Supplementary Material
Costly incentives design from an institutional perspective: cooperation, sustainability, and affluence

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Supplementary part 1. Population equilibrium

To predict the effect of incentive mechanism, **evolutionary game theory** which describes the population of agents engaging in pairwise interaction, has been generally accepted as a common framework to model and interpret the evolution of cooperation in a social dilemma. This part shows the derivation process of the analytical results of the population in equilibrium.

Assume a finite population of size N, M of them participant a market play PDG, two types of strategies for cooperation C and defection D are well mixed. Let $\pi(C)$ denote the expected payoff of strategy C, $\pi(D)$ donate that of strategy D, and $\bar{\pi}(\mathbf{x})$ denote the average payoff of the whole population:

$$\pi(C) = (1 - c_0 + R_{CC})x + (-T - c_0 + R_{CD})y \tag{1}$$

$$\pi(D) = (T - c_0 - F_{CD})x + (0 - c_0 - F_{DD})y \tag{2}$$

$$\bar{\pi}(\mathbf{x}) = \pi(C)x + \pi(D)y \tag{3}$$

The replicator dynamics of cooperators and defectors is:

$$\dot{x} = x(1-x)[\pi(C) - \pi(D).] \tag{4}$$

With the equations (1), (2), (3), and (4), let $\dot{x} = 0$,

$$(x^2 - x)[x + xR_{CC} + (1-x)R_{CD} + xF_{CD} + (1-x)F_{DD} - T] = 0 \tag{5}$$

Thus, the fixed points are $x^* = 0$, $x^* = 1$, and $x^* = \frac{-F_{DD} + T - R_{CD}}{1 + R_{CC} - R_{CD} + F_{CD} - F_{DD}}$.

We next discuss if the fixed point is the Nash equilibrium (NE) and satisfy the requirement of being as the evolutionary stable strategy (ESS).

Let $\mathbf{s}^* = (x^*, 1 - x^*)$, $\mathbf{s} = (p, 1-p)$, $\mathbf{s} \neq \mathbf{s}^*$, an ESS can be defined as a mixed strategy $\mathbf{s}^*$, such that for any strategy $\mathbf{s}$ and any sufficient small $\varepsilon > 0$,

$$\pi[\mathbf{s}^*, (1-\varepsilon)\mathbf{s}^* + \varepsilon\mathbf{s}] > \pi[\mathbf{s}, (1-\varepsilon)\mathbf{s}^* + \varepsilon\mathbf{s}]. \tag{6}$$

Using the linearity in probability of expected payoffs reduces 6 to:

$$(1-\varepsilon)\pi(\mathbf{s}^*, \mathbf{s}^*) + \varepsilon\pi(\mathbf{s}^*, \mathbf{s}) > (1-\varepsilon)\pi(\mathbf{s}, \mathbf{s}^*) + \varepsilon\pi(\mathbf{s}, \mathbf{s}). \tag{7}$$

If for all small $\varepsilon > 0$ and for all $\mathbf{s}$,

$$\pi(\mathbf{s}^*, \mathbf{s}^*) \geq \pi(\mathbf{s}, \mathbf{s}^*), \tag{8}$$

then $\mathbf{s}^*$ is a symmetric NE. Further, if

$$\pi(\mathbf{s}^*, \mathbf{s}) > \pi(\mathbf{s}, \mathbf{s}) \tag{9}$$

whenever $\pi(\mathbf{s}^*, \mathbf{s}^*) = \pi(\mathbf{s}, \mathbf{s}^*)$, $\mathbf{s}^*$ is an ESS [1].

1) fixed point at $x^* = 0$, $\mathbf{s}^* = (0,1), \mathbf{s} = (p, 1-p)$ $(0 < p \leq 1)$, we have:

$$\pi(\mathbf{s}^*, \mathbf{s}^*) = -c_0 - F_{DD} \tag{8a}$$

$$\pi(\mathbf{s}, \mathbf{s}^*) = p(-T - c_0 + R_{CD}) + (1-p)(0 - c_0 - F_{DD}) \tag{8b}$$

$$\pi(\mathbf{s}^*, \mathbf{s}) = p(T - c_0 - F_{CD}) + (1-p)(0 - c_0 - F_{DD}) \tag{9a}$$

$$\begin{aligned}\pi(\mathbf{s}, \mathbf{s}) =& p^2(1 - c_0 + R_{CC}) + p(1-p)(-T - c_0 + R_{CD}) \\ &+ p(1-p)(T - c_0 - F_{CD}) + (1-p)^2(0 - c_0 - F_{DD})\end{aligned} \tag{9b}$$

By 8a - 8b, we have

$$\begin{aligned}8a - 8b &= -p(-T - c_0 + R_{CD}) + p(0 - c_0 - F_{DD}) \\ &= p(T - F_{DD} - R_{CD}),\end{aligned} \tag{8c}$$

and by 9a - 9b, we have

$$9a - 9b = p^2(T - F_{CD} - R_{CC}) + p(1-p)(-F_{DD} + T - R_{CD}). \tag{9c}$$

The Table 1 shows the requirements for $x^* = 0$ being a NE or an ESS under different incentive policies.

