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Multiple Equilibria and Limit Cycles in Evolutionary Games with Logit Dynamics*

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23rd February 2011

Abstract

This note shows, by means of two simple, three-strategy games, the existence of stable periodic orbits and of multiple, interior steady states in a smooth version of the Best Response Dynamics, the Logit Dynamics. The main finding is that, unlike Replicator Dynamics, *generic* Hopf bifurcation and thus, stable limit cycles, occur under the Logit Dynamics, even for three strategy games. We also show that the Logit Dynamics displays another bifurcation which *cannot* occur under the Replicator Dynamics: the *fold bifurcation*, with *non-monotonic* creation and disappearance of steady states.

JEL classification: C72, C73

Keywords: Evolutionary games, Logit dynamics, Hopf bifurcation, fold bifurcation

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1 Introduction

A large body of the research on evolutionary game dynamics has focused on identifying classes of games and dynamics that ensure convergence to point-attractors such as the Nash equilibrium (Hofbauer and Weibull (1996), Hofbauer and Sandholm (2002), Sandholm (2005)). However, unfolding the mechanism through which non-convergent behavior may emerge is important for economic situations as, for instance, stable cyclical patters have already been noticed experimentally (Cason and Friedman (2003)) in game-theoretical models of price dispersion (Burdett and Judd (1983)). Hofbauer et al. (1980) investigate the phase portraits from three-strategy games under the replicator dynamics and conclude that only 'simple' behaviour - sinks, sources, centers, saddles - can occur. In general, evolutionary dynamics in a $n-$strategy game define a proper $n-1$ dynamical system on the $n-1$ simplex. An important result (Zeeman (1980)), is that there are no *generic* Hopf bifurcations, under Replicator Dynamics, on the 2-simplex: *"When $n = 3$ all Hopf bifurcations are degenerate"*[1]. In Replicator Dynamics only the "hairline" case of a continuum of cycles occurs, which are non-generic and disappear by slightly perturbing the payoff parameters. Another possibility in Replicator Dynamics is a so-called heteroclinic cycle consisting of saddle steady-states on the boundary of the simplex and their connecting saddle paths. Generic stable limit cycles do not arise in 3-strategy games under Replicator Dynamics. Although periodic and chaotic behaviour is documented in the literature for the Replicator Dynamics, there is much less evidence for such complicated behaviour in classes of evolutionary dynamics that may be more appropriate for human interaction (e.g. fictitious play or best response dynamics.). Shapley (1964) constructs an example of a non-zero sum game with a "polygon" attractor under fictitious play[2] whereas Berger and Hofbauer (2006) find stable periodic behaviour - two limit cycles bounding an asymptotically stable annulus under the Brown-von Neumann Nash (BNN) dynamic.

Our goal is to investigate the generic possibility of complicated dynamics (i.e. stable limit cycles, multiple steady states) in simple, well-known, three strategies games such as Rock-Paper-Scissors and Coordination games under a smoothed version of the Best Response

[1] Zeeman (1980), pp. 493.

[2] The Shapley attractor, under fictitious play and (perturbed) best-response dynamics, is also discussed in Gaunersdorfer and Hofbauer (1995) and Benaim et al. (2009).

dynamics, the Logit Dynamics. As already discussed above, Replicator Dynamics does not give rise to a stable limit cycle but to heteroclinic cycles in 3×3 circulant RSP games (Zeeman (1980)). We obtain an alternative proof of this classical result based on the computation of the first Lyapunov coefficient in the normal form of the vector field induced by the Replicator Dynamics, and show that all Hopf bifurcations are degenerate. Via the same technique, *generic* Hopf bifurcations are shown to occur under the Logit Dynamics for 3×3 circulant RSP games and, moreover, all these bifurcations are supercritical, i.e. the dynamical system exhibits *stable* limit cycles. In addition to analytical results on bifurcations, we use the advanced bifurcation software Matcont (Dhooge et al. (2003)) to provide a "computer-assisted proof" of the existence of bifurcation curves in the parameter space. Knowledge of these bifurcation curves provides key insight how complicated dynamics can arise in a 2-D parameter space, of the payoff and behavioral parameters. In particular, we show that in the pure 3×3 Coordination game, the transition to multiple equilibria may be *non-monotonic* as a single parameter is increased, e.g. a bifurcation route from 1-3-5-7-5-7 equilibria.

The note is organized as follows: Section 2 introduces the Logit Dynamics, while Section 3 gives a brief overview of the Hopf bifurcation theory. In Section 4 the Logit Dynamics is implemented on Rock-Scissors-Paper and in Section 5 on a 3×3 pure Coordination game. Concluding remarks are included in Section 6.

