Bounded rationality and heterogeneous expectations in macroeconomics

Massaro, D.

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This thesis studies the effect of individual bounded rationality on aggregate macroeconomic dynamics. Boundedly rational agents are specified as using simple heuristics in their decision making. An important aspect of the type of bounded rationality described in this thesis is that the population of agents is heterogeneous, that is, actors can choose from different rules to solve the same economic problem. The set of rules is disciplined by an evolutionary selection mechanism where the best performing rule, measured according to some fitness metric, attracts the higher number of agents. An important role in triggering switching between rules is played by the dynamic feedback between individual expectations of macroeconomic variables and their aggregate realizations. The macroeconomic models with heuristics switching developed in the thesis are used to evaluate standard policy advice and to explain aggregate time series data as well as experimental data on individual expectations and aggregate macro behavior.

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Bounded Rationality and Heterogeneous Expectations in Macroeconomics

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Domenico Massaro
Amsterdam, January 2012
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Chapter 1

Introduction

Individual expectations about future aggregate outcomes are the key feature that distinguishes social sciences and economics from the natural sciences. In every segment of macroeconomics expectations play a central role. Modern macroeconomic theory builds on a dynamic, stochastic, general equilibrium (DSGE) framework based on optimizing behavior. Structural relations derived within this framework and used for policy analysis are obtained from first-order conditions (Euler equations) that characterize optimal private-sector behavior. These conditions explicitly involve private-sector expectations about the future evolution of endogenous variables. Individual economic decisions today thus depend upon expectations about the future state of the global economy. Through these decisions, expectations feed back into the actual realizations of economic variables. Markets are therefore expectations feedback systems and any dynamic economic model depends crucially on its underlying expectations hypothesis.

Since the seminal works of Muth (1961), Lucas (1972) and Sargent (1973), rational expectations (RE) have become the leading paradigm on modeling expectations in economics. The idea of rational expectations has two components: first, that the behavior of each individual can be described as the outcome of maximizing an objective function subject to perceived constraints; and second, that the constraints perceived by everybody in the economy are mutually consistent. In such a framework, all agents are the same, expectations are model consistent and coincide on average with realizations, without systematic forecasting errors. Early arguments in
favor of the representative rational agent framework have been provided by leading scholars and date back to Alchian (1950) and Friedman (1953). The general underlying idea is that natural selection driven by realized profits will eventually eliminate non-rational behavior and lead to a market outcome dominated by rational profit maximizing agents. Therefore the economy can be described “as if” all agents were perfectly rational.

The RE approach has the important advantage of imposing strong discipline on individual forecasting behavior, minimizing the number of free parameters to explain data. However, RE models rest on the unrealistic assumption of perfect knowledge of the economy. Sargent (1993), for example, notes that “when implemented numerically or econometrically, rational expectations models impute much more knowledge to the agents within the model (who use the equilibrium probability distributions in evaluating their Euler equations) than is possessed by an econometrician, who faces estimation and inference problems that the agents in the model have somehow solved”. Sargent’s argument underscores that, not only the agents have to be endowed with a substantial amount of information in order to form RE, but even if perfect knowledge of the market were available, RE requires extremely strong computing abilities of the agents to solve the model, i.e., to make decisions such that all predictions and beliefs are consistent with the outcome of all agents’ choices.

Several tests of RE models, conducted in the last thirty years, have shown that their predictions are often at odds with empirical observations. Much evidence has been collected against the practice of describing human behavior as rational. Conlisk (1996) classifies this evidence as either “direct”, concerning rationality tests on individuals cognitive abilities relevant to economic decisions, or “confounded”, concerning tests in which the rationality hypothesis is tested jointly with other hypotheses in economic settings. Direct evidence against rationality consists, for example, in showing that individual responses to simple economic decisions typically present systematic errors and psychological biases (see, e.g., Tversky and Kahneman (1974), Grether and Plott (1979), Tversky and Thaler (1990), and Kahneman, Knetsch, and Tversky (1991)). Indirect evidence against rationality has been collected from empirical tests of the pre-
dictions of economic models built under the assumptions of RE. Few examples include house-
holds consumption data, which are often at odds with standard life cycle theory (see, e.g., Thaler
(1990), Flavin (1993), Carroll (1994), and Shea (1995)), survey data on expectations of inflation
and other variables, which commonly reject the unbiasedness and efficiency predictions of RE
(see, e.g., Frankel and Froot (1987), De Bondt and Thaler (1990), Ito (1990), and Capistran and
Timmermann (2009)), and the anomalous behavior of asset prices (see, e.g., Mehra and Prescott
(1985), Fama and French (1988), Lee, Shleifer, and Thaler (1991), and Schiller (2000)).

Bounded Rationality

In recent years, a significant part of economics witnessed a paradigm shift to an alternative,
behavioral view, where agents are boundedly rational (see, e.g., Conlisk (1996) for a survey).
Generally speaking, a boundedly rational agent is modeled as being able to choose what he
perceives as the best alternative in a decision making process, but he does not know the exact
structure of the economic environment. In standard optimizing theory, agents act as if they per-
form exhaustive searches over all possible decisions and then pick the best. However, Simon
(1955, 1957) emphasizes that individuals are limited in their knowledge and in their computing
abilities, and moreover that they face search costs to obtain sophisticated information in order to
pursue optimal decision rules. Simon argues that, because of these limitations, bounded ratio-
nality with agents using simple satisficing rules of thumb for their decisions under uncertainty,
is a more accurate and realistic description of human behavior than perfect rationality with fully
optimal decision rules. Kahneman and Tversky (1973) and Tversky and Kahneman (1974) pro-
vide evidence from psychology laboratory experiments that, in simple decision problems under
uncertainty, individual behavior can be described by simple heuristics which may lead to sig-
nificant biases. An interesting discussion on the use of simple heuristics as opposed to rational
behavior is contained in the Nobel Memorial Lectures in Simon (1979), while a more recent
overview on bounded rationality can be found in Kahneman (2003). When predicting future
variables, bounded rationality implies that individual agents do not know the true equilibrium
distribution of aggregate variables, therefore ex-ante predictions and ex-post outcomes need not to coincide on average. Hence, a boundedly rational agent is not able to solve for the equilibrium of the expectational feedback system. In contrast, he uses simple heuristics and keeps on updating his strategies as he learns about the economic environment through feedbacks about his past decisions. Researchers adopting RE models often argue that rationality is a useful assumption to describe the equilibrium outcome of this trial-and-error processes. According to this view, the repeated interaction of boundedly rational agents whose beliefs co-evolve with the economic environment in a dynamic feedback system, leads to the same outcome “as if” agents were perfectly rational. Convergence to rational behavior has been the topic of investigation of many theoretical papers on bounded rationality. In macroeconomics, much work has been done on adaptive learning, see e.g. Sargent (1993) and Evans and Honkapohja (2001) for detailed overviews. Boundedly rational agents do not know the true law of motion of the economy, but instead use time series observations to form expectations based on their own perceived law of motion, trying to learn the model parameters as new observations become available. Much of this literature focused on the stability of RE equilibria and addressed the possibility of agents learning to form rational expectations. In fact, adaptive learning may enforce convergence to RE equilibria but it may also lead to non-RE equilibria, such as the learning equilibria in Bullard (1994). As synthesized by Bullard (1996), “some rational expectations equilibria are learnable while others are not. Furthermore, convergence will in general depend on all the economic parameters of the system, including policy parameters”. Learnability of RE equilibria depends on the structure of the feedback system between individual expectations and the economic environment, as the signals received from the market might be deceptive to agents trying to acquire rational expectations through learning.

**Heterogeneous Expectations**

Although adaptive learning has become increasingly popular as an alternative paradigm to model private-sector expectations, most models still assume a representative agent who is learn-
ing about the economy. Kirman (1992, 2006) and Hommes (2006) summarize some of the arguments in support of heterogeneous expectations. One commonly referred to reason is the “no trade” argument, which states that in a world where all agents are rational and it is common knowledge that everyone is rational, there will be no trade. Several no trade theorems have been obtained in the literature (see, e.g., Milgrom and Stockey (1982) and Fudenberg and Tirole (1991)). However, no trade theorems are in sharp contrast with the high daily trading observed in real markets, and this reinforces the idea of heterogeneous expectations. Moreover, heterogeneity in individual expectations has been abundantly documented empirically. For example, Frankel and Froot (1987, 1990), Allen and Taylor (1990) and Taylor and Allen (1992) find that financial experts use different forecasting strategies to predict exchange rates. More recently, Carroll (2003), Mankiw, Reis, and Wolfers (2003), Branch (2004) and Pfajfar and Santoro (2010) provided supporting evidence for heterogeneous beliefs using survey data on inflation expectations, while Hommes, Sonnemans, Tuinstra, and van de Velden (2005), Adam (2007), Pfajfar and Zakelj (2010) and Hommes (2011) find evidence for heterogeneity in learning to forecast laboratory experiments.

In this thesis we depart from standard models by relaxing, as suggested by Sargent (1993), the second component of RE, i.e., mutual consistency of perceptions. In fact, in the models considered in the thesis, we fully maintain the assumption that individual choices are made optimally, given subjective expectations. To be more precise, we assume that agents only have knowledge of their objectives and of the constraints that they face, but they do not have a complete economic model of determination of aggregate variables. Individual decisions are then taken optimally on the basis of subjective expectations of future evolution of endogenous variables. In the light of the theoretical arguments and empirical evidence mentioned above, we assume that individual expectations are heterogeneous. Therefore different agents will, in general, take different decisions when facing the same economic problem as a function of their prediction rules. Prediction rules can differ in terms of sophistication, where the most sophisticated rule corresponds to the RE predictor. Starting from the idea of costly information pro-
cessing, shared with the literature on “rational inattention” (see e.g. Sims (2003)), we assume that the higher the sophistication of a rule, the higher the deliberation cost an agent pays to use it. Instead of considering the fractions of agents employing each rule as fixed and exogenously given, we let them evolve over time as a function of their \textit{fitness}. We thus employ an evolutionary approach where a Darwinian “survival of the fittest” mechanism is at work. A strategy that has performed better according to some measure, e.g. related to its forecasting error, is more likely to be adopted by a higher fraction of agents.\footnote{Empirical evidence that proportion of heterogeneous forecasters evolve over time as a reaction to forecast errors has been provided, among others, by Frankel and Froot (1991), Bloomfield and Hales (2002), Branch (2004), and Hommes (2011), using survey data as well as experimental data.} In order to model the updating process for fractions we use a discrete choice mechanism (see Manski and McFadden (1981)), introduced in the learning literature by Brock and Hommes (1997) to describe the endogenous selection among heterogeneous expectation rules.

\textbf{Thesis Outline}

This thesis presents applications of a bounded rationality and heterogeneous expectations framework to (New Keynesian) macroeconomic models of inflation and output dynamics. The results of the analysis are articulated in four complementary chapters. Chapters 2 and 3 focus on theoretical monetary policy considerations in the presence of heterogeneous beliefs. Chapter 2 presents a simple macro model with heterogeneous expectations while Chapter 3 provides a micro-foundation of heterogeneous expectations in a New Keynesian setting. Chapters 4 and 5 are devoted to the empirical validation of the heterogeneous expectations framework, using time series data (Chapter 4) as well as experimental data (Chapter 5). A working paper has been extracted from each chapter: Anufriev, Assenza, Hommes, and Massaro (2008) is based on Chapter 2, Massaro (2011) is based on Chapter 3, Assenza, Heemeijer, Hommes, and Massaro (2011) is based on Chapter 4, and Cornea, Hommes, and Massaro (2011) is based on Chapter 5.

Chapter 2 studies inflation dynamics under heterogeneous expectations and investigates the
robustness of monetary policies when agents use simple heuristics to predict future inflation, and they update their beliefs based on past forecasting performance. In this chapter we use a simple frictionless DSGE framework, which allows us to obtain analytical results about global dynamics in the model. The results of the analysis show how macroeconomic stability depends on the composition of the set of forecasting strategies and on the policy reaction coefficient of a Taylor-type interest rate rule. A result of particular interest is that the Taylor principle is no longer sufficient to guarantee uniqueness and global stability of the RE equilibrium, as multiple equilibria and non-rational beliefs may survive evolutionary competition.

Chapter 3 extends the analysis performed in Chapter 2 in two important respects. First, it derives a New-Keynesian framework consistent with heterogeneous expectations starting from the micro-foundations of the model. We then use this framework characterized by monopolistic competition, nominal rigidities and heterogeneous beliefs for the analysis of monetary policy and macroeconomic stability. Second, it introduces the sophisticated RE predictor in the set of forecasting strategies available to the agents in the economy. Due to the increased level of complexity, this chapter uses computational methods to derive policy implications for a monetary authority aiming at stabilizing the dynamic feedback system where macroeconomic variables and heterogeneous expectations co-evolve over time. A salient result is that policy attempts to achieve determinacy under RE may destabilize the economy even when only a small fraction of boundedly rational agents are present in the system.

Chapter 4 focuses on the empirical validation of the heterogeneous expectation framework. In this chapter we estimate a behavioral model of inflation dynamics with monopolistic competition, staggered price setting, and heterogeneous firms using U.S. time series data. In our stylized framework there are two types of price-setters, fundamentalists and naive. The estimation results show statistically significant behavioral heterogeneity and substantial time variation in the weights of different forecasting rules.

Chapter 5 investigates the individual expectation formation process in laboratory experiments with human subjects within a New Keynesian framework. Our data show that individuals
use simple heuristics to forecast aggregate macroeconomic variables within the experimental economies, and that individual learning takes the form of switching towards the best performing heuristics. We then use a simple model of individual learning with a performance-based evolutionary selection among forecasting strategies to explain experimental outcomes under different monetary policy regimes.

