mathbb{R}^X , in the following way.

Definition 1.7. Let $X \subseteq \mathcal{T}$ be a set. Let $\mathcal{X} \subseteq \mathcal{T}$ be the semiring generated by X . A homomorphism $\xi : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$ is determined by the values $(\xi(x) : x \in X) \in \mathbb{R}_{\geq 0}^X$.

We may thus identify the asymptotic spectrum of the semiring generated by X with a subspace of $\mathbb{R}_{\geq 0}^X$.

- (1) $\Delta(X) = \{\xi \in \mathbb{R}^X \text{ generating a } \geq\text{-monotone homomorphism } \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}\}$.
- (2) $\phi(a)(\xi) = \xi(a)$.

We call $(\Delta(X), \phi)$ the asymptotic spectrum of X , and write $\Delta(X)$. When $X = \{a\}$ we will denote $\Delta(X)$ by $\Delta(a)$.

We finish by stating the important relationship between the asymptotic spectrum and the asymptotic (sub)rank, which follows from Theorem 1.2 (see [59, Theorem 3.8]).

Proposition 1.8. *Let $\mathcal{X} \subseteq \mathcal{T}$ be a semiring. Let (Δ, ϕ) be an asymptotic spectrum of \mathcal{X} . Let $a \in \mathcal{X}$. Then*

$$\underline{Q}(a) = \min_{\xi \in \Delta} \phi(a)(\xi) \quad \underline{R}(a) = \max_{\xi \in \Delta} \phi(a)(\xi).$$

Remark 1.9. Obviously \underline{R} and \underline{Q} are \geq -monotones and have value n on $\langle n \rangle$. They are not universal spectral points however. Namely, the asymptotic rank of $e_1 \otimes e_1 \otimes 1 + e_2 \otimes e_2 \otimes 1$, $e_1 \otimes 1 \otimes e_1 + e_2 \otimes 1 \otimes e_2$ and $1 \otimes e_1 \otimes e_1 + 1 \otimes e_2 \otimes e_2$ is 2, whereas the tensor product equals the matrix multiplication tensor whose asymptotic rank is strictly smaller than 2^3 . With the same tensors one shows that asymptotic subrank is not multiplicative.

1.3. Our results.

Main results. The main results of this paper are summarised as follows.

- (1) We construct for the first time a non-trivial family of maps

$$F_\theta : \{k\text{-tensors}\} \rightarrow \mathbb{R}_{\geq 0},$$

named quantum functionals, that are nonincreasing under restriction \geq , normalised on the unit tensor $\langle r \rangle$, additive under direct sum \oplus and multiplicative under tensor product \otimes , i.e. universal spectral points, advancing Strassen's theory of asymptotic spectra.

- (2) We connect Strassen's asymptotic spectra to entanglement polytopes and the quantum marginal problem. This is to our knowledge the first time that information about asymptotic transformations is obtained from entanglement polytopes, as opposed to information about single-copy transformations.
- (3) We characterise asymptotic slice rank in terms of the quantum functionals and show that asymptotic slice rank coincides with asymptotic subrank for tight 3-tensors.

Paper overview. In Section 2 we set the scene by reviewing the theory of the Strassen upper and lower support functionals ζ^θ and ζ_θ introduced in [60]. This is important background material for our work. The support functionals allow us to compute elements in the asymptotic spectrum for a class of tensors called oblique tensors. Our exposition follows closely the exposition in [60] with the exception that we consider multilinear maps $V_1 \times \cdots \times V_k \rightarrow \mathbb{F}$ with $k \geq 3$ instead of bilinear maps $V_1 \times V_2 \rightarrow V_3$. Strassen already observed that his results generalise to the multilinear regime, but we will make this explicit for the sake of understanding and comparison to our results.

Section 3 contains our main result. Here we introduce two new families of functionals over the complex numbers. These functionals are denoted by F^θ and F_θ and called the *upper and lower quantum functionals*. Both functionals are \geq -monotone. The upper quantum functional F^θ is sub-additive and sub-multiplicative when θ is what we call noncrossing. The lower quantum functional F_θ is super-additive and super-multiplicative. For singleton θ the two functionals coincide and thus yield an additive, multiplicative \sqsupseteq -monotone, a universal spectral point.

In Section 4 we consider several families of tensors. We introduce the family of free tensors. We show that for free tensors, the Strassen upper support functional coincides with the quantum functionals for singleton θ . This is useful computationally, since the upper support functional is defined as a minimisation while the quantum functionals are defined as a maximisation. We next compute generic values of the quantum functionals for tensor formats in which a quantum state with completely mixed marginals exists. Our results extend over \mathbb{C} the results by Verena Tobler [66] on the generic value of the upper support functional.

In Section 5 we show that asymptotically the Strassen upper support functional and the upper quantum functional are upper bounds for slice rank and partition rank, recently introduced \geq -monotones that are neither super-multiplicative nor sub-multiplicative. As a consequence, we find that slice rank coincides asymptotically with subrank for tight 3-tensors. We additionally show that the quantum functionals characterise asymptotic slice rank for complex tensors.

2. STRASSEN SUPPORT FUNCTIONALS

In this section the field \mathbb{F} is arbitrary. After having introduced the concept of the asymptotic spectrum of tensors in [59], Strassen constructed a non-trivial family of spectral points in the asymptotic spectrum of *oblique* tensors in [60]. Oblique tensors are tensors for which the support is an antichain in some basis. In plain English, Strassen constructed a family of functions from k -tensors to \mathbb{R} that are \geq -monotone, are normalised to attain value r at $\langle r \rangle$, and, when restricted to oblique tensors, are additive under \oplus and multiplicative under \otimes . He calls his functions the *support functionals*, following the terminology of support functions of convex bodies in convex geometry. Not all tensors are oblique, and obliqueness is not a generic property. Many tensors that are of interest in the field of algebraic complexity theory, however, turn out to be oblique, notably the structure tensor of the algebra of $n \times n$ matrices. This section is devoted to explaining the construction of these spectral points for oblique tensors. We do this not only to set the scene and provide benchmarks for our new functionals, but also because the Strassen support functionals remain relevant today, for example in the context of combinatorial problems like the cap set problem, as we will make clear in Example 4.14.

The construction goes in four steps. First we define for probability distributions θ on $\{1, 2, \dots, k\}$ the *upper support functional* ζ^θ , which is \geq -monotone, normalised at $\langle r \rangle$, additive under \oplus and sub-multiplicative under \otimes . Second we define the *lower support functional* ζ_θ , which is \geq -monotone, normalised at $\langle r \rangle$, super-additive under \oplus and super-multiplicative under \otimes . Third we show that $\zeta^\theta(t) \geq \zeta_\theta(t)$ for any k -tensor t . Fourth we show that for oblique tensors t the upper support functional $\zeta^\theta(t)$ and lower support functional $\zeta_\theta(t)$ coincide, which gives a spectral point.

Notation. We use the following standard notation. For any natural number n we write $[n]$ for the set $\{1, 2, \dots, n\}$. For any finite set X , let $\mathcal{P}(X)$ be the set of

all probability distributions on X . For any probability distribution $P \in \mathcal{P}(X)$ the *Shannon entropy* of P is defined as $H(P) = -\sum_{x \in X} P(x) \log_2 P(x)$ with $0 \log_2 0$ understood as 0. The *support* of P is $\text{supp } P = \{x \in X \mid P(x) \neq 0\}$. Given finite sets I_1, \dots, I_k and a probability distribution $P \in \mathcal{P}(I_1 \times \dots \times I_k)$ on the product set $I_1 \times \dots \times I_k$ we denote the *marginal distribution* of P on I_i by P_i , that is, $P_i(a) = \sum_{x: x_i=a} P(x)$ for any $a \in I_i$.

We set some specific notation for this text. Let $k \geq 3$. For $i \in [k]$, let V_i be a vector space over \mathbb{F} of dimension n_i . Let I_i be a finite set. Let $f : V_1 \times \dots \times V_k \rightarrow \mathbb{F}$ be a multilinear map. We order the set $[n_i]$ naturally by $1 < 2 < \dots < n_i$. When the sets I_1, \dots, I_k are ordered, we give $I_1 \times \dots \times I_k$ the *product order*, defined by $x \leq y$ iff for all $i \in [k]$: $x_i \leq y_i$.

2.1. Upper support functional. We begin with introducing the upper support functional. We naturally define the notion of the support of a multilinear map $V_1 \times \dots \times V_k \rightarrow \mathbb{F}$.

Definition 2.1. Let $f : V_1 \times \dots \times V_k \rightarrow \mathbb{F}$ be a multilinear map. Let $\mathcal{B}(f)$ denote the set of k -tuples of ordered bases for V_1, \dots, V_k .³ That is, each element $B \in \mathcal{B}(f)$ is a k -tuple

$$((v_{1,1}, v_{1,2}, \dots, v_{1,n_1}), \dots, (v_{k,1}, v_{k,2}, \dots, v_{k,n_k}))$$

in which $(v_{i,1}, v_{i,2}, \dots, v_{i,n_i})$ is an ordered basis of V_i .⁴ Define the *support of f with respect to B* as

$$\text{supp}_B f := \{(\alpha_1, \dots, \alpha_k) \in [n_1] \times \dots \times [n_k] \mid f(v_{1,\alpha_1}, \dots, v_{k,\alpha_k}) \neq 0\}.$$

Definition 2.2. Let $\theta \in \mathcal{P}([k])$. Let $P \in \mathcal{P}(I_1 \times \dots \times I_k)$ be a probability distribution. Define $H_\theta(P)$ as the θ -weighted average of the Shannon entropies of the marginal distributions P_i of P ,

$$H_\theta(P) = \sum_{i \in [k]} \theta(i) H(P_i).$$

Let $\Phi \subseteq I_1 \times \dots \times I_k$ be a nonempty subset. Define $H_\theta(\Phi)$ as the maximum $H_\theta(P)$ over all probability distributions $P \in \mathcal{P}(\Phi)$,

$$(1) \quad H_\theta(\Phi) = \max_{P \in \mathcal{P}(\Phi)} H_\theta(P).$$

Definition 2.3 (Strassen upper support functional). Let $\theta \in \mathcal{P}([k])$. Let f be a nonzero multilinear map. Define $\rho^\theta(f)$ by

$$\rho^\theta(f) = \min_{B \in \mathcal{B}(f)} H_\theta(\text{supp}_B f)$$

and define ζ^θ by

$$\zeta^\theta(f) = 2^{\rho^\theta(f)}.$$

We call ζ^θ the *Strassen upper support functional* and we call ρ^θ the *logarithmic Strassen upper support functional*. When f is 0 we naturally define $\zeta^\theta(f) = 0$ and $\rho^\theta(f) = -\infty$.

The key properties of the upper support functional are as follows.

³Of course the set of k -tuples of basis for V_1, \dots, V_k depends on the choice of the V_i and not on the choice of multilinear map $f : V_1 \times \dots \times V_k \rightarrow \mathbb{F}$. However, we use the short notation $\mathcal{B}(f)$ since often we will be working with multilinear maps f for which the spaces V_i are left implicit.

⁴For brevity we will just say “bases” from now on rather than “ordered bases”.

Theorem 2.4 ([60]). *Let $f : V_1 \times \cdots \times V_k \rightarrow \mathbb{F}$ and $g : W_1 \times \cdots \times W_k \rightarrow \mathbb{F}$ be multilinear maps. Let $\theta \in \mathcal{P}([k])$. The Strassen upper support functional ζ^θ has the following properties.*

- (1) $\zeta^\theta(\langle r \rangle) = r$ for $r \in \mathbb{N}$.
- (2) $\zeta^\theta(f \oplus g) = \zeta^\theta(f) + \zeta^\theta(g)$.
- (3) $\zeta^\theta(f \otimes g) \leq \zeta^\theta(f)\zeta^\theta(g)$.
- (4) If $f \geq g$, then $\zeta^\theta(f) \geq \zeta^\theta(g)$.
- (5) $0 \leq \zeta^\theta(f) \leq (\dim V_1)^{\theta(1)} \cdots (\dim V_k)^{\theta(k)}$.

(Note that ζ^θ is not just sub-additive, but even additive.) One verifies directly that statements (1) and (5) of Theorem 2.4 are true. Statements (2)–(4) can be proved with generalisations of the arguments in [60, Section 2].

In the rest of this section we discuss an important alternative characterisation of the upper support functional in terms of filtrations. This alternative definition we need later to relate the upper and lower support functionals.

Definition 2.5. A *proper filtration* or *complete flag* of a vector space U is a sequence $(U_\alpha)_{1 \leq \alpha \leq a}$ of subspaces of U with $U = U_1 \supseteq \cdots \supseteq U_a = \{0\}$ and $\dim U_\alpha / U_{\alpha+1} = 1$ for $1 \leq \alpha < a$. (In this text all flags will be decreasing.) We denote by $\mathcal{F}_c(f)$ the set of k -tuples of complete flags $F = ((V_{1,\alpha_1})_{\alpha_1}, \dots, (V_{k,\alpha_k})_{\alpha_k})$ of V_1, \dots, V_k . For $F \in \mathcal{F}_c(f)$ we define the support of f with respect to F as

$$\text{supp}_F f = \{(\alpha_1, \dots, \alpha_k) \mid f(V_{1,\alpha_1} \times \cdots \times V_{k,\alpha_k}) \neq 0\} \subseteq [n_1] \times \cdots \times [n_k].$$

The alternative characterisation of the upper support functional in terms of filtrations is as follows. We refer to [60, Section 2] for the proof.

Proposition 2.6. $\rho^\theta(f) = \min_{F \in \mathcal{F}_c(f)} H_\theta(\text{supp}_F f)$.

Remark 2.7. The characterisation of ρ^θ as a minimisation over filtrations can be thought of as an asymptotic version of the filtration method explained in [18, Exercise 15.13].

2.2. Lower support functional. The upper support functional ζ^θ has a companion called the lower support functional ζ_θ which we introduce here. We shift focus from the support to the maximal points in the support.

Definition 2.8. Let $\Phi \subseteq [n_1] \times \cdots \times [n_k]$ be a subset. We define the *maximal points* of Φ with respect to the product order of the natural orders on $[n_1], \dots, [n_k]$ as

$$\max \Phi = \{\alpha \in \Phi \mid \nexists \beta \in \Phi : \alpha < \beta\}.$$

Definition 2.9. Let f be a nonzero multilinear map. Given $B \in \mathcal{B}(f)$ we set the notation $M_B f = \max_{\text{supp}_B} f$. Define $\rho_\theta(f)$ by

$$\rho_\theta(f) = \max_{B \in \mathcal{B}(f)} H_\theta(M_B f)$$

and define $\zeta_\theta(f)$ by

$$\zeta_\theta(f) = 2^{\rho_\theta(f)}.$$

We call ζ_θ the *Strassen lower support functional* and we call ρ_θ the *logarithmic Strassen lower support functional*. When f is 0 we naturally define $\zeta_\theta(f) = 0$ and $\rho_\theta(f) = -\infty$.

Theorem 2.10 ([60]). *Let $f : V_1 \times \cdots \times V_k \rightarrow \mathbb{F}$ and $g : W_1 \times \cdots \times W_k \rightarrow \mathbb{F}$. Let $\theta \in \mathcal{P}([k])$. The Strassen lower support functional ζ_θ has the following properties.*

- (1) $\zeta_\theta(\langle r \rangle) = r$ for $r \in \mathbb{N}$.
- (2) $\zeta_\theta(f \oplus g) \geq \zeta_\theta(f) + \zeta_\theta(g)$.
- (3) $\zeta_\theta(f \otimes g) \geq \zeta_\theta(f)\zeta_\theta(g)$.
- (4) *If $f \geq g$, then $\zeta_\theta(f) \geq \zeta_\theta(g)$.*
- (5) $0 \leq \zeta_\theta(f) \leq (\dim V_1)^{\theta(1)} \cdots (\dim V_k)^{\theta(k)}$.

Remark 2.11. Regarding statement (2) in Theorem 2.10, Bürgisser [17] shows that the lower support functional ζ_θ is not in general additive under the direct sum when $\theta_i > 0$ for all i . See also [60, Comment (iii)]. In particular, this implies that the upper support functional $\zeta^\theta(f)$ and the lower support functional $\zeta_\theta(f)$ are not equal in general, the upper support functional being additive. In fact, to show that the lower support functional is not additive, Bürgisser first shows that (with \mathbb{F} algebraically closed) the *typical value* of ζ_θ on $\mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$ equals $(1 - \min_i \theta_i) \log_2 n + o(n)$; on the other hand, Tobler [66] shows that the typical value of ζ^θ on $\mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$ equals $\log_2 n$. (So even generically ζ^θ and ζ_θ are different on $\mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$.) We discuss typical values in Section 4.3.

One verifies directly that statements (1) and (5) of Theorem 2.10 are true. For the proofs of statements (2)–(4) we refer to [60, Section 3].

We finish this section by giving an alternative definition of ρ_θ in terms of complete flags, in the same spirit as Proposition 2.6 for the upper support functional. Let us first make some general remarks about how (the maximal points in) the support with respect to a basis and the support with respect to a flag are related.

Definition 2.12. Let $\Phi \subseteq [n_1] \times \cdots \times [n_k]$ be a subset. We define the *downward closure* of Φ with respect to the product of the natural orders on $[n_1], \dots, [n_k]$ as

$$\downarrow \Phi = \{ \alpha \in [n_1] \times \cdots \times [n_k] \mid \exists \beta \in \Phi : \alpha \leq \beta \}.$$

To any $B = ((v_{1,\alpha_1}), \dots, (v_{k,\alpha_k})) \in \mathcal{B}(f)$ we can naturally associate a k -tuple of complete flags $F = ((V_{1,\alpha_1}), \dots, (V_{k,\alpha_k})) \in \mathcal{F}_c(f)$, by defining the subspace V_{i,α_i} as $\text{Span}\{v_{i,\alpha_i}, v_{i,\alpha_i+1}, \dots\}$.

The following key properties of the downward closure and the maximal points follow directly from the definitions.

Lemma 2.13. *Let $B \in \mathcal{B}(f)$ and $F \in \mathcal{F}_c(f)$ be associated. Then*

$$\begin{aligned} \downarrow \text{supp}_B f &= \text{supp}_F f, \\ \max \text{supp}_B f &= \max \text{supp}_F f. \end{aligned}$$

For $B \in \mathcal{B}(f)$ we have set the notation $M_B f = \max \text{supp}_B f$. For $F \in \mathcal{F}_c(f)$ we similarly set the notation $M_F f = \max \text{supp}_F f$.

Proposition 2.14. $\rho_\theta(f) = \max_{F \in \mathcal{F}_c(f)} H_\theta(M_F f)$ for $\theta \in \mathcal{P}([k])$.

Proof. Let $B \in \mathcal{B}(f)$ and $F \in \mathcal{F}_c(f)$ be associated. Then $M_B f = M_F f$ by Lemma 2.13 and hence $H_\theta(M_B f) = H_\theta(M_F f)$. Since any B is associated to some F and vice versa we conclude that

$$\max_{B \in \mathcal{B}(f)} H_\theta(M_B f) = \max_{F \in \mathcal{F}_c(f)} H_\theta(M_F f). \quad \square$$

2.3. Comparing the support functionals. Strassen calls his functionals *upper* and *lower* because he proves that the upper support functional is at least the lower support functional [60, Corollary 4.3].

Theorem 2.15. $\rho^\theta(f) \geq \rho_\theta(f)$.

This is a useful property when doing computations, since ρ^θ is defined as a minimisation and ρ_θ is defined as a maximisation.

In fact, in [60, Corollary 4.2] Strassen first proves $\zeta^\theta(f \otimes g) \geq \zeta^\theta(f)\zeta_\theta(g)$ and then proves Theorem 2.15 as a corollary. The purpose of this section is to give a more direct proof of Theorem 2.15 for the benefit of the reader. The proof essentially comes down to Proposition 2.16.

Proposition 2.16. *Let $f : V_1 \otimes \cdots \otimes V_k \rightarrow \mathbb{F}$. Let $F, G \in \mathcal{F}_c(f)$. There are permutations $\phi_i : [n_i] \rightarrow [n_i]$ ($i \in [k]$) such that $(\phi_1 \times \cdots \times \phi_k)M_G f \subseteq \text{supp}_F f$.*

Proof. Let V be some n -dimensional vector space, let $(V_\alpha)_\alpha$ and $(W_\beta)_\beta$ be complete flags of V , and let v_1, \dots, v_n and w_1, \dots, w_n be bases of V such that

$$\begin{aligned} V_\alpha &= \text{Span}\{v_\alpha, v_{\alpha+1}, \dots\}, \\ W_\beta &= \text{Span}\{w_\beta, w_{\beta+1}, \dots\}. \end{aligned}$$

Define the map

$$(2) \quad \phi : [n] \rightarrow [n] : \beta \mapsto \max\{\alpha \in [n] : V_\alpha \cap (w_\beta + W_{\beta+1}) \neq \emptyset\}.$$

We now show that ϕ is injective. Let $\beta, \gamma \in [n]$ with $\beta \leq \gamma$ and suppose $\alpha = \phi(\beta) = \phi(\gamma)$. Then (2) gives

$$\begin{aligned} (3) \quad & (\mathbb{F}^\times v_\alpha + V_{\alpha+1}) \cap (w_\beta + W_{\beta+1}) \neq \emptyset, \\ (4) \quad & V_{\alpha+1} \cap (w_\beta + W_{\beta+1}) = \emptyset, \\ (5) \quad & (\mathbb{F}^\times v_\alpha + V_{\alpha+1}) \cap (w_\gamma + W_{\gamma+1}) \neq \emptyset. \end{aligned}$$

If $\beta < \gamma$, then using (3) and (5) we may obtain a contradiction to (4). We conclude that $\beta = \gamma$.