2 Logit Dynamics

The set of evolutionary dynamics roughly splits into two classes[3]: imitative dynamics and pairwise comparison (belief-based or *'competent'* play). The first class is represented by the famous Replicator Dynamic (Taylor and Jonker (1978)) that captures the basic Darwinian tenet that strategies that fare better than average spread in the population. Formally, given a normal form game matrix $A[n \times n]$, the fractions x_i of each strategy $E_i \in \{E_1, E_{2,}...E_n\}$ evolve in the $n-1$ dimensional simplex $\mathbf{\Delta}^{n-1} = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1, x_i \geq 0\}$ according to:

$$\dot{x}_i = x_i[f_i(\mathbf{x}) - \bar{f}(\mathbf{x})] = x_i[(A\mathbf{x})_i - \mathbf{xAx}] \tag{1}$$

[3]See Sandholm (2008) for the microfoundations of these two classes of *evolutionary dynamics* as derived from the aggregation of *individual players' choices*.

where $f_i(\mathbf{x})$ is the payoff to strategy E_i when state of the population is $\mathbf{x}$ and $\bar{f}(\mathbf{x}) = \mathbf{xAx}$ is the average payoff. From the class of belief-based dynamics we focus on a smooth approximation of the Best Reply dynamics, the Logit dynamics introduced by Fudenberg and Levine (1998) and parameterised by the *intensity of choice* β (Brock and Hommes (1997)):

$$\dot{x}_i = \frac{\exp[\beta A\mathbf{x})_i]}{\sum_k \exp[\beta A\mathbf{x})_k]} - x_i. \tag{2}$$

When $\beta \approx \infty$ the probability of switching to the discrete 'best response' E_j is one while for a very low intensity of choice ($\beta \approx 0$) the switching rate is (almost) independent of the actual performance of the alternative strategies (almost equal probability mass is put on each of them). The quantal response equilibria of McKelvey and Palfrey (1995), also called 'logit equilibria' are fixed points of the Logit Dynamics.

3 Hopf and degenerate Hopf bifurcations

For the convenience of the general reader we briefly review the main bifurcation route towards a stable limit cycle, the Hopf bifurcation. In a one-parameter family of continuous-time systems, the only generic bifurcation through which a limit cycle is created or disappears is the non-degenerate Hopf bifurcation. Assume we are given a parameter-dependent, two dimensional system (as in, for example, Kuznetsov (1995)):

$$\dot{\mathbf{x}} = f(\mathbf{x}, \alpha), \mathbf{x} \in \mathbb{R}^2, \alpha \in R, \; f \quad \text{smooth}, \tag{3}$$

with a steady state at $\mathbf{x}^* = 0$, i.e. $f(0, \alpha) = 0$ and the Jacobian matrix evaluated at the fixed point $\mathbf{x}^* = 0$ having a pair of purely imaginary, complex conjugate eigenvalues at the bifurcation value $\alpha = 0$, i.e. $\lambda_{1,2} = \mu(\alpha) \pm i\omega(\alpha)$ with $\mu(\alpha) < 0$ for $\alpha < 0, \mu(0) = 0$ and $\mu(\alpha) > 0$ for $\alpha > 0$.

If , in addition, the following *genericity*[4] conditions are satisfied:

[4] *Genericity* usually refer to transversality and non-degeneracy conditions. Rougly speaking, the transversality condition means that complex eigenvalues cross the real line at non-zero speed. The nondegeneracy condition implies non-zero higher-order coefficients in Eq. (5) below. It ensures that the singularity $\mathbf{x}^*$ is typical (i.e. 'nondegenerate') for a class of singularities satisfying certain bifurcation conditions. See Kuznetsov (1995) pp. $89 - 98$ for a complete mathematical description of the Hopf bifurcation.

(i) $\left[\frac{\partial\mu(\alpha)}{\partial\alpha}\right]_{\alpha=0} \neq 0$ - *transversality* condition

(ii) $l_1(0) \neq 0$, where $l_1(0)$ is the *first* Lyapunov coefficient[5] - *nondegeneracy* condition,

then the system (3) undergoes a Hopf bifurcation at $\alpha = 0$. As α increases the steady state changes stability from a stable focus into an unstable focus.

There are two types of Hopf bifurcation, depending on the sign of the first Lyapunov coefficient $l_1(0)$:

(a) If $l_1(0) < 0$ then the Hopf bifurcation is *supercritical*: the stable focus x becomes *unstable* for $\alpha > 0$ and is surrounded by an *isolated, stable* closed orbit (limit cycle).