Each chapter is self-contained, with its own introduction, conclusion, notes and appendices as needed. For this reason, each chapter can be read independently from the others. A common bibliography is collected at the end of the thesis.
Chapter 2

A Simple Model of Inflation Dynamics under Heterogeneous Expectations

2.1 Introduction

The rational representative agent approach is still the core assumption in macro-economics. In contrast, in behavioral finance models with bounded rationality and heterogeneous expectations have been developed as a concrete alternative to the standard rational representative agent approach. These heterogeneous agents models mimic important observed stylized facts in asset returns, such as fat tails, clustered volatility and long memory, as discussed, e.g., in the extensive surveys of LeBaron (2006) and Hommes (2006). Although bounded rationality and adaptive learning have become increasingly important in macroeconomics, most models still assume a representative agent who is learning about the economy (see, e.g., Evans and Honkapohja (2001) and Sargent (1999) for extensive overviews) and thus ignore the possibility of heterogeneity in expectations and its consequences for monetary policy and macroeconomic stability. Some recent examples of macro models with heterogeneous expectations include Brock and de Fontnouvelle (2000), Evans and Honkapohja (2003, 2006), Branch and Evans (2006), Honkapohja and Mitra (2006), Berardi (2007), Tuinstra and Wagener (2007),

The importance of managing expectations for conducting monetary policy has been recognized and stressed, e.g., in Woodford (2003) (p. 15). However, the question how to manage expectations when forecasting rules are heterogeneous has hardly been addressed. The aim of our paper is to investigate whether the Central Bank can enhance macroeconomic stability, in the presence of heterogeneous expectations about future inflation, by implementing simple interest rate rules. In particular, we investigate how the ecology of potential forecasting rules affects the stabilizing properties of a simple Taylor rule. Moreover we study how, in a world where expectations are heterogeneous, the aggressiveness of the monetary authority in responding to fluctuations of the inflation rate affects these stabilizing properties. See also De Grauwe (2010) for a recent discussion on how heterogeneous expectations may affect monetary policy.

In order to study the potential (de-)stabilizing role of heterogeneous expectations we use a simple frictionless model of inflation. In our stylized model agents form expectations about the future rate of inflation using different forecasting rules. We employ the heterogeneous expectations framework of Brock and Hommes (1997), where the ecology of forecasting rules is disciplined by endogenous, evolutionary selection of strategies with agents switching between forecasting rules on the basis of their past performance.

Our paper relates to the literature on interest rate rules and price stability under learning dynamics. Howitt (1992) pointed out that interest rate rules that do not react aggressively to inflation are deceptive to people trying to acquire rational expectations through learning. Indeed, in a world in which any departure of expected inflation from its equilibrium level causes an
overreaction of actual inflation and generates a misleading signal for the agents, a forecasting rule that tries to learn from past mistakes will lead the economy away from equilibrium causing a cumulative process of accelerating inflation or deflation.\textsuperscript{1} Howitt (1992) shows that the cumulative process arises for any plausible backward looking learning rule\textsuperscript{2} in a homogeneous expectations setting. Moreover he shows that, by reacting more than point for point to inflation when setting the interest rate, the monetary authority can avoid the cumulative process. This monetary policy rule has become known as the “Taylor principle”, after Taylor (1993).

The present paper investigates the dynamical consequences of committing to an interest rate feedback rule in a world with endogenously evolving heterogeneous expectations. As we will see, the answer whether a Taylor rule can stabilize the cumulative process depends in interesting ways on the ecology of forecasting rules and on how aggressively the monetary authority adjusts the interest rate in response to inflation.

To illustrate the empirical relevance, we performed stochastic simulations of our model in

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{inflation_time_series.png}
\caption{Inflation time series. \textbf{Left:} US annualized inflation, quarterly data for a period 1960-2007. \textbf{Right:} Simulated data.}
\end{figure}

\textsuperscript{1}Friedman (1968) argued that interest rate pegging is not a sound monetary policy, even if the chosen interest rate is consistent with the Wicksellian natural rate of interest, and hence consistent with a rational expectation equilibrium. For even in that case, any small discrepancy between inflation expectations of the public and the ones required for the realization of the rational expectations equilibrium will drive expectations even farther from consistency with the rational expectations equilibrium. Howitt (1992) reformulates Friedman’s argument as a failure of convergence of learning dynamics to rational expectations equilibrium.

\textsuperscript{2}Adaptive learning in the sense of Evans and Honkapohja (2001) belongs to the class of learning rules considered by Howitt (1992).
order to reproduce some qualitative features of US inflation time series. In Fig. 2.1 we confront the simulated dynamics of a stochastic version of our model buffeted with shocks to economic fundamentals (right panel) with actual time series of US inflation (left panel). The model is simulated for 192 periods corresponding to quarters. An immediate observation is that the simulated inflation series is highly persistent. Hence, even in a frictionless DSGE model heterogeneous expectations rules may lead to highly persistent inflation (e.g. Milani (2007)). The monetary policy rule in the simulation exhibits a structural break in period 80, when the Central Bank changes the coefficient (measuring its aggressiveness in responding to actual inflation) of the interest rate rule. This break corresponds to the policy shift instituted by Fed chairman Paul Volcker in 1979. Before the structural break, in setting the interest rate the Central Bank responds relatively weakly to inflation. In our nonlinear model with heterogeneous expectations, when the Central Bank only responds weakly to inflation multiple steady states arise and, as a consequence, self-fulfilling expectations contribute to and reinforce a strong rise in inflation initially triggered by shocks to fundamentals, consistent with US data. In period $T = 80$, after the structural break, the Central Bank modifies the monetary policy rule to respond more aggressively, i.e., adapts the nominal interest rate more than point for point in response to inflation. Because of this policy change some of the high level steady states disappear and inflation stabilizes to low levels, consistent with US data.\(^3\) Our model, thus, explains the strong rise in US inflation between 1960 and 1980 as being triggered by shocks to economic fundamentals (such as the Oil shocks in 1973 and 1979), reinforced by evolutionary selection among heterogeneous forecasting rules under a too weakly responding Taylor rule in the pre-Volcker period, and the subsequent strong decline in US inflation data between 1980 and 2007 (the Great Moderation) enforced by a more aggressive interest rate rule.

The paper is organized as follows. Section 2.2 briefly recalls the ideas behind the cumula-

\(^3\)In our model under the Taylor rule inflation does not necessarily converge to the RE level, as different co-existing equilibria may persist. This result differs from the standard representative agent adaptive learning literature, where the interest rate rules that satisfy the Taylor principle lead to a unique, E-stable RE equilibrium, see, e.g., Bullard and Mitra (2002). Preston (2005) shows that, under adaptive learning, monetary policy rules obeying the Taylor principle lead to a unique, E-stable RE equilibrium when agents have heterogeneous but symmetric information sets.
tive process and recalls the micro-founded benchmark model. The model with heterogeneous expectations is introduced in Section 2.3, where we test the validity of the Taylor principle both in the case of a small number of constant forecasting rules (e.g., 3 or 5) and in the case of an arbitrarily large number of rules, applying the notion of Large Type Limit (Brock, Hommes, and Wagener (2005)). In Section 2.4 we discuss a calibration of our model to US inflation data. Finally, Section 2.5 concludes.

2.2 Interest Rate Rules and Cumulative Process

In this section we recall the instability problem implied by the Wicksellian cumulative process. We follow Benhabib, Schmitt-Grohé, and Uribe (2002), Woodford (2003), Cochrane (2005, 2010) and many others in describing a frictionless economy. Consumers maximize expected present discounted value of utility

$$\max \hat{E}_t \sum_{j=0}^{\infty} \delta^j u(C_{t+j}),$$

where $\hat{E}_t C_{t+j}$ indicates subjective expectations of private agents in period $t$ regarding consumption $C_{t+j}$ in period $t+j$, and $0 < \delta < 1$ is a discount factor. Consumers face a budget constraint given by

$$P_tC_t + B_t = (1 + i_{t-1})B_{t-1} + P_tY,$$

where $P_t$ is the price of the good, $B_t$ represents holdings of one-period bonds, $i_t$ is the nominal interest rate and $Y$ is a constant nonstorable endowment. We assume that the government issues no debt so that $B_t = 0$ and that public expenditure is equal to zero. The market clearing condition thus requires $C_t = Y$.

The first order condition for the optimization problem is given by the Euler equation

$$u_c(C_t) = \delta (1 + i_t) \hat{E}_t u_c(C_{t+1}) \left( \frac{P_t}{P_{t+1}} \right),$$
together with the budget constraint holding as an equality at each date.\(^4\) Using the consumers’ Euler equation and the market clearing condition we have that the interest rate follows a Fisher relation
\[
\frac{1}{1 + i_t} = \delta \hat{E}_t \left( \frac{P_t}{P_{t+1}} \right),
\]
which can be linearized to get
\[
i_t = r + \hat{E}_t \pi_{t+1},
\]
(2.1)
where \(\pi_{t+1}\) is the inflation rate and \(r\) is the constant real interest rate.\(^5\)

Assume that the monetary authority responds to the inflation rate according to the following Taylor rule:
\[
i_t = r + \phi_\pi \pi_t.
\]
(2.2)
We can solve the model by substituting out the nominal interest rate, in order to get the equilibrium condition
\[
\pi_t = \frac{1}{\phi_\pi} \hat{E}_t \pi_{t+1}.
\]
(2.3)
Following Howitt (1992) it is possible to show that interest rate rules with a reaction coefficient \(\phi_\pi < 1\) lead to a cumulative process when expectations are revised in an adaptive, boundedly rational way. To illustrate the failure of interest rules reacting less than point for point to inflation, let us assume that the economy is in the zero steady state and people expect a small amount of inflation. Equilibrium condition (2.3) implies that realized inflation will be even higher than expected when \(\phi_\pi < 1\). This means that the signal that agents receive from the market is misleading. Even though inflation was overestimated with respect to the equilibrium level, realized inflation suggests that agents underestimated it. Any reasonable rule that tries to learn from past mistakes will then lead agents to expect even higher inflation, causing a cumulative process of

\(^4\)As standard, we require that the agent’s subjective transversality condition, given by
\[
\lim_{j \to \infty} \hat{E}_t \delta^j u_c(C_{t+j}) \frac{B_{t+j}}{P_{t+j}} = 0,
\]
is satisfied ex-post. See Evans, Honkapohja, and Mitra (2003) for a discussion.

\(^5\)In equilibrium we have that the real interest rate is constant and given by \(r = \delta^{-1} - 1\).
accelerating inflation. Similarly, if people expect a deflation, an interest rate rule with a reaction coefficient $\phi_\pi < 1$ will lead to a cumulative process of accelerating deflation.

The actual inflation dynamics depends, of course, on the forecasting rule that agents use to form their expectations. As an illustrative example, consider the case of naive expectations, i.e., when agents expect that past inflation will persist in the future, $\hat{E}_t \pi_{t+1} = \pi_{t-1}$. Using (2.3) we can describe the dynamics under naive expectations by the linear equation

$$\pi_t = \frac{1}{\phi_\pi} \pi_{t-1},$$  
(2.4)

whose unique steady state corresponds to the RE equilibrium, $\pi^* = 0$. This steady state is, however, unstable, and thus any initial non-equilibrium level of inflation will lead to a cumulative process.

When the Central Bank implements a monetary policy rule that makes the nominal interest rate respond to the rate of inflation more than point for point, i.e., it obeys the “Taylor principle”, the cumulative process can be avoided. Assume that in the example above the Central Bank adopts a policy rule with a reaction coefficient $\phi_\pi > 1$. It is immediately clear that for such a Taylor rule the RE equilibrium is globally stable and the cumulative process will not arise.

Finally, notice that under RE the expected inflation coincides with the actual inflation so that dynamics can be described by:

$$\pi_t = \frac{1}{\phi_\pi} \pi_{t+1}.$$

It’s easy to see that when $\phi_\pi < 1$, the RE steady state $\pi^* = 0$ is indeterminate, i.e., it is approached by many RE paths, while when $\phi_\pi > 1$ the RE equilibrium is determinate.\(^6\)

\(^6\)This result is well known also for the full version of the New Keynesian model with frictions, see Woodford (2003).
2.3 Interest Rate Feedback Rules with Fundamentalists and Biased Beliefs

Will the cumulative process arise in an economy where agents have heterogeneous expectations about the future level of the inflation rate? Will an interest rate rule that obeys the Taylor principle succeed in stabilizing inflation? To address these questions we employ the framework of Adaptive Belief Systems proposed in Brock and Hommes (1997) to model heterogeneous expectations. Assume that agents can form expectations choosing from \( H \) different forecasting rules. We denote by \( \hat{E}_{h,t} \pi_{t+1} \) the forecast of inflation by rule \( h \). The fraction of agents using forecasting rule \( h \) at time \( t \) is denoted by \( n_{h,t} \). Assuming linear aggregation of individual expectations,\(^7\) actual inflation in equation (2.3) is given by

\[
\pi_t = \frac{1}{\phi_H} \sum_{h=1}^{H} n_{h,t} \hat{E}_{h,t} \pi_{t+1}.
\] (2.5)

The evolutionary part of the model describes the updating of beliefs over time. Fractions are updated according to an evolutionary fitness measure. The fitness measures of all strategies are publicly available, but subject to noise. Fitness is derived from a random utility model and given by

\[
\tilde{U}_{h,t} = U_{h,t} + \varepsilon_{h,i,t},
\]

where \( U_{h,t} \) is the deterministic part of the fitness measure and \( \varepsilon_{h,i,t} \) represent IID idiosyncratic noise at date \( t \), across types \( h = 1, \ldots, H \) and agents \( i \). Assuming that the noise \( \varepsilon_{h,i,t} \) is drawn from a double exponential distribution, in the limit as the number of agents goes to infinity, the probability that an agent chooses strategy \( h \) is given by the well known discrete choice fractions

\(^7\)Averaging of individual forecasts represents a first-order approximation to general nonlinear aggregation of heterogeneous expectations. Recent papers following the same approach include Adam (2007), Arifovic, Bullard, and Kostyshyna (2007), Brazier, Harrison, King, and Yates (2008), Branch and McGough (2010) and De Grauwe (2010). Branch and McGough (2009) developed a New Keynesian model with heterogeneous expectations. The model used in this paper can be considered as a frictionless version of their model.
(see Manski and McFadden (1981)):

$$n_{h,t} = \frac{e^{\beta U_{h,t-1}}}{\sum_{h=1}^{H} e^{\beta U_{h,t-1}}}.$$  \hspace{1cm} (2.6)

Note that the higher the fitness of a forecasting rule $h$, the higher the probability that an agent will select strategy $h$. The parameter $\beta$ is called the \textit{intensity of choice} and reflects the sensitivity of the mass of agents to selecting the optimal prediction strategy. The intensity of choice $\beta$ is inversely related to the variance of the noise term. The case $\beta = 0$ corresponds to the situation of infinite variance in which differences in fitness can not be observed, so agents do not switch between strategies and all fractions are constant and equal to $1/H$. The case $\beta = \infty$ corresponds to the situation without noise in which the deterministic part of the fitness can be observed perfectly and in every period all agents choose the best predictor. A natural performance measure is past squared forecast errors

$$U_{h,t-1} = - (\pi_{t-1} - \hat{\pi}_{h,t-2})^2 - C_h,$$ \hspace{1cm} (2.7)

where $C_h$ is the per period information gathering cost of predictor $h$.