We turn to F and G . Write

$$\begin{aligned} F &= ((V_{1,\alpha_1}), \dots, (V_{k,\alpha_k})), \\ G &= ((W_{1,\beta_1}), \dots, (W_{k,\beta_k})) \end{aligned}$$

and for each pair of complete flags $(V_{i,\alpha_i}), (W_{i,\beta_i})$ of V_i we define the permutation $\phi_i : [n_i] \rightarrow [n_i]$ in the same way as above. We prove $(\phi_1 \times \cdots \times \phi_k)M_G f \subseteq \text{supp}_F f$. Let $(\beta_1, \dots, \beta_k)$ be in $M_G f$. Let $(\alpha_1, \dots, \alpha_k) = (\phi_1 \times \cdots \times \phi_k)(\beta_1, \dots, \beta_k)$. Then by definition of the ϕ_i the intersection $V_{i,\alpha_i} \cap (w_{i,\beta_i} + W_{i,\beta_i+1})$ is not empty. Choose

$$\overline{w_{i,\beta_i}} \in V_{i,\alpha_i} \cap (w_{i,\beta_i} + W_{i,\beta_i+1}).$$

Since f is multilinear we have for some $x_{\beta'} \in \mathbb{F}$

$$f(\overline{w_{1,\beta_1}}, \dots, \overline{w_{k,\beta_k}}) = f(w_{1,\beta_1}, \dots, w_{k,\beta_k}) + \sum_{\beta' > \beta} x_{\beta'} f(w_{1,\beta'_1}, \dots, w_{k,\beta'_k})$$

with the sum over tuples $\beta' = (\beta'_1, \dots, \beta'_k)$ that are strictly larger than β in the product order. Since β is a maximal element in $\text{supp}_G f$ with respect to the product order, the sum over $\beta' > \beta$ equals zero. We conclude that $f(\overline{w_{1,\beta_1}}, \dots, \overline{w_{k,\beta_k}}) = f(w_{1,\beta_1}, \dots, w_{k,\beta_k}) \neq 0$. Therefore, $f(V_{1,\alpha_1} \times \cdots \times V_{k,\alpha_k})$ is not zero and thus $(\alpha_1, \dots, \alpha_k) \in \text{supp}_F f$. \square

The above proof of Proposition 2.16 may more naturally be phrased in the language of Schubert cells, when $\mathbb{F} = \mathbb{C}$, as follows. We use the notation and definitions of [15] with the difference that our complete flags are decreasing instead of increasing so we need to use the group of lower unitriangular matrices U^- instead of the group of upper unitriangular matrices U .

Proof. Let $U_n^- \subseteq \text{GL}(\mathbb{C}^n)$ be the unipotent subgroup of lower triangular $n \times n$ matrices with ones on the diagonal. Let Y be the variety of complete (decreasing) flags in \mathbb{C}^n . The group $\text{GL}(\mathbb{C}^n)$ naturally acts on Y . Let the symmetric group S_n act on Y via its natural action on the standard basis of \mathbb{C}^n . It is well-known (see e.g. [15, Proposition 1.2.1]) that the variety Y is the disjoint union of the U_n^- -orbits $C^\phi := U_n^- \phi x$ over all permutations ϕ in the symmetric group S_n , where x is the standard flag $\text{Span}\{e_1, \dots, e_n\} \supset \text{Span}\{e_2, \dots, e_n\} \supset \dots \supset \text{Span}\{e_n\}$ in \mathbb{C}^n . These orbits are called the *Schubert cells*.

We turn to the k -tuples of flags F and G . Write again

$$F = ((V_{1,\alpha_1}), \dots, (V_{k,\alpha_k})),$$

$$G = ((W_{1,\beta_1}), \dots, (W_{k,\beta_k})).$$

Without loss of generality we may assume that each (W_{i,β_i}) is the standard flag in \mathbb{C}^{n_i} . By the previous paragraph, for each $i \in [k]$ there is a permutation $\phi_i \in S_{n_i}$ such that the flag $(V_{i,\phi_i(\beta_i)})$ is in the $U_{n_i}^-$ -orbit of the flag (W_{i,β_i}) , that is, there is a group element $g_i \in U_{n_i}^-$ such that $V_{i,\phi_i(\beta_i)} = g_i W_{i,\beta_i}$ for every $\beta_i \in [n_i]$. Let $(\beta_1, \dots, \beta_k)$ be in $\text{supp}_G f$, so

$$f(W_{1,\beta_1} \times \dots \times W_{k,\beta_k}) \neq \{0\}.$$

Suppose $(\beta_1, \dots, \beta_k)$ is maximal. Then since $g_i \in U_{n_i}^-$ and since f is multilinear,

$$f(W_{1,\beta_1} \times \dots \times W_{k,\beta_k}) = f(g_1 W_{1,\beta_1} \times \dots \times g_k W_{k,\beta_k}).$$

By our choice of g_i ,

$$f(g_1 W_{1,\beta_1} \times \dots \times g_k W_{k,\beta_k}) = f(V_{1,\phi_1(\beta_1)} \times \dots \times V_{k,\phi_k(\beta_k)}).$$

We conclude that $(\phi_1(\beta_1), \dots, \phi_k(\beta_k))$ is in $\text{supp}_F f$. □

Proof of Theorem 2.15. By Proposition 2.6 and Proposition 2.14 we have

$$(6) \quad \rho^\theta(f) = \min_{F \in \mathcal{F}_c(f)} H_\theta(\text{supp}_F f),$$

$$(7) \quad \rho_\theta(f) = \max_{G \in \mathcal{F}_c(f)} H_\theta(M_G f).$$

Let $F \in \mathcal{F}_c(f)$ minimise (6) and let $G \in \mathcal{F}_c(f)$ maximise (7). It suffices to show

$$(8) \quad H_\theta(M_G f) \leq H_\theta(\text{supp}_F f).$$

Let $P \in \mathcal{P}(M_G f)$ such that $H_\theta(P) = H_\theta(M_G f)$. Proposition 2.16 gives permutations $\phi_i : [n_i] \rightarrow [n_i]$ such that

$$(\phi_1 \times \dots \times \phi_k) M_G f \subseteq \text{supp}_F f.$$

Let $Q \in \mathcal{P}(\text{supp}_F f)$ be defined by $Q(\alpha) = P((\phi_1^{-1} \times \dots \times \phi_k^{-1})(\alpha))$. Then $H_\theta(P) = H_\theta(Q)$. This proves (8). □

2.4. Robust and oblique tensors; spectral points. At this point we have the upper support functional ζ^θ , which is additive and sub-multiplicative, we have the lower support functional ζ_θ , which is super-additive and super-multiplicative, and we know that $\zeta^\theta(t) \geq \zeta_\theta(t)$ for any t . When the functionals ζ^θ and ζ_θ coincide we have an additive and multiplicative \leq -monotone, a spectral point. In general (in fact generically for certain formats) the functionals ζ^θ and ζ_θ do *not* coincide, however (Remark 2.11). To get spectral points from the support functionals we should thus restrict to smaller families of tensors $X \subseteq \mathcal{T}$. Definition 2.17 just gives a name to those tensors for which the support functionals do coincide.

Definition 2.17 (Robust). For $\theta \in \mathcal{P}([k])$ we say f is θ -robust if $\zeta^\theta(f) = \zeta_\theta(f)$. We say f is robust if f is θ -robust for all $\theta \in \mathcal{P}([k])$.

We try to understand what robust tensors look like. Let f be nonzero. Clearly f is θ -robust if and only if

$$(9) \quad \zeta^\theta(f) \leq \zeta_\theta(f).$$

Being θ -robust is closed under \oplus and \otimes , since $\zeta^\theta(f \oplus g) = \zeta^\theta(f) + \zeta^\theta(g) = \zeta_\theta(f) + \zeta_\theta(g) \leq \zeta_\theta(f \oplus g)$, and $\zeta^\theta(f \otimes g) \leq \zeta^\theta(f)\zeta^\theta(g) = \zeta_\theta(f)\zeta_\theta(g) \leq \zeta_\theta(f \otimes g)$. Equation (9) means precisely that there exist $B, D \in \mathcal{B}(f)$ and $P \in \mathcal{P}(M_D f)$ such that

$$(10) \quad H_\theta(\text{supp}_B f) \leq H_\theta(P).$$

In this case we have $\zeta_\theta(f) = \zeta^\theta(f) = 2^{H_\theta(P)}$. In particular, f is θ -robust if there is a $B \in \mathcal{B}(f)$ such that the maximisation $H_\theta(\text{supp}_B f)$ is attained by a probability distribution $P \in \mathcal{P}(M_B f)$. This criterion is automatically satisfied for all θ when in some basis the support of f equals the maximal points in the support of f . Recall that a subset Φ of a partially ordered set is called an *antichain* if all elements in Φ are pairwise incomparable.

Definition 2.18 (Oblique). We say f is *oblique* if there exists a $B \in \mathcal{B}(f)$ such that $\text{supp}_B f = M_B f$, that is, such that $\text{supp}_B f$ is an antichain.

Note that the family of oblique tensors is a semiring under \oplus and \otimes . Suppose f is oblique. Let $\text{supp}_B f$ be an antichain. Then f is robust and we have simply $\rho_\theta(f) = \rho^\theta(f) = H_\theta(\text{supp}_B f)$. In the language of asymptotic spectra this means the following.

Theorem 2.19 (Strassen). Let $\mathcal{X} \subseteq \mathcal{T}$ be the family of all oblique k -tensors. Let $\theta \in \mathcal{P}([k])$. Given $f \in \mathcal{X}$, let $B_f \in \mathcal{B}(f)$ be any bases in which $\text{supp}_{B_f} f$ is an antichain. Then

$$\zeta^\theta(f) = \zeta_\theta(f) = 2^{H_\theta(\text{supp}_{B_f} f)}$$

and thus $\zeta^\theta : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$ is a restriction monotone, that is normalised on $\langle r \rangle$, additive under direct sum \oplus and multiplicative under tensor product \otimes . So ζ^θ is an element of the asymptotic spectrum $\Delta(\mathcal{X})$.

Definition 2.20 (Support simplex). For any family of oblique tensors \mathcal{Y} , we call the image of $\mathcal{P}([k]) \rightarrow \Delta(\mathcal{Y}) : \theta \rightarrow \zeta_\theta$ the *support simplex* in the asymptotic spectrum of \mathcal{Y} . We denote the support simplex by $\zeta_{\mathcal{P}([k])}(\mathcal{Y})$.

We stress again that the lower support functionals are not additive and can thus not be universal spectral points. The upper support functionals may be universal spectral points, but this can, however, not be shown with the help of the lower support functionals.

3. QUANTUM FUNCTIONALS

In this section we let the base field \mathbb{F} be the complex numbers \mathbb{C} . The goal of this section is to construct an explicit family of universal spectral points in the asymptotic spectrum of tensors $\Delta(\mathcal{T})$. In other words, we construct maps $\{\text{complex } k\text{-tensors}\} \rightarrow \mathbb{R}_{\geq 0}$ that are \geq -monotone, multiplicative under \otimes , additive under \oplus and normalised on the unit tensor $\langle n \rangle$. In fact, we will prove \geq -monotonicity of our maps by proving \supseteq -monotonicity (where \supseteq denotes degeneration, cf. Remark 1.3).⁵

To achieve this we introduce two families of functionals. In Section 3.1 we introduce the *lower quantum functionals* F_θ and prove their basic properties. This family of functionals is defined in terms of quantum entropy and can be thought of as a quantum entropy version of the upper support functionals. We discuss it first because it requires the least background to discuss. In Section 3.2 we introduce the *upper quantum functionals* F^θ and prove their basic properties. This family of functionals is defined in terms of isotypic projections for the Schur–Weyl decomposition of tensor space, and can be thought of as a version of the upper support functional in which the decomposition along the standard basis of tensor space is replaced by the Schur–Weyl decomposition. In Section 3.3 and Section 3.4 we relate the lower and upper quantum functionals to find universal spectral points.

Before going into definitions and detailed discussion, let us briefly go into how the upper and lower quantum functionals will be parametrized and how they will relate to Strassen’s upper and lower support functionals of Section 2, on a high level. One way in which our construction of upper and lower quantum functionals differs from the upper and lower support functionals is that the quantum functionals will initially have a larger parameter space. Recall that the support functionals ζ^θ and ζ_θ are parametrised by the probability distributions θ on $[k]$. Each probability distribution θ induces a probability distribution, which we also call θ , on all bipartitions $\{S, [k] \setminus S\}$ with $S \subseteq [k]$ that is supported on the bipartitions of the form $\{\{j\}, [k] \setminus \{j\}\}$. Namely, $\{\{j\}, [k] \setminus \{j\}\}$ gets probability $\theta(j)$ for every $j \in [k]$. Our functionals F^θ and F_θ will be parametrised by the probability distributions θ on *all* bipartitions $\{S, [k] \setminus S\}$ of $[k]$ rather than just the ones of the form $\{\{j\}, [k] \setminus \{j\}\}$.

We have results on the quantum functionals that hold for all distributions θ on the bipartitions of $[k]$ and (stronger) results that hold for all those distributions θ that are supported on the bipartitions of the form $\{\{j\}, [k] \setminus \{j\}\}$. We show that the upper functional is at least the lower functional, $F^\theta(t) \geq F_\theta(t)$, for every distribution θ on bipartitions of $[k]$ (Section 3.3). When θ is supported on bipartitions of the form $\{\{j\}, [k] \setminus \{j\}\}$ we show that $F^\theta(t)$ and $F_\theta(t)$ are equal and are thus universal spectral points, and moreover $\zeta^\theta(t) \geq F^\theta(t)$ (Section 3.4). Finally,

⁵We note that, although monotonicity under degeneration is seemingly stronger than monotonicity under restriction, any multiplicative restriction-monotone ϕ is automatically degeneration-monotone. The main ingredient to see this is that if $s \trianglelefteq t$ then $s \lesssim t$. This follows from [18, Prop. 15.26] which says that if $s \trianglelefteq t$ with “approximation degree” q , then $s \leq \langle q^2 \rangle \otimes t$. That is, degeneration implies restriction with an overhead factor depending on the approximation degree. Importantly, approximation degree is easily seen to be *additive* under powering, so that $s \leq t$ with approximation degree q implies $s^{\otimes n} \leq t^{\otimes n}$ with approximation degree nq , and thus $s^{\otimes n} \leq \langle (nq)^2 \rangle \otimes t^{\otimes n+o(n)} \leq t^{\otimes n+o(n)}$, since $(nq)^2$ is a polynomial in n . Using this we find that if $s \trianglelefteq t$, then $s^{\otimes n} \leq t^{\otimes n+o(n)}$, which implies, using monotonicity and multiplicativity of ϕ , that $\phi(s)^n \leq \phi(t)^{n+o(n)}$ and hence $\phi(s) \leq \phi(t)$.

we show that the regularised upper support functional equals the upper quantum functional, $\lim_{n \rightarrow \infty} \zeta^\theta(t^{\otimes n})^{1/n} = F^\theta(t)$, which implies that $F^\theta(t) \geq \zeta_\theta(t)$ (Section 3.4).

Notation. We set some notation for this and later sections. As usual $[k]$ denotes the set of integers $\{1, 2, \dots, k\}$. For any subset $S \subseteq [k]$ we define the complement $\bar{S} = [k] \setminus S$ and for any element $j \in [k]$ we define the complement $\bar{j} = [k] \setminus \{j\}$. As before, V_1, \dots, V_k are vector spaces over the base field, which is the complex numbers in this section. For any subset $S \subseteq [k]$ we define the vector space $V_S = \bigotimes_{j \in S} V_j$.

A *bipartition* of $[k]$ is an unordered pair $\{S, \bar{S}\}$ with $S \subseteq [k]$, $S \neq \emptyset$ and $\bar{S} \neq \emptyset$. We say that the bipartitions $\{S, \bar{S}\}$ and $\{T, \bar{T}\}$ are *noncrossing* if $S \subseteq T$ or $T \subseteq S$ or $S \cap T = \emptyset$. In other words (more symmetrically), the bipartitions $\{S, \bar{S}\}$ and $\{T, \bar{T}\}$ are noncrossing if $S \cap \bar{T} = \emptyset$ or $T \cap \bar{S} = \emptyset$ or $S \cap T = \emptyset$. The set of bipartitions of $[k]$ is denoted by Π . We will consider two subsets of the set $\mathcal{P}(\Pi)$ of probability distributions on Π . Let $\mathcal{P}_s(\Pi)$ be the set of distributions supported on bipartitions of the form $\{\{j\}, \bar{j}\}$. We say a distribution $\theta \in \mathcal{P}(\Pi)$ is *noncrossing* if θ is supported on pairwise noncrossing partitions. Let $\mathcal{P}_{nc}(\Pi)$ be the set of noncrossing distributions. Note that the latter is not a convex set when k is at least 4 (i.e. as soon as non-crossing bipartitions exist). Clearly, we have the inclusions $\mathcal{P}_s(\Pi) \subseteq \mathcal{P}_{nc}(\Pi) \subseteq \mathcal{P}(\Pi)$, with equality if and only if $k \leq 3$.

3.1. Lower quantum functional. We introduce the lower quantum functional in this section and introduce the upper quantum functional in the next section. To define the lower quantum functional we need concepts from quantum information theory. We work with finite-dimensional Hilbert spaces. A *state* or *density operator* on a Hilbert space \mathcal{H} is a positive semidefinite linear map $\rho : \mathcal{H} \rightarrow \mathcal{H}$ satisfying $\text{Tr } \rho = 1$. Let $\mathcal{S}(\mathcal{H})$ be the set of states on \mathcal{H} . The *von Neumann entropy* or *quantum entropy* of ρ is defined as $H(\rho) = -\text{Tr } \rho \log \rho$. If $\mathcal{H} = \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_k$, then the j th *marginal* of a state $\rho \in \mathcal{S}(\mathcal{H})$ is $\rho_j = (\text{Tr}_{\mathcal{H}_1} \otimes \dots \otimes \text{Tr}_{\mathcal{H}_{j-1}} \otimes \text{Id} \otimes \text{Tr}_{\mathcal{H}_{j+1}} \otimes \dots \otimes \text{Tr}_{\mathcal{H}_k})\rho$. More generally, if $S \subseteq [k]$, then $\rho_S = (\bigotimes_{j \in S} \text{Id}_{\mathcal{H}_j} \otimes \bigotimes_{j \in \bar{S}} \text{Tr}_{\mathcal{H}_j})\rho \in \mathcal{S}(\mathcal{H}_S)$. A state ρ is *pure* if ρ has rank one as a linear map. If $\rho \in \mathcal{S}(\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_k)$ is pure and $S \subseteq [k]$, then ρ_S and $\rho_{\bar{S}}$ are unitarily equivalent up to enlarging the underlying Hilbert spaces $\mathcal{H}_1, \dots, \mathcal{H}_k$, and therefore the entropy of ρ_S equals the entropy of $\rho_{\bar{S}}$.

Remark 3.1. A probability distribution $P \in \mathcal{P}(X)$ gives rise to a state $\rho \in \mathcal{S}(\mathbb{C}^X)$ as $\rho = \sum_{x \in X} P(x) |x\rangle\langle x|$. This gives a bijection between probability distributions on X and those states on \mathbb{C}^X which are diagonal in the standard basis. In this way $\mathcal{P}(X)$ can be identified with a subset of $\mathcal{S}(\mathbb{C}^X)$, and this identification is compatible with the notations of marginal distribution/marginal state and Shannon entropy/von Neumann entropy.

Definition 3.2. Let $\mathcal{H} = \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_k$ be a Hilbert space. Given $\theta \in \mathcal{P}(\Pi)$ and a nonzero vector $\psi \in \mathcal{H}$, define

$$H_\theta(\psi) := \sum_{S \in \Pi} \theta(S) H\left(\left(\frac{1}{\langle \psi | \psi \rangle} |\psi\rangle\langle \psi|\right)_S\right).$$

Definition 3.3. Let $t \in V_1 \otimes \dots \otimes V_k$. Choose inner products on each V_j so that they become inner product spaces. For any $\theta \in \mathcal{P}(\Pi)$ we define

$$E_\theta(t) = \sup_{A_1, \dots, A_k} H_\theta((A_1 \otimes \dots \otimes A_k)t),$$

where the supremum is over invertible linear maps $A_j \in \text{GL}(V_j)$. For $t = 0$ we set $E_\theta(t) = -\infty$. We also define $F_\theta(t) = 2^{E_\theta(t)}$ if $t \neq 0$ and $F_\theta(0) = 0$. We call E_θ the *logarithmic lower quantum functional* and we call F_θ the *lower quantum functional*.

Remark 3.4. The supremum in the definition of E_θ is independent of the chosen inner products, since different inner products on a vector space are related by invertible linear maps. We could equivalently define $E_\theta(t)$ as a supremum of $H_\theta(t)$ over the choice of local inner products. We may as well restrict to those k -tuples of inner products for which $\langle t|t \rangle$ equals one.

The definition of E_θ is not sensitive to embedding each V_j in some larger vector space. In fact, we get another equivalent definition of E_θ by choosing large enough inner product spaces $\mathcal{H}_1, \dots, \mathcal{H}_k$ and taking the supremum of $H_\theta((A_1 \otimes \dots \otimes A_k)t)$ over all injective linear maps $A_j : V_j \rightarrow \mathcal{H}_j$.

