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## Charged spin textures over the Moore–Read quantum Hall state

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**Abstract.** We study charged spin textures (CSTs) over the Moore–Read quantum Hall state at filling factor  $5/2$ . We develop an algebraic framework and show that the pairing condition that is inherent in the Moore–Read state naturally leads to a class of CST, labeled by winding numbers  $[w_I, w_{II}]$ . The fundamental CSTs, with labels  $[1, 0]$  and electric charge  $e/4$ , is identified with the polar core vortex known in the spin-1 Bose–Einstein condensates literature. The spin texture carried by the fusion product of fundamental CSTs is correlated with the fusion channel of underlying non-Abelian quasiholes.

Fractional quantum Hall (fqH) systems are the best-known realizations of topological order in nature. Among all observed fqH states, the one at  $\nu = 5/2$  is highly special. Its proposed theoretical descriptions, the Pfaffian (Moore–Read (MR)) [1] and anti-Pfaffian [2] states in particular, are based on a pairing mechanism and, as a consequence, admit excitations with non-Abelian braid statistics.

In fqH systems, the magnetic and the Coulomb energies are of the same order and it is not *a priori* clear whether the electron spin plays a role in understanding these phases of matter. This leaves several possibilities. The first is that the ground state itself can be non-polarized. In a second possible scenario, the ground state is polarized, but the excitations involve overturned spins. In this paper, we analyze spin-full excitations over the MR state, which by itself is fully polarized. We analyze *charged spin textures* (CSTs) carrying the fundamental fractional charge  $q = e/4$  and study the spin textures that arise upon fusing these excitations<sup>4</sup>.

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<sup>4</sup> While our results are specific to the MR state, we expect that the essential insights carry over to a broader class of paired wave functions, such as the anti-Pfaffian state.

Experiments have confirmed that the lowest energy excitations in  $\nu = 1$  integer quantum Hall (iqH) systems are CSTs. They are well described by wave functions of the form [3]

$$\psi_{\text{skyrmion}}^{(\lambda)} = \psi_{\text{B}}^{(\lambda)} \psi_{\text{iqH}}, \quad (1)$$

where  $\psi_{\text{iqH}}$  is the ground state wave function for  $\nu = 1$  and  $\psi_{\text{B}}^{(\lambda)}$  is a wave function for bosons. These excitations are referred to as skyrmions, because in an effective field theory approach they correspond to field configurations carrying nontrivial topological charge. This is measured by the Pontryagin index [4], which is identified with the electric charge carried by the skyrmion. A typical spin profile (Pontryagin index +1, charge  $+e$  skyrmion) reads

$$\mathbf{S}(r, \phi) = (\sqrt{1 - \sigma^2} \cos \phi, -\sqrt{1 - \sigma^2} \sin \phi, \sigma), \quad (2)$$

where  $\sigma(r)$  is  $-1$  at the origin and  $+1$  at infinity.

The recent paper [5], see also [6, 7], took up the study of spin-full excitations over the MR state, stressing that the results may shed light on some of the experimental findings regarding the spin polarization at  $\nu = 5/2$ . A particular suggestion is that specific experimental probes aimed at detecting a spin polarization at  $\nu = 5/2$  [8] may excite spin-full excitations, thereby depolarizing the system. The authors of [5] report an extensive numerical study of up to  $N = 20$  particles in spherical geometry, where angular momentum ( $L$ ) and total spin ( $S$ ) are good quantum numbers. They identified low-lying states on the diagonal  $L = S$  as well as spin-full ground states with  $L = 0$  or  $L = 1$ . The  $L = S$  states are associated with charge  $2q = e/2$  skyrmions. The  $L = S = 0$  state in particular is well described by a product state of the form  $\psi_{\text{skyrmion}} = \psi_{\text{B}}^{(L=S=0)} \psi_{\text{MR}}$ . The spin-full states with  $L = 0(1)$  are naturally interpreted as being built from spatially separated charge  $q$  CSTs. The paper [5] presents a phase diagram specifying the nature of spin-full excitations (skyrmions versus separated CSTs) as a function of Zeeman splitting and lateral harmonic confinement strength, which is used to model disorder.