Consider an environment in which agents can choose between different constant “steady state” predictors to forecast future inflation. This represents a situation in which agents roughly know the fundamental steady state of the economy, but they are boundedly rational and disagree about the correct value of the fundamental inflation rate. Forecasting the RE equilibrium value of inflation, $\pi^* = 0$, requires some cognitive efforts and information gathering costs, which will be incorporated in the cost $C \geq 0$.\footnote{In our model formulation the fundamental steady state is deterministic, but the model can be reformulated with a stochastic fundamental. The costs $C \geq 0$ then represent information gathering costs of a time-varying fundamental steady state.} Realized inflation and expectations will co-evolve over time and evolutionary selection based on reinforcement learning will decide which forecasting rule performs better and will survive in the evolutionary environment. The class of constant forecasts is extremely simple, but it should be emphasized that it is also broad because it includes
all possible point-predictions of next period’s inflation level. Moreover, learning to forecast laboratory experiments with human subjects show that individuals use very simple rules, including constant predictors (see Assenza, Heemeijer, Hommes, and Massaro (2011), Hommes (2011)). For this simple class of rules it will be possible to obtain analytical results under heterogeneous expectations. We will consider simple examples with only a few rules as well as examples with a large number, even a continuum of rules, representing an ecology of predictors including all possible steady state predictions.

2.3.1 Evolutionary Dynamics with Few Constant Belief Types

As a first step we consider the simplest scenario in which agents can choose between three different forecasting rules:

\[
\begin{align*}
\hat{E}_1,t \pi_{t+1} &= 0, \\
\hat{E}_2,t \pi_{t+1} &= b, \\
\hat{E}_3,t \pi_{t+1} &= -b,
\end{align*}
\]

with bias parameter \( b > 0 \). Type 1 agents believe that the inflation rate will always be at its RE level. Type 2 agents have a positive bias, expecting that inflation will be above its fundamental level, while type 3 agents have a negative bias, expecting an inflation level below the fundamental value.\(^9\) Assuming that the equilibrium predictor is available at cost \( C \geq 0 \) and substituting the forecasting rules of the three types into (2.5) we get

\[
\pi_t = \frac{1}{\phi_n} (n_{2,t} b - n_{3,t} b) = f_\beta(\pi_{t-1}),
\]

\(^9\)Notice that this example has “symmetric” beliefs, in the sense that the positive and negative biases are exactly balanced around the REE. The main reason why we assume symmetry of the belief types is that under such an assumption the REE is among the steady states of the dynamical system. Thus, with symmetric belief types we can address the important question of stability of the REE. We stress, however, that symmetry of beliefs is not essential for many qualitative features of the model, e.g., bifurcations of multiple steady state with increase of intensity of choice. The insight of the model can therefore be used to study the consequences of policy changes (after which the symmetry would be lost since the belief types would not respond to the policy shift immediately), as we do in Section 2.4.
where fractions are updated according to the discrete choice model (2.6), that is,

\[ n_{2,t} = \frac{e^{-\beta(\pi_{t-1}-b)^2}}{Z_{t-1}}, \quad n_{3,t} = \frac{e^{-\beta(\pi_{t-1}+b)^2}}{Z_{t-1}}, \]

and

\[ Z_{t-1} = e^{-\beta(\pi_{t-1}^2+C)} + e^{-\beta(\pi_{t-1}-b)^2} + e^{-\beta(\pi_{t-1}+b)^2}. \]

Dynamics in (2.8) is described by a 1-dimensional map. This map \( f_\beta \) is increasing, bounded and symmetric w.r.t. point \( \pi = 0 \), see Appendix 2.A. It implies that the dynamics always have a steady state \( \pi^* = 0 \), which is the RE equilibrium. However, this RE steady state may not be globally or even locally stable. In some cases the dynamics may converge to other stable steady states, which will be denoted as \( \pi^+ > 0 \) and \( \pi^- = -\pi^+ < 0 \). In what follows we provide a complete analysis of global dynamics of (2.8) and show how these dynamics depend on parameters \( b, C, \beta \) and \( \phi_\pi \). We will distinguish between two cases: a “low cost” case, when \( C < b^2 \), which includes the case of a freely available equilibrium predictor, and a “high cost” case, when \( C \geq b^2 \).

Let us start with the case in which the fundamental predictor has small or even zero costs. We introduce two constants \( \phi^W_\pi(b, C) = \phi^W_\pi < \phi^a_\pi = \phi^a_\pi(b, C) \), described in Eqs. (2.19) and (2.20) of Appendix 2.B, respectively. Three different situations can be distinguished on the basis of the “strength” of the policy reaction coefficient \( \phi_\pi \). When \( \phi_\pi < \phi^W_\pi \), we define the monetary policy implemented by the Central Bank as weak. The corresponding dynamics are described in Proposition 2.3.1 below. When \( \phi^W_\pi < \phi_\pi < \phi^a_\pi \), the monetary policy is defined as moderate, whereas when \( \phi_\pi > \phi^a_\pi \), the monetary policy is defined as aggressive.\(^\text{10}\) These cases are analyzed in Propositions 2.3.2 and Proposition 2.3.3, respectively.

**Proposition 2.3.1.** Let \( C < b^2 \) (“low costs”) and \( \phi_\pi < \phi^W_\pi \) (“weak policy”). Then values

\[ 0 < \beta^*_1 \leq \beta^*_2 < \beta^*_3 \leq \beta^*_4 \]

exist such that

\(^\text{10}\)When \( C = 0 \), the threshold values of the monetary policy reaction coefficient \( \phi^W_\pi \) and \( \phi^a_\pi \) are independent of \( b \) (see Appendix 2.B). In particular we have that, when \( C = 0 \), \( \phi^W_\pi = 0.93 \) and \( \phi^a_\pi = 2. \)
Figure 2.2: Low information costs case. The map $f_\beta$ in the system with 3 belief types for different values of $\beta$. The parameters values are $b = 1, C = 0$, and $\phi_\pi = 0.5$.

- for $\beta < \beta_1^*$ the RE steady state is unique and globally stable.

- for $\beta_2^* < \beta < \beta_3^*$ three steady states exist, the unstable RE steady state $\pi^*$, and two other stable non-RE steady states, $\pi^+$ and $\pi^-$.

- for $\beta > \beta_4^*$ five steady states exist, three steady states are locally stable ($\pi^*, \pi^+$ and $\pi^-$) and two other steady states are unstable;

Proof. See Appendix 2.B, where we also provide numerical evidence that $\beta_3^* = \beta_4^*$ and that when $\phi_\pi$ is small enough, $\beta_1^* = \beta_2^*$.

Fig. 2.2 shows the maps $f_\beta$ under a weak monetary policy for small, medium, and high values of the intensity of choice $\beta$. We set costs $C = 0$ and the policy reaction coefficient $\phi_\pi = 0.5$. When the intensity of choice is relatively low, there exists only one steady state, the RE steady state, which is globally stable. For low intensity of choice agents are more or less evenly distributed over the different forecasting rules, thus realized inflation will remain relatively close to the fundamental steady state. As the intensity of choice increases, the RE steady state loses stability and two new stable non-fundamental steady states are created. However, as $\beta$ increases further, we have that the RE steady state becomes stable again and two additional unstable steady states are created. In the case of low costs for fundamentalists, we thus have
three stable steady states, $\pi^+ > 0$, $\pi^- < 0$ and also $\pi^* = 0$, for high values of the intensity of choice $\beta$. The economic intuition behind the fact that non-fundamental steady states exist for high intensity of choice is simple (cf. Proposition 2.B.2 in Appendix 2.B). Suppose that the intensity of choice is high and that, at time $t$, inflation rate $\pi_t$ is close to the optimistic belief, that is, $\pi_t \approx b$. The positive bias forecast will perform better than the negative bias and the fundamental belief. Therefore, when the intensity of choice is high, almost all agents will forecast inflation with the positive bias, i.e., $n_{2,t+1} \approx 1$, implying that $\pi_{t+1} \approx b/\phi_{\pi}$. The same intuition explains existence of a negative non-fundamental steady state for high intensity of choice. However, with low costs $C$, when the system is close to the fundamental steady state, a relatively cheap fundamental rule is the best predictor, causing more agents to switch to the fundamental rule and leading the dynamics to converge to the RE equilibrium $\pi^* = 0$.

When the Central Bank implements a “moderate” interest rate rule, the following applies

**Proposition 2.3.2.** Let $C < b^2$ (“low costs”) and $\phi_{\pi}^w < \phi_{\pi} < \phi_{\pi}^a$ (“moderate policy”). Then values $0 < \beta_1^* \leq \beta_2^*$ exist such that

- for $\beta < \beta_1^*$ the RE steady state is unique and globally stable;
- for $\beta > \beta_2^*$ five steady states exist, three steady states ($\pi^*$, $\pi^+$ and $\pi^-$) are locally stable and two other steady states are unstable;

**Proof.** See Appendix 2.B, where we also provide numerical evidence that $\beta_1^* = \beta_2^*$. □

Fig. 2.3 shows the maps $f_\beta$ under a moderate monetary policy for small, medium, and high values of the intensity of choice $\beta$. We set costs $C = 0$ and the policy reaction coefficient $\phi_{\pi} = 1.5$. As before, when the intensity of choice $\beta$ is relatively low we have a unique globally stable fundamental steady state $\pi^* = 0$. When the intensity of choice increases, the RE equilibrium remains locally stable and four additional steady states, two stable and two unstable are created.

The difference with the previous case, i.e., when the monetary authority implements a “weak” interest rate rule, is that the zero steady state does not lose local stability. Therefore, a relatively strong reaction of the interest rate to inflation in a neighborhood of the RE steady state, lead the
Figure 2.3: **Low information costs case.** The map \( f_\beta \) in the system with 3 belief types for different values of \( \beta \). The parameters values are \( b = 1 \), \( C = 0 \), and \( \phi_\pi = 1.5 \).

dynamics to converge to the fundamental equilibrium. However, when the intensity of choice is high and inflation is out of the basin of attraction of \( \pi^* \) and close, for example, to the optimistic belief, the implemented policy is not aggressive enough in reacting to inflation causing more and more agents to adopt the positive bias forecast and leading the economy to converge to the positive non-fundamental steady state. The same intuition explains the existence of a negative non-fundamental steady state.

When the Central Bank implements an “aggressive” monetary policy we have the following

**Proposition 2.3.3.** Let \( C < b^2 \) (“low costs”) and \( \phi_\pi > \phi_\pi^* \) (“aggressive policy”). Then the RE steady state is unique and globally stable for any \( \beta \).

**Proof.** See Appendix 2.B.

Fig. 2.4 shows the maps \( f_\beta \) under an aggressive monetary policy for small, medium, and high values of the intensity of choice \( \beta \). We set costs \( C = 0 \) and the policy reaction coefficient \( \phi_\pi = 2.5 \). By reacting aggressively to inflation, the monetary authority manages to avoid multiplicity of equilibria and keeps the RE equilibrium globally stable. The intuition for this result is simple. Consider the limiting case \( \beta = \infty \) and suppose that, at time \( t \), inflation rate \( \pi_t \) is close to the optimistic belief, that is, \( \pi_t \approx b \). When \( \beta = \infty \), all agents will forecast inflation with the positive bias, i.e., \( n_{2,t+1} = 1 \), implying that \( \pi_{t+1} = b/\phi_\pi \). When costs \( C = 0 \) the
Figure 2.4: **Low information costs case.** The map $f_\beta$ in the system with 3 belief types for different values of $\beta$. The parameters values are $b = 1$, $C = 0$, and $\phi_\pi = 2.5$.

threshold value $\phi_\pi^a = 2$. For $\phi_\pi > 2$ we have that $\pi_{t+1} < b/2$, so that the zero predictor will be the closest to realized inflation. This leads all agents to adopt the fundamental forecasting rule, driving thus the system to the RE equilibrium. The result in Proposition 2.3.3 implies that, given parameters $b$ and $C$, the Central Bank can always implement an interest rate rule that satisfies $\phi_\pi > \phi_\pi^a > 1$ and keep the RE equilibrium unique and globally stable.

Consider now the case in which the fundamental predictor has relatively high costs. Then

**Proposition 2.3.4.** Let $C \geq b^2$ (“high costs”). Then there exists $\beta^*$ such that

- for $\beta < \beta^*$ the RE steady state is unique and globally stable;

- for $\beta > \beta^*$ three steady state exist, the unstable RE steady state $\pi^*$, and two other locally stable non-RE steady states, $\pi^+$ and $\pi^-$.  

*Proof.* See Appendix 2.B.