The key properties of the lower quantum functional are as follows.

Theorem 3.5. *Let $\theta \in \mathcal{P}(\Pi)$. Let $s \in V_1 \otimes \dots \otimes V_k$, $t \in W_1 \otimes \dots \otimes W_k$.*

- (1) $F_\theta(\langle r \rangle) = r$ for any $r \in \mathbb{N}$.
- (2) $F_\theta(s \oplus t) \geq F_\theta(s) + F_\theta(t)$.
- (3) $F_\theta(s \otimes t) \geq F_\theta(s)F_\theta(t)$.
- (4) If $s \succeq t$, then $F_\theta(s) \geq F_\theta(t)$.
- (5) $0 \leq F_\theta(s) \leq \prod_{\{S, \bar{S}\} \in \Pi} \min\{\dim(V_S), \dim(V_{\bar{S}})\}^{\theta(\{S, \bar{S}\})}$.

Statement (5) is easy to prove. In the rest of this section we prove statements (1)–(4).

Lemma 3.6. *$F_\theta(\langle r \rangle) = r$ for every $r \in \mathbb{N}$ and $\theta \in \mathcal{P}(\Pi)$.*

Proof. Let $\psi = \langle r \rangle$. If we make the local bases orthonormal, then $|\psi\rangle\langle\psi|_S$ is a projector of rank r , and the entropy of the normalised spectrum equals $\log_2(r)$, for any $\emptyset \neq S \subsetneq [k]$. This gives the lower bound. On the other hand, the quantum entropy of a density matrix ρ is at most the logarithm of the matrix rank of ρ . Therefore, $H(|\psi\rangle\langle\psi|_S) \leq \log_2 R(|\psi\rangle\langle\psi|_S) \leq \log_2 R(\text{flatten}_S(\psi))$, where $\text{flatten}_S(\psi)$ is the matrix obtained from the k -tensor ψ by grouping the tensor legs indexed by S together and the tensor legs indexed by the complement \bar{S} together. This gives the upper bound. \square

Lemma 3.7. *If $s \succeq t$, then $F_\theta(s) \geq F_\theta(t)$.*

Proof. We may without loss of generality assume that s and t are elements of the same space $V_1 \otimes \dots \otimes V_k$ (Remark 3.4). Let $G = \text{GL}(V_1) \times \dots \times \text{GL}(V_k)$. By assumption, there is a sequence $(A_{1,i}, \dots, A_{k,i}) \in G$ such that $\lim_{i \rightarrow \infty} (A_{1,i} \otimes \dots \otimes A_{k,i})s = t$. For any k -tuple $(B_1, \dots, B_k) \in G$ we have therefore

$$\begin{aligned} E_\theta(s) &= \sup_{C_1, \dots, C_k} H_\theta((C_1 \otimes \dots \otimes C_k)s) \\ &\geq \lim_{i \rightarrow \infty} H_\theta((B_1 A_{1,i} \otimes \dots \otimes B_k A_{k,i})s) \\ &= H_\theta((B_1 \otimes \dots \otimes B_k)t) \end{aligned}$$

by continuity of H_θ . Now take the supremum over all $(B_1, \dots, B_k) \in G$ to obtain $E_\theta(s) \geq E_\theta(t)$ and thus $F_\theta(s) \geq F_\theta(t)$. \square

To continue we need two basic properties of entropy.

Lemma 3.8 (Recursion property of quantum entropy, e.g. [5, Eq. 12.19]). *Let ρ_i be density matrices with support in orthogonal subspaces \mathcal{H}_i of a Hilbert space $\mathcal{H} = \bigoplus_{i=1}^M \mathcal{H}_i$. Then the density matrix $\rho = \sum_i p_i \rho_i$ has quantum entropy $H(\rho) = H(p) + \sum_{i=1}^M p_i H(\rho_i)$.*

Lemma 3.9 (See e.g. [60, Eq. 2.13]). *Let $x, y \in \mathbb{R}_{\geq 0}$. Then*

$$\max_{0 \leq p \leq 1} 2^{px+(1-p)y+h(p)} = 2^x + 2^y.$$

Lemma 3.10. $F_\theta(s \oplus t) \geq F_\theta(s) + F_\theta(t)$.

Proof. Choose inner products $\langle \cdot, \cdot \rangle_j$ on V_j and $\langle \cdot, \cdot \rangle'_j$ on W_j in such a way that s and t have norm one, and consider $H_\theta(s)$ and $H_\theta(t)$ with respect to these inner products (cf. Remark 3.4). Equip $V_j \oplus W_j$ with the direct sum of the inner products. Let $u = \sqrt{p}s \oplus \sqrt{1-p}t \in (V_1 \oplus W_1) \otimes \cdots \otimes (V_k \oplus W_k)$. For any $p \in [0, 1]$ the vector u has norm one and $u \cong s \oplus t$. For any subset $S \subseteq [k]$ the reduced states are related as $|u\rangle\langle u|_S = p|s\rangle\langle s|_S \oplus (1-p)|t\rangle\langle t|_S$. Therefore, by recursivity of quantum entropy (Lemma 3.8),

$$H(|u\rangle\langle u|_S) = pH(|s\rangle\langle s|_S) + (1-p)H(|t\rangle\langle t|_S) + h(p).$$

Taking the θ -weighted average of both sides gives

$$H_\theta(u) = pH_\theta(s) + (1-p)H_\theta(t) + h(p).$$

Since $E_\theta(s \oplus t) \geq H_\theta(u)$ we may take the supremum over the choices of inner products to get

$$E_\theta(s \oplus t) \geq pE_\theta(s) + (1-p)E_\theta(t) + h(p).$$

Apply Lemma 3.9 to see that $F_\theta(s \oplus t) \geq F_\theta(s) + F_\theta(t)$. □

Lemma 3.11. $F_\theta(s \otimes t) \geq F_\theta(s)F_\theta(t)$.

Proof. Choose inner products $\langle \cdot, \cdot \rangle_j$ on V_j and $\langle \cdot, \cdot \rangle'_j$ on W_j in such a way that s and t have norm one, and consider $H_\theta(s)$ and $H_\theta(t)$ with respect to these inner products. Equip $V_j \otimes W_j$ with the tensor product of the inner products. Then $s \otimes t \in (V_1 \otimes W_1) \otimes \cdots \otimes (V_k \otimes W_k)$ has norm one. For any subset $S \subseteq [k]$ the reduced states are related as $|s \otimes t\rangle\langle s \otimes t|_S = |s\rangle\langle s|_S \otimes |t\rangle\langle t|_S$. Therefore $H(|s \otimes t\rangle\langle s \otimes t|_S) = H(|s\rangle\langle s|_S) + H(|t\rangle\langle t|_S)$. Taking the θ -weighted average of both sides gives $H_\theta(s \otimes t) = H_\theta(s) + H_\theta(t)$. Since $E_\theta(s \otimes t) \geq H_\theta(s \otimes t)$, taking the supremum over the choice of inner products gives $E_\theta(s \otimes t) \geq E_\theta(s) + E_\theta(t)$ and thus $F_\theta(s \otimes t) \geq F_\theta(s)F_\theta(t)$. □

3.2. Upper quantum functional. To define the upper quantum functional we need concepts from representation theory.

An *integer partition* is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$ of nonnegative integers satisfying the nonincreasing property $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d$. We say that λ is a partition of n and write $\lambda \vdash n$ if $\lambda_1 + \dots + \lambda_d = n$. We denote the number of nonzero parts of λ by $\ell(\lambda)$. We write $\lambda \vdash_d$ if $\ell(\lambda) \leq d$. We write $\lambda \vdash_d n$ if $\lambda \vdash n$ and $\ell(\lambda) \leq d$. Let \mathcal{P}_n be the set of partitions of n . For a partition $\lambda \vdash n$ let $\bar{\lambda}$ be the sequence $(\frac{\lambda_1}{n}, \dots, \frac{\lambda_d}{n})$. For $e \geq d$ we may view $\bar{\lambda}$ as a probability distribution on $[e]$, and we can consider the Shannon entropy $H(\bar{\lambda})$ of $\bar{\lambda}$.

Let V be a complex vector space of dimension d . Let the symmetric group S_n act on the tensor power $V^{\otimes n}$ by permuting the tensor legs and let the general linear group $GL(V)$ act on $V^{\otimes n}$ by acting on the n tensor legs simultaneously. These

actions commute, so $V^{\otimes n}$ is a $\mathrm{GL}(V) \times S_n$ -module. It decomposes into irreducible modules as

$$(11) \quad V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} \mathbb{S}_\lambda(V) \otimes [\lambda],$$

where $[\lambda]$ is an irreducible S_n -module and $\mathbb{S}_\lambda(V)$ is an irreducible $\mathrm{GL}(V)$ -module if $\ell(\lambda) \leq d$ and 0 otherwise. Decomposition (11) is referred to as Schur–Weyl duality. Let $P_\lambda^V \in \mathrm{End}(V^{\otimes n})$ be the equivariant projection onto the subspace isomorphic to $\mathbb{S}_\lambda(V) \otimes [\lambda]$. Recall that for $S \subseteq [k]$ we defined V_S as $\bigotimes_{j \in S} V_j$. Generalising the above discussion, for any subset $S \subseteq [k]$, the group $\mathrm{GL}(V_S) \times S_n$ acts naturally on $(V_1 \otimes \cdots \otimes V_k)^{\otimes n}$, which decomposes as

$$(V_1 \otimes \cdots \otimes V_k)^{\otimes n} \cong \bigoplus_{\lambda \vdash n} \mathbb{S}_\lambda(V_S) \otimes [\lambda] \otimes (V_{\overline{S}})^{\otimes n}$$

with $\mathrm{GL}(V_S)$ acting trivially on $(V_{\overline{S}})^{\otimes n}$. We define $P_\lambda^{V_S} \in \mathrm{End}((V_{[k]})^{\otimes n})$ to be the equivariant projection onto the subspace isomorphic to $\mathbb{S}_\lambda(V_S) \otimes [\lambda] \otimes (V_{\overline{S}})^{\otimes n}$.

We are interested in powers $t^{\otimes n}$ so we want to restrict $(V_{[k]})^{\otimes n}$ to the symmetric subspace $\mathbb{S}_{(n)}(V_{[k]})$. The following observation is well-known.

Lemma 3.12. *Let $S \subseteq [k]$. Let $\lambda \vdash n$.*

- (1) *The projections $P_{(n)}^{V_{[k]}}$ and $P_\lambda^{V_S}$ commute.*
- (2) *The projections $P_\lambda^{V_S} P_{(n)}^{V_{[k]}}$ and $P_\lambda^{V_{\overline{S}}} P_{(n)}^{V_{[k]}}$ are equal.*

Proof. For the first statement, the projector $P_{(n)}^{V_{[k]}}$ is just the symmetriser $\frac{1}{n!} \sum_{\pi \in S_n} \pi$, which commutes with the S_n -equivariant projector $P_\lambda^{V_S}$. We now prove the second statement. By Schur–Weyl duality,

$$(V_S)^{\otimes n} \otimes (V_{\overline{S}})^{\otimes n} \cong \bigoplus_{\lambda, \mu} \mathbb{S}_\lambda(V_S) \otimes [\lambda] \otimes \mathbb{S}_\mu(V_{\overline{S}}) \otimes [\mu].$$

Considering the symmetric subspace gives

$$\begin{aligned} ((V_S \otimes V_{\overline{S}})^{\otimes n})^{S_n} &\cong \bigoplus_{\lambda, \mu} \mathbb{S}_\lambda(V_S) \otimes \mathbb{S}_\mu(V_{\overline{S}}) \otimes ([\lambda] \otimes [\mu])^{S_n} \\ &\cong \bigoplus_{\lambda} \mathbb{S}_\lambda(V_S) \otimes \mathbb{S}_\lambda(V_{\overline{S}}) \end{aligned}$$

since $([\lambda] \otimes [\mu])^{S_n} \cong ([\lambda]^* \otimes [\mu])^{S_n} \cong \mathrm{Hom}_{S_n}([\lambda], [\mu])$ is one-dimensional if $\lambda = \mu$ and the zero space otherwise. The image of $P_\lambda^{V_S} P_{(n)}^{V_{[k]}}$ and the image of $P_\lambda^{V_{\overline{S}}} P_{(n)}^{V_{[k]}}$ are both equal to the subspace of $(V_S \otimes V_{\overline{S}})^{\otimes n}$ isomorphic to $\mathbb{S}_\lambda(V_S) \otimes \mathbb{S}_\lambda(V_{\overline{S}})$. \square

Lemma 3.12 says that we can unambiguously label isotypic projections on the symmetric subspace by a pair $(b, \lambda) \in \Pi \times \mathcal{P}_n$. For $b = \{S, \overline{S}\} \in \Pi$ and a partition $\lambda \vdash n$, define $P_\lambda^{V_b} := P_\lambda^{V_S} P_{(n)}^{V_{[k]}}$.

Lemma 3.13. *If $b_1, b_2 \in \Pi$ are noncrossing bipartitions and $\lambda^{(1)}, \lambda^{(2)}$ are partitions, then $P_{\lambda^{(1)}}^{V_{b_1}}$ and $P_{\lambda^{(2)}}^{V_{b_2}}$ commute.*

Proof. Without loss of generality we may say $b_1 = \{S_1, \overline{S_1}\}$ and $b_2 = \{S_2, \overline{S_2}\}$ with $S_1 \cap S_2 = \emptyset$. Clearly the projectors $P_{\lambda^{(1)}}^{V_{S_1}}$ and $P_{\lambda^{(2)}}^{V_{S_2}}$ commute, since the intersection $S_1 \cap S_2$ is empty. By statement (1) in Lemma 3.12 we obtain

$$P_{\lambda^{(1)}}^{V_{b_1}} P_{\lambda^{(2)}}^{V_{b_2}} = P_{\lambda^{(1)}}^{V_{S_1}} P_{(n)}^{V_{[k]}} P_{\lambda^{(2)}}^{V_{S_2}} P_{(n)}^{V_{[k]}} = P_{\lambda^{(2)}}^{V_{b_2}} P_{\lambda^{(1)}}^{V_{b_1}}. \quad \square$$

We can now define the upper quantum functional.

Definition 3.14. Let $t \in V_1 \otimes \cdots \otimes V_k$. Let $\theta \in \mathcal{P}(\Pi)$. Define

$$E^\theta(t) = \sup_{(\lambda^{(b)})} \sum_{b \in \text{supp } \theta} \theta(b) H(\overline{\lambda^{(b)}})$$

with the supremum over all tuples $(\lambda^{(b)})_{b \in \Pi}$ of partitions $\lambda^{(b)} \vdash n$ such that

$$\prod_{b \in \text{supp } \theta} P_{\lambda^{(b)}}^{V_b} t^{\otimes n} \neq 0.$$

We define $F^\theta(t) = 2^{E^\theta(t)}$ if $t \neq 0$ and $F^\theta(0) = 0$. We call F^θ the *upper quantum functional* and E^θ the *logarithmic upper quantum functional*.

Remark 3.15. The definition of $E^\theta(t)$ in Definition 3.14 depends on the order of the product $\prod_{b \in \text{supp } \theta} P_{\lambda^{(b)}}^{V_b}$, since the projectors in general do not commute. For $\theta \in \mathcal{P}_{\text{nc}}$, however, the projectors do commute and the order does not matter. Since we will mostly focus on $\theta \in \mathcal{P}_{\text{nc}}$ we will leave the order of the elements of Π implicit.

Theorem 3.16. Let $\theta \in \mathcal{P}_{\text{nc}}(\Pi)$. Let $s \in V_1 \otimes \cdots \otimes V_k$ and $t \in W_1 \otimes \cdots \otimes W_k$.

- (1) $F^\theta(\langle r \rangle) = r$ for $r \in \mathbb{N}$.
- (2) $F^\theta(s \oplus t) \leq F^\theta(s) + F^\theta(t)$.
- (3) $F^\theta(s \otimes t) \leq F^\theta(s)F^\theta(t)$.
- (4) If $s \geq t$, then $F^\theta(s) \geq F^\theta(t)$.

Statement (4) is a special case of Lemma 3.17.

Lemma 3.17. Let $\theta \in \mathcal{P}(\Pi)$. If $s \geq t$, then $F^\theta(s) \geq F^\theta(t)$.

Proof. We may assume s and t are not zero. Let $(\lambda^{(b)})_b$ be a tuple of partitions of n . It is sufficient to show

$$\prod_{b \in \text{supp } \theta} P_{\lambda^{(b)}}^{W_b} t^{\otimes n} \neq 0 \implies \prod_{b \in \text{supp } \theta} P_{\lambda^{(b)}}^{V_b} s^{\otimes n} \neq 0,$$

since then $E^\theta(s)$ is a supremum over a larger set than $E^\theta(t)$. Suppose that

$$\prod_{b \in \text{supp } \theta} P_{\lambda^{(b)}}^{V_b} s^{\otimes n} = 0.$$

Let $A_i : V_i \rightarrow W_i$ ($i \in [k]$) be linear maps. For any $S \subseteq [k]$ and $\lambda \vdash n$

$$P_\lambda^{W_S}(A_1 \otimes \cdots \otimes A_k)^{\otimes n} = (A_1 \otimes \cdots \otimes A_k)^{\otimes n} P_\lambda^{V_S}.$$

Therefore

$$\prod_{b \in \text{supp } \theta} P_{\lambda^{(b)}}^{W_b}(A_1 \otimes \cdots \otimes A_k)^{\otimes n} s^{\otimes n} = (A_1 \otimes \cdots \otimes A_k)^{\otimes n} \prod_{b \in \text{supp } \theta} P_{\lambda^{(b)}}^{V_b} s^{\otimes n} = 0,$$

and by continuity $\prod_{b \in \text{supp } \theta} P_{\lambda^{(b)}}^{W_b} t^{\otimes n} = 0$. □

In the rest of this section we prove statements (2) and (3), for which we need bounds on the dimension of irreducible $\text{GL}(V)$ - and S_n -representations, and the semigroup property of Kronecker coefficients and Littlewood–Richardson coefficients.

Remark 3.18 (Dimension bounds). The number of partitions of n into at most d parts is at most $(n + 1)^d$. For $\lambda \vdash n$ the dimension of the irreducible S_n -module $[\lambda]$ is given by the hook-length formula

$$\dim [\lambda] = \frac{n!}{\prod_{(i,j) \in Y(\lambda)} \text{hook}(i, j)},$$

where $Y(\lambda)$ is the Young diagram of shape λ , and $\text{hook}(i, j)$ equals the number of boxes in $Y(\lambda)$ directly below (i, j) plus the number of boxes directly to the right of (i, j) plus one, and the product is over all coordinates of $Y(\lambda)$. For $\lambda \vdash_d n$ the dimension of the irreducible $\text{GL}(V)$ -module $\mathbb{S}_\lambda(V)$ is given by the formula

$$\dim \mathbb{S}_\lambda(V) = \prod_{1 \leq i < j \leq d} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

We will make use of the following estimates:

$$(12) \quad \frac{n!}{\prod_{\ell=1}^d (\lambda_\ell + d - \ell)!} \leq \dim [\lambda] \leq \frac{n!}{\prod_{\ell=1}^d \lambda_\ell!},$$

$$(13) \quad \dim \mathbb{S}_\lambda(V) \leq (n + 1)^{d(d-1)/2}.$$

In order to set notation on specific representation-theoretic coefficients used in this work, we recall Definition 3.19.

Definition 3.19. Let $\mu, \nu \vdash n$ be partitions. Restrict the irreducible $S_n \times S_n$ -representation $[\mu] \otimes [\nu]$ to an S_n -representation via $S_n \rightarrow S_n \times S_n : \pi \mapsto (\pi, \pi)$ and consider the isotypic decomposition,

$$(14) \quad [\mu] \otimes [\nu] \downarrow_{S_n}^{S_n \times S_n} \cong \bigoplus_{\lambda \vdash n} \mathbb{C}^{g_{\lambda, \mu, \nu}} \otimes [\lambda],$$

where $\mathbb{C}^{g_{\lambda, \mu, \nu}}$ is the multiplicity space for the irreducible representation $[\lambda]$ in its isotypic component. The number $g_{\lambda, \mu, \nu}$ is called Kronecker coefficient of the symmetric group or *Kronecker coefficient* for short. (Equivalently, $g_{\lambda, \mu, \nu} = \dim([\lambda] \otimes [\mu] \otimes [\nu])^{S_n}$ which justifies the symmetric notation.) Let $\lambda \vdash_{a+b}$ be a partition. Restrict the irreducible GL_{a+b} -module $\mathbb{S}_\lambda(\mathbb{C}^{a+b})$ to a $\text{GL}_a \times \text{GL}_b$ -module via the block diagonal embedding $\text{GL}_a \times \text{GL}_b \rightarrow \text{GL}_{a+b}$, and consider the isotypic decomposition,

$$(15) \quad \mathbb{S}_\lambda(\mathbb{C}^{a+b}) \downarrow_{\text{GL}_a \times \text{GL}_b}^{\text{GL}_{a+b}} \cong \bigoplus_{\substack{\mu \vdash_a \\ \nu \vdash_b}} \mathbb{C}^{c_{\mu, \nu}^\lambda} \otimes \mathbb{S}_\mu(\mathbb{C}^a) \otimes \mathbb{S}_\nu(\mathbb{C}^b),$$

where $\mathbb{C}^{c_{\mu, \nu}^\lambda}$ is the multiplicity space for the irreducible module $\mathbb{S}_\mu(\mathbb{C}^a) \otimes \mathbb{S}_\nu(\mathbb{C}^b)$ in its isotypic component. The number $c_{\mu, \nu}^\lambda$ is called a *Littlewood–Richardson coefficient*.