(b)If $l_1(0) > 0$ then the Hopf bifurcation is *subcritical*: for $\alpha < 0$ the basin of attraction of the stable focus $\mathbf{x}^*$ is surrounded by an *unstable* cycle which shrinks and disappears as α crosses the critical value $\alpha = 0$ while the system diverges quickly from a neighbourhood of $\mathbf{x}^*$. In case (a) the stable cycle is created immediately *after* α reaches the critical value and thus the Hopf bifurcation is called *super*critical, while in case (b) the unstable cycle already exists *before* the critical value, i.e. a *sub*critical Hopf bifurcation. The supercritical Hopf is also known as *a soft* or *non-catastrophic* bifurcation because , even when the system becomes unstable, it still lingers within a small neighbourhood of the equilibrium bounded by the limit cycle, while the subcritical case is a *sharp/catastrophic* bifurcation as the system moves far away from the unstable equilibrium. If the first Lyapunov coefficient $l_1(0) = 0$ then there is a degeneracy in the third order terms of the normal form and we have a degenerate Hopf bifurcation which may display richer behavior: e.g. a continuum of cycles, a limit cycle *bifurcating* into two or more cycles, or the *coexistence* of stable and unstable cycles.

For the planar case, the first Lyapunov coefficient $l_1(0)$ can be computed *without* explicitly deriving the normal form, from the Taylor coefficients of a transformed version of the original vector field. The computation of $l_1(0)$ for higher dimensional systems makes use of the Center Manifold Theorem by which the orbit structure of the original system near $(\mathbf{x}^*, \alpha)$, is fully determined by its restriction to the two-dimensional center manifold[6]. On the center manifold

[5]This is the coefficient of the third order term in the normal form of the Hopf bifurcation (Eq. (5) below).
[6]The center manifold is the invariant manifold spanned by the eigenvectors corresponding to the eigenvalues with zero real part (Kuznetsov (1995) pp. 157).

(3) takes the form (Wiggins (2003)):

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \operatorname{Re}\lambda(\alpha) & -\operatorname{Im}\lambda(\alpha) \\ \operatorname{Im}\lambda(\alpha) & \operatorname{Re}\lambda(\alpha) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f^1(x,y,\alpha) \\ f^2(x,y,\alpha) \end{pmatrix} \tag{4}$$

where $\lambda(\alpha)$ is an eigenvalue of the linearized vector field around the steady state and the nonlinear functions $f^1(x,y,\alpha), f^2(x,y,\alpha)$ of order $O(|x|^2)$ are derived from the original vector field. At the Hopf bifurcation point α, $\lambda_{1,2} = \pm i\omega$ and the first Lyapunov coefficient is (Wiggins (2003)):

$$l_1(\alpha)=\frac{1}{16}[f^1_{xxx}+f^1_{xyy}+f^2_{xxy}+f^2_{yyy}]+\frac{1}{16\omega}[f^1_{xy}(f^1_{xx}+f^1_{yy}) - f^2_{xy}(f^2_{xx}+f^2_{yy}) - f^1_{xx}f^2_{xx}+f^1_{yy}f^2_{yy}] \tag{5}$$

4 Rock-Scissors-Paper Games

The Rock-Paper-Scissors class of games (or games of cyclical dominance) formalize strategic interactions where each strategy E_i is an unique best response to strategy E_{i+1} for $i = 1,2$ and E_3 is a best response to E_1:

$$A = \begin{pmatrix} & E_1 & E_2 & E_3 \\ E_1 & 0 & \delta & -\varepsilon \\ E_2 & -\varepsilon & 0 & \delta \\ E_3 & \delta & -\varepsilon & 0 \end{pmatrix} ; \delta, \varepsilon \geq 0 \tag{6}$$

4.1 Circulant RSP Game and Replicator Dynamics

Letting $\mathbf{x(t)} = (x(t), y(t), z(t))$ denote the population state at time instance t define a point from the 2-dimensional simplex, the replicator equation (1) with the game matrix (6) induce on the 2-simplex the vector field:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} x[y\delta - z\varepsilon - (x\,(y\delta - z\varepsilon) + y\,(-x\varepsilon + z\delta) + z\,(x\delta - y\varepsilon))] \\ y[-x\varepsilon + z\delta - (x\,(y\delta - z\varepsilon) + y\,(-x\varepsilon + z\delta) + z\,(x\delta - y\varepsilon))] \\ z[x\delta - y\varepsilon - (x\,(y\delta - z\varepsilon) + y\,(-x\varepsilon + z\delta) + z\,(x\delta - y\varepsilon))] \end{bmatrix} \tag{7}$$

Hofbauer and Sigmund (2003) use the Poincare-Bendixson theorem together with the Dulac criterion to prove that limit cycles cannot occur in games with three strategies under the replicator dynamics. As an illustration of the Hopf bifurcation method, we present the following alternative proof of Zeeman (1980) non-genericity result:

Proposition 1 *All Hopf bifurcations are degenerate in the circulant Rock-Scissor-Paper game under Replicator Dynamics.*

Proof. Substituting $z = 1-x-y$ into (7) yields a 2-dim system on the simplex of the form (4) with $f^1(x,y) = -x\delta + 2xy\delta + x^2\delta$ and $f^2(x,y) = y\delta - 2xy\delta - y$. For these quadratic f^1 and f^2 it follows immediately from (5) that the first Lyapunov coefficient is $l_1(\varepsilon^{Hopf} = \delta^{Hopf}) = 0$ implying a first degeneracy in the third order terms from the Taylor expansion of the normal form. The detected bifurcation is a *degenerate Hopf* bifurcation[7]. ■

Although, in general, the orbital structure at a degenerate Hopf bifurcation may be extremely complicated, for our particular vector field induced by the Replicator Dynamics it can be shown by Lyapunov function techniques (Hofbauer and Sigmund (2003), Zeeman (1980)) that a continuum of cycles is born *exactly* at the critical parameter value.