Supplementary table 1. NE and ESS analysis under different conditions when $x^* = 0$

Incentive Policy	Scale of parameters	NE	ESS
Reward	$0 < R_{CC} + R_{CD}$, $F_{CD} = F_{DD} = 0$	When $T \geq R_{CD}$, $8c \geq 0$	When $T > R_{CD}$, $9c > 0$
Punishment	$R_{CD} = R_{DD} = 0$, $0 < F_{CD} + F_{DD}$	When $T \geq F_{DD}$, $8c \geq 0$	When $T \geq F_{CD}$, $9c > 0$
Mixed incentives	$R_{CD} + R_{CC} \neq 0$, $F_{CD} + F_{DD} \neq 0$	When $T \geq R_{CD} + F_{DD}$, $8c \geq 0$	When $T \geq \max[R_{CD} + F_{DD}, R_{CC} + F_{CD}]$, $9c > 0$

2) Consider the fixed point at $x^* = 1$, $\mathbf{s}^* = (1,0), \mathbf{s} = (1-p, p)$ $(0 < p \leq 1)$, we have:

$$\pi(\mathbf{s}^*, \mathbf{s}^*) = 1 - c_0 + R_{CC} \tag{8d}$$

$$\pi(\mathbf{s}, \mathbf{s}^*) = (1-p)(1 - c_0 + R_{CC}) + p(T - c_0 - F_{CD}) \tag{8e}$$

$$\pi(\mathbf{s}^*, \mathbf{s}) = (1-p)(1 - c_0 + R_{CC}) + p(-T - c_0 + R_{CD}) \tag{9d}$$

$$\begin{aligned}\pi(\mathbf{s}, \mathbf{s}) =& (1-p)^2(1 - c_0 + R_{CC}) + p(1-p)(-T - c_0 + R_{CD}) \\ &+ p(1-p)(T - c_0 - F_{CD}) + p^2(0 - c_0 - F_{DD})\end{aligned} \tag{9e}$$

By 8d - 8e, we have

$$\begin{aligned}8d - 8e &= p(1 - c_0 + R_{CC}) - p(T - c_0 - F_{CD}) \\ &= p(1 - T + R_{CC} + F_{CD}),\end{aligned} \tag{8f}$$

and by 9d - 9e, we have

$$\begin{aligned}9d - 9e =& p(1-p)(1 - c_0 + R_{CC}) - p(1-p)(T - c_0 - F_{CD}) \\ &+ p^2(-T - c_0 + R_{CD}) - p^2(0 - c_0 - F_{DD}) \\ =& p(1-p)(1 - T + R_{CC} + F_{CD}) + p^2(-T + R_{CD} + F_{DD}).\end{aligned} \tag{9f}$$

The Table 2 shows the requirements for $x^* = 1$ being a NE or an ESS under different incentive policies.

Supplementary table 2. NE and ESS analysis under different conditions when $x^* = 1$

Incentive Policy	Scale of parameters	NE	ESS
Reward	$0 < R_{CC} + R_{CD}$, $F_{CD} = F_{DD} = 0$	When $R_{CC} \geq T - 1$, $8f \geq 0$	When $R_{CC} \geq T$, $9f > 0$
Punishment	$R_{CD} = R_{DD} = 0$, $0 < F_{CD} + F_{DD}$	When $F_{CD} \geq T - 1$, $8f \geq 0$	When $F_{DD} > T$,$9f > 0$
Mixed incentives	$R_{CD} + R_{CC} \neq 0$, $F_{CD} + F_{DD} \neq 0$	When $R_{CC} + F_{CD} \geq T - 1$, $8f \geq 0$	When $\min[R_{CD} + F_{DD}, R_{CC} + F_{CD}] \geq T$, $9f > 0$