Remark 3.20 (Semigroup property). If λ and λ' are partitions, then $\lambda + \lambda'$ is defined by elementwise addition. A fundamental property of the Kronecker coefficients and the Littlewood–Richardson coefficients is the well-known *semigroup property* (see e.g. [21]):

$$\begin{aligned} &\text{if } g_{\lambda, \mu, \nu} > 0 \text{ and } g_{\alpha, \beta, \gamma} > 0, \text{ then } g_{\lambda + \alpha, \mu + \beta, \nu + \gamma} > 0; \\ &\text{if } c_{\mu, \nu}^\lambda > 0 \text{ and } c_{\beta, \gamma}^\alpha > 0, \text{ then } c_{\mu + \beta, \nu + \gamma}^{\lambda + \alpha} > 0. \end{aligned}$$

In other words, the triples of partitions for which the Kronecker coefficients are nonzero form a semigroup under elementwise addition, and the same is true for Littlewood–Richardson coefficients.

The semigroup properties can be used to prove Lemma 3.21. Of this lemma, the first statement can be found in [23], while we do not know of any source that explicitly states the second statement. For the convenience of the reader we give the proofs of both statements. Let $H(P)$ denote the Shannon entropy and let $h(p)$ denote the binary entropy function.

Lemma 3.21. *Let λ, μ, ν be integer partitions.*

- (1) *If $g_{\lambda, \mu, \nu}$ is nonzero, then $H(\bar{\lambda}) \leq H(\bar{\mu}) + H(\bar{\nu})$.*
- (2) *If $c_{\mu, \nu}^\lambda$ is nonzero, then $H(\bar{\lambda}) \leq \frac{|\mu|}{|\nu|+|\mu|}H(\bar{\mu}) + \frac{|\nu|}{|\nu|+|\mu|}H(\bar{\nu}) + h\left(\frac{|\mu|}{|\nu|+|\mu|}\right)$.*

Proof. We begin with the first statement. Suppose $g_{\lambda, \mu, \nu}$ is nonzero. The semigroup property of Kronecker coefficients implies $g_{N\lambda, N\mu, N\nu} \neq 0$ for any positive integer N , where $N\lambda$ is the partition with parts $N\lambda_i$. Then the irreducible representation $[N\lambda]$ is isomorphic to a subspace of $[N\mu] \otimes [N\nu]$ by (14). So obviously $\dim [N\lambda] \leq \dim [N\mu] \dim [N\nu]$. We use the dimension bounds (12) to see that $NnH(\bar{\lambda}) - o(N) \leq NnH(\bar{\mu}) + NnH(\bar{\nu})$ when $N \rightarrow \infty$. We thus obtain the required inequality $H(\bar{\lambda}) \leq H(\bar{\mu}) + H(\bar{\nu})$.

We prove the second statement. By Schur–Weyl duality, as an $S_n \times \text{GL}_{a+b}$ -representation $(\mathbb{C}^{a+b})^{\otimes n}$ decomposes as

$$(\mathbb{C}^{a+b})^{\otimes n} \cong \bigoplus_{\lambda \vdash_{a+b} n} [\lambda] \otimes \mathbb{S}_\lambda(\mathbb{C}^{a+b})$$

and for the restriction via $\text{GL}_a \times \text{GL}_b \rightarrow \text{GL}_{a+b}$ (denoted by \downarrow) we obtain

$$(\mathbb{C}^{a+b})^{\otimes n} \downarrow \cong \bigoplus_{\lambda \vdash_{a+b} n} \bigoplus_{\substack{\mu \vdash_a \\ \nu \vdash_b}} [\lambda] \otimes \mathbb{C}^{c_{\mu, \nu}^\lambda} \otimes \mathbb{S}_\mu(\mathbb{C}^a) \otimes \mathbb{S}_\nu(\mathbb{C}^b).$$

On the other hand, we may first expand $(\mathbb{C}^a \oplus \mathbb{C}^b)^{\otimes n}$ and then apply Schur–Weyl duality for GL_a and GL_b separately,

$$\begin{aligned} (\mathbb{C}^{a+b})^{\otimes n} &\cong (\mathbb{C}^a \oplus \mathbb{C}^b)^{\otimes n} \cong \bigoplus_{k=0}^n \mathbb{C}^{\binom{n}{k}} \otimes (\mathbb{C}^a)^{\otimes k} \otimes (\mathbb{C}^b)^{\otimes n-k} \\ &\cong \bigoplus_{k=0}^n \bigoplus_{\substack{\mu \vdash_a k \\ \nu \vdash_b n-k}} \mathbb{C}^{\binom{n}{k}} \otimes [\mu] \otimes \mathbb{S}_\mu(\mathbb{C}^a) \otimes [\nu] \otimes \mathbb{S}_\nu(\mathbb{C}^b). \end{aligned}$$

Suppose $c_{\mu, \nu}^\lambda$ is nonzero. Then the irreducible representation $[\lambda]$ is isomorphic to a subspace of $\mathbb{C}^{\binom{n}{|\mu|}} \otimes [\mu] \otimes [\nu]$. So obviously $\dim [\lambda] \leq \binom{n}{|\mu|} \dim [\mu] \dim [\nu]$. Because of the semigroup property we may repeat everything for $N\lambda, N\mu, N\nu$ instead of λ, μ, ν and obtain $\dim [N\lambda] \leq \binom{Nn}{N|\mu|} \dim [N\mu] \dim [N\nu]$. We use the dimension bounds (12) to see that $NH(\bar{\lambda}) - o(N) \leq Nh\left(\frac{|\mu|}{n}\right) + \frac{|\mu|}{n}NH(\bar{\mu}) + \frac{|\nu|}{n}NH(\bar{\nu})$ when $N \rightarrow \infty$, where $h(p)$ denotes the binary entropy function. \square

Let V and W be vector spaces of dimensions d and e . The tensor power $(V \oplus W)^{\otimes n}$ is both a $\text{GL}(V) \times \text{GL}(W)$ -module and a $\text{GL}(V \oplus W)$ -module. Let $\lambda \vdash_{d+e} n + m$, $\mu \vdash_d n$ and $\nu \vdash_e m$. The projections $P_\lambda^{V \oplus W}$ and $P_\mu^V \otimes P_\nu^W$ commute, and their product is nonzero if and only if the Littlewood–Richardson coefficient $c_{\mu, \nu}^\lambda$ is nonzero.

Lemma 3.22. *Let $\theta \in \mathcal{P}_{\text{nc}}(\Pi)$. Then $F^\theta(s \oplus t) \leq F^\theta(s) + F^\theta(t)$.*

Proof. Let $(\lambda^{(b)})_{b \in \text{supp } \theta}$ be a tuple of partitions of n such that

$$(16) \quad \prod_{b \in \text{supp } \theta} P_{\lambda^{(b)}}^{(V \oplus W)_b} (s \oplus t)^{\otimes n} \neq 0.$$

Since $P_{\lambda^{(b)}}^{(V \oplus W)_b} = P_{\lambda^{(b)}}^{(V \oplus W)_b} P_{(n)}^{(V \oplus W)}$ statement (16) is equivalent to

$$(17) \quad \sum_{m=0}^n \binom{n}{m} \prod_{b \in \text{supp } \theta} P_{\lambda^{(b)}}^{(V \oplus W)_b} (s^{\otimes m} \otimes t^{\otimes (n-m)}) \neq 0.$$

At least one of the summands in (17) is nonzero, i.e. for some m

$$\prod_{b \in \text{supp } \theta} P_{\lambda^{(b)}}^{(V \oplus W)_b} (s^{\otimes m} \otimes t^{\otimes (n-m)}) \neq 0.$$

We may write $s^{\otimes m} \otimes t^{\otimes (n-m)}$ as

$$s^{\otimes m} \otimes t^{\otimes (n-m)} = \sum_{\substack{(\mu^{(b)}) \\ (\nu^{(b)})}} \prod_{b \in \text{supp } \theta} P_{\mu^{(b)}}^{V_b} s^{\otimes m} \otimes \prod_{b \in \text{supp } \theta} P_{\nu^{(b)}}^{W_b} t^{\otimes (n-m)},$$

where the sum is over tuples of partitions $\mu^{(b)} \vdash m$ and $\nu^{(b)} \vdash n - m$. There thus exist tuples $(\mu^{(b)})_b$ and $(\nu^{(b)})_b$ such that

$$(18) \quad P_{\lambda^{(b)}}^{(V \oplus W)_b} (P_{\mu^{(b)}}^{V_b} \otimes P_{\nu^{(b)}}^{W_b}) \neq 0 \quad \text{for all } b \in \text{supp } \theta,$$

$$(19) \quad \prod_{b \in \text{supp } \theta} P_{\mu^{(b)}}^{V_b} s^{\otimes m} \neq 0,$$

$$(20) \quad \prod_{b \in \text{supp } \theta} P_{\nu^{(b)}}^{W_b} t^{\otimes n-m} \neq 0.$$

Eq. (18) implies that the Littlewood–Richardson coefficient $c_{\mu^{(b)}, \nu^{(b)}}^{\lambda^{(b)}}$ is nonzero and thus by Lemma 3.21, $H(\lambda^{(b)}) \leq \frac{m}{n} H(\mu^{(b)}) + (1 - \frac{m}{n}) H(\nu^{(b)}) + h(\frac{m}{n})$. Setting $p = m/n$ we can therefore write

$$\sum_{b \in \text{supp } \theta} \theta(b) H(\lambda^{(b)}) \leq p \sum_{b \in \text{supp } \theta} \theta(b) H(\mu^{(b)}) + (1 - p) \sum_{b \in \text{supp } \theta} \theta(b) H(\nu^{(b)}) + h(p)$$

which because of (19) and (20) is at most $pE^\theta(s) + (1 - p)E^\theta(t) + h(p)$. Taking the supremum over the admissible tuples $(\lambda^{(b)})_b$ gives the inequality $E^\theta(s \oplus t) \leq pE^\theta(s) + (1 - p)E^\theta(t) + h(p)$. We are done by applying Lemma 3.9. \square

The tensor power $(V \otimes W)^{\otimes n}$ is both a $\text{GL}(V) \times \text{GL}(W)$ -module and a $\text{GL}(V \otimes W)$ -module. Let $\lambda \vdash_{de} n$, $\mu \vdash_d n$ and $\nu \vdash_e n$. The projections $P_\lambda^{V \otimes W}$ and $P_\mu^V \otimes P_\nu^W$ commute, and their product is nonzero if and only if the Kronecker coefficient $g_{\lambda, \mu, \nu}$ is nonzero.

Lemma 3.23. *Let $\theta \in \mathcal{P}_{\text{nc}}(\Pi)$. Then $F^\theta(s \otimes t) \leq F^\theta(s)F^\theta(t)$.*

Proof. Let $(\lambda^{(b)})_b$ be a tuple of partitions of n such that

$$\prod_{b \in \text{supp } \theta} P_{\lambda^{(b)}}^{(V \otimes W)_b} (s \otimes t)^{\otimes n} \neq 0.$$

We may write

$$s^{\otimes n} \otimes t^{\otimes n} = \sum_{\substack{(\mu^{(b)})_b: \mu^{(b)} \vdash n \\ (\nu^{(b)})_b: \nu^{(b)} \vdash n}} \prod_{b \in \text{supp } \theta} P_{\mu^{(b)}}^{V_b} s^{\otimes n} \otimes \prod_{b \in \text{supp } \theta} P_{\nu^{(b)}}^{W_b} t^{\otimes n}.$$

There exist tuples $(\mu^{(b)})_b$ and $(\nu^{(b)})_b$ such that

$$(21) \quad P_{\lambda^{(b)}}^{(V \otimes W)_b} (P_{\mu^{(b)}}^{V_b} \otimes P_{\nu^{(b)}}^{W_b}) \neq 0 \quad \text{for all } b \in \text{supp } \theta,$$

$$(22) \quad \prod_{b \in \text{supp } \theta} P_{\mu^{(b)}}^{V_b} s^{\otimes n} \neq 0,$$

$$(23) \quad \prod_{b \in \text{supp } \theta} P_{\nu^{(b)}}^{W_b} t^{\otimes n} \neq 0.$$

By (21) the Kronecker coefficient $g_{\lambda^{(b)}, \mu^{(b)}, \nu^{(b)}}$ is nonzero and so by Lemma 3.21 we have $H(\overline{\lambda^{(b)}}) \leq H(\overline{\mu^{(b)}}) + H(\overline{\nu^{(b)}})$. Therefore

$$\sum_{b \in \text{supp } \theta} \theta(b) H(\overline{\lambda^{(b)}}) \leq \sum_{b \in \text{supp } \theta} \theta(b) H(\overline{\mu^{(b)}}) + \sum_{b \in \text{supp } \theta} \theta(b) H(\overline{\nu^{(b)}})$$

which because of (22) and (23) is at most $E^\theta(s) + E^\theta(t)$. Taking the supremum over the admissible tuples $(\lambda^{(b)})_b$ gives $E^\theta(s \otimes t) \leq E^\theta(s) + E^\theta(t)$. \square

3.3. Comparing the quantum functionals. In this section we show that in the general regime $\theta \in \mathcal{P}(\Pi)$ the upper quantum functional is at least the lower quantum functional, justifying the names.

Theorem 3.24. *Let $\theta \in \mathcal{P}(\Pi)$. Then*

$$E^\theta(t) \geq E_\theta(t).$$

To prove Theorem 3.24 we use the *gentle measurement lemma* and the *spectrum estimation theorem*. A proof of the gentle measurement lemma can be found in [73]; see also [50, Lemma 5] for the specific bound that we use. Keyl and Werner proved the spectrum estimation theorem [40]; see also [37] and [23, Theorem 1] for a more succinct proof. Let $\mathcal{S}_{\leq}(\mathcal{H})$ be the set of positive semidefinite operators $\rho : \mathcal{H} \rightarrow \mathcal{H}$ satisfying $\text{Tr } \rho \leq 1$.

Lemma 3.25 (Gentle measurement). *Let $\rho \in \mathcal{S}_{\leq}(\mathcal{H})$. Let $X : \mathcal{H} \rightarrow \mathcal{H}$ be a linear map such that $0 \leq X \leq I$. Then*

$$\|X\rho X - \rho\|_1 \leq 2\sqrt{1 - \text{Tr}(X\rho X)}.$$

Corollary 3.26. *Let $\rho_0 \in \mathcal{S}(\mathcal{H})$ and $0 \leq X_1, \dots, X_n \leq I$. Suppose that for each $i \in [n]$ the inequality $\text{Tr}(X_i \rho_0 X_i) \geq 1 - \varepsilon$ holds. Then*

$$(24) \quad \|X_n \cdots X_2 X_1 \rho_0 X_1 X_2 \cdots X_n - \rho_0\|_1 \leq 2n\sqrt{\varepsilon}.$$

In particular,

$$\text{Tr}(X_n \cdots X_2 X_1 \rho_0 X_1 X_2 \cdots X_n) \geq 1 - 2n\sqrt{\varepsilon}.$$

Proof. Let $\rho_i = X_i \rho_{i-1} X_i$ for $i \in [n]$. By the Hölder inequality and the gentle measurement lemma (Lemma 3.25), we have

$$\begin{aligned} \|\rho_i - \rho_0\|_1 &\leq \|\rho_i - X_i \rho_0 X_i\|_1 + \|X_i \rho_0 X_i - \rho_0\|_1 \\ &= \|X_i(\rho_{i-1} - \rho_0)X_i\|_1 + \|X_i \rho_0 X_i - \rho_0\|_1 \\ &\leq \|\rho_{i-1} - \rho_0\|_1 \cdot \|X_i\|_\infty^2 + 2\sqrt{1 - \text{Tr}(X_i \rho_0 X_i)} \\ &\leq \|\rho_{i-1} - \rho_0\|_1 + 2\sqrt{\varepsilon}. \end{aligned}$$

The statement follows by induction on i . □

Theorem 3.27 (Spectrum estimation). *Let \mathcal{H} have dimension d . Let $\rho \in \mathcal{S}(\mathcal{H})$ be a density matrix. Let $r \in \mathbb{R}^d$ be the sequence of eigenvalues of ρ ordered nonincreasingly. Let $\lambda \vdash n$. Then*

$$\text{Tr}(P_\lambda^\mathcal{H} \rho^{\otimes n}) \leq (n + 1)^{d(d-1)/2} 2^{-nD(\lambda|r)},$$

where $D(p||q) = \sum_i p_i(\log_2 p_i - \log_2 q_i)$ is the Kullback–Leibler divergence of probability distributions p and q .

Corollary 3.28. *Let $\rho \in \mathcal{S}(\mathcal{H})$. Let $\varepsilon > 0$. Let $X = \sum P_\lambda^\mathcal{H}$ where the sum is over partitions $\lambda \vdash n$ such that $H(\bar{\lambda}) \geq H(\rho) - \varepsilon$. Then $\lim_{n \rightarrow \infty} \text{Tr}(\rho^{\otimes n} X) = 1$.*

Proof. We refer to Corollary 2 in [23]. □

Proof of Theorem 3.24. Let $\ell = |\text{supp } \theta|$. Let $\varepsilon > 0$ be arbitrary. Choose inner products on V_1, \dots, V_k such that t is a unit vector and such that the θ -weighted average of the marginal quantum entropies of t is at least $E_\theta(t) - \varepsilon$. Let $(r_b)_{b \in \Pi}$ be the tuple of marginal quantum entropies of t . By Corollary 3.28 there is an n_0 such that for every $n \geq n_0$ and $i \in [\ell]$ the inequality $\text{Tr}(|t\rangle\langle t|^{\otimes n} X_i) \geq 1 - \varepsilon$ holds, where

$$X_i = \sum_{\substack{\lambda \vdash n: \\ H(\bar{\lambda}) \geq r_{\pi(i)} - \varepsilon}} P_\lambda^\mathcal{H}.$$

According to Corollary 3.26 this in turn implies

$$\text{Tr}(X_\ell \cdots X_2 X_1 (|t\rangle\langle t|)^{\otimes n} X_1 X_2 \cdots X_\ell) \geq 1 - 2\ell\sqrt{\varepsilon}.$$

In particular, $X_\ell \cdots X_2 X_1 t^{\otimes n} \neq 0$ for ε small enough and n large enough. By the definition of the operators X_i this implies that there is a tuple $(\lambda^{(i)})_{i \in [\text{supp } \theta]}$ of partitions $\lambda^{(i)} \vdash n$ such that

$$P_{\lambda^{(\pi(\ell))}}^{V_{\pi(\ell)}} \cdots P_{\lambda^{(\pi(2))}}^{V_{\pi(2)}} P_{\lambda^{(\pi(1))}}^{V_{\pi(1)}} t^{\otimes n} \neq 0$$

and such that $H(\bar{\lambda}^{(i)}) \geq r_{\pi(i)} - \varepsilon$ for each $i \in [\ell]$. Therefore,

$$E^\theta(t) \geq \sum_{b \in \text{supp } \theta} \theta(b) r_b - \varepsilon \geq E_\theta(t) - 2\varepsilon.$$

This holds for any $\varepsilon > 0$, so $E^\theta(t) \geq E_\theta(t)$. □

3.4. Universal spectral points. We have the upper quantum functional F^θ which is sub-additive and sub-multiplicative, we have the lower quantum functional F_θ which is super-additive and super-multiplicative, both are \leq -monotone and normalised on $\langle r \rangle$, and we know that $F^\theta(t) \geq F_\theta(t)$. (See Theorem 3.16, Theorem 3.5 and Theorem 3.24.) In this section we show that $F^\theta(t)$ and $F_\theta(t)$ in fact coincide for the singleton regime $\theta \in \mathcal{P}_s(\Pi)$ and for any tensor t , which shows that $F^\theta = F_\theta$ is a universal spectral point. The ingredient for the proof is an object called the entanglement polytope and two characterisations for this object.

Let $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$ be a Hilbert space. Let $G = \text{GL}(\mathcal{H}_1) \times \cdots \times \text{GL}(\mathcal{H}_k)$. Let $|\psi\rangle \in \mathcal{H}$ be a unit vector. Then $|\psi\rangle\langle\psi| \in \mathcal{S}(\mathcal{H})$ and therefore we can consider the marginals $|\psi\rangle\langle\psi|_j$ of $|\psi\rangle\langle\psi|$ for $j \in [k]$. For each $j \in [k]$, let $r_j(\psi)$ be the nonincreasingly ordered tuple of eigenvalues of the j th marginal. The *entanglement polytope* of a nonzero tensor $t \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$ is the subset $\Delta_t \subseteq \mathbb{R}^{\dim \mathcal{H}_1 + \cdots + \dim \mathcal{H}_k}$ defined as

$$\Delta_t = \{ (r_1(\psi), \dots, r_k(\psi)) \mid \psi \in \overline{G \cdot t} \text{ and } \|\psi\| = 1 \}.$$

Note that Δ_t does not depend on the choice of inner products on \mathcal{H}_j . To relate F^θ to F_θ we need another important characterisation of Δ_t . The following characterization of Δ_t appears in [70, Theorem 1] and [51] and is a specialization of a result by Brion [14].