Here we follow a purely algebraic approach to obtain an explicit expression for a variety of CSTs over the MR state. It rests on two observations. The first [9] is that the MR wave function, which is most commonly written in terms of a Pfaffian determinant [1], can be expressed as (we focus on the bosonic version with filling  $\nu = 1$  and fundamental charge  $q = e/2$ ; its fermionic counterpart with  $\nu = 1/2$  and  $q = e/4$  can be obtained by including an overall Jastrow factor)

$$\psi_{\text{MR}}(\{z_i\}) = \text{Symm}_{\text{I,II}} [\psi_{\text{I}}^L \psi_{\text{II}}^L]. \quad (3)$$

Here we divide the particles into two groups I, II of equal size, write a bosonic Laughlin wave function  $\psi_{\text{I,II}}^L(\{z_j\}) = \prod_{i < j \in \text{I,II}} (z_i - z_j)^2$  for each group, and then symmetrize over all ways to distribute the particles over groups I and II. The second observation [3] is that in the lowest Landau level (LLL), the most general wave function for  $N$  spin-full fermions in  $N + 1$  available one-particle orbitals that vanishes when two particles are placed at the same position<sup>5</sup> factorizes as  $\psi_{\text{B}} \psi_{\text{iqH}}$ , where  $\psi_{\text{B}}$  describes  $N$  spin-1/2 bosons in two orbitals. These orbitals can be viewed as the  $L_z = \pm 1/2$  components of an  $L = 1/2$  doublet of angular momentum. The combined orbital and spin angular momenta give rise to an  $\text{SU}(4)$  symmetry, with the four one-particle

<sup>5</sup> The Pauli principle only dictates that the wave function should vanish when two particles of identical spin are placed at the same position. The restriction to states that have the additional property that they vanish when two particles of opposite spin are placed at the same position makes the whole construction far more transparent. The price to pay is that not all possible states are reachable by this construction.

**Table 1.** Multiplicities of  $(L, S)$  multiplets in state space for  $N = 4$  bosonic spin-1/2 particles on the sphere, subjected to MR pairing condition and in the presence of flux  $N_\phi$ . The  $L = S = 0$  state at  $N_\phi = 1$  is the bosonic NASS state; the state with  $L = 0, S = 2$  at  $N_\phi = 2$  is the bosonic MR state.

						$N_\phi = 3$	$S = 0$	$S = 1$	$S = 2$
		$N_\phi = 2$	$S = 0$	$S = 1$	$S = 2$	$L = 0$	1	0	1
$N_\phi = 1$	$S = 0$	$L = 0$	1	0	1	$L = 1$	0	2	0
$L = 0$	1	$L = 1$	0	1	0	$L = 2$	2	1	1
		$L = 2$	1	0	0	$L = 3$	0	1	0
						$L = 4$	1	0	0

states ( $L = S = 1/2$ ) corresponding to the fundamental (vector) representation, with Dynkin labels  $[1\ 0\ 0]$ <sup>6</sup>. The partition sum for  $N$  such bosons becomes

$$Z_B = \text{SU}(4) \text{ irrep}[N\ 0\ 0]$$

$$= \sum_{K=0}^{N/2} (L = N/2 - K, S = N/2 - K). \quad (4)$$

For large  $N$ , the highest weight states (HW) of each of the ( $L = S = N/2 - K$ ) multiplets simplify to become

$$|\psi_B^K\rangle = \text{HW}(L = S = N/2 - K) \rightarrow |\downarrow_K, \uparrow_{N-K}\rangle, \quad (5)$$

with the left (right) position in the ket corresponding to the  $L_z = -1/2$  ( $+1/2$ ) orbital. The expression for a size- $\lambda$  skyrmion in the disc geometry is then obtained as a weighted sum over these states [3, 11],

$$\left( \sum_K \lambda^K \psi_B^K \right) \psi_{\text{iqH}} = \prod_{i < j} (z_i - z_j) \prod_l (z_l | \uparrow \rangle_l + \lambda | \downarrow \rangle_l) e^{-(|z|^2)/4}, \quad (6)$$

leading to the spin texture equation (2).