3) Consider the fixed point at $x^* = \frac{T - R_{CD} - F_{DD}}{1 + R_{CC} - R_{CD} + F_{CD} - F_{DD}} = \frac{B}{A}$. Let $q = \frac{B}{A}$, $\mathbf{s}^* = (q, 1-q), \mathbf{s} = (p, 1-p)$ $(0 \leq p \leq 1, p \neq q)$, we have:

$$\begin{aligned}\pi(\mathbf{s}^*,\mathbf{s}^*) = q^2(1 - c_0 + R_{CC}) + q(1-q)(-T - c_0 + R_{CD}) \\ + q(1-q)(T - c_0 - F_{CD}) + (1-q)^2(0 - c_0 - F_{DD})\end{aligned} \tag{8g}$$

$$\begin{aligned}\pi(\mathbf{s},\mathbf{s}^*) = pq(1 - c_0 + R_{CC}) + p(1-q)(-T - c_0 + R_{CD}) \\ + (1-p)q(T - c_0 - F_{CD}) + (1-p)(1-q)(0 - c_0 - F_{DD})\end{aligned} \tag{8h}$$

$$\begin{aligned}\pi(\mathbf{s}^*,\mathbf{s}) = pq(1 - c_0 + R_{CC}) + (1-p)q(-T - c_0 + R_{CD}) \\ + p(1-q)(T - c_0 - F_{CD}) + (1-p)(1-q)(0 - c_0 - F_{DD})\end{aligned} \tag{9g}$$

$$\begin{aligned}\pi(\mathbf{s},\mathbf{s}) = p^2(1 - c_0 + R_{CC}) + p(1-p)(-T - c_0 + R_{CD}) \\ + p(1-p)(T - c_0 - F_{CD}) + (1-p)^2(0 - c_0 - F_{DD})\end{aligned} \tag{9h}$$

By 8g - 8h, we have

$$\begin{aligned}8g - 8h =& q(q-p)(1 - c_0 + R_{CC} - T + c_0 + F_{CD}) + (1-q)(q-p)(-T - c_0 + R_{CD} + c_0 + F_{DD}) \\ =& q(q-p)(1 - T + R_{CC} + F_{CD}) + (1-q)(q-p)(-T + R_{CD} + F_{DD}) \\ =& (q-p)[q(1 + R_{CC} + F_{CD} - R_{CD} - F_{DD}) - T + R_{CD} + F_{DD} \\ =& (q-p)[q(1 + R_{CC} + F_{CD}) + (1-q)(R_{CD} + F_{DD}],\end{aligned} \tag{8i}$$

and by 9g - 9h, we have

$$\begin{aligned}9g - 9h =& p(q-p)(1 - c_0 + R_{CC} - T + c_0 + F_{CD}) + (1-p)(q-p)(-T - c_0 + R_{CD} + c_0 + F_{DD}) \\ =& p(q-p)(1 - T + R_{CC} + F_{CD}) + (1-p)(q-p)(-T + R_{CD} + F_{DD} \\ =& (q-p)[p(1 + R_{CC} + F_{CD} - R_{CD} - F_{DD}) - T + R_{CD} + F_{DD}] \\ =& (q-p)[p(1 + R_{CC} + F_{CD}) + (1-p)(R_{CD} + F_{DD}) - T].\end{aligned} \tag{9i}$$

The Table 3 shows the requirements for $x^* = q$ being a NE or an ESS under different incentive policies.

Supplementary table 3. NE and ESS analysis under different conditions when $x^* = q$

Incentive Policy	Scale of parameters	NE	ESS
Reward	$0 < R_{CC} + R_{CD}$, $F_{CD} = F_{DD} = 0$	$8i = 0$, thus always hold	$9i < 0$, thus the ESS is not hold
Punishment	$R_{CD} = R_{DD} = 0$, $0 < F_{CD} + F_{DD}$	$8i = 0$, thus always hold	$9i < 0$, thus the ESS is not hold
Mixed incentives	$R_{CD} + R_{CC} \neq 0$, $F_{CD} + F_{DD} \neq 0$	$8i = 0$, thus always hold	$9i < 0$, thus the ESS is not hold

Supplementary part 2. Rate of R_{CC} (F_{DD}) in $R_{CC}+F_{CD}$ ($R_{CD}+F_{DD}$)

Let α be the rate, $k = R_{CC}+F_{CD}$, to minimize the difference between reward($R_{CD}+R_{CC}$) and punishment ($F_{CD}+F_{DD}$), meanwhile satisfy the constraints are $R_{CD} \geq R_{CC}, F_{CD} \geq F_{DD}$, we have:

$$\begin{aligned} \min_{\alpha} \quad & k\alpha + (k+1)\alpha - [k(1-\alpha)+(1+k)\alpha] \\ \text{s.t.} \quad & (1-\alpha)2 \geq 3\alpha \\ & (1-\alpha)1 \geq 4\alpha \end{aligned} \tag{10}$$

The solution is $\alpha = 0.2$. Consequently, these four parameters are set as shown in Table 4.