Theorem 3.29. *The entanglement polytope Δ_t is the Euclidean closure of the set*

$$\{ (\overline{\lambda^{(1)}}), \dots, \overline{\lambda^{(k)}} \mid (P_{\lambda^{(1)}}^{\mathcal{H}_1} \otimes \cdots \otimes P_{\lambda^{(k)}}^{\mathcal{H}_k}) t^{\otimes n} \neq 0 \}.$$

We now return to the quantum functionals.

Theorem 3.30. *Let $\theta \in \mathcal{P}_s(\Pi)$. Then $E^\theta(t) = E_\theta(t)$.*

Proof. Using the simple fact that the quantum marginal entropy can be defined in terms of the usual entropy of the spectrum, and since $\theta \in \mathcal{P}_s(\Pi)$, the value of $E_\theta(t)$ is related to the entanglement polytope Δ_t as follows

$$(25) \quad E_\theta(t) = \sup \{ \theta(1)H(r_1) + \cdots + \theta(k)H(r_k) \mid (r_1, \dots, r_k) \in \Delta_t \}.$$

Using the characterisation in Theorem 3.29 we can write (25) as

$$\sup \{ \theta(1)H(\overline{\lambda^{(1)}}) + \cdots + \theta(k)H(\overline{\lambda^{(k)}}) \mid (P_{\lambda^{(1)}}^{V_1} \otimes \cdots \otimes P_{\lambda^{(k)}}^{V_k}) t^{\otimes n} \neq 0 \}$$

which equals $E^\theta(t)$. □

By the discussion at the beginning of this section the following is immediate.

Corollary 3.31. *Let $\theta \in \mathcal{P}_s(\Pi)$. Then $F^\theta = F_\theta$ is a point in the asymptotic spectrum $\Delta(\mathcal{T})$, a universal spectral point.*

Definition 3.32 (Quantum simplex). Analogous to the support simplex (Definition 2.20), we define for any family \mathcal{X} the *quantum simplex* $F_{\mathcal{P}([k])}(\mathcal{X})$ as the image of the map $\mathcal{P}([k]) \rightarrow \Delta(\mathcal{X}) : \theta \mapsto F^\theta$.

We relate the Strassen upper support functional ρ^θ to the new upper quantum functional E^θ . Recall that ρ^θ is defined for $\theta \in \mathcal{P}_s(\Pi) \subseteq \mathcal{P}(\Pi)$, that is, for probability distributions θ supported on bipartitions of the form $\{\{j\}, \bar{j}\}$. Let $t \in V_1 \otimes \cdots \otimes V_k$.

Remark 3.33 (Majorisation property). Let v_1, \dots, v_d be a basis of V . Let $T \subseteq \text{GL}(V)$ be the subgroup of linear maps having diagonal matrix in this basis, a maximal torus for $\text{GL}(V)$. For a tuple $i = (i_1, \dots, i_n) \in [d]^n$ let the vector $v_i \in V^{\otimes n}$ be defined as

$$v_i := v_{i_1} \otimes \dots \otimes v_{i_n}.$$

Let $i \in [d]^n$ and let Q be the type of i , that is, Q is the probability distribution on $[d]$ so that for every $x \in [d]$ the probability $Q(x)$ is proportional to the number of times that x occurs in i . Then the vector v_i is a *weight vector* for T with weight nQ , meaning that for any $t = \text{diag}(t_1, \dots, t_d) \in T$ we have $t \cdot v_i = \prod_{j=1}^d (t_j)^{nQ(j)} v_i$. The module $\mathbb{S}_\lambda(V)$ has so-called highest weight λ , and thus $P_\lambda^V v_i \neq 0$ implies that $nQ \preceq \lambda$, where \preceq is the dominance order or majorisation order on partitions. We call this the *majorisation property*. In particular, $P_\lambda^V v_i \neq 0$ implies the inequality of Shannon entropies $H(Q) \geq H(\bar{\lambda})$, since the Shannon entropy is Schur-concave.

Theorem 3.34. *Let $\theta \in \mathcal{P}_s(\Pi)$. Then $\rho^\theta(t) \geq E^\theta(t)$.*

Proof. Choose a k -tuple of bases $B = ((v_{1,\alpha_1}), \dots, (v_{k,\alpha_k}))$ in $\mathcal{B}(t)$ such that $\rho^\theta(t) = H_\theta(\text{supp}_B t)$. Suppose $(P_{\lambda^{(1)}}^{V_1} \otimes \dots \otimes P_{\lambda^{(k)}}^{V_k}) t^{\otimes N} \neq 0$ for some k -tuple of partitions $(\lambda^{(1)}, \dots, \lambda^{(k)})$ and $N \in \mathbb{N}$. Given an N -type $Q \in \mathcal{P}_N([n_1] \times \dots \times [n_k])$, define the vector $v_Q \in V_1^{\otimes N} \otimes \dots \otimes V_k^{\otimes N}$ as

$$v_Q := \sum_x \bigotimes_{m=1}^N v_{1,x_{m,1}} \otimes v_{2,x_{m,2}} \otimes \dots \otimes v_{k,x_{m,k}},$$

where the sum is over all N -tuples $x = (x_1, \dots, x_N) \in T_Q^N$ of type Q , and define the element $t_Q \in \mathbb{C}$ as $t_Q := \prod_{i \in \text{supp } Q} (t_{i_1, \dots, i_k})^{NQ(i)}$. Then the tensor power $t^{\otimes N}$ can be written as

$$t^{\otimes N} = \sum_Q t_Q v_Q,$$

where the sum is over $Q \in \mathcal{P}_N(\text{supp}_B t)$. Then $(P_{\lambda^{(1)}}^{V_1} \otimes \dots \otimes P_{\lambda^{(k)}}^{V_k}) v_Q \neq 0$ for some $Q \in \mathcal{P}_N(\text{supp}_B t)$. By the majorization property this implies for all $j \in [k]$ that $NQ_j \preceq \lambda^{(j)}$ and thus $H(Q_j) \geq H(\bar{\lambda}^{(j)})$. This shows that

$$\rho^\theta(t) = H_\theta(\text{supp}_B t) \geq H_\theta(Q) = \sum_{j=1}^k \theta(j) H(Q_j) \geq \sum_{j=1}^k \theta(j) H(\bar{\lambda}^{(j)}).$$

Take the supremum of the right hand side over the possible $(\lambda^{(1)}, \dots, \lambda^{(k)})$ to get the required inequality $\rho^\theta(t) \geq E^\theta(t)$. \square

By Theorem 3.34, for $\theta \in \mathcal{P}([k])$, $E^\theta(t) \leq \rho^\theta(t) \leq n_1^{\theta(1)} \dots n_k^{\theta(k)}$. Therefore, if $E^\theta(t) = n_1^{\theta(1)} \dots n_k^{\theta(k)}$, then $\rho^\theta(t) = n_1^{\theta(1)} \dots n_k^{\theta(k)}$. We will show the converse of this in Theorem 4.31, that is, when $\theta_i > 0$ for all i , if $E^\theta(t) < n_1^{\theta(1)} \dots n_k^{\theta(k)}$, then $\rho^\theta(t) < n_1^{\theta(1)} \dots n_k^{\theta(k)}$.

The final result of this section is that the regularisation of the upper support functional equals the upper quantum functional.

Theorem 3.35. *Let $t \in V_1 \otimes \dots \otimes V_k$, $\theta \in \mathcal{P}([k])$. Then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \rho^\theta(t^{\otimes n}) = E^\theta(t).$$

Proof. By Theorem 3.34, $E^\theta(t) \leq \rho^\theta(t)$, so $\lim_{n \rightarrow \infty} \frac{1}{n} E^\theta(t^{\otimes n}) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \rho^\theta(t^{\otimes n})$. We know $E^\theta(t) = \lim_{n \rightarrow \infty} \frac{1}{n} E^\theta(t^{\otimes n})$ by Corollary 3.31. Therefore

$$\lim_{n \rightarrow \infty} \frac{1}{n} \rho^\theta(t^{\otimes n}) \geq E^\theta(t).$$

We now prove \leq . Let $n \in \mathbb{N}$ and consider the decomposition

$$V_i^{\otimes n} = \bigoplus_{\lambda \vdash n} \mathbb{S}_\lambda(V_i) \otimes [\lambda].$$

In each direct summand on the right hand side choose a basis with (disjoint) index sets $I_{i,\lambda}$, $\lambda \vdash n$. Together they form a basis of $V_i^{\otimes n}$. Let $B \in \mathcal{B}(t^{\otimes n})$ be this k -tuple of bases. Define the set

$$\Phi = \bigcup \{ I_{1,\lambda^{(1)}} \times \cdots \times I_{k,\lambda^{(k)}} \mid \lambda^{(1)}, \dots, \lambda^{(k)} : \sum_{i=1}^k \theta(i) H(\overline{\lambda^{(i)}}) \leq E^\theta(t) \}.$$

By definition, $\text{supp}_B t^{\otimes n} \subseteq \Phi$, and therefore

$$\rho^\theta(t^{\otimes n}) \leq H_\theta(\text{supp}_B t^{\otimes n}) \leq H_\theta(\Phi).$$

To estimate $H_\theta(\Phi)$, note that by concavity of the entropy and invariance of Φ under permutations within each $I_{i,\lambda}$, the maximum is attained at a distribution which is uniform within each block $I_{1,\lambda^{(1)}} \times \cdots \times I_{k,\lambda^{(k)}}$. Let P be such a distribution with $H_\theta(\Phi) = H_\theta(P)$, and let the total probability within block $I_{1,\lambda^{(1)}} \times \cdots \times I_{k,\lambda^{(k)}}$ be $Q(\lambda^{(1)}, \dots, \lambda^{(k)})$. Thus Q describes a ‘‘coarse’’ structure of P , which is uniform conditioned on Q . The marginals of P inherit a similar structure: P_i is uniform within each block $I_{i,\lambda^{(i)}}$ and the total probability of such a block is $Q_i(\lambda^{(i)})$. The recursion property allows us to express $H(P_i)$ as

$$\begin{aligned} H(P_i) &= H(Q_i) + \sum_{\lambda^{(i)}} Q_i(\lambda^{(i)}) \log_2 |I_{i,\lambda^{(i)}}| \\ &= H(Q_i) + \sum_{\lambda^{(1)}, \dots, \lambda^{(k)}} Q(\lambda^{(1)}, \dots, \lambda^{(k)}) \log_2 |I_{i,\lambda^{(i)}}|, \end{aligned}$$

and therefore

$$(26) \quad H_\theta(P) = H_\theta(Q) + \sum_{\lambda^{(1)}, \dots, \lambda^{(k)}} Q(\lambda^{(1)}, \dots, \lambda^{(k)}) \sum_{i=1}^k \theta(i) \log_2 |I_{i,\lambda^{(i)}}|.$$

We use the estimates

$$|I_{i,\lambda}| = \dim(\mathbb{S}_\lambda(V_i) \otimes [\lambda]) \leq (n+1)^{d_i(d_i-1)/2} \binom{n}{\lambda_1, \dots, \lambda_{d_i}} \leq 2^{nH(\bar{\lambda})+o(n)}$$

and

$$H_\theta(Q) \leq H(Q) \leq d_1 d_2 \cdots d_k \log_2(n+1) = o(n).$$

In the sum over $\lambda^{(1)}, \dots, \lambda^{(k)}$ in (26) only those terms can have nonzero probability which satisfy

$$\sum_{i=1}^k \theta(i) H(\overline{\lambda^{(i)}}) \leq E^\theta(t).$$

For these terms we have the upper bound

$$\sum_{i=1}^k \theta(i) \log_2 |I_{i,\lambda^{(i)}}| \leq n \sum_{i=1}^k \theta(i) H(\overline{\lambda^{(i)}}) + o(n) \leq nE^\theta(t) + o(n).$$

The sum of all $Q(\lambda^{(1)}, \dots, \lambda^{(k)})$ equals 1, and therefore

$$\rho^\theta(t^{\otimes n}) \leq H_\theta(P) \leq nE^\theta(t) + o(n),$$

which implies $\lim_{n \rightarrow \infty} \frac{1}{n} \rho^\theta(t^{\otimes n}) \leq E^\theta(t)$. □

Remark 3.36. In the language of comment 7.(ii) in [60], our Theorem 3.35 shows that the set Z^θ is a single point.

Using Theorem 3.35 we can see that the quantum functionals are at least the lower support functional.

Corollary 3.37. *Let $t \in V_1 \otimes \dots \otimes V_k$, $\theta \in \mathcal{P}_s(\Pi)$. Then $E^\theta(t) \geq \rho_\theta(t)$.*

Proof. By Theorem 3.35,

$$F^\theta(t) = \lim_{n \rightarrow \infty} \zeta^\theta(t^{\otimes n})^{1/n}.$$

We know $\zeta^\theta(t) \geq \zeta_\theta(t)$ by Theorem 2.15 and thus

$$\lim_{n \rightarrow \infty} \zeta^\theta(t^{\otimes n})^{1/n} \geq \lim_{n \rightarrow \infty} \zeta_\theta(t^{\otimes n})^{1/n}.$$

The lower support functional ζ_θ is super-multiplicative under \otimes (Theorem 2.10), so

$$\lim_{n \rightarrow \infty} \zeta_\theta(t^{\otimes n})^{1/n} \geq \zeta_\theta(t).$$

Combining these statements proves the theorem. □

4. SEVERAL FAMILIES OF TENSORS

Now that we have in place the definitions and key properties of Strassen's support functionals (Section 2) and the new quantum functionals (Section 3), we will focus on studying various basic classes of tensors. Not only are these classes very natural, they also correspond to important problems in combinatorics and complexity. Focusing on these classes of tensors will allow us to understand more precisely the absolute and relative power of the support functionals and the quantum functionals.

We begin our discussion with the most special (i.e. smallest) class of tensors, namely that of *tight* tensors (Section 4.1). Many important tensors, most notably the matrix multiplication tensors, are tight, but not all tensors are tight. In particular, tight tensors form a sub-semiring of the oblique tensors that we have seen earlier (Definition 2.18). Strassen proved the surprising result that for tight 3-tensors the support functionals are sufficiently powerful to compute the asymptotic subrank. Namely, the asymptotic subrank is precisely their pointwise minimum. We introduce a new general technique in this context (Theorem 4.11) using combinatorial degeneration and discuss an important related example in combinatorics (the cap set problem, Example 4.14) as an application of both the support functionals to tight tensors and combinatorial degeneration. This discussion and connections in Section 4.1 give a good idea of the power of the support functionals, and looking ahead, they lead up to the later Section 5, in which we will prove a similar, but much more general, pointwise minimum characterization for the *quantum* functionals (namely that the pointwise minimum of the quantum functionals precisely characterizes the asymptotic slice rank of *any* tensor).

Next, in order to understand better the *relative* power of the support functionals and quantum functionals, we discuss *free* tensors (Section 4.2), a class of tensors

whose simple definition is originated in representation theory. The previously discussed oblique tensors (which contains the tight tensors) form a sub-semiring of the free tensors. (Again, not all tensors are free.) We prove the strong connection that for free tensors (over the complex numbers) the upper support functionals and the quantum functionals *coincide* (in the singleton regime, naturally, where these functionals have the same parameter space $\mathcal{P}([k])$). This in particular implies that the upper support functionals are spectral points for free tensors (over the complex numbers).

We finish by discussing *generic* (i.e. random) tensors (Section 4.3). This is a totally different regime than the previous two, which allows us to gain extra insight into the absolute and relative power of the support and quantum functionals. In particular, we compute the value of the quantum functionals on generic tensors and we compare these values to the known results for the support functionals (upper and lower). Related to this we apply the Hilbert–Mumford criterion to see that if the upper quantum functional is not maximal, then the upper support functional is not maximal.

4.1. Tight tensors. In this section we let \mathbb{F} be an arbitrary field. Strassen defined the family of tight tensors as follows.

Definition 4.1 (Tight). Let $\Phi \subseteq I_1 \times \dots \times I_k$. We say that the set Φ is *tight* if there are injective maps $u_i : I_i \rightarrow \mathbb{Z}$ ($i \in [k]$) such that for every $(\alpha_1, \dots, \alpha_k) \in \Phi$ we have

$$(27) \quad u_1(\alpha_1) + \dots + u_k(\alpha_k) = 0.$$

We say a tensor f is *tight* if there is a $B \in \mathcal{B}(f)$ such that the set $\text{supp}_B f$ is tight.

Remark 4.2. Clearly, tight tensors are oblique. To summarise the families of tensors that we have defined up to now, we have

$$\{\text{tight}\} \subseteq \{\text{oblique}\} \subseteq \{\text{robust}\} \subseteq \{\theta\text{-robust}\}.$$

Recall that the families of oblique, robust and θ -robust tensors each forms a semiring under \otimes and \oplus . Tight tensors have the same property [60, Section 5]. Another property is that any subset of a tight set is tight.

Example 4.3. Let $k \geq 3$ be fixed. For any integer $n \geq 1$ and $c \in [n]$ the following set is clearly tight:

$$\Phi_n(c) = \{\alpha \in \{0, \dots, n-1\}^k \mid \alpha_1 + \dots + \alpha_k = c\}.$$

For any integer $n \geq 2$ and any $c \in [n]$ the following set is *not* tight (cf. Exercise 15.20 in [18]):

$$\Psi_n(c) = \{\alpha \in \{0, \dots, n-1\}^k \mid \alpha_1 + \dots + \alpha_k = c \pmod n\}.$$

The *tensor* $f_n = \sum_{\alpha \in \Psi_n(n-1)} e_{\alpha_1} \otimes \dots \otimes e_{\alpha_k} \in (\mathbb{F}^n)^{\otimes k}$ is tight, however, when the ground field \mathbb{F} contains a primitive n th root of unity ζ . Namely, the elements $v_j = \sum_{i=1}^n \zeta^{ij} e_i$ ($j \in [n]$) form a basis of \mathbb{F}^n . Let $B \in \mathcal{B}(f_n)$ be the corresponding product basis. We have $f_n = \sum_{j=1}^n v_j \otimes \dots \otimes v_j$, and thus $\text{supp}_B f_n = \{\alpha \in [n]^k \mid \alpha_1 = \dots = \alpha_k\}$ which is clearly tight. (See also [18, Exercise 15.25].) When the characteristic of \mathbb{F} equals n the tensor f_n is also tight, as we will see in Example 4.14.

We care about tight tensors because of the following remarkable theorem for tight 3-tensors, proved by Strassen [60, Lemma 5.1] using the method of Coppersmith and Winograd [28]. We need the concept of asymptotic subrank of a set (cf. [60, Section 5]). We say $D \subseteq I_1 \times \cdots \times I_k$ is a *diagonal* when any two distinct $\alpha, \beta \in D$ are distinct in all k coordinates. In other words, for elements in D , the value at one coordinate uniquely determines the value at the other $k - 1$ coordinates. Let $\Phi \subseteq I_1 \times \cdots \times I_k$. We say a diagonal $D \subseteq I_1 \times \cdots \times I_k$ is *free for Φ* or simply $D \subseteq \Phi$ is a *free diagonal* if $D = \Phi \cap (D_1 \times \cdots \times D_k)$, where $D_i = \{x_i \mid (x_1, \dots, x_k) \in D\}$. Define the *subrank* $\mathbb{Q}(\Phi)$ as the size of the largest free diagonal $D \subseteq \Phi$. For $\Phi \subseteq I_1 \times \cdots \times I_k$ and $\Psi \subseteq J_1 \times \cdots \times J_k$ we naturally define the product $\Phi \times \Psi \subseteq (I_1 \times J_1) \times \cdots \times (I_k \times J_k)$. Define the *asymptotic subrank* $\underline{\mathbb{Q}}(\Phi) = \lim_{n \rightarrow \infty} \mathbb{Q}(\Phi^{\times n})^{1/n}$. Let $t \in \mathbb{F}^{n_1} \otimes \cdots \otimes \mathbb{F}^{n_k}$ and let Φ be the support of t in the standard basis. Then $\mathbb{Q}(\Phi) \leq \mathbb{Q}(t)$ and $\underline{\mathbb{Q}}(\Phi) \leq \underline{\mathbb{Q}}(t)$. The number $\mathbb{Q}(\Phi)$ may be interpreted as the largest number n such that $\langle n \rangle$ can be obtained from t using a restriction that consists of matrices that have at most one nonzero entry in each row and in each column. (This is called M-restriction in [58, Section 6] which stands for monomial restriction.)

Let $\Phi \subseteq [n_1] \times \cdots \times [n_k]$ and let $t \in \mathbb{F}^{n_1} \otimes \cdots \otimes \mathbb{F}^{n_k}$ be any tensor with support equal to Φ . Then

$$\mathbb{Q}(\Phi) \leq \mathbb{Q}(t), \text{ and } \underline{\mathbb{Q}}(\Phi) \leq \underline{\mathbb{Q}}(t).$$

So we may upper bound $\underline{\mathbb{Q}}(\Phi)$ by choosing the field \mathbb{F} and the tensor t cleverly and upper bounding $\underline{\mathbb{Q}}(t)$.

Theorem 4.4. *Let $\Phi \subseteq I_1 \times I_2 \times I_3$ be tight. Then*

$$(28) \quad \underline{\mathbb{Q}}(\Phi) = \max_{P \in \mathcal{P}(\Phi)} \min\{2^{H(P_1)}, 2^{H(P_2)}, 2^{H(P_3)}\}.$$

The amazing consequence of Theorem 4.4 is that the Strassen support functionals are sufficiently powerful to compute the asymptotic subrank of tight 3-tensors.