The bosonic MR state is uniquely characterized as the highest density spin-polarized LLL state that is annihilated by the pairing Hamiltonian  $H_{\text{pair}} = \sum_{i < j < k} \delta(z_i - z_j) \delta(z_j - z_k)$ . We restrict ourselves to spin-full states that satisfy the very same MR pairing condition  $H_{\text{pair}} \psi = 0$ . For spin-1/2 bosons, the highest density state with this ‘MR pairing’ property is the non-Abelian spin singlet (NASS) state [12] with filling fraction  $\nu = 4/3$ . In spherical geometry, the NASS state is realized for  $N_\phi^{\text{NASS}} = 3/4N - 2$  flux quanta. The total space of paired states for  $N_\phi$  in the vicinity of  $N_\phi^{\text{MR}} = N - 2$  can be understood through counting formulas for spin-full quasiholes over the NASS state [12]. In table 1, we list the dimensions of each of the  $(L, S)$  subspaces for  $N = 4$  particles for  $N_\phi = N_\phi^{\text{NASS}} = 1$ ,  $N_\phi = N_\phi^{\text{MR}} = 2$ ,  $N_\phi = N_\phi^{\text{MR}} + 1 = 3$ .

The idea is now to extend the trial states equation (3) to the spin-full case by including factors of type  $\psi_B$  separately in both group I and group II. All states generated in this way

<sup>6</sup> For an excellent review of the theory of Lie algebras, see [10]. This work also contains lists of irreducible representations of Lie algebras that arise in physical systems.

satisfy the pairing property. They constitute a subset of all paired states at  $N_\phi = N_\phi^{\text{MR}} + 1$ , as listed in table 1.

We first analyze the symmetric product of the states  $\psi_B$  for groups I and II. In SU(4) group theory,

$$Z_B^{I,II} = [N/2\ 0\ 0] \otimes_{\text{sym}} [N/2\ 0\ 0] = \sum_{l=0}^{N/4} [N - 4l\ 2l\ 0]. \quad (7)$$

For  $N = 4$ , the  $l = 0$  contribution has  $(L, S) = (2, 2)$ ,  $(1, 1)$  and  $(0, 0)$ , totaling 35 states, whereas the  $l = 1$  term comprises  $(L, S) = (2, 0)$ ,  $(1, 1)$ ,  $(0, 2)$ ,  $(0, 0)$ , totaling 20 states. For general  $N$ ,  $l$ , the representation  $[N - 4l\ 2l\ 0]$  contains fully polarized states ( $S = N/2$ ) at  $L = N/2 - 2l$ . Note that if we were to fully symmetrize over all  $N$  particle coordinates, only the  $l = 0$  term would survive, reducing the construction to the states  $|\psi_B^K\rangle$  describing the iqH skyrmion.

However, we first perform what we call the LLL lift of  $\psi_B^{I,II}$ ,

$$\psi_B^{I,II} \rightarrow \psi_B^{I,II}(\{x_{i \in I \cup II}\}) \psi_I^L(\{x_{j \in I}\}) \psi_{II}^L(\{x_{k \in II}\}),$$

where the sets of coordinates  $\{x_i\}$  contain both up-spins  $z_i$  and down-spins  $w_i$ . The polynomial  $\psi_B$  is then symmetric with respect to exchanging  $z_i$  and  $z_j$  if  $i, j$  are in the same group, and similarly for the down-spin coordinates; we require no further symmetries. If we only then symmetrize over all  $N$  particles, we keep a much bigger set of spin-full states satisfying the pairing condition. In general, states obtained for two orbitals lift to independent states at  $N_\phi = N - 1$ . One exception is  $(L, S) = (0, 0)$  at  $N = 4$ , where the LLL-lifts from the  $l = 0, 1$  multiplets coincide.

We now argue that the states that survive after the symmetrization step with  $l > 0$  can be viewed as charge  $q$  CSTs separated by a distance set by  $l$ , where  $l = N/4$  corresponds to the situation that two  $q$  CSTs sit on opposite poles of the sphere. This is most easily seen by focusing on the fully polarized ( $S = N/2$ ) states, where the expressions can be compared to explicit formulae describing spin-less charge  $q$  quasiholes. The states (in disc geometry) for  $N$  paired, spin-polarized bosons at  $N_\phi = N - 1$  are obtained by expanding the two-quasihole wave function

$$\text{Symm}_{I,II} \prod_{i \in I} (\eta_1 - z_i) \prod_{i < j \in I} (z_i - z_j)^2 \prod_{k \in II} (\eta_2 - z_k) \prod_{k < l \in II} (z_k - z_l)^2 \quad (8)$$

on a basis of symmetric polynomials in  $\eta_1, \eta_2$ , where the powers of the  $\eta_s$  indicate the location of the two quasiholes. On the sphere, the resulting angular momenta are (for  $N$  a multiple of 4)  $L = N/2, N/2 - 2, \dots, 0$ . To leading order in  $1/N$ , the state at  $L = 0$  corresponds to  $\eta_1^{N/2} \eta_2^0 + \eta_1^0 \eta_2^{N/2}$ , indicating that indeed the two quasiholes are on opposite sides of the sphere. For the spin-full case, one similarly finds that two-CST states with  $L \ll N/2$  correspond to well-separated CSTs.