Fig. 2.5 shows the maps $f_\beta$ when costs $C$ are relatively high for small, medium, and high values of the intensity of choice $\beta$. We set costs $C = 1$ and the policy reaction coefficient to $\phi_\pi = 1.5$. When the intensity of choice is relatively low, there exists only one steady state, the RE steady state, which is globally stable. As before, for low intensity of choice agents are more or less evenly distributed over the different forecasting rules, thus realized inflation will remain
Figure 2.5: **High information costs case.** The map $f_\beta$ in the system with 3 belief types for different values of $\beta$. The parameters values are $b = 1$, $C = 1$, and $\phi_\pi = 1.5$.

relatively close to the fundamental steady state. As the intensity of choice increases, the RE steady state loses stability and two new stable non-fundamental steady state are created. When costs for the fundamental predictor are high, the Central Bank can not implement a monetary policy that keeps the RE equilibrium stable when the intensity of choice is relatively high. In fact, the costs for the fundamental predictor are so high that they overcome the forecasting error of the biased beliefs, even when inflation is close to the RE equilibrium. However, the more aggressive the monetary policy, the higher the bifurcation value $\beta^*$ and the smaller the distance between the stable non-fundamental equilibria and the RE steady state.

A similar analysis can be made for other examples with larger number of constant beliefs. Fig. 2.6 illustrates graphs of the 1-D map when there are five strategy types $b_h \in \{-1, -1/2, 0, 1/2, 1\}$, the costs $C$ of the fundamental predictor are low, and the monetary policy rule is such that multiple equilibria exist but the RE steady state remains locally stable. We also show the creation of five multiple steady state equilibria as the intensity of choice increases by means of the bifurcation diagram.

For small and medium values of $\beta$ the bifurcation scenario is similar to the three types case. However for high values of the intensity of choice, four additional steady states, two stable and two unstable, are created. The intuition for the appearance of the new stable steady
Figure 2.6: **Top panels:** The map $f_\beta$ in the system with 5 belief types, $b_k \in \{-1, -1/2, 0, 1/2, 1\}$, for different values of $\beta$. The parameter values are $b = 1$, $C = 0$, and $\phi = 1$. **Lower panel:** Bifurcation diagram for this system with the same parameter values with respect to the intensity of choice. Solid lines indicate stable equilibria and dashed lines unstable equilibria. For high values of $\beta$, 9 different steady states co-exist, 5 stable separated by 4 unstable steady states.

states is similar as before. Any available predictor would give the most precise forecast if the past inflation rate is sufficiently close to it. A high intensity of choice causes a large group of agents to choose this successful predictor, locking the inflation dynamics into a self-fulfilling stable equilibrium steady state close to that predictor. One can construct similar examples for any finite (odd) number, $H = 2K + 1$, of forecasting strategies generating $H$ multiple stable
equilibria. A finite class of forecasting rules seems reasonable as boundedly rational agents may exhibit “digit preference” and restrict their inflation predictions to values in integer numbers, e.g., 2%, 3%, or to half percentages, e.g., 2.5% or 3.5%, within the range of historically observed values from say −5% to +15%.11

The results in this section show that, in the presence of few belief types and evolutionary selection between different predictors, the Taylor principle is no longer sufficient to guarantee uniqueness and stability of the RE steady state. As a concrete example, the reaction coefficient suggested by Taylor (1993), i.e., \( \phi_\pi = 1.5 \) might not be sufficient to guarantee the determinacy (uniqueness and stability) of the RE equilibrium (see Figs. 2.3 and 2.5).

### 2.3.2 Many Belief Types

The previous analysis shows that in an economy with an ecology of 3 or 5 fundamentalists and biased beliefs, a cumulative process leading to accelerating inflation or deflation does not arise. Rather, for high intensity of choice and depending on the strength of policy reactions to inflation, the system might lock in one of multiple steady state equilibria, with a majority of agents using the forecasting rule with the smallest error at that equilibrium steady state. A natural question is: what happens when the number of constant forecasting rules increases and approaches infinity? As we will see, if agents select beliefs from a continuum of forecasting rules, representing an ecology containing all constant predictions, the cumulative process will reappear when the policy rule does not satisfy the Taylor principle.

Suppose there are \( H \) belief types \( b_n \), all available at zero costs. The evolutionary dynamics

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11Digit preference has been observed in both survey measures of expectations and experimental data. Curtin (2005) and Duffy and Lunn (2009) find evidence for digit preference respectively in the Michigan Survey data and in the EU Consumer Survey for Ireland. Assenza, Heemeijer, Hommes, and Massaro (2011) observed digit preference in learning-to-forecast-experiments adopting the New Keynesian model. In particular they find that about 14% of predictions of participants are integers and about 32% of predictions have a precision of 1 decimal. Overall, about 99.6% of predictions have a maximum precision of 2 decimals.
with $H$ belief types is given by

$$
\pi_t = \frac{1}{\phi} \cdot \frac{\sum_{h=1}^{H} b_h e^{-\beta(\pi_{t-1} - b_h)^2}}{\sum_{h=1}^{H} e^{-\beta(\pi_{t-1} - b_h)^2}} =: f_{\beta}^{H}(\pi_{t-1}). \tag{2.9}
$$

The dynamics of the system with $H$ belief types $b_h$ is described by a 1-D map $f_{\beta}^{H}$. What can be said about the dynamical behavior when $H$ is large? In general, it is difficult to obtain analytical results for systems with many belief types. We apply the concept of Large Type Limit (LTL henceforth) introduced in Brock, Hommes, and Wagener (2005) to approximate the evolutionary system with many belief types in (2.9). Suppose that at the beginning of the economy, i.e., at period $t = 0$, all $H$ belief types $b = b_h \in \mathbb{R}$ are drawn from a common initial distribution with density $\psi(b)$. We can derive the LTL of the system as follows. Divide both numerator and denominator of (2.9) by $H$ and rewrite the “$H$-type system” as

$$
\pi_t = \frac{1}{\phi} \cdot \frac{1}{H} \sum_{h=1}^{H} b_h e^{-\beta(\pi_{t-1} - b_h)^2} \int b e^{-\beta(\pi_{t-1} - b)^2} \psi(b) db \quad =: F_{\beta}(\pi_{t-1}). \tag{2.10}
$$

The LTL is obtained by replacing the sample mean with the population mean in both the numerator and the denominator, yielding

$$
\pi_t = \frac{1}{\phi} \cdot \frac{1}{H} \sum_{h=1}^{H} b_h e^{-\beta(\pi_{t-1} - b_h)^2} \int e^{-\beta(\pi_{t-1} - b)^2} \psi(b) db =: F_{\beta}(\pi_{t-1}). \tag{2.10}
$$

As shown in Brock, Hommes, and Wagener (2005), when the number of strategies $H$ is sufficiently large, the LTL dynamical system (2.10) is a good approximation of the dynamical system with $H$ belief types given by (2.9). In particular, if $H$ is large then with high probability the steady states and their local stability conditions coincide for both the LTL map $F_{\beta}$ and the $H$-belief system map $f_{\beta}^{H}$. In other words, properties of the evolutionary dynamical system with many types of agents can be studied using the LTL system.

For suitable distributions $\psi(b)$ of initial beliefs, the LTL (2.10) can be computed explic-
Figure 2.7: Graphs of the LTL map $F_\beta$ in (2.11) when the Taylor principle is not satisfied for a normal distribution $\psi(b) \simeq N(0, 0.25)$ of initial beliefs. The reaction coefficient is $\phi_\pi = 0.5$.

**Left:** $\beta = 1$. **Right:** $\beta = 10$.

ity. As an illustrative example consider the case when $\psi(b)$ is a normal distribution, $\psi(b) \simeq N(m, s^2)$. Plugging the normal density in (2.10), a straightforward computation shows that the LTL map $F_\beta$ is linear with slope increasing in $\beta$, given by

$$F_\beta(\pi) = \frac{1}{\phi_\pi} \cdot \frac{m + 2\beta s^2 \pi}{1 + 2\beta s^2}.$$

(2.11)

In particular, when the initial beliefs distribution is centered around $m = 0$, the unique steady state of the LTL map is the RE equilibrium, $\pi^* = 0$. This case is illustrated in Fig. 2.7, where we show the LTL map for different values of the intensity of choice when the interest rate rule does not satisfy the Taylor principle.

For $\beta = \beta^* = \frac{\phi_\pi}{2s^2(1 - \phi_\pi)}$ the slope of the linear map is exactly 1. Hence, the RE equilibrium is globally stable for $\beta < \beta^*$ and unstable otherwise. When $m \neq 0$, i.e., when the initial belief distribution is not symmetric with respect to the fundamental equilibrium, the unique steady state of the LTL system is not the REE, but the stability result and critical value $\beta^*$ do not change (cf. footnote 9).

We can conclude, therefore, that when initial beliefs are drawn from a normal distribution centered around the REE and the number of belief types is sufficiently high, an increase in the
intensity of choice, beyond the bifurcation value $\beta^*$, leads to instability of the system. Indeed, when $\beta$ is low, agents are more or less equally distributed among predictors. This means that the average expected inflation will be close to zero. Hence realized inflation will be close to the steady state value, more agents will adopt the steady state predictor and inflation will converge. However, when the intensity of choice increases and agents can switch faster to better predictors, the system becomes unstable. This is so because, for example, when the inflation rate is above its steady state value most agents will switch to an even more positive bias belief, leading to an even higher realized inflation rate. A cumulative process of ever increasing inflation arises again.

Note that increasing the variance $s^2$ of the normal distribution of initial beliefs has exactly the same effect on the LTL dynamics (2.11) as increasing the intensity of choice. When $\phi_\pi < 1$ we have that for $s^2 < \frac{\phi_\pi}{2\beta(1 - \phi_\pi)}$ the LTL map is globally stable, and it is unstable otherwise. Hence, when many initial beliefs are drawn from a normal distribution with small variance, the system will be stable, otherwise it will be unstable and a cumulative process will arise. The spread of initial beliefs is therefore an important element for the stability of the economy.

In the previous example we have assumed a normal distribution $\psi(b)$ of initial beliefs. Applying the results derived in Hommes and Wagener (2010), similar conclusions can be obtained for general distribution functions of initial beliefs. In fact, for systems with many belief types $b_h$ and initial beliefs drawn from a fixed strictly positive distribution function, when the intensity of choice becomes sufficiently large, a cumulative process arises with high probability.\(^{12}\)

To get some intuition for this result, it will be instructive to look at the limiting case $\beta = \infty$. When there is a continuum of beliefs, the best predictor in every period, according to past forecast error, will be the predictor that exactly coincides with last period’s inflation realization, $b_h = \pi_{t-1}$. For $\beta = \infty$, all agents will switch to the optimal predictor. Hence, for $\beta = \infty$, the economy with heterogeneous agents updating their beliefs through reinforcement learning

\(^{12}\)This result follows by applying Lemma 1, p. 31 of Hommes and Wagener (2010), stating that for any strictly positive distribution function $\psi$ describing initial beliefs, as the intensity of choice goes to infinity, the corresponding LTL map converges to a linear map with slope $1/\phi_\pi$.\[29\]
behaves exactly the same as an economy with a representative naive agent, for which we have shown that a cumulative process will arise when the monetary policy rule does not satisfy the Taylor principle (see Section 2.2, eq. (2.4)).

Finally consider a monetary authority implementing an interest rate rule that obeys the Taylor principle, i.e., \( \phi_\pi > 1 \). It should be clear that the “unstable” situation shown in the right panel of Fig. 2.7 cannot occur. Indeed, in this case we will have that

\[
\lim_{\beta \to \infty} F_{\beta}(\pi) = \frac{1}{\phi_\pi} \pi.
\]

Hence an interest rate rule that responds aggressively to actual inflation, i.e., \( \phi_\pi > 1 \), will fully stabilize the system, for all values of the intensity of choice \( \beta \). In contrast, if the policy rule of the Central Bank is not sufficiently aggressive, i.e., \( \phi_\pi < 1 \), then inflation dynamics will only be stable for small values of the intensity of choice, but the cumulative process will reappear when the intensity of choice is large. Therefore, when the model is indeterminate under rational expectations, there is a cumulative process for both the representative agent adaptive learning specification and the heterogeneous expectations case with many belief types. The same result holds for a normal initial distribution of beliefs centered around \( m \neq 0 \), even though the steady state of the dynamics will differ from the RE equilibrium in this case.

### 2.4 Stochastic Simulations

In this section we discuss stochastic simulations of our nonlinear model with heterogeneous expectations in order to match some characteristics of US inflation quarterly data over the period 1960-2007. We consider an ecology of \( H = 12 \) forecasting rules, \( b_h \in \{1, \ldots, 11\} \),\(^\text{13}\) so that the dynamics of inflation is given by

\[
\pi_t = f^{H}_{\phi_\pi}(\pi_{t-1}) + \varepsilon_t,
\]

\(^{13}\)The constant forecasting rules considered are within the range of integer predictors described by Curtin (2005) in the Michigan Survey data on inflation expectations.
where the map $f^H_{\phi_\pi}$ is defined as\footnote{The presence of constant $c$ in map (2.13) stems from the fact that, in general, the constant term in the Taylor rule (2.2) might differ from the equilibrium interest rate $r$. In our simulations we assume a Taylor rule of the form $i_t = \phi_\pi \pi_t$, so that $c = r/\phi_\pi$. Assuming a standard discount factor $\delta = 0.99$ we have that $c \approx 0.01/\phi_\pi$.}

$$f^H_{\phi_\pi}(\pi_{t-1}) = \frac{1}{\phi_\pi} \cdot \frac{\sum_{h=1}^H b_h e^{-\beta(\pi_{t-1}-b_h)^2}}{\sum_{h=1}^H e^{-\beta(\pi_{t-1}-b_h)^2}} + c,$$

and the exogenous random shocks $\varepsilon_t$ are drawn from a normal distribution with mean 0 and standard deviation $\sigma_\varepsilon = 0.5$. The notation $f^H_{\phi_\pi}(\pi_{t-1})$ stresses the fact that the nonlinear map

Figure 2.8: **Top Panels:** Simulated inflation time series. **Left:** Simulated inflation for a particular realization of the stochastic shocks. **Right:** Average inflation (solid line) and its variance (dashed line) over 1000 simulations. **Bottom Panels:** Steady states of the dynamics before and after the structural break. **Left:** The steady states of the dynamics as intersection points of the 45-degree line with the map $f^H_{0.98}(\pi)$ before the structural break (solid) and the map $f^H_{1.05}(\pi)$ after the structural break (dashed). **Right:** Plot of the maps $f^H_{\phi_\pi}(\pi) - \pi$. The same steady states are now clearly visible as intersections with the horizontal axis. The stable (unstable) steady states are marked with black (white) dots. The basin of attraction of a stable steady state is the interval between two adjacent unstable steady states.
depends on the monetary policy parameter $\phi_\pi$, the coefficient in the Taylor rule. In all stochastic simulations there is a structural break in period $T = 80$ in the Central Bank’s reaction function. In the first part of the simulations the policy rule reacts weakly to inflation (i.e., $\phi_\pi < 1$), while in the second part the interest rate reacts more than point to point to inflation (i.e., $\phi_\pi > 1$). In particular we consider a reaction coefficient $\phi_\pi = 0.98$ for periods $t = 1 − 79$ and a reaction coefficient $\phi_\pi = 1.05$ for periods $t = 80 − 192$.