Corollary 4.5 ([60, Proposition 5.4]). *Let $f \in V_1 \otimes V_2 \otimes V_3$ be tight. Then*

$$\underline{\mathbb{Q}}(f) = \min_{\theta \in \mathcal{P}(\{3\})} \zeta^\theta(f).$$

Moreover, if Φ is a tight support for f , then $\underline{\mathbb{Q}}(f) = \underline{\mathbb{Q}}(\Phi)$.

Remark 4.6. Strassen conjectured in [61, Conjecture 5.3] that for the family \mathcal{Y} of tight 3-tensors the support functionals give all spectral points in the asymptotic spectrum $\Delta(\mathcal{Y})$, that is, the support simplex and the asymptotic spectrum coincide for \mathcal{Y} . In [60] numerous examples are given of subfamilies of \mathcal{Y} for which this is the case. We focus on one such example in Example 4.14.

Remark 4.7. The assumption $\Phi \subseteq I_1 \times I_2 \times I_3$ in Theorem 4.4 is crucial. Namely, statement (28) becomes false when instead we let $\Phi \subseteq I_1 \times \cdots \times I_k$ with $k \geq 4$ and we let the right-hand side of the equation be $\max_{P \in \mathcal{P}(\Phi)} \min_i 2^{H(P_i)}$; see [24, Example 1.1.38]. In [24] the construction of Theorem 4.4 is extended to obtain a lower bound for $\underline{\mathbb{Q}}(\Phi)$ when $k \geq 4$. This lower bound is not known to be tight in general. We give an illustration in Example 4.8 of a case in which the lower bound is tight.

Example 4.8 (Type tensors). Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition of k . Define the set Φ_λ and the *type tensor* D_λ as

$$\begin{aligned} \Phi_\lambda &:= \{\alpha \in [n]^k \mid \text{type}(\alpha) = \lambda\}, \\ D_\lambda &:= \sum_{\alpha \in \Phi_\lambda} e_{\alpha_1} \otimes \dots \otimes e_{\alpha_k} \in (\mathbb{F}^n)^{\otimes k}, \end{aligned}$$

where $\text{type}(\alpha) = \lambda$ means that the k -tuple α is a permutation of the k -tuple consisting of λ_1 ones, λ_2 twos, etc. In quantum information theory (let $\mathbb{F} = \mathbb{C}$) each type tensor D_λ encodes a so-called *Dicke state of type λ* . The type tensor $D_{(k-1,1)}$ encodes what is called the (*generalised*) *W state*. Clearly D_λ is oblique, since the support in the standard basis Φ_λ is an antichain (in fact, Φ_λ is clearly tight), and therefore

$$\rho^\theta(D_\lambda) = \rho_\theta(D_\lambda) = H_\theta(\Phi_\lambda)$$

(and over \mathbb{C} the quantum functionals $E^\theta(D_\lambda)$ and $E_\theta(D_\lambda)$ take the same value as $\rho^\theta(D_\lambda)$). One verifies that $H_\theta(\Phi_\lambda) \geq H((\frac{\lambda_1}{k}, \dots, \frac{\lambda_\mu}{k}))$, with equality when θ is uniform. Of course $H_\theta(\Phi_\lambda) \leq n$, with equality when θ equals one of $(1, 0, 0, \dots, 0)$, $(0, 1, 0, \dots, 0)$, etc. It is shown in [69] and [24], and independently in [35], that for all $k \geq 3$ we have $\underline{Q}(D_{(1,k-1)}) = \underline{Q}(\Phi_{(1,k-1)}) = 2^{h(k-1)}$, where $h(p)$ denotes the binary entropy function. In [24] it is shown that $\underline{Q}(D_{(2,2)}) = \underline{Q}(\Phi_{(2,2)}) = 2$ which in [3] is extended to, for all even $k \geq 4$, $\underline{Q}(D_{(k/2,k/2)}) = \underline{Q}(\Phi_{(k/2,k/2)}) = 2$.

We are going to extend Corollary 4.5. Suppose $\Psi \subseteq I_1 \times I_2 \times I_3$ is not tight, but has a tight subset $\Phi \subseteq \Psi$. In the rest of this section we focus on obtaining a lower bound on $\underline{Q}(\Psi)$ via Φ . This has an application in the context of tri-colored sum-free sets (Example 4.14) for example. We begin with the following standard notion.

Definition 4.9. Let $\Phi \subseteq \Psi \subseteq I_1 \times \dots \times I_k$. We say that Φ is a *combinatorial degeneration of Ψ* , and write $\Psi \triangleright \Phi$, if there are maps $u_i : I_i \rightarrow \mathbb{Z}$ ($i \in [k]$) such that for all $\alpha \in I_1 \times \dots \times I_k$, if $\alpha \in \Psi \setminus \Phi$, then $\sum_{i=1}^k u_i(\alpha_i) > 0$, and if $\alpha \in \Phi$, then $\sum_{i=1}^k u_i(\alpha_i) = 0$. Note that the maps u_i need not be injective.

Combinatorial degeneration gets its name from the following standard proposition [18, Proposition 15.30].

Proposition 4.10. *Let $f \in V_1 \otimes \dots \otimes V_k$. Let $\Psi = \text{supp}_B f$. Let $\Phi \subseteq \Psi$ such that $\Psi \triangleright \Phi$. Then $f \triangleright f|_\Phi$.*

Proposition 4.10 brings us only slightly closer to our goal. Namely, given $f \in V_1 \otimes \dots \otimes V_k$ with $\Psi = \text{supp}_B f$, and given $\Phi \subseteq \Psi$ such that $\Psi \triangleright \Phi$, it follows directly from Proposition 4.10 that $f \triangleright f|_\Phi$ and thus $\underline{Q}(f) \geq \underline{Q}(f|_\Phi)$. This, however, does not give us a lower bound on $\underline{Q}(\Psi)$. Theorem 4.11, which extends the technique of [41], does.

Theorem 4.11. *Let $\Phi \subseteq \Psi \subseteq I_1 \times \dots \times I_k$. If $\Psi \triangleright \Phi$, then*

$$\underline{Q}(\Psi) \geq \underline{Q}(\Phi).$$

Lemma 4.12. *Let $S \subseteq T \subseteq I_1 \times \dots \times I_k$, and $W_i \subseteq I_i$. Let $n \in \mathbb{N}$.*

- *If $T \triangleright S$, then $T^{\times n} \triangleright S^{\times n}$.*
- *If S is a free diagonal in T , then $S^{\times n}$ is a free diagonal in $T^{\times n}$.*

- If S is a free diagonal in T , then $S \cap (W_1 \times \cdots \times W_k)$ is a free diagonal in $T \cap (W_1 \times \cdots \times W_k)$.
- If S is a free diagonal in $T \cap (W_1 \times \cdots \times W_k)$, then S is a free diagonal in T .

The proof of Lemma 4.12 is straightforward and left to the reader. First we lower bound $\underline{Q}(\Psi)$ by the (nonasymptotic) subrank $Q(\Phi)$.

Lemma 4.13. *Let $\Phi \subseteq \Psi \subseteq I_1 \times \cdots \times I_k$. If $\Psi \supseteq \Phi$, then $\underline{Q}(\Psi) \geq Q(\Phi)$.*

Proof. Pick maps $u_i : I_i \rightarrow \mathbb{Z}$ such that

$$\sum_{i=1}^k u_i(\alpha_i) = 0 \quad \text{for } \alpha \in \Phi,$$

$$\sum_{i=1}^k u_i(\alpha_i) > 0 \quad \text{for } \alpha \in \Psi \setminus \Phi.$$

Let D be a free diagonal in Φ with $|D| = Q(\Phi)$ and let

$$w_i = \sum_{x \in D_i} u_i(x).$$

Let $n \in \mathbb{N}$ and define

$$W_i = \left\{ (x_1, \dots, x_{n|D|}) \in I_i^{\times n|D|} \mid \sum_{j=1}^{n|D|} u_i(x_j) = nw_i \right\}.$$

Then

$$\Psi^{\times n|D|} \cap (W_1 \times \cdots \times W_k) = \Phi^{\times n|D|} \cap (W_1 \times \cdots \times W_k).$$

The inclusion \supseteq is clear. To show \subseteq , let $(x_1, \dots, x_k) \in \Psi^{\times n|D|} \cap (W_1 \times \cdots \times W_k)$. Write $x_i = (x_{i,1}, x_{i,2}, \dots, x_{i,n|D|})$ and consider the $n|D| \times k$ matrix of evaluations

$$\begin{matrix} u_1(x_{1,1}) & u_2(x_{2,1}) & \cdots & u_k(x_{k,1}) \\ u_1(x_{1,2}) & u_2(x_{2,2}) & \cdots & u_k(x_{k,2}) \\ \vdots & \vdots & \ddots & \vdots \\ u_1(x_{1,n|D|}) & u_2(x_{2,n|D|}) & \cdots & u_k(x_{k,n|D|}). \end{matrix}$$

The sum of the i th column is nw_i by definition of W_i , and $\sum_{i=1}^k nw_i = 0$. The row sums are nonnegative by definition of the maps u_1, \dots, u_k . We conclude that the row sums are zero. Therefore (x_1, \dots, x_k) is an element of $\Phi^{\times n|D|}$.

Since D is a free diagonal in Φ , $D^{\times n|D|}$ is a free diagonal in $\Phi^{\times n|D|}$, and also $D^{\times n|D|} \cap (W_1 \times \cdots \times W_k)$ is a free diagonal in $\Phi^{\times n|D|} \cap (W_1 \times \cdots \times W_k)$, which in turn is equal to $\Psi^{\times n|D|} \cap (W_1 \times \cdots \times W_k)$. Therefore $D^{\times n|D|} \cap (W_1 \times \cdots \times W_k)$ is also a free diagonal in $\Psi^{\times n|D|}$, i.e.

$$Q(\Psi^{\times n|D|}) \geq |D^{\times n|D|} \cap (W_1 \times \cdots \times W_k)|.$$

In the set $D^{\times n|D|}$ consider the strings with uniform type, i.e. where all $|D|$ elements of D occur exactly n times. These are clearly in $W_1 \times \cdots \times W_k$, and their number is $\binom{n|D|}{n, \dots, n}$. Therefore

$$Q(\Psi^{\times n|D|}) \geq \binom{n|D|}{n, \dots, n} = |D|^{n|D| - o(n)},$$

which implies $\underline{Q}(\Psi) = \lim_{n \rightarrow \infty} \underline{Q}(\Psi^{\times n|D|})^{\frac{1}{n|D|}} \geq |D|$. □

Proof of Theorem 4.11. Clearly, we may write $\underline{Q}(\Psi) = \lim_{n \rightarrow \infty} \underline{Q}(\Psi^{\times n})^{1/n}$. By Lemma 4.13,

$$\lim_{n \rightarrow \infty} \underline{Q}(\Psi^{\times n})^{1/n} \geq \lim_{n \rightarrow \infty} \underline{Q}(\Phi^{\times n})^{1/n}.$$

The right-hand side equals $\underline{Q}(\Phi)$ by definition. □

Example 4.14 (The cap set problem). We mention one application in combinatorics of the support functionals (Section 2) and the ideas in this section about tight tensors (Corollary 4.5) and combinatorial degeneration (Theorem 4.11). (For an extended discussion related to this example we refer to [74].) A subset $A \subseteq (\mathbb{Z}/3\mathbb{Z})^n$ is called a *cap set* if any line in A is a point, a line being a triple of points of the form $(u, u+v, u+2v)$. The *cap set problem* is to decide whether the maximal size of a cap set in $(\mathbb{Z}/3\mathbb{Z})^n$ grows like $3^{n-o(n)}$ or like $c^{n+o(n)}$ for some $c < 3$ when $n \rightarrow \infty$. In a breakthrough result Ellenberg and Gijswijt [32], inspired by the work of Croot, Lev and Pach [29], settled this problem, showing that $c \leq 3(207 + 33\sqrt{33})^{1/3}/8 \approx 2.755$. Tao [64] phrased the cap set problem as the problem of computing the size of the largest main diagonal in powers of the *cap set tensor* $\sum_{\alpha} e_{\alpha_1} \otimes e_{\alpha_2} \otimes e_{\alpha_3}$ where the sum is over $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}/3\mathbb{Z}$ with $\alpha_1 + \alpha_2 + \alpha_3 = 0$. Here main diagonal refers to a subset A of the basis elements such that restricting to $A \times A \times A$ gives the tensor $\sum_{v \in A} v \otimes v \otimes v$. A main diagonal is essentially a unit tensor, so it is sufficient to upper bound the asymptotic subrank of the cap set tensor interpreted as a tensor over \mathbb{F}_3 . It follows from a standard isomorphism that the cap set tensor is in the $\text{GL}_3(\mathbb{F}_3)^{\times 3}$ -orbit of the structure tensor of the algebra $\mathbb{F}_3[x]/(x^3)$ (and in particular is thus a tight tensor). This implies that the asymptotic spectrum of the cap set tensor and the asymptotic spectrum of $\mathbb{F}_3[x]/(x^3)$ coincide. The tensor $\mathbb{F}_3[x]/(x^3)$ is one of the main examples in [60] for which Strassen computes the whole asymptotic spectrum using the support functionals. The minimal point in this asymptotic spectrum, which corresponds to the asymptotic subrank, is $3(207 + 33\sqrt{33})^{1/3}/8 \approx 2.755$ (see [60, Table 1]). This reproves the result of Ellenberg–Gijswijt. The matching lower bound for the tricolored version of the cap set problem can be obtained fairly directly from combining Corollary 4.5 and Theorem 4.11.

4.2. Free tensors. In this section we let \mathbb{F} be the field of complex numbers \mathbb{C} . We consider a family of tensors called free tensors. These were introduced by Franz in [33]. Recall that for any $\theta \in \mathcal{P}_s(\Pi)$ and any tensor t

$$E^\theta(t) = E_\theta(t) \quad (\text{Theorem 3.30})$$

and

$$\rho^\theta(t) \geq E^\theta(t) \quad (\text{Theorem 3.34}).$$

We care about free tensors, because for any free tensor t we can show that the three values $\rho^\theta(t)$, $E^\theta(t)$, $E_\theta(t)$ coincide for $\theta \in \mathcal{P}_s(\Pi)$.

Definition 4.15 (Free). Let $\Phi \subseteq I_1 \times \cdots \times I_k$. We say Φ is *free* if the following holds. For any $x, y \in \Phi$, if $x \neq y$, then the tuples x and y differ in at least two positions. Let $t \in V_1 \otimes \cdots \otimes V_k$. We say t is *free* if $\text{supp}_B t$ is free for some choice of bases $B \in \mathcal{B}(t)$.

Remark 4.16. Clearly, oblique tensors are free. We thus have $\{\text{tight}\} \subseteq \{\text{oblique}\} \subseteq \{\text{free}\}$. Free tensors form a semigroup under \otimes and \oplus , like the tight tensors and oblique tensors.

Remark 4.17. We prove there exist tensors that are not free in $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ for $n \geq 5$. First we upper bound the maximal size of a free support. Let $\Phi \subseteq [n] \times [n] \times [n]$ be free. Then $|\Phi| = |\{(\alpha_1, \alpha_2) : \alpha \in \Phi\}| \leq n^2$. Second we apply a lemma of Bürgisser [17], which is as follows. Let

$$Z_n = \{t \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n : \exists B \in \mathcal{B}(t) \mid \text{supp}_B t \leq n^3 - 3n^2\}.$$

Let $Y_n = \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n \setminus \overline{Z_n}$. Then the set Y_n is Zariski open and nonempty. Now let $n \geq 5$ and let $t \in Y_n$. Then $\forall B \in \mathcal{B}(t) \mid \text{supp}_B t \geq n^3 - 3n^2 > n^2$. We conclude t is not free.

Theorem 4.18. *Let t be free. Let $\theta \in \mathcal{P}_s(\Pi)$. Then $\rho^\theta(t) = E^\theta(t) = E_\theta(t)$.*

Proof. By Theorem 3.34 and Theorem 3.30 we have $\rho^\theta(t) \geq E^\theta(t) = E_\theta(t)$. We prove $\rho^\theta(t) \leq E_\theta(t)$. Let $B \in \mathcal{B}(t)$ such that $\text{supp}_B t$ is free. It is sufficient to prove the following claim. For any $P \in \mathcal{P}(\text{supp}_B t)$ the tuple of ordered marginals of P is in the entanglement polytope Δ_t . This result can be found in [33, 53, 54]; see also [71]. Then $\rho^\theta(t) \leq \max_{P \in \mathcal{P}(\text{supp}_B t)} H_\theta(P) \leq E_\theta(t)$. We give a proof of the above claim. Write $B = ((e_{\alpha_1}), \dots, (e_{\alpha_k}))$. Let $T \subseteq G = \text{SL}_{n_1} \times \dots \times \text{SL}_{n_k}$ be the corresponding torus. Let μ_G map any density matrix ρ to its reduced density matrices,

$$\mu_G : \rho \mapsto (\rho_1, \dots, \rho_k).$$

Let μ_T map any density matrix to its reduced density matrices projected onto diagonal matrices by setting the off-diagonal entries to zero,

$$\mu_T : \rho \mapsto (P_{\text{diag}} \rho_1, \dots, P_{\text{diag}} \rho_k).$$

The maps μ_G and μ_T can be identified with the moment map for $K = \text{SU}_{n_1} \times \dots \times \text{SU}_{n_k}$ and abelian moment map for $T \subseteq K$ in the sense of [14, 54]. We write $\mu_G(\psi)$ for $\mu_G(|\psi\rangle\langle\psi|)$ and we write $\mu_T(\psi)$ for $\mu_T(|\psi\rangle\langle\psi|)$. Let D_\downarrow be the set of k -tuples of diagonal marginals with ordered diagonal entries. It is a non-trivial fact that $\mu_T(\overline{T \cdot t})$ is a convex polytope, and that $\mu_G(\overline{G \cdot t}) \cap D_\downarrow$ is a convex polytope, namely the entanglement polytope Δ_t [14]. The elements of $\mu_T(\overline{T \cdot t})$ and $\mu_G(\overline{G \cdot t}) \cap D_\downarrow$ are tuples of diagonal matrices whose diagonal entries are a probability distribution. We will identify these elements with these tuples of probability distributions. Let $\Phi = \text{supp}_B t$ be the support of t . Let $\alpha \in \Phi$. Then

$$e_{\alpha_1} \otimes \dots \otimes e_{\alpha_k} \in \overline{T \cdot t}.$$

Therefore, the k -tuple of point distributions $(\delta_{\alpha_1}, \dots, \delta_{\alpha_k})$ is in $\mu_T(\overline{T \cdot t})$. Since the set $\mu_T(\overline{T \cdot t})$ is convex, the convex hull $\text{conv}\{(\delta_{\alpha_1}, \dots, \delta_{\alpha_k}) \mid \alpha \in \Phi\}$ is a subset of $\mu_T(\overline{T \cdot t})$. Observe that $\text{conv}\{(\delta_{\alpha_1}, \dots, \delta_{\alpha_k}) \mid \alpha \in \Phi\}$ equals the set $M = \{(P_1, \dots, P_k) \mid P \in \mathcal{P}(\text{supp}_B t)\}$. So

$$M \subseteq \mu_T(\overline{T \cdot t}).$$

Let $M_\downarrow \subseteq M$ consist of the tuples of ordered marginal distributions, then

$$M_\downarrow \subseteq \mu_T(\overline{T \cdot t}) \cap D_\downarrow.$$

Now suppose that $\text{supp}_B t$ is free. The freeness implies that the marginal density matrices of t are diagonal matrices and thus $\mu_G(\overline{T \cdot t})$ consists of diagonal matrices, i.e. $\mu_T(\overline{T \cdot t}) = \mu_G(\overline{T \cdot t})$. Clearly, $\mu_G(\overline{T \cdot t}) \subseteq \mu_G(\overline{G \cdot t})$. Intersecting with D_\downarrow gives

$$M_\downarrow \subseteq \mu_T(\overline{T \cdot t}) \cap D_\downarrow \subseteq \mu_G(\overline{G \cdot t}) \cap D_\downarrow = \Delta_t.$$

We conclude that for any $P \in \mathcal{P}(\text{supp}_B t)$ the tuple of ordered marginals of P is in the entanglement polytope Δ_t . \square

Remark 4.19. In the special case of t being oblique, the statement in our Theorem 4.18 corresponds to Corollary 12 in the paper [62] of Strassen.

Example 4.20. Coppersmith and Winograd in [28] obtained the upper bound $\omega \leq 2.41$ on the matrix multiplication exponent ω by analysing the *easy Coppersmith–Winograd tensor*, which for $q \geq 1$, is defined as

$$\text{CW}_q = \frac{1}{\sqrt{3q}} \left(\sum_{i=1}^q |0ii\rangle + |i0i\rangle + |ii0\rangle \right) \in \mathbb{C}^{q+1} \otimes \mathbb{C}^{q+1} \otimes \mathbb{C}^{q+1}.$$

This tensor is free, so $E_\theta(\text{CW}_q)$ equals $\rho^\theta(\text{CW}_q)$. We compute for any $\theta \in \mathcal{P}([3])$ the lower bound

$$\begin{aligned} E_\theta(\text{CW}_q) &\geq H\left(\frac{q}{3q} |0\rangle\langle 0| + \frac{2}{3q} \sum_{j=1}^q |j\rangle\langle j|\right) \\ &= H\left(\frac{1}{3}, \underbrace{\frac{2}{3q}, \dots, \frac{2}{3q}}_q\right) \\ &= \frac{2}{3} \log_2 q + h\left(\frac{1}{3}\right). \end{aligned}$$

On the other hand, let $\theta = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ be uniform. Then using the KKT-conditions (see [60, Proposition 2.1]) and the recursion property of Shannon entropy we compute the upper bound

$$\begin{aligned} \rho\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)(\text{CW}_q) &\leq H\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \left(\{(0, i, i), (i, 0, i), (i, i, 0) : i \in [q]\}\right) \\ &= H\left(\frac{1}{3}, \underbrace{\frac{2}{3q}, \dots, \frac{2}{3q}}_q\right) \\ &= \frac{2}{3} \log_2 q + h\left(\frac{1}{3}\right). \end{aligned}$$

We conclude that $\rho^\theta(\text{CW}_q) \geq \frac{2}{3} \log_2 q + h(\frac{1}{3})$ with equality when θ is uniform. Obviously, $\rho^\theta(\text{CW}_q) \leq \log_2(q+1)$ with equality when θ equals $(1, 0, 0)$ or $(0, 1, 0)$ or $(0, 0, 1)$.