For working toward explicit expressions, it is most convenient to perform the LLL lift and subsequent symmetrization in second quantization. Within each group, the LLL lift amounts to an embedding of a state defined on two orbitals to one on  $N$  orbitals, with coefficients set by the expansion of the corresponding Laughlin factor. It has the important property that both  $L$  and  $S$  quantum numbers are preserved. For the simple example of the two-particle, polarized  $\nu = 1/2$  Laughlin state (corresponding to one of the groups I, II for  $N = 4$  particles and  $N_\phi = 3$

flux quanta), the LLL lift takes the form

$$\begin{aligned} |\uparrow_2, 0\rangle &\rightarrow 2\sqrt{6}|\uparrow, 0, \uparrow, 0\rangle - 4|0, \uparrow_2, 0, 0\rangle, \\ |\uparrow, \uparrow\rangle &\rightarrow 6|\uparrow, 0, 0, \uparrow\rangle - 2|0, \uparrow, \uparrow, 0\rangle, \\ |0, \uparrow_2\rangle &\rightarrow 2\sqrt{6}|0, \uparrow, 0, \uparrow\rangle - 4|0, 0, \uparrow_2, 0\rangle. \end{aligned} \quad (9)$$

We will work out the second line above as an example. In the first quantization, this state corresponds to the lift of  $(z_1 + z_2)$ ,

$$(z_1 + z_2)(z_1 - z_2)^2 = (z_1^3 + z_2^3) - (z_1 z_2^2 + z_1^2 z_2), \quad (10)$$

where we have expanded on a basis of symmetric monomials. When going from first to second quantization on the sphere, particles in orbital  $l = 0, \dots, N_\phi$  obtain an additional factor  $\sqrt{l!} \sqrt{(N_\phi - l)!}$ , giving

$$\begin{aligned} (z_1^3 + z_2^3) &\rightarrow \sqrt{6}\sqrt{6}|\uparrow, 0, 0, \uparrow\rangle, \\ (z_1 z_2^2 + z_1^2 z_2) &\rightarrow \sqrt{2}\sqrt{2}|0, \uparrow, \uparrow, 0\rangle, \end{aligned}$$

which leads to (9) after we plug in the relative coefficients found in expansion (10). The symmetrization over groups I and II is easily done through the step

$$|\dots, m_I, \dots\rangle \times_s |\dots, m_{II}, \dots\rangle \rightarrow \sqrt{\frac{(m_I + m_{II})!}{m_I! m_{II}!}} |\dots, m_I + m_{II}, \dots\rangle, \quad (11)$$

where  $m_I, m_{II}$  indicate the occupation number of a given orbital (including its spin label).

As an explicit example, we present the two  $L = S = 1$  states for  $N = 4$  particles with  $N_\phi = 3$ . In the first step, we identify the  $L = S = 1$  HW states within the two distinct  $SU(4)$  multiplets with Dynkin labels  $[4\ 0\ 0]$  and  $[0\ 2\ 0]$ ,

$$\begin{aligned} \psi_B^{[4\ 0\ 0]} &\propto [\sqrt{2}|\uparrow, \uparrow\uparrow\downarrow\rangle + |\uparrow, \downarrow\uparrow_2\rangle - 3|\downarrow, \uparrow\uparrow_2\rangle] + [\text{I} \leftrightarrow \text{II}], \\ \psi_B^{[0\ 2\ 0]} &\propto [-\sqrt{2}|\uparrow, \uparrow\uparrow\downarrow\rangle + 2|\uparrow, \downarrow\uparrow_2\rangle] + [\text{I} \leftrightarrow \text{II}], \end{aligned} \quad (12)$$