Supplementary table 4. Mixed incentives setup

$R_{CC}+F_{CD}$	R_{CC}	F_{CD}	$R_{CD}+F_{DD}$	R_{CD}	F_{DD}
1	0.2	0.8	2	1.6	0.4
1.25	0.25	1	2.25	1.8	0.45
1.5	0.3	1.2	2.5	2	0.5
1.75	0.35	1.4	2.75	2.2	0.55
2	0.4	1.6	3	2.4	0.6
2.25	0.45	1.8	3.25	2.6	0.65
2.5	0.5	2	3.5	2.8	0.7
2.75	0.55	2.2	3.75	3	0.75
3	0.6	2.4	4	3.2	0.8

Supplementary part 3. Accumulated wealth of the third-party

From equation 4, we have

$$\begin{aligned} \dot{x} = \frac{dx}{dt} =& (1+R_{CC}-R_{CD}+F_{CD}-F_{DD})x^3 \\ &+ (2R_{CD}+2F_{DD}-F_{CD}-R_{CC}-T-1)x^2 \\ &- (R_{CD}+F_{DD}-T)x \end{aligned} \tag{11}$$

Let $a = 1+R_{CC}-R_{CD}+F_{CD}-F_{DD}$, $b = 2R_{CD}+2F_{DD}-F_{CD}-R_{CC}-T-1$, $c = T-R_{CD}-F_{DD}$, we have:

$$\frac{dx}{dt} = ax^3+bx^2+cx \tag{12}$$

We can then solve the differential equation:

$$\begin{aligned} \int \frac{1}{ax^3+bx^2+cx}dx = \int 1dt \\ =& \int \frac{1}{x(ax^2+bx+c)}dx \\ =& \int \frac{1}{cx} + \frac{-ax-b}{c(ax^2+bx+c)}dx \\ =& \frac{1}{c}\int \frac{1}{x}dx - \frac{a}{c}\int \frac{x}{ax^2+bx+c}dx - \frac{b}{c}\int \frac{1}{ax^2+bx+c}dx \\ =& \frac{1}{c}\ln x - \frac{a}{c}\left(\frac{\ln(ax^2+bx+c)}{2a} - \frac{b}{a\sqrt{4ac-b^2}}\tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)\right) \\ &- \frac{b}{c}\frac{2}{a\sqrt{4ac-b^2}}\tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right) + C \\ =& t \end{aligned} \tag{13}$$

$x^{(t)}$ is the inverse function of 13, let $x^{(t)} := F(t,a,b,c)$. The wealth of the third party at time step t is shown as 2.3

$$\begin{aligned}
W_T^{(t)} = M^{(t)} \Big(& c_0 + x^{(t)}\big(1-x^{(t)}\big)F_{CD} + \big(1-x^{(t)}\big)^2 F_{DD} - \big(x^{(t)}\big)^2 R_{CC} - x^{(t)}\big(1-x^{(t)}\big)R_{CD} \\
& - \alpha\Big(x^{(t)}\big(1-x^{(t)}\big)F_{CD} + \big(1-x^{(t)}\big)^2 F_{DD}\Big)\Big) \qquad (14)
\end{aligned}$$

$$\begin{aligned}
= M^{(t)} \Big(& \big(x^{(t)}\big)^2 \big((1-\alpha)(F_{DD}-F_{CD}) - R_{CC} + R_{CD}\big) + x^{(t)}\big((1-\alpha)(F_{CD}-2F_{DD}) - R_{CD}\big) \\
& + c_0 + (1-\alpha)F_{DD}\Big) \qquad (15)
\end{aligned}$$

$$\begin{aligned}
= M^{(t)} \Big(& F^2(t,a,b,c)\big((1-\alpha)(F_{DD}-F_{CD}) - R_{CC} + R_{CD}\big) \qquad (16) \\
& + F(t,a,b,c)\big((1-\alpha)(F_{CD}-2F_{DD}) - R_{CD}\big) + c_0 + (1-\alpha)F_{DD}\Big) \qquad (17)
\end{aligned}$$

Thus,

$$W_T = \int W_T^{(t)}\,dt \qquad (18)$$

$$\begin{aligned}
= & \big((1-\alpha)(F_{DD}-F_{CD}) - R_{CC} + R_{CD}\big) \int M^{(t)} F^2(t,a,b,c)\,dt \qquad (19) \\
& + \big((1-\alpha)(F_{CD}-2F_{DD}) - R_{CD}\big) \int M^{(t)} F(t,a,b,c)\,dt \\
& + \big(c_0 + (1-\alpha)F_{DD}\big) \int M^{(t)}\,dt
\end{aligned}$$

References

1. Crawford, V.P. Learning and mixed-strategy equilibria in evolutionary games. *J.Theor.Biol*. **140**, 537–550 (1989).