Let $q = 1$. Then the tensor CW_1 is tight and therefore oblique, which implies that $\rho_\theta(\text{CW}_1) = \rho^\theta(\text{CW}_1) \geq h(\frac{1}{3})$ with equality when θ is uniform. In fact CW_1 equals the tensor $D_{(2,1)}$ (the W state) from Example 4.8.

Remark 4.21. As shown in Example 4.20, the support and quantum functionals give for all $q \geq 2$ the lower bound $\underline{\mathbb{R}}(\text{CW}_q) \geq q + 1$. If the asymptotic rank $\underline{\mathbb{R}}(\text{CW}_2)$ equals 3, then the exponent of matrix multiplication ω equals 2 (see e.g. [18, Remark 15.44] and [10, Section 9]). Bläser and Lysikov in [11] obtained the border rank lower bound $\underline{\mathbb{R}}(\text{CW}_q^{\otimes n}) \geq (q+1)^n + 2^n - 1$ which asymptotically also implies $\underline{\mathbb{R}}(\text{CW}_q) \geq q + 1$, and thus leaves open the possibility that $\underline{\mathbb{R}}(\text{CW}_q) = q + 1$.

Remark 4.22. We summarise the known relationships among the support and quantum functionals.

- (1) $F^\theta(t) \geq F_\theta(t)$ if $\theta \in \mathcal{P}(\Pi)$ (Theorem 3.24)
- (2) $\zeta^\theta(t) \geq \lim_{n \rightarrow \infty} \zeta^\theta(t^{\otimes n})^{1/n} = F^\theta(t) = F_\theta(t) \geq \lim_{n \rightarrow \infty} \zeta_\theta(t^{\otimes n})^{1/n} \geq \zeta_\theta(t)$
if $\theta \in \mathcal{P}_s(\Pi)$ (Theorems 3.30, 3.34 and 3.35)
- (3) $\zeta^\theta(t) = \zeta_\theta(t)$ if $\theta \in \mathcal{P}_s(\Pi)$ and t is oblique (Theorem 2.19)
- (4) $\zeta^\theta(t) = F^\theta(t) = F_\theta(t) \geq \zeta_\theta(t)$ if $\theta \in \mathcal{P}_s(\Pi)$ and t is free (Theorem 4.18)

4.3. Generic tensors. The quantum functionals over \mathbb{C} and the support functionals over algebraically closed fields, when restricted to tensors of a specific format, have a generic value, that is, there is a Zariski open set on which the value of the functional is constant. Recall that in an irreducible affine variety (e.g. $\mathbb{C}^{n_1 \times n_2 \times n_3}$) the intersection of any two nonempty Zariski open sets is nonempty, and therefore in an irreducible affine variety any nonempty Zariski open set is Zariski dense.

4.3.1. Generic value of the Strassen support functionals. Let \mathbb{F} be algebraically closed. For any format (n_1, n_2, n_3) there is a nonempty Zariski open subset of $\mathbb{F}^{n_1 \times n_2 \times n_3}$ on which ζ^θ and ζ_θ each has a constant value. (Indeed, a constructible set X in an affine variety Y contains a subset that is open and dense in the closure of X in Y ; see e.g. [13]. There are only finitely many possible supports for tensors in $V = \mathbb{F}^{n_1 \times n_2 \times n_3}$. The map ζ^θ therefore attains only finitely many values on V and thus has finitely many fibres, each of which is a constructible set. At least one of these fibres is dense in V by irreducibility of V . This fibre contains a subset that is open and dense in V . The same argument holds for ζ_θ .)

Definition 4.23. The above value of ζ^θ and ζ_θ is called *typical* or *generic*, and is denoted by $\zeta^\theta(n_1, n_2, n_3)$ and $\zeta_\theta(n_1, n_2, n_3)$ respectively.

Obviously, $0 \leq \zeta_\theta(n_1, n_2, n_3) \leq \zeta^\theta(n_1, n_2, n_3) \leq n_1^{\theta(1)} n_2^{\theta(2)} n_3^{\theta(3)}$.

Definition 4.24. Let $\Psi \subseteq [n_1] \times [n_2] \times [n_3]$. We call Ψ *comfortable* if there is a subset $\Phi \subseteq \Psi$ and a probability distribution $P \in \mathcal{P}(\Phi)$ such that Φ is an antichain and each marginal P_i is uniform on $[n_i]$. We call a format (n_1, n_2, n_3) comfortable if $[n_1] \times [n_2] \times [n_3]$ itself is comfortable. (In [66] comfortable is called *bequem*.)

Example 4.25. The format (n, n, n) is comfortable. Namely, let Φ be the set $\{(\alpha, \alpha, n+1-\alpha) \mid \alpha \in [n]\}$ and let $P \in \mathcal{P}(\Phi)$ be the uniform probability distribution. Clearly Φ is an antichain and each marginal P_i is uniform on $[n_i]$. The format (ab, bc, ca) is comfortable. Namely let Φ be the support of the matrix multiplication tensor $\langle a, b, c \rangle$ and let $P \in \mathcal{P}(\Phi)$ be the uniform probability distribution. The set Φ is tight and hence an antichain with respect to some product order, and P_1 is uniform on $[ab]$, P_2 is uniform on $[bc]$ and P_3 is uniform on $[ca]$.

Theorem 4.26 ([17, 66]). *Let $\theta \in \mathcal{P}([3])$.*

- (1) $\zeta^\theta(n_1, n_2, n_3) = n_1^{\theta(1)} n_2^{\theta(2)} n_3^{\theta(3)}$ when (n_1, n_2, n_3) is comfortable.
- (2) $\zeta_\theta(n, n, n) = n^{1 - \min_i \theta(i) + o(1)}$ when $n \rightarrow \infty$.

Recall that the upper support functional is additive. An interesting byproduct of Bürgisser's proof of statement 2 (see [17, Theorem 3.1]) is that the lower support functional is not additive.

Proposition 4.27. *Let $\theta \in \mathcal{P}([3])$ with $\theta_i > 0$ for all i . Then ζ_θ is not additive.*

4.3.2. *Generic value of the quantum functionals.* Let \mathbb{F} be the complex numbers \mathbb{C} . We consider the singleton regime $\theta \in \mathcal{P}([k])$, so that we can apply the theory of entanglement polytopes. Namely, recall that by Theorem 3.30, in this regime, for $t \in \mathbb{C}^{n_1} \otimes \dots \otimes \mathbb{C}^{n_k}$, the value of $E_\theta(t)$ is obtained by the following maximisation over the entanglement polytope Δ_t ,

$$E_\theta(t) = E^\theta(t) = \max\{\theta(1)H(r_1) + \dots + \theta(k)H(r_k) \mid (r_1, \dots, r_k) \in \Delta_t\}.$$

Fixing the tensor format (n_1, \dots, n_k) , there exist finitely many covariants P_1, \dots, P_m such that, with $(\lambda^{(j,1)}, \dots, \lambda^{(j,k)})$ being the type of P_j , the entanglement polytope of any $t \in \mathbb{C}^{n_1} \otimes \dots \otimes \mathbb{C}^{n_k}$ equals the convex hull

$$\Delta_t = \text{conv}\{\overline{(\lambda^{(j,1)}, \dots, \lambda^{(j,k)})} \mid P_j(t) \neq 0\},$$

see [70, Theorem 2]. This implies that there are only finitely many possible entanglement polytopes Δ_t associated to $t \in \mathbb{C}^{n_1 \times \dots \times n_k}$ and thus F_θ takes only finitely many values. Moreover there is a nonempty Zariski open subset U of $\mathbb{C}^{n_1 \times \dots \times n_k}$ on which F_θ has a constant value. Namely let U be the Zariski open set of tensors for which all generators are nonzero,

$$U = \bigcap_{j=1}^m \{t \in \mathbb{C}^{n_1 \times \dots \times n_k} \mid P_j(t) \neq 0\}.$$

For every $t \in U$ the entanglement polytope Δ_t is maximal, that is, equal to the convex hull

$$\text{conv}\{\overline{(\lambda^{(j,1)}, \dots, \lambda^{(j,k)})} \mid j \in [m]\}.$$

Definition 4.28. The above value of F_θ is called *typical* or *generic*, and is denoted by $F_\theta(n_1, \dots, n_k)$.

Obviously, $0 \leq F_\theta(n_1, \dots, n_k) \leq n_1^{\theta(1)} \dots n_k^{\theta(k)}$.

We say that $|\psi\rangle \in \mathbb{C}^{n_1} \otimes \dots \otimes \mathbb{C}^{n_k}$ has *completely mixed marginals* if for each $i \in [k]$ the marginal quantum entropy $H(|\psi\rangle\langle\psi|_i)$ equals the maximal value $\log_2 n_i$. Bryan et al. in [16] give a characterisation of the formats (n_1, \dots, n_k) for which a state with completely mixed marginals exists. (The earlier characterisation by Klyachko and Shumovsky in [42, Corollary 2.5.2.2] appears to be incorrect.)

Theorem 4.29. *Let $\theta \in \mathcal{P}([k])$ such that $\theta_i > 0$ for all i . We have*

$$F_\theta(n_1, \dots, n_k) = n_1^{\theta(1)} \dots n_k^{\theta(k)}$$

if and only if $\mathbb{C}^{n_1} \otimes \dots \otimes \mathbb{C}^{n_k}$ contains a pure quantum state with completely mixed marginals.

Proof. Let (n_1, \dots, n_k) be a format such that $\mathbb{C}^{n_1} \otimes \dots \otimes \mathbb{C}^{n_k}$ contains a pure quantum state with completely mixed marginals. Let U be the Zariski open set defined above. Let $t \in U$. Then the entanglement polytope Δ_t is the maximal entanglement polytope and thus the maximisation

$$E_\theta(t) = E^\theta(t) = \max\{\theta(1)H(r_1) + \dots + \theta(k)H(r_k) \mid (r_1, \dots, r_k) \in \Delta_t\}$$

achieves the value $\theta(1) \log_2 n_1 + \dots + \theta(k) \log_2 n_k$. We conclude that $F_\theta(n_1, \dots, n_k)$ equals the upper bound $n_1^{\theta(1)} \dots n_k^{\theta(k)}$. On the other hand, if $\mathbb{C}^{n_1} \otimes \dots \otimes \mathbb{C}^{n_k}$ does not contain a pure quantum state with completely mixed marginals, then $((\frac{1}{n_1}, \dots, \frac{1}{n_1}), \dots, (\frac{1}{n_k}, \dots, \frac{1}{n_k}))$ is not in the maximal entanglement polytope and

thus $\max\{\theta(1)H(r_1) + \dots + \theta(k)H(r_k) \mid (r_1, \dots, r_k) \in \Delta_t\}$ is strictly smaller than $\theta(1) \log_2 n_1 + \dots + \theta(k) \log_2 n_k$ for $t \in U$ (and in fact for any t). \square

Remark 4.30. In particular, $F_\theta(n_1, \dots, n_k) = n_1^{\theta(1)} \dots n_k^{\theta(k)}$ for any comfortable format (n_1, \dots, n_k) . Indeed, if (n_1, \dots, n_k) is comfortable, then by definition there is a set $\Phi \subseteq [n_1] \times \dots \times [n_k]$ and a probability distribution $P \in \mathcal{P}(\Phi)$ such that Φ is an antichain and each marginal P_i is uniform on $[n_i]$. Define the tensor

$$t = \sum_{\alpha \in \Phi} \sqrt{P(\alpha)} e_{\alpha_1} \otimes \dots \otimes e_{\alpha_k} \in \mathbb{C}^{n_1} \otimes \dots \otimes \mathbb{C}^{n_k}.$$

Since Φ is an antichain and thus free, t has completely mixed marginals. Following the proof of Theorem 4.29 the value of $F_\theta(n_1, \dots, n_k)$ equals $n_1^{\theta(1)} \dots n_k^{\theta(k)}$.

4.3.3. Hilbert–Mumford criterion and instability. We have seen that generically for certain formats E^θ and ρ^θ are maximal. We give a related application of the Hilbert–Mumford criterion. Let $t \in \mathbb{C}^{n_1} \otimes \dots \otimes \mathbb{C}^{n_k}$. Let $\theta \in \mathcal{P}([k])$. We know that

$$E^\theta(t) \leq \rho^\theta(t) \leq n_1^{\theta(1)} \dots n_k^{\theta(k)}.$$

Therefore,

$$E^\theta(t) = n_1^{\theta(1)} \dots n_k^{\theta(k)} \implies \rho^\theta(t) = n_1^{\theta(1)} \dots n_k^{\theta(k)}.$$

We show the following converse.

Theorem 4.31. *Let $t \in \mathbb{C}^{n_1} \otimes \dots \otimes \mathbb{C}^{n_k}$. Let $\theta \in \mathcal{P}([k])$ such that $\theta_i > 0$ for all i . If $E^\theta(t) < n_1^{\theta(1)} \dots n_k^{\theta(k)}$, then $\rho^\theta(t) < n_1^{\theta(1)} \dots n_k^{\theta(k)}$.*

We will use terminology and results from [49]. The point in the entanglement polytopes corresponding to quantum states that are locally maximally mixed will be called the *origin* and denoted by O .

Lemma 4.32. *$O \notin \mu_G(\overline{G \cdot t})$ if and only if t is G -unstable i.e. $0 \in \overline{G \cdot t}$.*

Proof. According to [49, Lemma 2.3], if t is G -semistable, then $\overline{G \cdot t}$ contains a closed orbit. This implies according to [49, Theorem 2.2(ii)] and [49, Definition 2.1(iii)] that $O \in \mu_G(\overline{G \cdot t})$. If t is G -unstable, then all G -invariants vanish, which implies $O \notin \mu_G(\overline{G \cdot t})$. \square

Proof of Theorem 4.31. Let $G = \text{SL}(\mathbb{C}^{n_1}) \times \dots \times \text{SL}(\mathbb{C}^{n_k})$ act naturally on $V = \mathbb{C}^{n_1} \otimes \dots \otimes \mathbb{C}^{n_k}$. Let $t \in V$. If $E^\theta(t) < n_1^{\theta(1)} \dots n_k^{\theta(k)}$, then $O \notin \Delta_t$. This means t is G -unstable (Lemma 4.32) i.e. $0 \in \overline{G \cdot t}$. By the Hilbert–Mumford criterion [49, Theorem 3.1] this implies that there exists a one-parameter subgroup λ such that t is λ -unstable. This implies that there exists a maximal torus $T \subseteq G$ such that t is T -unstable. By Lemma 4.32 this implies that $O \notin \mu_T(\overline{T \cdot t})$. Let $B \in \mathcal{B}(t)$ be a choice of bases compatible with T . As shown in the proof of Theorem 4.18,

$$(29) \quad M = \{(P_1, \dots, P_k) \mid P \in \mathcal{P}(\text{supp}_B t)\} \subseteq \mu_T(\overline{T \cdot t}).$$

Therefore $O \notin M$. Noting that $H_\theta(\text{supp}_B t)$ equals $n_1^{\theta(1)} \dots n_k^{\theta(k)}$ if and only if $O \in M$, we see that $H_\theta(\text{supp}_B t) < n_1^{\theta(1)} \dots n_k^{\theta(k)}$. Therefore we conclude $\rho^\theta(t) \leq \max_{P \in \text{supp}_B t} H_\theta(P) < n_1^{\theta(1)} \dots n_k^{\theta(k)}$. \square

We prove a quantitative version of Theorem 4.31 in terms of instability, which moreover holds over all fields. Our result improves a result of Blasiak et al. in [12] when the tensor format (n_1, \dots, n_k) is nonuniform enough.

Instability is a standard notion in geometric invariant theory. Let \mathbb{F} be algebraically closed. Let $t \in \mathbb{F}^{n_1} \otimes \cdots \otimes \mathbb{F}^{n_k}$. Let $G = \mathrm{SL}_{n_1}(\mathbb{F}) \times \cdots \times \mathrm{SL}_{n_k}(\mathbb{F})$. The tensor t is called unstable if 0 is in the orbit closure $\overline{G \cdot t}$, with the closure taken in the Zariski topology, and otherwise t is called semistable.

We use the quantitative notion of instability for a tensor $t \in \mathbb{F}^{n_1} \otimes \cdots \otimes \mathbb{F}^{n_k}$ defined in [12], with \mathbb{F} an arbitrary field,

$$\mathrm{instab}(t) = \max_{B \in \mathcal{B}(t)} \max_{w_1, \dots, w_k} \sup \left\{ \varepsilon \geq 0 \mid \forall a \in \mathrm{supp}_B t : \sum_{i=1}^k w_i(a_i) \leq \sum_{i=1}^k \left(\frac{1}{|I_i|} \sum_{x \in I_i} w_i(x) - \varepsilon \max_x w_i(x) \right) \right\},$$

where the first maximum is over the choice of bases of $\mathbb{F}^{n_1}, \dots, \mathbb{F}^{n_k}$, with index sets I_1, \dots, I_k , and the second maximum is over weight functions $w_i : I_i \rightarrow \mathbb{R}_{\geq 0}$ that are not identically zero. When \mathbb{F} is algebraically closed, the Hilbert–Mumford criterion says that t is unstable if and only if $\mathrm{instab}(t) > 0$.

For probability distributions $P, Q \in \mathcal{P}(X)$, define the total variation distance

$$V(P, Q) = \frac{1}{2} \sum_{x \in X} |P(x) - Q(x)|$$

and define the Kullback-Leibler divergence

$$D(P||Q) = \sum_{x \in X} P(x) \log_2(P(x)/Q(x)).$$

We will use Pinsker’s inequality, which says

$$(30) \quad V(P, Q) \leq \sqrt{\frac{\ln 2}{2} D(P||Q)},$$

see e.g. [67, Lemma 2.5].

Theorem 4.33. *For $t \in \mathbb{F}^{n_1} \otimes \cdots \otimes \mathbb{F}^{n_k}$ and $\theta \in \mathcal{P}([k])$,*

$$(31) \quad \rho^\theta(t) \leq \sum_{i=1}^k \theta(i) \log_2 n_i - \left(\frac{2}{\ln 2} \min_i \theta(i) \right) \mathrm{instab}(t)^2.$$

Proof. The two sides of (31) are continuous in θ , so we may without loss of generality assume that $\theta(i) > 0$ for all $i \in [k]$. Choose bases $B \in \mathcal{B}(t)$, weights w_1, \dots, w_k , and $\varepsilon \geq 0$, such that for every $a \in \mathrm{supp}_B t$

$$\sum_{i=1}^k w_i(a_i) \leq \sum_{i=1}^k \left(\frac{1}{|I_i|} \sum_{x \in I_i} w_i(x) - \varepsilon \max_x w_i(x) \right),$$

i.e. for every $a \in \mathrm{supp}_B t$

$$\varepsilon \sum_{i=1}^k \max_x w_i(x) \leq \sum_{i=1}^k \frac{1}{|I_i|} \sum_{x \in I_i} w_i(x) - \sum_{i=1}^k w_i(a_i).$$

Let $P \in \mathcal{P}(\text{supp}_B t)$ be a probability distribution on $\text{supp}_B t$. Then

$$\begin{aligned} \varepsilon \sum_{i=1}^k \max_x w_i(x) &\leq \sum_{i=1}^k \frac{1}{|I_i|} \sum_{x \in I_i} w_i(x) - \sum_{a \in \text{supp}_B t} P(a) \sum_{i=1}^k w_i(a_i) \\ &= \sum_{i=1}^k \sum_{x \in I_i} \left(\frac{1}{|I_i|} - P_i(x) \right) w_i(x) \\ &\leq \frac{1}{2} \sum_{i=1}^k \sum_{x \in I_i} \left| \frac{1}{|I_i|} - P_i(x) \right| \max_x w_i(x). \end{aligned}$$

Let U_i be the uniform probability distribution on I_i . We apply Pinsker's inequality (30) to get

$$\frac{1}{2} \sum_{i=1}^k \sum_{x \in I_i} \left| \frac{1}{|I_i|} - P_i(x) \right| \max_x w_i(x) \leq \sum_{i=1}^k \sqrt{\frac{\ln 2}{2} D(P_i \| U_i)} \max_x w_i(x).$$

Next we apply the Cauchy-Schwarz inequality to obtain

$$\begin{aligned} &\sum_{i=1}^k \sqrt{\frac{\ln 2}{2} D(P_i \| U_i)} \max_x w_i(x) \\ &\leq \sqrt{\frac{\ln 2}{2}} \left(\sum_{i=1}^k D(P_i \| U_i) \right)^{1/2} \left(\sum_{i=1}^k \max_x w_i(x)^2 \right)^{1/2} \\ &\leq \sqrt{\frac{\ln 2}{2}} \left(\sum_{i=1}^k D(P_i \| U_i) \right)^{1/2} \sum_{i=1}^k \max_x w_i(x) \\ &\leq \sqrt{\frac{\ln 2}{2} \frac{1}{\min_i \theta(i)}} \left(\sum_{i=1}^k \theta(i) D(P_i \| U_i) \right)^{1/2} \sum_{i=1}^k \max_x w_i(x) \\ &= \sqrt{\frac{\ln 2}{2} \frac{1}{\min_i \theta(i)}} \left(\sum_{i=1}^k \theta(i) \log_2 |I_i| - \theta(i) H(P_i) \right)^{1/2} \sum_{i=1}^k \max_x w_i(x). \end{aligned}$$

We conclude

$$\varepsilon^2 \leq \frac{\ln 2}{2} \frac{1}{\min_i \theta(i)} \left(\sum_{i=1}^k \theta(i) \log_2 |I_i| - \theta(i) H(P_i) \right)$$

and thus

$$\sum_{i=1}^k \theta(i) H(P_i) \leq \sum_{i=1}^k \theta(i) \log_2 |I_i| - \frac{2}{\ln 2} \min_i \theta(i) \varepsilon^2.$$

Now take the supremum over P , and the infimum over ε , w_i and B . \square

5. CHARACTERIZATION OF ASYMPTOTIC SLICE RANK

This section is about two variations on tensor rank, namely slice rank, which was introduced by Tao [64], and partition rank, which was introduced by Naslund [48]. We know that subrank is asymptotically upper bounded by the quantum functionals. It is known that subrank is at most slice rank and partition rank. The result of this section is that, remarkably, even slice rank and partition rank are asymptotically upper bounded by the quantum functionals. In fact, we show

that the asymptotic slice rank $\lim_{n \rightarrow \infty} \text{SR}(t^{\otimes n})^{1/n}$ exists and equals the minimum $\min_{\theta \in \mathcal{P}([k])} F^\theta(t)$.⁶

5.1. Slice rank, partition rank and asymptotic versions. Recall that a simple tensor is a tensor of the form $v_1 \otimes \cdots \otimes v_k \in V_1 \otimes \cdots \otimes V_k$ with $v_i \in V_i$ for $i \in [k]$, and that the rank $R(t)$ of a tensor $t \in V_1 \otimes \cdots \otimes V_k$ is the smallest number r such that t can be written as a sum of r simple tensors.