4.2 Circulant RSP Game and Logit Dynamics

The Logit evolutionary dynamics (2) applied to the circulant normal form game (6) leads to:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \frac{\exp(\beta(y\delta - z\varepsilon))}{\exp(\beta(y\delta-z\varepsilon))+\exp(\beta(-x\varepsilon+z\delta))+\exp(\beta(x\delta-y\varepsilon))} - x \\ \frac{\exp(\beta(-x\varepsilon+z\delta))}{\exp(\beta(y\delta-z\varepsilon))+\exp(\beta(-x\varepsilon+z\delta))+\exp(\beta(x\delta-y\varepsilon))} - y \\ \frac{\exp(\beta(x\delta-y\varepsilon))}{\exp(\beta(y\delta-z\varepsilon))+\exp(\beta(-x\varepsilon+z\delta))+\exp(\beta(x\delta-y\varepsilon))} - z \end{bmatrix} \tag{8}$$

By substituting $z = 1 - x - y$ into (8) we can reduce to a 2-D system

$$\begin{aligned} \frac{\exp(\beta(y\delta-\varepsilon(-x-y+1)))}{\exp(\beta(x\delta-y\varepsilon))+\exp(\beta(-x\varepsilon+\delta(-x-y+1)))+\exp(\beta(y\delta-\varepsilon(-x-y+1)))} - x = 0 \\ \frac{\exp(\beta(-x\varepsilon+\delta(-x-y+1)))}{\exp(\beta(x\delta-y\varepsilon))+\exp(\beta(-x\varepsilon+\delta(-x-y+1)))+\exp(\beta(y\delta-\varepsilon(-x-y+1)))} - y = 0 \end{aligned} \tag{9}$$

[7] Since all 3^{rd} and higher-order terms are zero, the Hopf bifurcation has, in fact, an "infinite number of degeneracies" with all higher order Lyapunov coefficients $l_i(\varepsilon^{Hopf} = \delta^{Hopf}) = 0, i \geq 2$. This explains why, for the Replicator Dynamics, a continuum of cycles exists after the Hopf bifurcation.

We are now able to state the main result:

Proposition 2 *The Logit Dynamics (8) on the circulant, bad (i.e. $\delta < \varepsilon$) Rock-Scissors-Paper game exhibits a generic Hopf bifurcation and, therefore, has limit cycle. Moreover, all Hopf bifurcations are supercritical, i.e. the limit cycle is born stable. The critical values for which the Hopf bifurcation arises are given by $\beta^{Hopf} = \frac{6}{\varepsilon-\delta}$ for $0 < \delta < \varepsilon$.*

Proof. The $2-$dim simplex barycentrum $[x = 1/3, y = 1/3, z = 1/3]$ remains a solution of (9) irrespective of the value of β. The Jacobian of (9) evaluated at this steady state is: $\begin{bmatrix} \frac{1}{3}\beta\varepsilon - 1 & \frac{1}{3}\beta\delta + \frac{1}{3}\beta\varepsilon \\ -\frac{1}{3}\beta\delta - \frac{1}{3}\beta\varepsilon & -\frac{1}{3}\beta\delta - 1 \end{bmatrix}$, with eigenvalues: $\lambda_{1,2} = \frac{1}{6}\beta(\varepsilon - \delta) - 1 \pm i\frac{1}{2}\sqrt{\frac{1}{3}}(\beta\delta + \beta\varepsilon)$.The Hopf bifurcation (necessary) condition $\mathrm{Re}(\lambda_{1,2}) = 0$ leads to:

$$\beta^{Hopf} = \frac{6}{\varepsilon - \delta}.(0 < \delta < \varepsilon) \tag{10}$$

Notice that for the zero-sum RSP game ($\varepsilon = \delta$) - unlike Replicator Dynamics which exhibited a degenerate Hopf at $\varepsilon = \delta$ - the barycentrum is always asymptotically stable ($\mathrm{Re}\,\lambda_{1,2} = -1$) under Logit Dynamics. For $\varepsilon < \delta$, i.e. the "good" RSP game, the interior steady state is always locally stable under Logit Dynamics. Condition (10) gives the necessary first-order condition for Hopf bifurcation to occur; in order to show that the Hopf bifurcation is non-degenerate we have to compute the first Lyapunov coefficient $l_1(\beta^{Hopf}, \varepsilon, \delta)$ according to (5) and check whether it is non-zero. For this, we first use equations (4) to obtain the nonlinear functions:

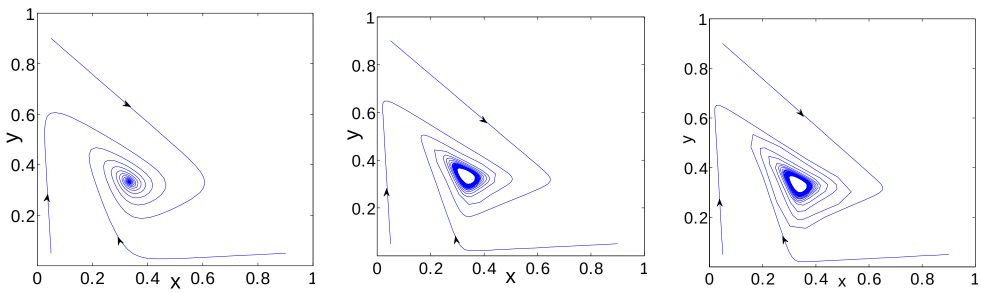
$$\begin{aligned} f_1(x,y) &= y\sqrt{3}\frac{\varepsilon+\delta}{\varepsilon-\delta} - x + \frac{\exp\left(\frac{6(y\delta-\varepsilon(-x-y+1))}{-\delta+\varepsilon}\right)}{\exp\left(\frac{6(x\delta-y\varepsilon)}{-\delta+\varepsilon}\right) + \exp\left(\frac{6(-x\varepsilon+\delta(-x-y+1))}{-\delta+\varepsilon}\right) + \exp\left(\frac{6(y\delta-\varepsilon(-x-y+1))}{-\delta+\varepsilon}\right)} \\ f_2(x,y) &= -x\sqrt{3}\frac{\varepsilon+\delta}{\varepsilon-\delta} - y + \frac{\exp\left(\frac{6(-x\varepsilon+\delta(-x-y+1))}{-\delta+\varepsilon}\right)}{\exp\left(\frac{6(x\delta-y\varepsilon)}{-\delta+\varepsilon}\right) + \exp\left(\frac{6(-x\varepsilon+\delta(-x-y+1))}{-\delta+\varepsilon}\right) + \exp\left(\frac{6(y\delta-\varepsilon(-x-y+1))}{-\delta+\varepsilon}\right)} \end{aligned}$$

Next, using Eq. (10) and first Lyapunov coefficient formula (5) we obtain:

$$l_1(\beta^{Hopf}, \varepsilon, \delta) = \left[\frac{1728\delta\varepsilon - 4320\delta^2 - 4320\varepsilon^2 - 4320\delta\varepsilon + 1728\delta^2 + 1728\varepsilon^2}{19\delta^2 - 38\delta\varepsilon + 19\varepsilon^2 - 16\delta\varepsilon + 8\delta^2 + 8\varepsilon^2}\right]$$
$$= \frac{-2592\delta\varepsilon - 2592\delta^2 - 2592\varepsilon^2}{27\delta^2 - 54\delta\varepsilon + 27\varepsilon^2} = -\frac{96(\delta\varepsilon + \delta^2 + \varepsilon^2)}{(\varepsilon - \delta)^2} < 0, \text{ for } \varepsilon > \delta > 0.$$

■

Computer simulations of this route to a stable cycle are shown in Fig. 1. As β increases from 10 to 35 (i.e. the noise level is decreasing) the interior stable steady state looses stability via a supercritical Hopf bifurcation and a small, stable limit cycle emerges around the unstable steady state[8]. Unlike Replicator Dynamics, *stable* cyclic behavior does occur under the Logit dynamics even for three-strategy games.



(a) Stable focus, $\beta = 10$. (b) Generic Hopf, $\beta = 30$. (c) Limit cycle, $\beta = 35$.

Figure 1: Rock-Scissors-Paper and Logit Dynamics for fixed game $\varepsilon = 1, \delta = 0.8$ and different values of the behavioral parameter β. Qualitative changes in the phase portraits: a stable interior fixed point (Panel (a)) loses stability at $\beta = 30$ i, via a supercritical Hopf bifurcation (Panel (b)); if β is pushed up even further, a stable limit cycle is born (Panel (c)).