The stochastic time series in Figure 2.8 replicates the observed pattern of a strong rise in US inflation until 1980 and a sharp decline and stabilization of inflation thereafter (see also Fig. 2.1). Of course the particular realization shown in the top left panel is affected by stochastic shocks, but this pattern is quite common and reproduced by the time series of average inflation, averaged over 1000 stochastic simulations, in Figure 2.8 (top right panel). The plot of the corresponding variance of the stochastic simulations shows that the variance is low after the structural break, implying that the strong decline in inflation after the structural break is a robust feature of the nonlinear model with heterogeneous beliefs and a monetary authority that obeys the Taylor principle. On the other hand, before the structural break the variance of the stochastic simulations is large, showing that the rise in inflation can be either slow or fast depending upon the realizations of the exogenous stochastic shocks. In particular, a few large positive shocks to inflation, such as large oil shocks, may trigger an increase in inflation which then becomes amplified by evolutionary pressure of self-fulfilling forecasting rules predicting high inflation.

The bottom panel in Figure 2.8 illustrates how the number of steady states in the nonlinear model with heterogeneous expectations changes when the monetary policy coefficient $\phi_\pi$ increases from 0.98 to 1.05. Before the structural break there are 21 steady states, 11 stable ones separated by 10 unstable steady states, ranging approximately from a low level of 1 to a high level of 11. A careful look at Figure 2.8 (bottom right panel) reveals an important asymmetry in the basins of attraction of each stable steady state: the basin of attraction (whose endpoints consist of the two neighboring unstable steady states) is relatively large to the left of the stable steady state and relatively small to the right. In the presence of (symmetric) stochastic shocks
to inflation, jumps to the basin of attraction of a higher stable steady states are therefore more likely than jumps to a lower level. This explains why for $\phi_\pi = 0.98$ on average inflation will rise from low levels to high levels as shown by the average inflation of the stochastic simulations.

After the structural break, when the Central Bank switches to a more aggressive Taylor rule, the number of steady states has decreased from 21 to 11, with 6 stable ones separated by 5 unstable steady states, ranging from approximately 1 to 6. Hence, an increase of the monetary policy parameter $\phi_\pi$ causes a number of high level steady states to disappear, implying more stable inflation dynamics in the stochastic nonlinear system as illustrated in the stochastic simulations after the structural break.

It is interesting to note that similar results occur when we allow for (infinitely) many constant prediction rules. Indeed our results concerning the LTL system in (2.10) in Section 2.3.2 show that, when agents are sensitive to difference in forecasting performance (i.e., for high values of the intensity of choice $\beta$), the inflation dynamics with an ecology of many steady state predictors drawn from a normal distribution of initial beliefs is well approximated by the linear map in (2.12), with slope $\frac{1}{\phi_\pi}$. This implies globally stable inflation dynamics approaching the RE equilibrium rate of inflation when $\phi_\pi > 1$, but exploding inflation dynamics when $\phi_\pi < 1$. Hence, in an ecology with many steady state predictors when the Central Bank uses a Taylor rule with $\phi_\pi < 1$ a cumulative process of rising inflation is very likely, while the monetary authority can manage heterogeneous expectations and achieve global macro economic stability by using a more aggressive Taylor rule with $\phi_\pi > 1$.

### 2.5 Concluding Remarks

We have used a simple frictionless DSGE model to study the role of heterogeneous expectations about future inflation and the potential (de-)stabilizing effect of different interest rate rules. We use the heterogeneous expectations framework of Brock and Hommes (1997), where the

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15 As $\phi_\pi$ increases from 0.98 to 1.05 the high level steady states disappear in pairs of two (one stable and one unstable) through a number of subsequent saddle-node bifurcations.
ecology of forecasting rules is disciplined by endogenous, evolutionary selection of strategies
with agents switching towards more successful rules.

Macroeconomic stability and inflation dynamics depend in interesting ways on the set of
forecasting strategies and the coefficient of an interest rate rule à la Taylor. When the monetary
authority responds weakly to inflation, heterogeneous agents trying to learn from their forecast
errors receive misleading signals from the market. Instead of leading the economy closer to the
equilibrium, these signals cause a cumulative process of rising inflation, triggered by exogenous
shocks to economic fundamentals and reinforced by self-fulfilling expectations of high inflation.
In contrast, when the nominal interest rate is adjusted more than point for point in response to
inflation, the monetary authority can manage heterogeneous expectations by sending signals
that help agents to correct their forecast errors instead of compounding them. The rationale
for an aggressive monetary policy is therefore rather different in the case of heterogeneous
beliefs compared to the homogeneous rational expectation case. In presence of heterogeneous
expectations, by reacting aggressively to inflation, the Central Bank sends correct signals for the
evolutionary selection of strategies and induces stable dynamics converging to the RE steady
state. Under rational expectations, by obeying the Taylor principle, the monetary authority
induces dynamics that will explode in any equilibrium but one. Ruling out explosive paths
guarantees then uniqueness of the equilibrium.16

However, while the Taylor principle is sufficient to ensure convergence to the RE steady
state in the case of a continuum of beliefs, the standard policy recommendation, i.e., \( \phi_{\pi} > 1 \),
is no longer sufficient to guarantee uniqueness and global stability of the RE steady state in
the case of finitely many belief types. In fact, in order to avoid multiple equilibria, the policy
rule must be aggressive enough (e.g., \( \phi_{\pi} > \phi^u_{\pi} \geq 2 \) in the 3 belief types example with small
costs of Section 2.3.1) to ensure that realized inflation is sufficiently close to the RE steady
state, so that the fundamental predictor will perform relatively better than other strategies in the
economy. The intuition is that with a continuum of beliefs and any \( \phi_{\pi} > 1 \), there is always

16See Cochrane (2010) for a discussion.
a belief type closer to the fundamental than current inflation performing better, thus pushing inflation gradually towards the fundamental. In contrast, with finitely many types and $\phi_\pi > 1$, but not large enough (i.e., $\phi_\pi < \phi^a_\pi$), the system may get locked in an almost self-fulfilling steady state belief. We would like to stress that the case of finitely many belief types seems empirically relevant as digit preference has been observed in experimental and survey data (see e.g., Assenza, Heemeijer, Hommes, and Massaro (2011), Curtin (2005) and Duffy and Lunn (2009)) and as our stochastic simulations illustrate.

Future work should further investigate the effect of heterogeneous expectations on the dynamics of aggregate output and inflation in models with frictions and nominal rigidities, such as the New Keynesian framework, and the conditions under which monetary policy rules may stabilize or may fail to stabilize aggregate macroeconomic variables.

In a recent paper Branch and McGough (2010) already studied heterogeneous expectations in the New Keynesian model. However, they only present numerical simulations showing instability and complex dynamics. An interesting topic for future work would be, for example, to apply our analytical framework to the fully-fledged New Keynesian model setting.
Appendix 2.A   Model with a finite number of types

In this appendix we consider the dynamics of the general model with $H$ types, as introduced in the beginning of Section 2.3. The dynamics are given by

$$
\pi_t = \frac{1}{\phi_\pi} \cdot \sum_{h=1}^{H} b_h \exp (\beta U_{t-1}) = \frac{1}{\phi_\pi} \cdot \sum_{h=1}^{H} b_h \exp (\beta (\pi_{t-1} - b_h^2 + C_h)).
$$

(2.14)

Recall that agents can choose one of the $H$ forecasting rules $\hat{E}_{t,h} = b_h$ available at cost $C_h$. The reaction coefficient $\phi_\pi > 0$ in the interest rate rule (2.2) measures the aggressiveness of the monetary policy.

The map $\pi_t = f(\pi_{t-1})$, where $f$ denotes the RHS of (2.14), defines a one dimensional dynamical system, whose properties are described in the following technical Lemma.

**Lemma 1.** Let $f : \mathbb{R} \to \mathbb{R}$ describe the dynamics of the system with finite number of types. Let $b = \min_h \{b_h\}$ and $\bar{b} = \max_h \{b_h\}$ denote the smallest and the largest available forecasts. Then:

1. For $\beta > 0$ function $f$ is strictly increasing.

2. $f(x) \to \frac{b}{\phi_\pi}$ for $x \to -\infty$ and $f(x) \to \frac{\bar{b}}{\phi_\pi}$ for $x \to \infty$.

**Proof.** To show that the map $f$ is increasing we use the same strategy as in Hommes and Wagener (2010). Multiplying both numerator and denominator by $\exp(-\beta \pi^2_{t-1})$, the map can be rewritten as

$$
f(\pi_{t-1}) = \frac{1}{\phi_\pi} \cdot \sum_{h=1}^{H} b_h \exp (-\beta [-2\pi_{t-1}b_h + b_h^2 + C_h]) \cdot \sum_{h=1}^{H} \exp (-\beta [-2\pi_{t-1}b_h + b_h^2 + C_h]).
$$

We write $x_h := \exp (-\beta [-2\pi_{t-1}b_h + b_h^2 + C_h])$, introduce the normalization factor $Z = \sum_{h=1}^{H} x_h$, and notice that $n_{h,t} = x_h/Z$. 

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Then the derivative $f'$ is given by

$$f'(\pi_{t-1}) = \frac{1}{\phi_\pi} \cdot \sum_{h=1}^{H} \frac{2\beta b_h^2 y_h Z - b_h y_h \sum_{h=1}^{H} 2\beta b_h y_h}{Z^2} =$$

$$= \frac{2\beta}{\phi_\pi} \cdot \left( \sum_{h} \frac{b_h^2 x_h}{Z} - \sum_{h} \frac{b_h x_h}{Z} \sum_{h} \frac{b_h x_h}{Z} \right) =$$

$$= \frac{2\beta}{\phi_\pi} \cdot \left( \sum_{h} b_h^2 n_{h,t} - \left( \sum_{h} b_h n_{h,t} \right)^2 \right).$$

(2.15)

The term in the last brackets is positive since it can be interpreted as the variance of a discrete stochastic variable $\xi$ taking values $b_h$ with probability $n_{h,t}$. Thus, $f'(\pi_{t-1}) > 0$ for $\beta > 0$, i.e., function $f$ is strictly increasing.

When $\pi_{t-1}$ is large enough, the forecasting rule with $\hat{E}_{t-2}\pi_{t-1} = \bar{b}$ has the highest performance measure, because the squared error term dominates constant costs. Hence, as $\pi_{t-1} \to \infty$, asymptotically all the agents use this forecasting rule, i.e., $n_{\bar{b}} \to 1$, while the fractions of all the other rules converge to 0. Since $f(\pi_{t-1}) = \sum_{h} b_h n_{h,t} / \phi_\pi$, we obtain that $f(\pi_{t-1}) \to \bar{b} / \phi_\pi$ for $\pi_{t-1} \to \infty$. The proof that $f(\pi_{t-1}) \to \bar{b} / \phi_\pi$ for $\pi_{t-1} \to -\infty$ is similar.

This Lemma implies that the dynamics (2.14) are quite simple. Independently of the initial condition, they monotonically converge to one of the finite number of steady states. For the generic case in which there are only hyperbolic steady states (i.e., no steady state $\pi^*$ exists with $f'(\pi^*) = 1$) the number of steady states is odd and the locally stable steady states alternate with the unstable steady states. The basin of attraction of a stable steady state is the largest possible interval containing a given steady state without any other steady states, see illustration in the lower left panel of Fig. 2.8. Furthermore, the dynamics are bounded within the interval $(\bar{b} / \phi_\pi, \bar{b} / \phi_\pi)$. 37
Appendix 2.B  Dynamics of the model with 3 types

This appendix investigates the global dynamics of the 3-type system considered in Section 2.3.1. We refer the reader to Kuznetsov (1995) for a detailed mathematical treatment of bifurcation theory.

With 3 forecasting rules, \( \hat{E}_{1,t} \pi_{t+1} = 0 \), \( \hat{E}_{2,t} \pi_{t+1} = b \) and \( \hat{E}_{3,t} \pi_{t+1} = -b \), where \( b > 0 \) is the bias parameter, system (2.14) becomes \( \pi_t = f_{\beta}(\pi_{t-1}) \), where

\[
 f_{\beta}(\pi) = \frac{b}{\phi_\pi} \cdot \frac{e^{-\beta(\pi+b)^2} + e^{-\beta(\pi-b)^2}}{e^{-\beta(\pi^2+C)} + e^{-\beta(\pi+b)^2} + e^{-\beta(\pi-b)^2}} = \frac{b}{\phi_\pi} \cdot \frac{1 - e^{-4\beta b \pi}}{1 + e^{-4\beta b \pi} + e^{-\beta(C-b^2+2b\pi)}}. \tag{2.16}
\]

The following result will be useful.

**Lemma 2.** Equation \( 2 + e^x - xe^x = 0 \) has a unique solution \( x^* \in (1, 2) \). For \( x < x^* \) we have \( 2 + e^x - xe^x > 0 \) and for \( x > x^* \) we have \( 2 + e^x - xe^x < 0 \).

**Proof.** Consider the function \( g(x) = 2 + e^x - xe^x \). Notice that \( \lim_{x \to -\infty} g(x) = 2 \), \( \lim_{x \to \infty} g(x) = -\infty \), \( g(0) = 3 \), and that derivative \( g'(x) = -xe^x \). Hence, for \( x \leq 0 \) function \( g \) increases from 2 to 3 and has no zeros. For \( x > 0 \) function \( g \) is strictly decreasing and has at most one zero.