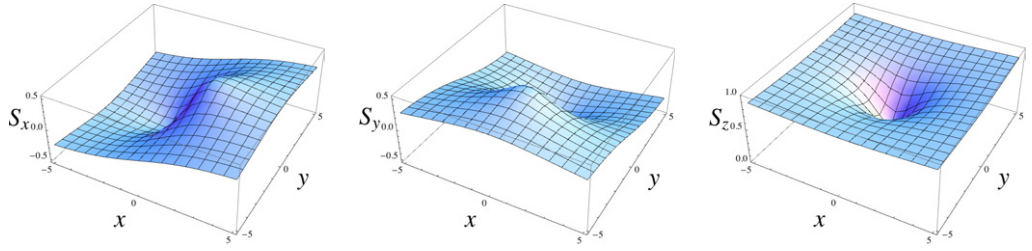
where we use single (double) arrows for indicating the particles in group I (II). In the second step, we perform the LLL lift and then symmetrize, leading to

$$\begin{aligned} \psi^{[4\ 0\ 0]} &\propto 5|0, \downarrow, \uparrow_3, 0\rangle - 3\sqrt{2}|0, \uparrow\downarrow, \uparrow, \uparrow\rangle + 7|0, \uparrow_2, \downarrow, \uparrow\rangle - \frac{5}{3}\sqrt{3}|0, \uparrow, \uparrow_2\downarrow, 0\rangle \\ &\quad - |0, \uparrow_2, \uparrow, \downarrow\rangle - \sqrt{3}|\downarrow, 0, \uparrow_2, \uparrow\rangle + 3|\downarrow, \uparrow, 0, \uparrow_2\rangle - \sqrt{6}|\uparrow, 0, \uparrow\downarrow, \uparrow\rangle \\ &\quad + 3\sqrt{3}|\uparrow, 0, \uparrow_2, \downarrow\rangle + 3|\uparrow, \downarrow, 0, \uparrow_2\rangle - 3\sqrt{2}|\uparrow, \uparrow, 0, \uparrow\downarrow\rangle, \end{aligned} \quad (13)$$

$$\begin{aligned} \psi^{[0\ 2\ 0]} &\propto 4|0, \downarrow, \uparrow_3, 0\rangle - \frac{3}{2}\sqrt{2}|0, \uparrow\downarrow, \uparrow, \uparrow\rangle + 2|0, \uparrow_2, \downarrow, \uparrow\rangle - \frac{4}{3}\sqrt{3}|0, \uparrow, \uparrow_2\downarrow, 0\rangle \\ &\quad + |0, \uparrow_2, \uparrow, \downarrow\rangle - 2\sqrt{3}|\downarrow, 0, \uparrow_2, \uparrow\rangle + 6|\downarrow, \uparrow, 0, \uparrow_2\rangle + \sqrt{6}|\uparrow, 0, \uparrow\downarrow, \uparrow\rangle \\ &\quad - 3|\uparrow, \downarrow, 0, \uparrow_2\rangle - \frac{3}{2}\sqrt{2}|\uparrow, \uparrow, 0, \uparrow\downarrow\rangle. \end{aligned}$$

We remark that while the construction correctly reproduces two linearly independent paired states at  $L = S = 1$ , the algebraic structure is not very transparent. For one thing, the  $SU(4)$  symmetry is lost in the LLL lift. In addition, the two states equation (13) is not orthogonal.

The  $(L, S)$  states constructed here can directly be compared to the numerical ground states found in [5]. In particular, where [5] finds  $L = 0, 1$  ground states for  $N = 12$  at  $S = 4, 5, 6$ , we expect good overlaps with the states presented here.



**Figure 1.** Spin components  $S_x$ ,  $S_y$  and  $S_z$  for configuration with single CST[1, 0] at the origin and a quasihole at infinity, after stereographic projection.