Figure 2 depicts the *curve* of Hopf bifurcations as defined by the first order condition (10) in the $(\beta, \varepsilon - \delta)$ parameter space. As we cross this Hopf curve from below the stable interior fixed point loses stability and a stable periodic attractor surrounds it. The picture summarizes the possible types of dynamical behavior for the "good" and the "bad" Rock-Paper-Scissors game. For $\varepsilon < \delta$ (i.e. "good" RSP game) the interior, fully mixed steady state is always locally[9] stable under Logit Dynamics, similar to the behavior of Replicator Dynamics on this

[8]Similar limit cycles can be detected in the payoff parameter space if the intensity of choice is kept constant and the game parameter δ is allowed to change (Ochea (2010)).
[9]Numerical simulations suggest that for $\varepsilon < \delta$ the steady state is even globally stable.

class of RSP games. For $\varepsilon > \delta$ (i.e. "bad" RSP game) the behavior depends on how sensitive players are to differences in fitness, and, unlike Replicator Dynamics, Logit displays richer dynamics: when the intensity of choice increases beyond a critical threshold a stable limit cycle arises after a Hopf bifurcation.

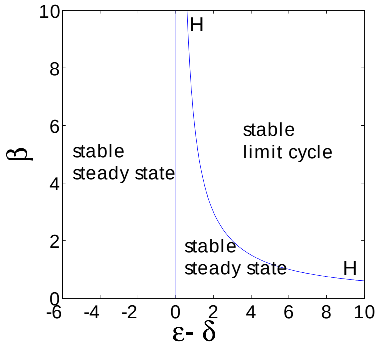


Figure 2: Rock-Scissors-Paper and Logit Dynamics: Supercritical Hopf curve in $(\beta, \varepsilon - \delta)$ parameter space, analytically computed.

5 Coordination Game

Using topological arguments, Zeeman (1980) shows that three-strategies games have at most one interior, isolated fixed point under Replicator Dynamics[10]. This implies that a fold[11] bifurcation in which two *isolated* fixed points collide and disappear when some parameter is varied, cannot occur *in the interior* of the simplex. In this section we show - by means of the classical coordination game - that multiple, isolated, interior steady-states may exist under Logit Dynamics and show that the fold catastrophe occurs when we alter the intensity of choice β. We use advanced numerical tools (Dhooge et al. (2003)) for detecting all the fold catastrophe bifurcation curves in the parameter space. Earlier simulations with the Mathematica package Dynamo[12] suggest the occurrence of multiple, interior logit equilibria in the pure 3×3 coordination. What we provide here is a "computer-assisted" proof for the existence of fold bifurcations and unveil the exact sequence of fold bifurcations through which such multiplicity arises. Knowledge of these bifurcation curves leads to a novel finding that,

[10] See Theorem 3 pp. 478 in Zeeman (1980).

[11] In a continuous-time dynamical system a fold bifurcation occurs when the Jacobian matrix evaluated at the critical equilibrium has a zero eigenvalue. Technically, other higher-order non-degeneracy conditions must hold, as well. See Kuznetsov (1995) pp. $81 - 84$ for a complete treatment of the fold bifurcation.

[12] See the Logit movie at http://www.ssc.wisc.edu/~whs/dynamo/gallery/logitmovie.html for a fixed, pure Coordination game with Logit Dynamics with $\beta \in [0.1, 1000]$.

depending on the payoff parameter ε, the transition from 1 to 7 steady states for increasing values of the intensity of choice may be *non-monotonic*. We consider the simplest version of a symmetric 3×3[13] pure coordination game, given by the following payoff matrix:

$$A = \begin{pmatrix} 1-\varepsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+\varepsilon \end{pmatrix}, 0 < \varepsilon < 1. \tag{11}$$

Logit Dynamics for the payoff matrix A of the Coordination game (11) generates the following vector field on the simplex of frequencies (x, y, z) of strategies E_1, E_2, E_3, respectively:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \frac{\exp((1-\varepsilon)\beta x)}{\exp((1-\varepsilon)\beta x)+\exp(\beta y)+\exp((1+\varepsilon)\beta z)} - x \\ \frac{\exp(\beta y)}{\exp((1-\varepsilon)\beta x)+\exp(\beta y)+\exp((1+\varepsilon)\beta z)} - y \\ \frac{\exp((1+\varepsilon)\beta z)}{\exp((1-\varepsilon)\beta x)+\exp(\beta y)+\exp((1+\varepsilon)\beta z)} - z \end{bmatrix} \tag{12}$$