On the other hand, \( g(1) = 2 > 0 \), while \( g(2) = 2 - e^2 < 0 \), because \( e^x > 1 + x \) for \( x = 2 \) becomes \( e^2 > 3 \). Applying the intermediate value theorem we obtain that there exists \( x^* \), zero of function \( g \), and that \( x^* \in (1, 2) \). \( \square \)

We proceed as follows. First, we derive two useful and important results. In Proposition 2.B.1 we analyze the local stability of the RE steady state, while in Proposition 2.B.2 we derive the dynamics for the limiting case \( \beta = +\infty \). These results will allow us to distinguish between the cases of high and low costs, and, in the latter case, between weak, moderate and aggressive monetary policy. Second, we study the concavity of function \( f_{\beta} \) in Lemma 3. Combining it with Proposition 2.B.1 we will then be able to give a full characterization of dynamics for the high cost case and to prove Proposition 2.3.4. Third, inspired by Proposition 2.B.2, we formalize in Lemma 4 the intuition that whenever non-fundamental steady states exist for
\( \beta = +\infty \), they will also exist for \( \beta \) high enough. Combining this Lemma with results of Proposition 2.B.1 we will derive Propositions 2.3.1 and 2.3.2. Finally, we will prove the important result of Proposition 2.3.3 on global stability in the low cost case with aggressive monetary policy.

The following result gives the conditions for the local stability of the RE steady state.

**Proposition 2.B.1** (Local stability of the RE steady state). *Consider the dynamics given by (2.16). Let \( x^* \) denote the unique solution of equation \( 2 + e^x - xe^x = 0 \). The following cases are possible:

1. \( C \geq b^2 \) (“high costs”). Then there exists a positive value \( \beta^* \), such that for \( \beta < \beta^* \) the RE steady state is locally stable, and for \( \beta > \beta^* \) the RE steady state is unstable.

2. \( C < b^2 \) (“low costs”). Then two cases are possible:
   1a. when \( \phi_\pi < 2(x^* - 1) \frac{b^2}{2 - C} \) two values \( 0 < \beta_1^* < \beta_2^* \) exist such that for \( \beta \notin [\beta_1^*, \beta_2^*] \) the RE steady state is locally stable, and for \( \beta \in (\beta_1^*, \beta_2^*) \) the RE steady state is unstable.
   1b. when \( \phi_\pi > 2(x^* - 1) \frac{b^2}{2 - C} \) the RE steady state is locally stable for any \( \beta \geq 0 \).

*Proof.* The fractions of three rules in the RE steady state are given by \( n_1^* = \frac{e^{-\beta C}}{Z} \) and \( n_2^* = n_3^* = \frac{e^{-\beta b^2}}{Z} \), where \( Z = e^{-\beta C} + 2e^{-\beta b^2} \). Substituting these fractions into (2.15), we find the derivative of map \( f_\beta \) in the RE steady state

\[
 f'_\beta(0) = \frac{4b^2 \beta}{\phi_\pi} \cdot \frac{e^{-\beta b^2}}{2e^{-\beta b^2} + e^{-\beta C}} = \frac{4b^2 \beta}{\phi_\pi} \cdot \frac{1}{2 + e^{-\beta(C-b^2)}}.
\]

The condition of local stability is given by \( f'_\beta(0) < 1 \), or, equivalently by \( h(\beta) < \phi_\pi \), where function \( h \) is defined as

\[
 h(\beta) = \frac{4b^2 \beta}{2 + e^{-\beta(C-b^2)}}. \tag{2.17}
\]
Notice that \( h(0) = 0 \) and the derivative of the function in \( \beta \) is given by

\[
h' = \frac{4b^2}{(2 + e^{-\beta(b^2)})^2} \left( 2 + e^{-\beta(b^2)} + \beta(b^2)e^{-\beta(b^2)} \right) = \frac{4b^2}{(2 + e^x)^2} (2 + e^x - xe^x),
\]

where we introduced the variable \( x = (b^2 - C)/\beta \).

In the high costs case, \( C \geq b^2 \), variable \( x \) is negative and, according to Lemma 2, function \( h \) strictly increases in \( \beta \) from 0 to \( \infty \). Thus, when \( \beta \) becomes higher than the bifurcation value \( \beta^* \) defined as \( \beta^* = h^{-1}(\phi_\pi) \), the RE steady state looses stability.

If \( C < b^2 \), variable \( x \) is positive and changes from 0 to \( \infty \) together with \( \beta \). We have then that function \( h \) is initially increasing in \( \beta \) and then decreasing. Function \( h \) takes its maximum value in the point where \( x = x^* \), i.e., when \( \beta = x^*/(b^2 - C) \). The value of function \( h \) in this point is given by

\[
h \left( \frac{x^*}{b^2 - C} \right) = \frac{4b^2}{2 + e^{x^*}} \cdot \frac{x^*}{b^2 - C} = \frac{4b^2}{2 + \frac{2}{x^*-1}} \cdot \frac{x^*}{b^2 - C} = (x^* - 1) \frac{2b^2}{b^2 - C}.
\]

The maximum value of \( h \) is positive according to Lemma 2. If it is larger than \( \phi_\pi \), then the two solutions of equation \( h(\beta) = \phi_\pi \) define an interval \((\beta_1, \beta_2)\) where \( h(\beta) > \phi_\pi \), and so the RE steady state is unstable. In the opposite case, if the maximum value of \( h \) is smaller than \( \phi_\pi \), then \( h(\beta) < \phi_\pi \) for any \( \beta \) and the RE steady state is always locally stable.

**Proposition 2.B.2** (Steady states for \( \beta = +\infty \)). Consider the dynamics given by (2.16) for the special case of \( \beta = +\infty \). Let us denote \( \pi^* = 0, \pi^+ = b/\phi_\pi \) and \( \pi^- = -b/\phi_\pi \). The following cases are possible:

1. \( C \geq b^2 \) ("high costs”). Then there are two locally stable steady states, \( \pi^+ \) and \( \pi^- \), with corresponding basins of attraction \((0, \infty)\) and \((-\infty, 0)\). The RE steady state \( \pi^* \) is unstable.

2. \( C < b^2 \) ("small costs”). Then two cases are possible:

   1.2a when \( \phi_\pi < \frac{2b^2}{b^2 - C} \) the system has three locally stable steady states, \( \pi^*, \pi^+ \) and \( \pi^- \).
The basin of attraction of the RE steady state is \( \left( -\frac{b^2-C}{2b}, \frac{b^2-C}{2b} \right) \).

(2b) when \( \phi > \frac{2b^2}{b^2-C} \) there exists a unique, globally stable RE steady state.

Proof. We will derive the map \( f_\infty \) governing the dynamics in case \( \beta = \infty \) explicitly. In this case the best performing rule will be chosen by all population at any period. The performances of three rules are given by

\[
U_{1,t-1} = -\pi_{t-1}^2 - C, \quad U_{2,t-1} = -\pi_{t-1}^2 + 2b\pi_{t-1} - b^2, \quad U_{3,t-1} = -\pi_{t-1}^2 - 2b\pi_{t-1} - b^2.
\]

Assume, first, that \( \pi_{t-1} > 0 \). Then the second rule (\( \hat{E}_{2,t}\pi_{t+1} = b \)) is always better than the third one (\( \hat{E}_{3,t}\pi_{t+1} = -b \)). Comparing the second rule with the first one (\( \hat{E}_{1,t}\pi_{t+1} = 0 \)) we find that

\[
U_{2,t-1} > U_{1,t-1} \iff 2b\pi_{t-1} - b^2 > -C \iff \pi_{t-1} > \frac{b^2 - C}{2b}.
\]

For \( C < b^2 \) the RHS of the last inequality gives a threshold after which the agents would use the non-RE forecasting rule. When \( C > b^2 \) the RE forecasting rule will never be used if \( \pi_{t-1} > 0 \).

Analogously we find that for \( \pi_{t-1} < 0 \) the third rule always outperforms the second one and it is better than the first if and only if \( \pi_{t-1} \) is less than the threshold \(-\frac{(b^2 - C)}{2b}\). When \( C > b^2 \) the RE forecasting rule will never be used if \( \pi_{t-1} < 0 \).

We conclude that function \( f_\infty \) has the following form. For \( C \geq b^2 \)

\[
\begin{array}{ll}
\frac{b}{\phi_n} & \text{if } \pi_{t-1} > 0 \\
0 & \text{if } \pi_{t-1} = 0 \\
-\frac{b}{\phi_n} & \text{if } \pi_{t-1} < 0,
\end{array}
\]

whereas for \( C < b^2 \)

\[
\begin{array}{ll}
\frac{b}{\phi_n} & \text{if } \pi_{t-1} > \frac{b^2-C}{2b} \\
0 & \text{if } \pi_{t-1} \in \left( -\frac{b^2-C}{2b}, \frac{b^2-C}{2b} \right) \\
-\frac{b}{\phi_n} & \text{if } \pi_{t-1} < -\frac{b^2-C}{2b}.
\end{array}
\]
In the high cost case two steady states \( \pi^+ \) and \( \pi^- \) always exist and function \( f_\infty \) is flat around them. This proves part (1) of the Lemma. In the low cost case the non-RE steady state \( \pi^+ \) exists if and only if the 45-degree line has an intersection with the upper horizontal parts of \( f_\infty \), i.e., when it intersects the line \( b/\phi_\pi \) at some \( \pi > (b^2 - C)/(2b) \). The condition for this is \( b/\phi_\pi > (b^2 - C)/(2b) \) or, equivalently, \( \phi_\pi < (2b^2)/(b^2 - C) \), which distinguishes cases 2(a) and 2(b) of Lemma. Lower non-RE steady state \( \pi^- \) exists if and only if the upper non-RE steady state \( \pi^+ \) exist due to symmetry of map \( f_\infty \).

Lemma 3. If \( C \geq b^2 \) function \( f_\beta \) is concave on the set \((0, \infty)\) for every \( \beta > 0 \). If \( C < b^2 \) function \( f_\beta \) defined on \((0, \infty)\) is concave for every \( 0 < \beta < \frac{\ln 4}{b^2 - C} \).

Proof. We find with direct computations that the second derivative of \( f_\beta \) is given by

\[
 f_\beta''(\pi) = -\frac{4b^2 \beta^2 (1 - e^{-4b\pi \beta}) e^{-(C+2b\pi)\beta} \phi_\pi (e^{C/\beta} + e^{b(b-2\pi)\beta} + e^{(C-4b\pi)\beta})^3}{(e^{(b^2-C)\beta} - e^{2(b^2-C-b\pi)\beta} + 8e^{-2b\pi\beta} + e^{(b^2-C-4b\pi)\beta})}.
\]

The fraction in this expression is positive for \( \pi > 0 \). The term between brackets can be rewritten as

\[
e^{(b^2-C)\beta} (1 + e^{-4b\pi \beta}) + e^{-2b\pi\beta} \left( 8 - e^{2(b^2-C)\beta} \right).
\]

In the high cost case, \( C \geq b^2 \), this expression is positive because \( e^{2(b^2-C)\beta} \leq 1 \). It implies that \( f_\beta''(\pi) < 0 \) for \( \pi > 0 \) and \( \beta > 0 \).

Consider the low cost case, \( C < b^2 \), and fix \( \beta \) such that \( 0 < \beta < \frac{\ln 4}{b^2 - C} \). When \( \pi = 0 \), the term between brackets becomes \( 8 + 2e^{(b^2-C)\beta} - e^{2(b^2-C)\beta} = 8 + 2x - x^2 \), where we introduced \( x = e^{(b^2-C)\beta} \). Notice that \( 1 < x < 4 \) for a given \( \beta \), and, therefore, the term is positive. Hence, by continuity of the second derivative, \( f_\beta''(\pi) < 0 \) for small \( \pi > 0 \). With a further increase of \( \pi \), the sign of the second derivative would change when the term in brackets is zero, i.e., when

\[
e^{(b^2-C)\beta} \left( e^{2(b^2-C)\beta} - e^{-2b\pi\beta} \right) = -\frac{e^{-2b\pi\beta}}{1 + e^{-4b\pi\beta}}.
\]

(2.18)

The left hand-side can be written as function \( x/(8 - x^2) \) and for \( x \in (1, 4) \) we have that the left
hand-side does not take values in the interval \([-0.5, 0]\). However, the right-hand side does take values only in this interval, as a function \(-t/(1 + t^2)\) of \(t = e^{-2bπβ} \in (0, 1]\). It means that there is no \(π\) to satisfy equality (2.18) and \(f''_β\) does not change its sign. We established that \(f''_β(π) < 0\) for a given \(β\) and for any \(π > 0\). This completes the proof.

**Proof of Proposition 2.3.4.** The previous Lemma shows that for the high cost case, \(C \geq b^2\), function \(f_β\) is concave for \(π > 0\) irrespectively of \(β > 0\). In Proposition 2.B.1(1) we found that there is a critical value \(β^*\) such that, when \(β < β^*\), the RE steady state is locally stable. For such \(β\) the global stability then follows immediately from concavity of \(f_β\) for positive \(π\) and symmetry of this function w.r.t. \(π = 0\). When the RE steady state is unstable, concavity together with boundedness of \(f\) implies that there exists a unique steady state with \(π^+ > 0\), and that this steady state is locally stable. By symmetry, there exists also a steady state \(π^- < 0\), also locally stable.

We comment that in the high cost case the region of instability always exists. The threshold is defined by \(β^* = h^{-1}(φ_π)\), where \(h\) is defined in (2.17). The RE steady state loses its stability at \(β = β^*\) through pitchfork bifurcation. Proposition 2.B.2 implies that for \(β \to ∞\) the non-fundamental and locally stable steady states \(π^+\) and \(π^-\) converge to \(b/φ_π\) and \(-b/φ_π\), respectively. Therefore, an aggressive policy of the Central Bank (i.e., high \(φ_π\)) reduces both the interval of instability and the deviations from the RE steady state.