The bosonic parts of the wave functions simplify considerably in the large- $N$  limit: the leading polarized states of the  $[N\ 0\ 0]$  and  $[0\ N/2\ 0]$  irreps are

$$\psi_B^{[N\ 0\ 0]} \rightarrow |0, \uparrow_{N/2} \uparrow_{N/2}\rangle, \quad \psi_B^{[0\ N/2\ 0]} \rightarrow |\uparrow_{N/2}, \uparrow_{N/2}\rangle. \quad (14)$$

We can then consider states with  $K$  overturned spins as in equation (5), separately in groups I and II. This leads to simple expressions for the situation where the spin textures in groups I and II have sizes  $\lambda_I$  and  $\lambda_{II}$ . Starting from  $\psi_B^{[0\ N/2\ 0]}$ , with group I and II textures on opposite sides of the two-orbital subspace, and then taking these expressions through the LLL lift and symmetrization, leads to two-CST configurations where the charge  $q$  CSTs sit on opposite sides of the sphere. This expression, expanded in powers of sizes  $\lambda_I$  and  $\lambda_{II}$ , symbolically reads

$$\sum_{K_I, K_{II}} \lambda_I^{K_I} \lambda_{II}^{K_{II}} \left( \text{Symm}_{I, II} \right) (\text{LLL lift}) |\uparrow_{N/2-K_I} \downarrow_{K_{II}}, \downarrow_{K_I} \uparrow_{N/2-K_{II}}\rangle. \quad (15)$$

From here on we will label our textures as CST[ $w_I, w_{II}$ ], where  $w_I, w_{II}$  are winding numbers (with respect to a given location on the sphere) of the skyrmions that would appear in groups I and II if the symmetrization step in our construction were not performed. The SU(4) label  $[N\ 0\ 0]$  corresponds to CST[1, 1] and  $[0\ N/2\ 0]$  gives two spatially separated CST[1, 0].

For a given quantum state, we measure the components of the spin field by acting with

$$\mathbf{S}(z, \bar{z}) = \sum_{l, l'} (a_{l, \alpha}^\dagger \sigma_{\alpha\beta} a_{l', \beta}) \phi_l^\dagger(\bar{z}) \phi_{l'}(z), \quad (16)$$

where the  $\phi_l(z)$  are single-particle wave functions that depend on the geometry and our normalization is such that a polarized system has  $S_z = 1$ . In figure 1, we plot the expectation value of the spin vector for a configuration with  $N = 8$ ,  $\lambda_I = 1.0$ ,  $\lambda_{II} = 0.0$ .

In a conformal field theory (CFT)-based description [1], the quasihole operator comes with an Ising  $\sigma$ -field. For  $N$  even, a collection of these  $\sigma$ -fields will fuse to the identity operator, whereas they fuse to the  $\psi$  sector for  $N$  odd. This is due to the fact that the electron operator carries a  $\psi$  field, so performing the contractions of an odd number of electron operators within the CFT correlator will always leave one  $\psi$  field: this one has to pair with the fusion product of all of the  $\sigma$ -fields in order for the total correlator to fuse to the identity.

It was shown in [14] that the density profile of the system on the sphere after the fusion of two charge  $q$  quasiholes to a charge  $2q$  quasihole differs between the two cases: in the case of  $N$ , even the density drops to zero at the location of the  $2q$  quasihole, whereas the density drop for  $N$  odd is wider and less deep.

This has consequences for the possible spin textures that may arise as a result of fusing elementary CST[1, 0]. Our construction recovers the polarized quasihole states in the limit



$\lambda \rightarrow 0$ . This means that we expect the density for up-particles to vanish in the core of the fusion product of two CST[1,0] for  $N$  even, but not for  $N$  odd.

This becomes rather obvious in the two-layer construction that we have been using throughout this paper. For  $N$  even, the particles can be divided equally into two groups. The  $K_I = K_{II} = 0$  term appearing in the expansion of two CSTs analogous to (15), but now for both CSTs at the same position, is then

$$|0, \uparrow_{N/2}\rangle \text{ for group I, } |0, \uparrow_{N/2}\rangle \text{ for group II.}$$

However, for  $N$  odd, we have to divide the particles unequally among the two groups. The division that requires the least amount of total flux is  $(N+1)/2, (N-1)/2$ . The  $N_\phi$  is equal for both groups, and is at least  $N-1$ : this is the highest power for a single particle appearing in the expansion of the Laughlin factor for the group containing  $(N+1)/2$  particles<sup>7</sup>. Note that  $N_\phi^{\text{MR}} = N-2$ , so that paired states with an odd number of particles will always have quasiholes present.