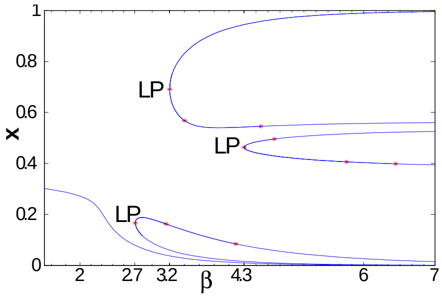
5.1 Bifurcations

We choose first a relatively large payoff perturbation $\varepsilon = 0.1$. Unlike Replicator, the Logit Dynamics displays multiple, interior isolated steady states created via a fold bifurcation. In a 3-strategy pure Coordination game, three interior stable steady states emerge through a sequence of two saddle-node bifurcations, as illustrated in Fig. 3. For small values of β the unique, interior stable steady state is close to the simplex barycentrum $(1/3, 1/3, 1/3)$. As β increases this steady state travels in the direction of the Pareto-superior equilibrium $(0, 0, 1)$. A first fold bifurcation occurs at $\beta = 2.77$ (see Fig. 3a) and two new fixed points are created, one stable and one unstable. If we increase β even further ($\beta \approx 3.26$) a second fold bifurcation takes place and two additional equilibria emerge, one stable and one unstable. Finally, two new fixed points arise at $\beta = 4.31$ via a saddle-source bifurcation[14]. Three stable steady states co-exist for large values of the intensity of choice β. Note that the three stable steady states coincide with the 'logit equilibria' of McKelvey and Palfrey (1995) that converge to

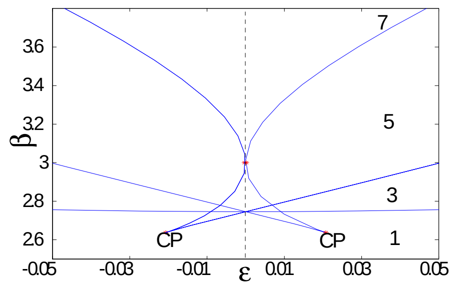
[13] The 2×2 pure coordination game also exhibits multiple, interior *logit* equilibria, but we restrict to the 3×3 case in order to contrast our Logit Dynamics results with Zeeman (1980) Replicator Dynamics analysis, on the same class of games.

[14] A *saddle-source* bifurcation is a fold bifurcation where two *unstable* steady states are created: one saddle fixed point (i.e. at least one eigenvalue with positive real part) and one source (i.e. both eigenvalues have positive real part).

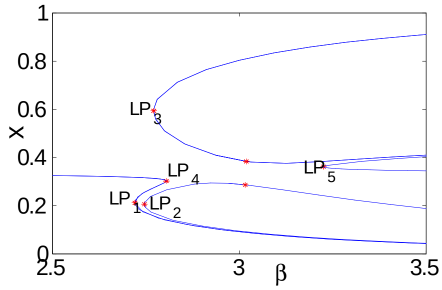
the pure strategy Nash equilibria when $\beta \to \infty$.



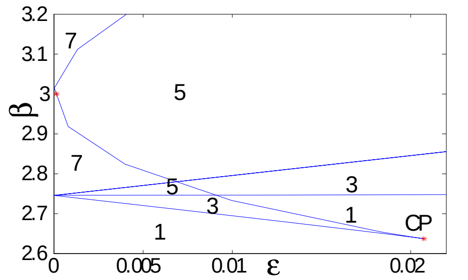
(a) Cascade of Fold bifurcations in (x,β) space for ε=0.1



(b) Cusp points (CP) organize the equilibria structure



(c) Cascade of Fold bifurcations in (x, β) space for $\varepsilon = 0.005$.



(d) Number of logit equilibria in the (β, ε) space for small ε.

Figure 3: Coordination Game and Logit Dynamics. Panel (a): curves of equilibria along with codimension I singularities, fold catastrophe (LP) points. Panel (b): curves of fold bifurcations along with detected codimension II singularities -cusp (CP) points - traced in the (ε, β) parameter space. Curves of equilibria (c) and of fold bifurcations (d) for small $\varepsilon = 0.005$.

Fig. 3b plots curves of fold bifurcations in the (β, ε) parameter space. The points labelled CP are codimension II[15] bifurcations and are important, because they act as "organizing centers" of the complete bifurcation diagram with co-dimension I bifurcation curves. The cusp points (CP) in Fig. 3b are endpoints of the saddle-node bifurcation curves along which multiple, interior steady states are created in the Coordination Game under the smoothed best response dynamics. When a fold bifurcation curve is crossed from below two additional equilibria are created: one stable and one unstable after a saddle-node bifurcation and two unstable steady states after a saddle-source bifurcation. If choice is virtually random ($\beta \approx 0$) there is an unique steady state while if β increases, the number of steady states increases

[15] The codimension of a bifurcation defines the number of parameters that needs to be varied in order for the bifurcation to occur generically (Kuznetsov (1995)).

from 1 to 3, then from 3 to 5 and, finally, from 5 to 7. For small values of ε the bifurcation scenario is more complicated. For $\varepsilon = 0.005$, Panel (c) in Fig. 3 shows a sequence of 5 saddle-node(source) $\beta-$bifurcations, while Panel (d) depicts a blow-up of the relevant, small ε region in the (ε, β) space of Fig. 3b . Notice that, as β increases, the number of steady states changes *non-monotonically* from $1 - 3 - 5 - 7 - 5 - 7$. These results illustrate the importance of the numerical detection of co-dimension I bifurcation curves together with the co-dimension II cusp bifurcation points in 2-D parameter space to fully understand how the transition to multiple steady states jointly depends on payoff and behavioral parameters.