**Low Cost Case**

Assume that we are in the low cost case, \(C < b^2\). Let us introduce the positive quantity

\[
φ_w^π(b, C) = 2(x^* - 1) \frac{b^2}{b^2 - C}.
\]

When the reaction coefficient \(φ_π < φ_w^π(b, C)\) we define the monetary policy as “weak”. In fact in this case the policy rule fails to ensure even local stability of the RE steady state when \(β\) increases (see Proposition 2.B.1(2)). We can find numerically \(x^* \approx 1.46\), so that \(φ_w^π(b, C)\) is approximately given by \(0.93 \frac{b^2}{b^2 - C}\). Given parameters \(b\) and \(C\), the Central Bank can implement
an interest rate rule with $\phi_\pi > \phi^\alpha_\pi(b, C)$ and keep the RE steady state locally stable.

However, as Proposition 2.B.2(2) shows, this value of $\phi_\pi$ can be not sufficient for the *global* stability of the RE steady state. Let us introduce another positive quantity

$$\phi^a_\pi(b, C) = \frac{2b^2}{b^2 - C}.$$  \hspace{1cm} (2.20)

When the reaction coefficient $\phi_\pi > \phi^a_\pi(b, C)$ we define the monetary policy as “aggressive”. According to Proposition 2.B.2(2), in this case the RE steady state is unique and globally stable, at least for $\beta = +\infty$. Of course, $\phi^w_\pi(b, C) < \phi^a_\pi(b, C)$ (it follows from Lemma 2), and when $\phi_\pi \in (\phi^w_\pi(b, C), \phi^a_\pi(b, C))$ we define the monetary policy as “moderate”. It is strong enough to guarantee local stability of RE steady state for every $\beta$ but it is not sufficiently aggressive to guarantee the global stability of the RE steady state for $\beta = +\infty$.

Intuitively, when $\beta$ gets larger, the map $f_\beta$ in (2.16) gets closer to the piece-wise map $f_\infty$ derived in the proof of Proposition 2.B.2. Therefore, when the monetary policy is weak or moderate, we expect to observe the non-RE steady states for $\beta$ high enough. The following result confirms this intuition.

**Lemma 4.** Suppose $C < b^2$ and $\phi < \phi^a_\pi(b, C)$. For high enough $\beta$, the dynamics given by (2.16) have locally stable steady states $\pi^+ > 0$ and $\pi^- = -\pi^+ < 0$.

**Proof.** We will prove the existence of $\pi^+$. The existence of $\pi^-$ will then follow from the symmetry of $f_\beta$.

Take $\varepsilon = 1/\phi_\pi - 1/\phi^a_\pi(b, C) = 1/\phi_\pi - (b^2 - C)/(2b^2) > 0$, define $\delta = \varepsilon(b^2 - C)/(2b) > 0$ and consider set $U = \{ \pi : \pi > (b^2 - C)/(2b) + \delta \}$. Set $U$ is bounded from below and on this set $f_\beta(\pi)$ converges to $f_\infty(\pi) = b/\phi_\pi$ from below when $\beta \to \infty$. Hence, for any $\pi \in U$ for sufficiently high $\beta$ it holds that

$$f_\beta(\pi) > \frac{b}{\phi_\pi} \cdot \frac{1 + \varepsilon}{1 + \frac{2\varepsilon}{b^2 - C} \varepsilon} = \frac{b}{\phi_\pi} \cdot \frac{1 + \varepsilon}{1 + \frac{2\varepsilon}{b^2 - C}} = \frac{b^2 - C}{2b} (1 + \varepsilon) = \frac{b^2 - C}{2b} + \delta.$$  

Thus, increasing and bounded from above, $f_\beta$ maps set $U$ into itself. Therefore, there should
Figure 2.9: Bifurcation diagram of dynamics in the low cost case, shown in coordinates \((\beta, \phi_{\pi})\). Parameters are \(C = 0\) and \(b = 1\). The dashed curve gives all the parameters of the pitchfork bifurcation, while the solid line gives the parameters of tangent bifurcation. The inset zooms the area where the curve of pitchfork bifurcation (dashed) intersects with the curve of tangent bifurcation (solid). The shaded area represents the region in the parameters space in which the RE is locally unstable.

exist a locally stable steady state within set \(U\).

**Proof of Proposition 2.3.1.** According to Proposition 2.B.1(2a) the RE steady state is locally stable for small \(\beta\) but loses and again acquires its local stability through two subsequent pitchfork bifurcations. Together with concavity of \(f_{\beta}\) proved in Lemma 3, it implies global stability of RE steady state for small \(\beta\). Consider now the moment of the first pitchfork bifurcation, which we denote as \(\beta = \beta^*_2\). At this moment the RE steady state loses its stability, but it might do it in two different ways. If at this instance, function \(f_{\beta}\) is concave for \(\pi > 0\), then the bifurcation at \(\beta^*_2\) is supercritical and two stable non-RE steady states are created (in this case \(\beta^*_1 = \beta^*_2\)). But if function \(f_{\beta}\) is not concave (and in particular it is convex for small \(\pi > 0\)), then this bifurcation is subcritical, which implies that two unstable steady states had to exist for \(\beta < \beta^*_2\). The only way in which they could be created is via tangent bifurcation at some smaller \(\beta = \beta^*_1\). Our numeric analysis for different values of \(C\) and \(b\) demonstrates that such scenario may happen for values of \(\phi_{\pi}\) which are very close to \(\phi_{\pi}^w\). See illustration in Fig. 2.9.
The RE steady state regains its stability at $\beta = \beta_3^*$, when function $f_\beta$ is convex for small $\pi > 0$. Therefore, at $\beta_3^*$ a subcritical pitchfork bifurcation takes place and two new unstable steady states are created. But this implies (given that $f_\beta$ is increasing and bounded) the existence of two other stable non-RE steady states. These five steady state will be also observed for high $\beta$, as we proved in Lemma 4. Thus, we suspect and never disproved through simulations that $\beta_3^* = \beta_4^*$.

Proof of Proposition 2.3.2. According to Proposition 2.B.1(2b) the RE steady state is always locally stable. It is unique and, therefore, globally stable, when function $f_\beta$ is concave, i.e., for small $\beta$ (see Lemma 3). On the other hand, when $\beta$ is high enough there are two other locally stable steady states, $\pi^+$ and $\pi^-$, see Lemma 4. These steady states could be created only via tangent bifurcation. Since we cannot rule out the possibility of a number of subsequent tangent bifurcations (when the non-RE steady states are created and disappeared), we denote as $\beta_1^*$ the instance of the first tangent bifurcation and as $\beta_2^*$ the instance of the last tangent bifurcation. However, in our numeric analysis for different values of $C$ and $b$ we never encountered a case in which $\beta_1^* \neq \beta_2^*$. See illustration in Fig. 2.9.

Proof of Proposition 2.3.3. Since for the aggressive monetary policy $\phi_\pi > \phi^a_\pi > \phi^w_\pi$, from Proposition 2.B.1(2b) it follows that the RE steady state is locally stable. In order to prove that it is globally stable for every $\beta$, we will show that it is the unique steady state of the dynamics. Since $f_\beta$ is an increasing function, uniqueness will imply global stability.

Assume that $\pi > 0$. Since function $f_\beta$ is bounded from above by a horizontal asymptote $b/\phi_\pi$, i.e., $f_\beta(\pi) < b/\phi_\pi$ for every $\pi$, no steady state can exist within the interval $[b/\phi_\pi, \infty)$. Let us consider $\pi \in (0, b/\phi_\pi)$ and show that $f_\beta(\pi) \in (0, b/(2\phi_\pi)]$. Since $f_\beta$ is an increasing map, $0 = f_\beta(0) \leq f_\beta(\pi) \leq f_\beta(b/\phi_\pi)$. Furthermore, since $\phi_\pi > \frac{2b^2}{b^2-C}$ and $b^2 - C > 0$, we have that $e^{-\beta(C-b^2+2b^2/\phi_\pi)} \geq 1$. Combining these inequalities we derive that

$$f_\beta(\pi) \leq f_\beta \left( \frac{b}{\phi_\pi} \right) = \frac{b}{\phi_\pi} \cdot \frac{1 - e^{-4\beta b^2/\phi_\pi}}{1 + e^{-4\beta b^2/\phi_\pi} + e^{-\beta(C-b^2+2b^2/\phi_\pi)}} \leq \frac{b}{\phi_\pi} \cdot \frac{1 - e^{-4\beta b^2/\phi_\pi}}{2 + e^{-4\beta b^2/\phi_\pi}} \leq \frac{b}{2\phi_\pi}.$$
We now showed that there is no fixed point of map \( f_\beta \) for \( \pi > \frac{b}{2\phi} \).

Suppose, finally, that \( 0 < \pi \leq \frac{b}{2\phi} \). Applying restrictions \( \phi_\pi > \frac{2\beta}{\pi^2 - C} \) and \( b^2 - C > 0 \) we find that the condition on \( \pi \) implies that \( \pi < \frac{b^2 - C}{4b} \), so that \( 4b\pi < (b^2 - C) \) and \( C - b^2 + 2b\pi < (C - b^2)/2 \). We obtain then the following estimate of dynamics on the interval \((0, b/(2\phi_\pi))\)

\[
f_\beta(\pi) = \frac{b}{\phi_\pi} \cdot \frac{1 - e^{-4\beta\pi}}{1 + e^{-4\beta\pi} + e^{-\beta(b^2 - 2b\pi)}} \leq \frac{b^2 - C}{2b} \cdot \frac{1 - e^{-4\beta\pi}}{1 + e^{-4\beta\pi} + e^{\beta(b^2 - C)/2}}. \tag{2.21}
\]

Let the function on the right-hand side be denoted as \( g(\pi) \). This function is, obviously, increasing in \( \pi \), and, with direct computations we find that its first two derivatives are given by:

\[
g'(\pi) = 2\beta \left( b^2 - C \right) \frac{e^{-4\beta\pi}}{1 + e^{-4\beta\pi} + e^{-\beta(b^2 - 2b\pi)}} \left( 2 + e^{\beta(b^2 - C)/2} \right),
\]

and

\[
g''(\pi) = -8\beta^2 b \left( b^2 - C \right) \left( 2 + e^{\beta(b^2 - C)/2} \right) e^{-4\beta\pi} \frac{1 - e^{-4\beta\pi} + e^{\beta(b^2 - C)/2}}{(1 + e^{-4\beta\pi} + e^{\beta(b^2 - C)/2})^2}.
\]

For positive \( \pi \) we have that \( e^{-4\beta\pi} \leq 1 \), and hence, the second derivative is always negative. Also notice that

\[
g'(0) = 2\beta \left( b^2 - C \right) \frac{1}{2 + e^{\beta(b^2 - C)/2}} = \frac{2x}{2 + e^{x/2}},
\]

where \( x = (b^2 - C)\beta \). But the function \( 2x/(2 + e^{x/2}) \) has a maximum in a point satisfying

\[
4e^{-x/2} + 2 - x = 0,
\]

which is \( x^* \), where \( x^* \) was defined in Lemma 2. The value of this maximum is then (approximately) 0.93, which is less than 1. Therefore, \( g'(0) < 1 \).

Since \( g \) is concave function for \( \pi > 0 \), the last condition implies that \( g(\pi) < \pi \) for every \( \pi > 0 \). Combining it with estimate in (2.21), we conclude that \( f_\beta(\pi) < \pi \) for all \( 0 < \pi \leq \frac{b}{2\phi_\pi} \).

This proves that there is no positive steady state for dynamics given by \( f_\beta \). Since the function \( f_\beta \) is odd it also implies that no negative steady state exists. This completes the proof. \( \square \)
Chapter 3

A Micro-Foundation of Heterogenous Expectations in New Keynesian Models

3.1 Introduction

Over the past decade the New Keynesian model has become increasingly popular in the analysis of monetary policy. This model is built under the hypothesis of rational expectations (RE) and assumes a representative agent structure. Although adaptive learning has become increasingly important as an alternative approach to modeling private sector expectations, most of these models still assume a representative agent who is learning about the economy (see e.g. Evans and Honkapohja (2001) and Sargent (1999) for extensive overviews). Carroll (2003), Mankiw, Reis, and Wolfers (2003), Branch (2004), Pfajfar and Santoro (2010) and Pfajfar (2008) recently provided empirical evidence in support of heterogeneous expectations using survey data on inflation expectations, while Hommes, Sonnemans, Tuinstra, and van de Velden (2005), Adam (2007), Pfajfar and Zakelj (2010), Assenza, Heemeijer, Hommes, and Massaro (2011), and Hommes (2011) find evidence for heterogeneity in learning to forecast laboratory experiments with human subjects. Some recent examples of models with heterogeneous expectations in macroeconomics include Brock and de Fontnouvelle (2000), Evans and Honkapohja (2003,

In the light of the empirical evidence, the primary interest of this paper is to incorporate heterogeneous beliefs into the micro-foundations of the New Keynesian framework. The first contribution of our work is thus the development of a micro-founded framework for monetary policy analysis consistent with heterogeneous, possibly boundedly rational, expectations.

In our model agents solve infinite horizon decision problems. The RE hypothesis requires that agents make fully optimal decisions given their beliefs and that agents know the true equilibrium distribution of variables that are beyond their control. Achieving the standard rationality requirements of RE models is especially difficult in a setting with heterogeneous agents. Individuals need to gather and process a substantial amount of information about the economy, including details about other agents in the market and their expectations, in order to derive the objective probability distributions of aggregate variables.

Starting from this observation, we assume that agents need to pay information gathering and processing costs in order to achieve RE. As an alternative, individuals can use simple and freely available prediction rules (heuristics) to forecast aggregate variables. Our modeling approach slightly departs from the standard RE benchmark. In fact we assume costly information processing without excluding the RE predictor from the set of forecasting rules available to the agents in the economy. Moreover, while we assume cognitive limitations of individual under-

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1 Models with this kind of approach have been studied by Marcet and Sargent (1989) and Sargent (1993) among others.