This extra flux in the system gives two extra orbitals to particles in the smaller group, whereas particles in the larger group have no additional orbitals. The first term in the expansion now becomes

$$|0, 0, \uparrow_{(N-1)/2}\rangle \text{ for group I, } |\uparrow_{(N+1)/2}\rangle \text{ for group II.}$$

Even without performing the whole calculation, one can already see that the density will not vanish since the particles in group II are spread homogeneously over the sphere. The natural texture

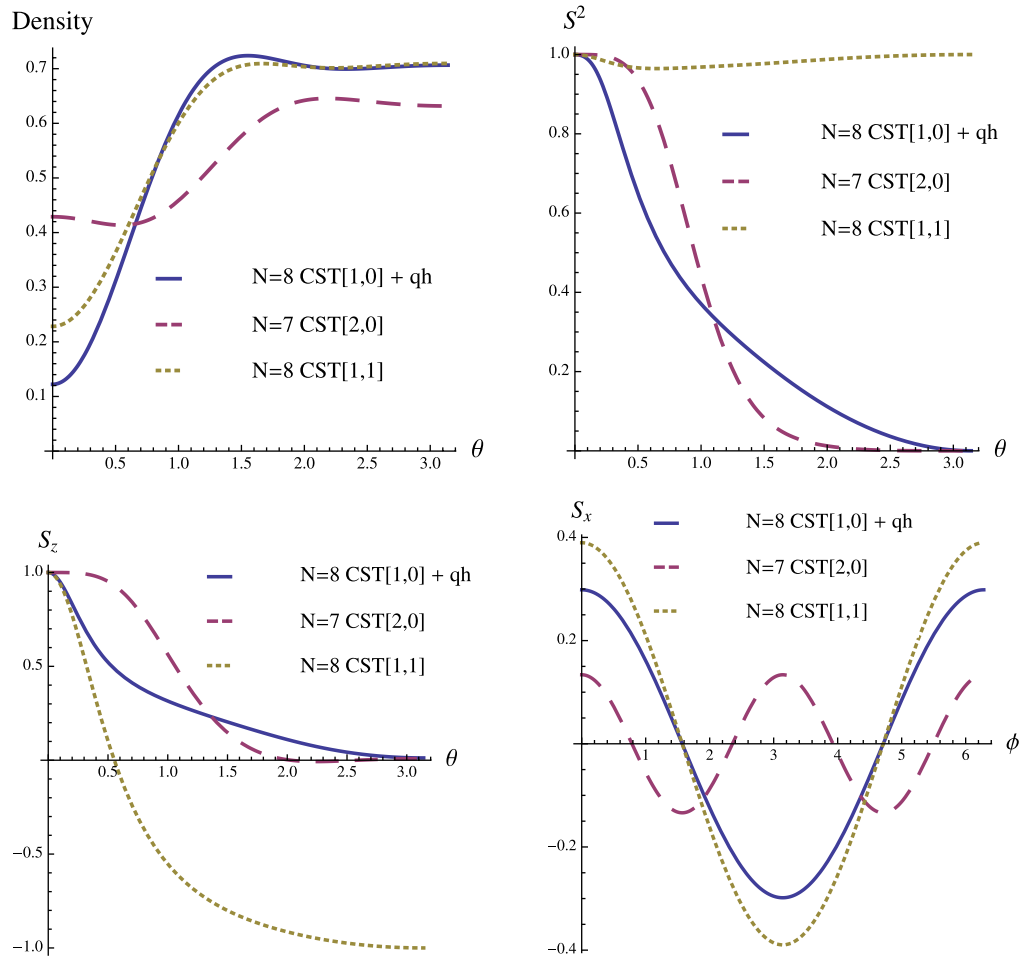
$$\sum_K \lambda^K \left( \text{Symm}_{I,II} \right) (\text{LLL lift}) |\downarrow_K, 0, \uparrow_{(N-1)/2-K}\rangle |\uparrow_{(N+1)/2}\rangle \quad (17)$$

has winding number 2 for group I and 0 for group II. Therefore, we argue that the simplest possible charge  $2q$  configuration has winding indices  $[2, 0]$  for  $N$  odd.