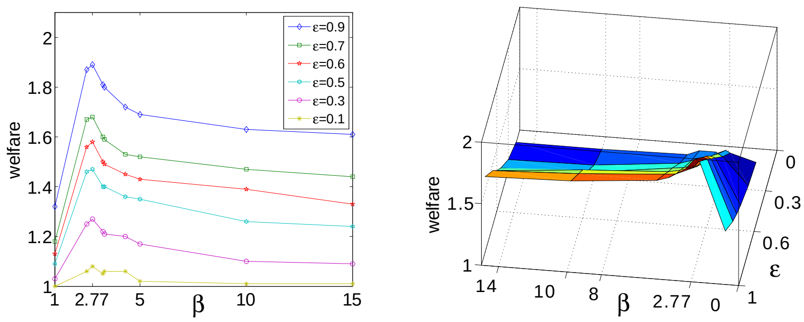
5.2 Welfare Analysis

The numerical computation of the basins of attraction for different equilibria reveals interesting properties of the Logit dynamics from a social welfare perspective. We construct a measure of long-run aggregate welfare as the payoff at the stable steady state weighted by the size of the corresponding basin of attraction. Whereas for $\beta \to \infty$ the basins of attraction are similar in size as in the Replicator Dynamics, for moderate levels of rationality the population manages to coordinate close to the Pareto optimal Nash equilibria[16].

Fig. 4 illustrates how the long-run average welfare depends on the parameter β for different levels of the payoff difference ε. Long run average welfare increases with the payoff difference ε but evolves *non-monotonically* with respect to the behavioural parameter β (Fig. 4ab). Long-run average welfare increases as the fully mixed equilibrium moves towards the Pareto optimal $(0, 0, 1)$ vertex, attains a maximum just before the first fold bifurcation occurs at $\beta^{LP_1} = 2.77$ and then decreases. As our measure of average welfare is constructed as payoffs at steady state weighted by the corresponding sizes of basins of attraction, there are two effects driving the welfare peak *before* β^{LP_1} is hit. First, the steady state payoff is higher the closer the steady state is to the Pareto optimal equilibrium. Second, there is a 'basin of attraction' effect: before the first fold bifurcation threshold $\beta = \beta^{LP_1}$ is reached the entire simplex is attracted by the unique steady state lying close to the optimal equilibrium. Intuitively, the noisy choices in the low-beta regime help players escape the path-dependency built into the

[16]see Ochea (2010) for details on the computation of the basins of attraction areas for both Replicator and Logit Dynamics and varying payoff and intensity of choice parameters.

game and coordinate close to the Pareto-optimal equilibrium. In the limiting case $\beta \to \infty$, the stable fixed points of the Logit Dynamics (i.e. the logit equilibria) coincide with the pure strategy Nash equilibria of the underlying game which, for this Coordination Game, are exactly the stable nodes of the Replicator Dynamics. Thus the analysis (stable fixed points, basins' of attraction sizes) of the 'unbounded' rationality case is identical to the one pertaining to the Replicator Dynamics in Coordination game.



(a) $W - \beta$ plot for different ε (b) surface plot (W, β, ε)

Figure 4: Coordination Game and Logit Dynamics-Long-run average welfare plots as function of payoff perturbation parameter ε and the intensity of choice β. As ε increases the long-run average welfare increases. The long-run average welfare is non-monotonic as a function of the intensity of choice β, with the maximum arising just before the first fold bifurcation.

6 Conclusions

The first goal of this note was to prove that, even for 'simple' three-strategy games, periodic *attractors* do occur under a rationalistic way of modelling evolution in games, the Logit dynamics. Identifying stable cyclic behaviour translates into proving that *generic*, non-degenerate Hopf bifurcations arise in these Logit evolutionary systems. By means of normal form computations, we demonstrated that a non-degenerate Hopf can *not* occur for Replicator Dynamics, when the number of strategies is three for games like Rock-Scissors-Paper. However, in Logit dynamics, even for the three strategy case, stable cycles are created, via a generic, non-degenerate, *supercritical* Hopf bifurcation. Secondly, multiple, interior steady states exist in a 3×3 Coordination game, under the Logit Dynamics and this multiplicity of equilibria is created through a sequence of fold bifurcations. Using the numerically detected bifurcation curves we showed that, as the sensitivity to payoff asymmetry β increases, the transition from 1 to 7 logit equilibria may be *non-monotonic*. Finally, a measure of aggregate welfare reaches a maximum for intermediate $\beta-$values, just before the first fold bifurcation occurs, when most of the population coordinates close to the Pareto-superior equilibrium. An interesting topic for future research would be to run laboratory experiments with human subjects to find out which class of evolutionary selection dynamics - either an imitation-based Replicator or the more involved, belief-based Logit process- is more relevant for players' actual learning behavior.

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