Definition 5.1. A *slice* is a tensor of the form $v \otimes w \in V_1 \otimes \cdots \otimes V_k$ where $v \in V_j$ and $w \in V_{\bar{j}}$ for some $j \in [k]$. A *multi-slice* is a tensor of the form $v \otimes w \in V_1 \otimes \cdots \otimes V_k$ where $v \in V_S$ and $w \in V_{\bar{S}}$ for some subset $S \subseteq [k]$. Let $t \in V_1 \otimes \cdots \otimes V_k$. The *slice rank* of t , denoted $\text{SR}(t)$, is the smallest number r such that t can be written as a sum of r slices. The *partition rank* of t , denoted $\text{PR}(t)$, is the smallest number r such that t can be written as a sum of r multi-slices.

Slice rank and partition rank are clearly \leq -monotones. The slice rank and partition rank of $\langle r \rangle$ equals r [48, 64]. It follows that subrank is at most partition rank which is at most slice rank,

$$Q(t) \leq \text{PR}(t) \leq \text{SR}(t).$$

Computing upper bounds on subrank $Q(t)$ and asymptotic subrank $\underline{Q}(t)$ was the main motivation for introducing slice rank in [64].

Example 5.2. While tensor rank is easily seen to be sub-multiplicative with respect to tensor products, and subrank to be super-multiplicative, slice rank and partition rank are neither sub-multiplicative nor super-multiplicative. For example, the tensors $\sum_{i=1}^n e_i \otimes e_i \otimes 1$, $\sum_{i=1}^n e_i \otimes 1 \otimes e_i$, $\sum_{i=1}^n 1 \otimes e_i \otimes e_i$ have slice rank one, while their tensor product equals the matrix multiplication tensor $\langle n, n, n \rangle$ which has slice rank n^2 (see [12, Remark 4.6]). This shows slice rank is not sub-multiplicative. To see slice rank is not super-multiplicative, take for example W to be the tensor $e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1$. The slice rank of W equals two. The value of the logarithmic upper support functional $\rho^{(1/3, 1/3, 1/3)}(W)$ equals $h(1/3) \approx 0.918296$ (see Example 4.8). In Theorem 5.9 we will show that for any $\theta \in \mathcal{P}([k])$ the value of $\text{SR}(t^{\otimes n})$ is at most $2^{\rho^\theta(t)n + o(n)}$ when $n \rightarrow \infty$. We thus obtain the upper bound $\text{SR}(W^{\otimes n}) \leq 2^{h(1/3)n + o(n)}$, which proves the claim. Now it is also clear that slice rank is not equal to subrank or border subrank in general, since subrank and border subrank are super-multiplicative.

Since slice rank and partition rank are not sub-multiplicative and not super-multiplicative, the limit $\lim_{n \rightarrow \infty} \text{SR}(t^{\otimes n})^{1/n}$ and the analogous limit for partition rank might not exist. We will show that $\lim_{n \rightarrow \infty} \text{SR}(t^{\otimes n})^{1/n}$ in fact does exist. For now define

$$\begin{aligned} \text{SR}^\sim &= \limsup_{n \rightarrow \infty} \text{SR}(t^{\otimes n})^{1/n}, \\ \text{PR}^\sim(t) &= \limsup_{n \rightarrow \infty} \text{PR}(t^{\otimes n})^{1/n}. \end{aligned}$$

We have

$$\underline{Q}(t) \leq \text{PR}^\sim(t) \leq \text{SR}^\sim(t).$$

⁶In recent work [22] the relation between slice rank and the quantum functionals was strengthened to give an asymptotic characterization of a *weighted* notion of slice rank. This correspondence gives rise to a proposal for universal spectral points over other fields than the complex numbers.

5.2. Characterization by support and quantum functionals. We show that slice rank and partition rank are asymptotically upper bounded by the quantum functionals.

Theorem 5.3. *Let \mathbb{F} be the complex numbers \mathbb{C} . Let $t \in V_1 \otimes \cdots \otimes V_k$. If $\theta \in \mathcal{P}(\Pi)$, then $\text{PR}^\sim(t) \leq F^\theta(t)$. If $\theta \in \mathcal{P}_s(\Pi)$, then $\text{SR}^\sim(t) \leq F^\theta(t)$.*

Proof. Let $c = E^\theta(t)$. Let $\ell = |\text{supp } \theta|$. Let $n \in \mathbb{N}$. Let $t_0 = t^{\otimes n}$. Define a sequence of tensors t_1, \dots, t_ℓ by

$$t_i = \left(\text{Id} - \sum_{\substack{\lambda^{(i)} \vdash n: \\ H(\lambda^{(i)}) \leq c}} P_{\lambda^{(i)}}^{V_i} \right) t_{i-1}.$$

Let

$$s_i = \sum_{\substack{\lambda^{(i)} \vdash n: \\ H(\lambda^{(i)}) \leq c}} P_{\lambda^{(i)}}^{V_i} t_{i-1}.$$

We can write t_ℓ in two ways. On the one hand,

$$t_\ell = t^{\otimes n} - s_1 - s_2 - \cdots - s_\ell.$$

On the other hand,

$$\begin{aligned} t_\ell &= \left(\text{Id} - \sum_{\substack{\lambda^{(\ell)} \vdash n: \\ H(\lambda^{(\ell)}) \leq c}} P_{\lambda^{(\ell)}}^{V_\ell} \right) \cdots \left(\text{Id} - \sum_{\substack{\lambda^{(1)} \vdash n: \\ H(\lambda^{(1)}) \leq c}} P_{\lambda^{(1)}}^{V_1} \right) t^{\otimes n} \\ &= \sum_{\substack{\lambda^{(\ell)} \vdash n: \\ H(\lambda^{(\ell)}) > c}} \cdots \sum_{\substack{\lambda^{(1)} \vdash n: \\ H(\lambda^{(1)}) > c}} P_{\lambda^{(\ell)}}^{V_\ell} \cdots P_{\lambda^{(1)}}^{V_1} t^{\otimes n} \end{aligned}$$

which is 0 by definition of E^θ . Therefore $t^{\otimes n} = \sum_{i=1}^\ell s_i$.

The partition rank of an element in the image of $P_{\lambda^{(i)}}^{V_i}$ is at most $2^{nH(\bar{\lambda})}$. Each s_i is in the image of $\sum P_{\lambda^{(i)}}^{V_i}$ where the sum is over partitions $\lambda \vdash n$ with $H(\bar{\lambda}) \leq c$ and with at most $d_1 d_2 \cdots d_k$ parts. There are at most $(n+1)^{d_1 d_2 \cdots d_k}$ such partitions. Therefore $\text{PR}(s_i) \leq (n+1)^{d_1 d_2 \cdots d_k} 2^{nc}$. This implies

$$\text{PR}^\sim(t) \leq \limsup_{n \rightarrow \infty} (\ell(n+1)^{d_1 d_2 \cdots d_k} 2^{nc})^{1/n} = F^\theta(t).$$

This proves the first statement. The proof of the second statement is the same, except that it uses the upper bound $\text{SR}(s_i) \leq (n+1)^{d_i} 2^{nE^\theta(t)}$. □

Let us focus on the singleton regime $\theta \in \mathcal{P}([k])$. We show that the asymptotic slice rank exists and equals the minimum value over F^θ with $\theta \in \mathcal{P}([k])$.

Lemma 5.4. *Let $t \in \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_k}$ be nonzero. For any $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ there are partitions $\lambda^{(1)}, \dots, \lambda^{(k)} \vdash n$ satisfying $(P_{\lambda^{(1)}} \otimes \cdots \otimes P_{\lambda^{(k)}}) t^{\otimes n} \neq 0$ and*

$$\min\{H(\frac{1}{n}\lambda^{(1)}), \dots, H(\frac{1}{n}\lambda^{(k)})\} \geq \min_{\theta \in \mathcal{P}([k])} E^\theta(t) - \varepsilon.$$

Proof. Let Δ_t be the moment polytope of t . By the minimax theorem

$$\begin{aligned} \min_{\theta \in \mathcal{P}([k])} E^\theta(t) &= \min_{\theta \in \mathcal{P}([k])} \max_{(r_1, \dots, r_k) \in \Delta_t} \sum_{j=1}^k \theta(j) H(r_j) \\ &= \max_{(r_1, \dots, r_k) \in \Delta_t} \min_{\theta \in \mathcal{P}([k])} \sum_{j=1}^k \theta(j) H(r_j) \\ &= \max_{(r_1, \dots, r_k) \in \Delta_t} \min_{j \in [k]} H(r_j). \end{aligned}$$

Let $\mu^{(1)}, \dots, \mu^{(k)} \vdash m$ be partitions such that

$$\min_{j \in [k]} H\left(\frac{1}{m} \mu^{(j)}\right) \geq \min_{\theta \in \mathcal{P}([k])} E^\theta(t) - \varepsilon/2$$

and $(P_{\mu^{(1)}} \otimes \dots \otimes P_{\mu^{(k)}}) t^{\otimes m} \neq 0$. This is possible since Δ_t is the closure of such rescaled partitions and the entropy is continuous. We use that $(P_{(1)} \otimes \dots \otimes P_{(1)}) t = t \neq 0$ and that the tuples of partitions $\lambda^{(1)}, \dots, \lambda^{(k)}$ satisfying $(P_{\lambda^{(1)}} \otimes \dots \otimes P_{\lambda^{(k)}}) t^{\otimes n} \neq 0$ form a semigroup.

Any $n \in \mathbb{N}$ can be written as $n = mq + r$ where $q, r \in \mathbb{N}$, $0 \leq r < m$. Let $\lambda^{(j)} = q\mu^{(j)} + (r)$. By the semigroup property, $(P_{\lambda^{(1)}} \otimes \dots \otimes P_{\lambda^{(k)}}) t^{\otimes n} \neq 0$. The entropies can be estimated using the concavity of entropy as

$$\begin{aligned} H\left(\frac{1}{n}(q\mu^{(j)} + (r))\right) &= H\left(\frac{qm}{n} \frac{1}{m} \mu^{(j)} + \frac{r}{n} \frac{1}{r} (r)\right) \geq \frac{qm}{n} H\left(\frac{1}{m} \mu^{(j)}\right) \\ &= \frac{m}{n} \lfloor \frac{n}{m} \rfloor H\left(\frac{1}{m} \mu^{(j)}\right) \geq \left(1 - \frac{m}{n}\right) H\left(\frac{1}{m} \mu^{(j)}\right). \end{aligned}$$

When n is large enough, this is greater than $H(\frac{1}{m} \mu^{(j)}) - \varepsilon/2$. Choose n_0 such that this is true for all $j \in [k]$. □

Lemma 5.5. *Let $t \in \mathbb{C}^{d_1} \otimes \dots \otimes \mathbb{C}^{d_k}$ and suppose that $(P_{\lambda^{(1)}} \otimes \dots \otimes P_{\lambda^{(k)}}) t^{\otimes n} \neq 0$ for some partitions $\lambda^{(1)}, \dots, \lambda^{(k)} \vdash n$. Then*

$$\text{SR}(t^{\otimes n}) \geq \min\{\dim[\lambda_1], \dots, \dim[\lambda_k]\}.$$

Proof. $t^{\otimes n}$ restricts to $(P_{\lambda^{(1)}} \otimes \dots \otimes P_{\lambda^{(k)}}) t^{\otimes n}$, which by assumption is nonzero. Choose rank one projections A_j in the vector spaces $\mathbb{S}_{\lambda^{(j)}}(\mathbb{C}^{d_j})$ such that

$$((\text{Id}_{[\lambda^{(1)]}} \otimes A_1) \otimes \dots \otimes (\text{Id}_{[\lambda^{(k)]}} \otimes A_k)) (P_{\lambda^{(1)}} \otimes \dots \otimes P_{\lambda^{(k)}}) t^{\otimes n} \neq 0.$$

The resulting tensor is invariant under the diagonal S_n -action; therefore the marginals are maximally mixed. In the GIT language, this means they are unstable; therefore the slice rank is the smallest local dimension by [12, Theorem 4.6]. □

Theorem 5.6. *For any tensor $t \in \mathbb{C}^{d_1} \otimes \dots \otimes \mathbb{C}^{d_k}$,*

$$\liminf_{n \rightarrow \infty} \text{SR}(t^{\otimes n})^{1/n} \geq \min_{\theta \in \mathcal{P}([k])} 2^{E^\theta(t)}.$$

Proof. Choose $\varepsilon > 0$ and for all large enough n choose $\lambda^{(1)}, \dots, \lambda^{(k)} \vdash n$ as in Lemma 5.4. By Lemma 5.5,

$$\begin{aligned} \text{SR}(t^{\otimes n}) &\geq \min\{\dim[\lambda^{(1)}], \dots, \dim[\lambda^{(k)}]\} \\ &\geq \min\{2^{nH(\frac{1}{n} \lambda^{(1)})}, \dots, 2^{nH(\frac{1}{n} \lambda^{(k)})}\} 2^{-o(n)} \\ &\geq 2^{n(\min_{\theta \in \mathcal{P}([k])} E^\theta(t) - \varepsilon)} 2^{-o(n)}, \end{aligned}$$

therefore

$$\liminf_{n \rightarrow \infty} \text{SR}(t^{\otimes n})^{1/n} \geq 2^{\min_{\theta \in \mathcal{P}([k])} E^\theta(t) - \varepsilon}.$$

This is true for any $\varepsilon > 0$, therefore

$$\liminf_{n \rightarrow \infty} \text{SR}(t^{\otimes n})^{1/n} \geq 2^{\min_{\theta \in \mathcal{P}([k])} E^\theta(t)}. \quad \square$$

Corollary 5.7. *For complex tensors, asymptotic slice rank exists and equals the minimum value of the support functionals over $\theta \in \mathcal{P}([k])$,*

$$\underline{\text{SR}}(t) := \lim_{n \rightarrow \infty} \text{SR}(t^{\otimes n})^{1/n} = \min_{\theta \in \mathcal{P}([k])} 2^{E^\theta(t)}.$$

Remark 5.8. Connections between asymptotic slice rank and moment polytopes have also been observed in [19, Corollary 6.5] and [25].

For tensors over arbitrary fields we have the following result. (See also [65] for a similar statement.)

Theorem 5.9. *Let $t \in V_1 \otimes \cdots \otimes V_k$. Let $\theta \in \mathcal{P}([k])$. Then*

$$\text{SR}^\sim(t) \leq \zeta^\theta(t).$$

Proof. Choose a k -tuple of bases $B = ((v_{1,1}, \dots, v_{1,d_1}), \dots, (v_{k,1}, \dots, v_{k,d_k}))$ in $\mathcal{B}(f)$. Let $n \in \mathbb{N}$. As in the proof of Theorem 3.34, given a distribution $Q \in \mathcal{P}_n([d_1] \times \cdots \times [d_k])$, define the vector $v_Q \in V_1^{\otimes n} \otimes \cdots \otimes V_k^{\otimes n}$ as

$$v_Q := \sum_x \bigotimes_{m=1}^n v_{1,x_{m,1}} \otimes v_{2,x_{m,2}} \otimes \cdots \otimes v_{k,x_{m,k}},$$

where the sum is over all n -tuples $x = (x_1, \dots, x_n) \in T_Q^n$, and define the element $t_Q \in \mathbb{C}$ as $t_Q := \prod_{i \in \text{supp } Q} (t_{i_1, \dots, i_k})^{nQ(i)}$. Then the tensor power $t^{\otimes n}$ can be written as

$$(32) \quad t^{\otimes n} = \sum_Q t_Q v_Q,$$

where the sum is over $Q \in \mathcal{P}_n(\text{supp}_B t)$. Let $W_1 \otimes \cdots \otimes W_k \subseteq V_1^{\otimes n} \otimes \cdots \otimes V_k^{\otimes n}$ be the subspace where W_j is spanned by all vectors $v_{j,i_1} \otimes v_{j,i_2} \otimes \cdots \otimes v_{j,i_k}$ such that the empirical distribution of $(i_1, \dots, i_k) \in [d_j]^n$ is equal to the marginal distribution Q_j . Then $v_Q \in W_1 \otimes \cdots \otimes W_k$. This implies that

$$\text{SR}(v_Q) \leq \min_{j \in [k]} 2^{nH(Q_j)} \leq 2^{n \sum_{j=1}^k \theta(j)H(Q_j)}.$$

The number of terms in (32) is at most $(n+1)^{|\text{supp}_B t|}$. Therefore,

$$\begin{aligned} \log_2 \text{SR}^\sim(t) &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \left(|\text{supp}_B t| \log_2(n+1) + n \max_P \sum_{j=1}^k \theta(j)H(P_j) \right) \\ &= \max_P \sum_{j=1}^k \theta(j)H(P_j) \end{aligned}$$

with both maximisations over $P \in \mathcal{P}(\text{supp}_B t)$. This inequality holds for any basis choice $B \in \mathcal{B}(f)$ and therefore we may minimise the right side over all basis choices. This proves the theorem. \square

Corollary 5.10. *Let t be a tight 3-tensor. Then $\underline{\text{SR}}(t) = \lim_{n \rightarrow \infty} \text{SR}(t^{\otimes n})^{1/n}$ exists and equals the asymptotic subrank $\underline{Q}(t) = \lim_{n \rightarrow \infty} Q(t^{\otimes n})^{1/n}$.*

Proof. We have

$$\underline{Q}(t) \leq \text{SR}^\sim(t) \leq \min_{\theta} \zeta^{\theta}(t)$$

and $\min_{\theta} \zeta^{\theta}(t)$ equals $\underline{Q}(t)$ since t is a tight 3-tensor (Corollary 4.5). \square

Remark 5.11. We summarise the relationships among the functionals and asymptotic subrank, rank and slice rank.

- (1) $\text{SR}(t) := \lim_{n \rightarrow \infty} \text{SR}(t^{\otimes n})^{1/n} = \min_{\theta \in \mathcal{P}_s(\Pi)} F^{\theta}(t)$ (Corollary 5.7)
- (2) $\underline{Q}(t) \leq \text{PR}^\sim(t) \leq \underline{\text{SR}}(t) \leq F^{\theta}(t) \leq \zeta^{\theta}(t)$ if $\theta \in \mathcal{P}_s(\Pi)$ (Theorems 5.3 and 5.9)
- (3) $\underline{Q}(t) = \text{PR}^\sim(t) = \underline{\text{SR}}(t) = \min_{\theta \in \mathcal{P}_s(\Pi)} F^{\theta}(t) = \min_{\theta \in \mathcal{P}_s(\Pi)} \zeta^{\theta}(t)$ if t tight order-3 (Corollary 5.10)
- (4) $\underline{Q}(t) \leq \text{PR}^\sim(t) \leq F^{\theta}(t)$ if $\theta \in \mathcal{P}(\Pi)$ (Theorem 5.3)
- (5) $\underline{\zeta}_{\theta}(t) \leq \underline{\mathbb{R}}(t)$ if $\theta \in \mathcal{P}_s(\Pi)$
- (6) $F_{\theta}(t) \leq \underline{\mathbb{R}}(t)$ if $\theta \in \mathcal{P}_{\text{nc}}(\Pi)$.

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