It is natural to ask the question of what spin textures arise when multiple CST[1, 0] are fused together. Depending on the path one takes through the Bratteli diagram of the CFT of the underlying quasiholes, multiple options are possible. In [13], the authors define the number of unpaired fermions  $F$  in order to write down wave functions in different fusion channels. For a general number of extra fluxes  $\Delta N_\phi$  above the ground state,  $0 \leq F \leq \Delta N_\phi$  and  $F = N \bmod 2$ . We obtain the same wave functions by dividing the particles as  $(N+F)/2$  in group I and  $(N-F)/2$  in group II. The associated spin textures satisfy the identity  $|w_I - w_{II}| = 2F$ . For  $\Delta N_\phi = 1$  and  $F = 0$  (1), this corresponds to the even (odd) discussion in the above paragraph. The relation between the formulation of wave functions in terms of this  $F$ , the multiple group construction and those obtained through CFT will be further presented in an upcoming work [17].

We have studied three representative cases in detail: a skyrmion CST[1, 1] for  $N = 8$ , a separated CST[1, 0]/quasihole pair for  $N = 8$  and a single CST[2, 0] for  $N = 7$ . The results are in figure 2. We have chosen these cases for the following reasons. The CST[1, 0] is the fundamental charge  $q$  spin texture, associated with the  $\sigma$ -field in the Ising CFT. The skyrmion CST[1, 1] is given because it shows that our construction includes the results of earlier

<sup>7</sup> Note that one can also divide the particles unequally in the case of  $N$  even. This leads to states with quasiholes: consider the division  $N/2+1, N/2-1$ . The particles in the first group require at least  $N_\phi = N$ , which means that the particles in the second group have four excess fluxes.



**Figure 2.** Density,  $S^2$  and  $S_z$  of an  $N = 8$  CST[1, 0]/quasihole pair, an  $N = 7$  CST[2, 0] and an  $N = 8$  skyrmion (CST[1, 1]) at  $\lambda = 2.0$  as a function of polar angle  $\theta$  at azimuthal angle  $\phi = 0$  and  $S_x$  for the same systems as a function of  $\phi$  at  $\theta = \frac{\pi}{2}$ . The spin textures are centered at  $\theta = \pi$ ; for the  $N = 8$  CST[1, 0] there is a quasihole at  $\theta = 0$ .

studies [3]. It is also the fusion product, following the discussion above, of two elementary CST[1, 0] in the trivial ( $N$  even) fusion channel. The CST[2, 0] is the fusion product of two CST[1, 0] when the overall fusion channel (in CFT language) is  $\psi$  or, alternatively, when the number of particles  $N$  is odd.

Two observations about the behavior of these textures are in place. First of all we see that CST[2, 0] has winding number 2 when the azimuthal angle runs from 0 to  $2\pi$ . Furthermore, the (expectation value of the) length of the spin vector vanishes in the core of CSTs of type [1, 0] and [2, 0]. For  $N$  large, the latter effect seems to hold for all CST[ $n$ , 0]. This behavior closely mimics that of the ‘polar core vortex’ appearing in rotating spin-1 Bose–Einstein condensates (BECs) [15]. The observation that the MR state carries an effective spin-1 field due to the pairing of electrons has been made in earlier studies [7, 16]. The BEC polar core vortices have the

following mean field spin vector expectation value,

$$\mathbf{S}(r, \phi) = (\sqrt{2\rho(1-\rho)} \cos n\phi, -\sqrt{2\rho(1-\rho)} \sin n\phi, \rho), \quad (18)$$

with  $\rho(r)$  equal to 0 at the origin and approaching 1 at infinity. The integrated Pontryagin density for these textures equals  $Q_{\text{top}} = \frac{n}{4}$ . Numerical values for our CST[1, 0] textures approach  $\frac{1}{4}$  for  $\lambda \gg 1$ . For a general texture CST[ $w_{\text{I}}, w_{\text{II}}$ ], the integrated Pontryagin density is no longer a topological index in the usual sense (the target space manifold is  $\mathbb{R}^3$  instead of  $S^2$ , so the integral does not have to be an integer). Also, the relation between electric and topological charge densities,  $\rho_{\text{elec}} = \nu e \rho_{\text{top}}$ , valid in Abelian quantum Hall states, takes a different form in general non-Abelian states, of which the MR state is a prototypical example.

Having completed this work, we have been informed of related but unpublished work of I Dimov and C Nayak. They considered first quantized expressions for CST based on the ‘Pfaffian’ expression of the MR state and examined the associated spin textures, observing as we did the vanishing of the spin vector in the core of the CST. It seems clear that for the purpose of numerically generating finite- $N$  trial states, the procedure proposed here (which avoids cumbersome expansions of expressions in first quantization) is particularly efficient.

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