2 The idea of costly information processing is shared with the literature on “rational inattention” (see Sims (2003) among others).
standing, we fully maintain the assumption that individual choices are made optimally given subjective, possibly non-rational, forecasts. The decision whether to pay the costs for the RE predictor will depend on the dynamic trade off between these costs and the forecasting performance of the heuristics along the lines of Brock and Hommes (1997). Agents’ beliefs determine aggregate outcomes and are subsequently updated based upon recent performances when new public information becomes available in a sort of Bayesian updating mechanism. Co-evolution of subjective expectations with observed macroeconomic outcomes emerges due to the ongoing evaluation of predictors in a dynamic feedback system (Diks and van der Weide (2005)).

As underscored by Preston (2005), when the micro-foundations underpinning the New Keynesian model are solved under the non-rational expectations assumption, the predicted aggregate dynamics depend on long horizon forecasts. Hence, in making current decisions about spending and pricing of their output, agents take into account forecasts of macroeconomic conditions over an infinite horizon. This is a direct consequence of the fact that individuals are assumed to only have knowledge of their own objectives and of the constraints they face, and they do not have a complete economic model of determination of aggregate variables. We derive aggregate demand and supply equations consistent with heterogeneous expectations by explicitly aggregating individual decision rules. The resulting reduced form model is analytically tractable and encompasses the representative rational agent benchmark as a special case. In fact, when RE agents are present in the economy, it is possible to reduce the aggregate equations depending on long horizon forecasts to a system where only one-period ahead forecasts matter by using the law of iterated expectations. When all agents have RE, the model with forecasts over the infinite horizon reduces to the standard New Keynesian model.

Within a general framework of heterogeneous expectations, we test the desirability of standard policy recommendations. We find that bounded rationality represents an important source of instability and indeterminacy. In fact, monetary policy rules that lead to determinacy in a world with a representative rational agent may destabilize the economy even when only a small

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3Cognitive limitations of individuals have been abundantly documented by psychologists and brain scientists. For recent surveys see Kahneman and Thaler (2006) and Della Vigna (2007).
fraction of boundedly rational agents is added to the system. Our results thus confirm the con-
cerns for private sector expectations of policy makers such as Bernanke (2004) and show the
importance of taking bounded rationality into account when designing monetary policies.

Closely related to our results is the parallel paper by Branch and McGough (2009) who also
introduce heterogeneous expectations in a New Keynesian framework. Differently from our
work, Branch and McGough (2009) start from the assumption that agents with subjective (non-

rationa) expectations choose optimal plans that satisfy the associated Euler equations instead
of looking at the perceived lifetime intertemporal budget constraint⁴. The consequence of this
behavioral assumption is that in Branch and McGough (2009) only one-period ahead forecasts
matter for aggregate dynamics. This represents the main difference with our framework where
individual decision rules as well as aggregate equations depend on long horizon forecasts.

The paper is organized as follows. Section 3.2 briefly recalls the standard New Keynesian
model with a representative rational agent. The general framework consistent with heteroge-

neous expectations is derived in Section 3.3. Section 3.4 presents an application to monetary
policy by considering determinacy issues in an economy with a continuum of boundedly ratio-
nal beliefs together with perfectly rational expectations. Finally, Section 3.5 concludes.

3.2 The model with a representative (rational) agent

In this section we briefly recall the standard New Keynesian model of output and inflation
determination under RE⁵.

3.2.1 Households

The preferences of the representative household are defined over a composite consumption good
cₜ and time devoted to market employment hₜ. Households maximize the expected present

⁴See Preston (2005) and Evans, Honkapohja, and Mitra (2003) for a discussion about the two different modeling
approaches.

⁵For a detailed presentation of the New Keynesian framework see Woodford (2003) and Galí (2008) among
others.
discounted value of utility:

\[
\max E_t \sum_{s=t}^{\infty} \delta^{s-t} \left( \frac{c_s^{1-\sigma}}{1-\sigma} - \chi h_s^{1+\gamma} \right).
\]

The composite consumption good consists of differentiated products produced by monopolistically competitive final good producers. There is a continuum of goods of measure 1, each of them produced by firms indexed by \( j \in [0, 1] \). The composite consumption good that enters the households’ utility function and the aggregate price index for consumption are the usual CES aggregators, defined as

\[
c_t = \left( \int_{0}^{1} c_{j,t}^{\eta} \, dj \right)^{\frac{1}{\eta}} \quad \text{and} \quad P_t = \left( \int_{0}^{1} P_{j,t}^{1-\eta} \, dj \right)^{\frac{1}{1-\eta}},
\]

where the parameter \( \eta \) governs the price elasticity of demands for individual goods. The budget constraint of the household can be written in real terms as

\[
c_t + b_t \leq w_t h_t + R_{t-1} \pi_{t-1}^{-1} b_{t-1} + d_t,
\]

where \( b_t \) represents holdings of one-period bonds, \( w_t \) is the real wage, \( R_t \) is the (gross) nominal interest rate, \( \pi_t \equiv P_t/P_{t-1} \) is the inflation between period \( t \) and period \( t-1 \) and \( d_t \) are dividends received from firms.

The first order conditions for the optimization problem are:

\[
c_t^{-\sigma} = E_t \left( \delta R_t \pi_{t+1}^{-1} c_{t+1}^{-\sigma} \right) \quad \text{(3.3)}
\]

\[
\chi h_t^{\gamma} = w_t c_t^{-\sigma} \quad \text{(3.4)}
\]

plus the budget constraint (3.2) holding as an equality and the transversality condition

\[
\lim_{s \to -\infty} \delta^{s-t} E_t c_s^{-\sigma} b_s = 0. \quad \text{(3.5)}
\]
3.2.2 Firms

We assume a continuum of firms indexed by $j \in [0, 1]$ producing differentiated goods. The production function is assumed to be linear in labor:

$$y_{j,t} = h_{j,t}.$$  \hfill (3.6)

The real profits of firm $j$ are

$$P_{j,t} - W_t = \left( \frac{P_{j,t}}{P_t} - W_t \right) \left( \frac{P_t}{P_t} \right)^{-\eta} c_t,$$

where we substituted out the demand for firm $j$’s output at time $t$ constraint

$$y_{j,t} = c_{j,t} = \left( \frac{P_{j,t}}{P_t} \right)^{-\eta} c_t.$$

Assuming a Calvo staggered price setting, a firm able to reset its price is choosing $P_{j,t}$ to solve

$$\max_t E_t \sum_{s=t}^{\infty} \omega^{s-t} Q_s \left( \frac{P_{j,t}}{P_s} - \frac{W_s}{P_s} \right) \left( \frac{P_s}{P_s} \right)^{-\eta} c_s,$$

where $\omega$ is the degree of price stickiness and $Q_s$ is the stochastic discount factor given by $\delta^{s-t} (c_s/c_t)^{-\sigma}$. Since all firms adjusting their price at time $t$ are facing the same problem, they will set the same price, $P^*_t$. The first order condition for firms adjusting their price is

$$E_t \sum_{s=t}^{\infty} \omega^{s-t} Q_s \left( \frac{1 - \eta}{P_s} + \eta \frac{1}{P_s} w_s \right) \left( \frac{P^*_t}{P_s} \right)^{-\eta} c_s = 0. \hfill (3.7)$$

Rearranging terms and using the definition of $Q_s$ we can rewrite equation (3.7) as

$$\frac{P^*_t}{P_t} = \frac{E_t \sum_{s=t}^{\infty} (\omega\delta)^{s-t} c_{s-\sigma} w_s \left( \frac{P_s}{P_t} \right)^{\eta}}{E_t \sum_{s=t}^{\infty} (\omega\delta)^{s-t} c_{s-\sigma} \left( \frac{P_s}{P_t} \right)^{\eta-1}}. \hfill (3.8)$$
Using the price index definition (3.1) we can write the aggregate price level as

\[ P_t = \left( \omega P_{t-1}^{1-\eta} + (1 - \omega) (P_t^*)^{1-\eta} \right)^{\frac{1}{1-\eta}}. \] (3.9)

### 3.2.3 Government

We assume that the government issues no debt so that

\[ b_t = 0, \quad \text{for all } t, \]

and that public expenditure is equal to zero. The market clearing condition thus requires

\[ y_t = c_t. \]

Moreover, aggregating over firms, the total dividends transferred to the representative household are given by

\[ d_t = y_t - w_t h_t. \]

### 3.2.4 Aggregate equations

In order to derive aggregate equations for output and inflation, we linearize the set of first order conditions around the steady state derived in Appendix 3.A. Log-linearizing the household’s first order condition (3.3) and using the market clearing condition we get the expression

\[ \hat{y}_t = E_t \hat{y}_{t+1} - \frac{1}{\sigma} \left( \hat{R}_t - E_t \hat{\pi}_{t+1} \right), \] (3.10)

where \( \hat{x}_t \equiv \ln(x_t/x) \) denotes the log deviation of \( x_t \) from its steady state value \( x \). Equation (3.10) is referred to as the **dynamic IS equation** (IS). 

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Log-linearizing equations (3.8) and (3.9) gives the expression for aggregate inflation

\[ \hat{\pi}_t = \delta E_t \hat{\pi}_{t+1} + \tilde{k} \hat{w}_t, \]

(3.11)

where \( \tilde{k} = \frac{(1-\omega)(1-\delta\omega)}{\omega} \). Using the linearized version of the equilibrium condition (3.4) and the linearity of the production function\(^6\) we can write \( \hat{w}_t = \gamma \hat{h}_t + \sigma \hat{y}_t = (\sigma + \gamma) \hat{y}_t \), so that (3.11) becomes

\[ \hat{\pi}_t = \delta E_t \hat{\pi}_{t+1} + k \hat{y}_t, \]

(3.12)

where \( k = (\sigma + \gamma) \frac{(1-\omega)(1-\delta\omega)}{\omega} \). Equation (3.12) is referred to as New Keynesian Phillips Curve (NKPC).

### 3.3 The model with heterogeneous expectations

We now depart from the benchmark model of Section 3.2 by introducing heterogeneous agents with possibly non-rational expectations. The framework is in fact very general, so that the sophisticated rational expectation predictor belongs to the class of available forecasting rules. In what follows we keep the assumption that the government issues no debt and public expenditure is equal to zero.

#### 3.3.1 Households

Consider an economy populated by a set of agents distributed uniformly along the unit interval with a total population mass of 1. That is, for \( i \in [0, 1] \) the probability distribution function is \( f(i) = 1 \) and \( f(i) = 0 \) elsewhere. The maximization problem of household \( i \) reads as

\[ \max \tilde{E}_{i,t} \sum_{s=t}^{\infty} \delta^{s-t} \left( \frac{1-\sigma}{1-\sigma} - \chi \frac{\hat{h}_{i,s}^{1+\gamma}}{1+\gamma} \right) \]

\(^6\)Note that \( \hat{y}_t = \hat{h}_t \) up to a first order approximation in a neighborhood of the zero inflation steady state, see Gali (2008).
subject to
\[ c_{i,t} + b_{i,t} \leq w_{i} h_{i,t} + R_{t-1}^{-1} b_{i,t-1} + d_{t}, \]
where \( \tilde{E}_i \) denotes the subjective expectation of household \( i \). The first order conditions for the problem are given by
\[
\hat{c}_{i,t} = \tilde{E}_{i,t} (\delta R_{t} \pi_{t+1}^{-1} \hat{c}_{i,t+1}) \\
\hat{h}_{i,t} = \gamma^{-1} (\hat{w}_{t} - \sigma \hat{c}_{i,t})
\]
which can be log-linearized around the steady state derived in Appendix 3.A in order to get
\[
\hat{c}_{i,t} = \tilde{E}_{i,t} \hat{c}_{i,t+1} - \sigma^{-1} \left( \delta R_{t} - \tilde{E}_{i,t} \pi_{t+1} \right) \\
\hat{h}_{i,t} = \gamma^{-1} (\hat{w}_{t} - \sigma \hat{c}_{i,t}) \quad (3.13)
\]
which can be log-linearized around the steady state derived in Appendix 3.A in order to get
\[
\tilde{b}_{i,t} - \beta^{-1} \tilde{b}_{i,t-1} - b \beta^{-1} \left( \hat{R}_{t-1} - \hat{\pi}_{t} \right) - (1 - \eta^{-1}) (\hat{w}_{t} + \hat{h}_{i,t}) - \eta^{-1} \hat{d}_{t} + \hat{c}_{i,t} = 0, \quad (3.15)
\]
where \( \hat{x}_{t} \equiv \ln(x_{t}/x) \) denotes the log deviation of \( x_{t} \) from its steady state value \( x \), while \( \bar{x}_{t} \equiv x_{t} - x \) is just the difference of variable \( x_{t} \) from its steady state \( x \). Since the budget constraint has to hold as an equality in every period we can iterate it forward from period \( t \) on to get the intertemporal budget constraint defined as
\[
\tilde{E}_{i,t} \sum_{s=t}^{\infty} \delta^{s-t} \hat{c}_{i,s} = \tilde{E}_{i,t} \sum_{s=t}^{\infty} \delta^{s-t} \left[ \delta^{-1} \tilde{b}_{i,t-1} + (1 - \eta^{-1}) (\hat{w}_{s} + \hat{h}_{i,s}) + \eta^{-1} \hat{d}_{s} \right], \quad (3.16)
\]
where we used that \( b = 0 \) and we imposed the No Ponzi constraint\(^7\)

\[
\lim_{s \to \infty} \tilde{E}_{i,t} \delta^{s-t} \tilde{b}_{i,s+1} = 0.
\]

As shown in Appendix 3.B, after some algebraic manipulations we can derive the following consumption rule for agent \( i \):

\[
\hat{c}_{i,t} = (1 - \delta) \gamma \left( \frac{1}{\delta (1 - \delta)} \tilde{b}_{i,t-1} + \tilde{E}_{i,t} \sum_{s=t}^{\infty} \delta^{s-t} \frac{(1 - \eta^{-1}) (1 + \gamma)}{\gamma} \hat{w}_s + \eta^{-1} \hat{d}_s \right) \\
- \frac{\delta}{\sigma} \tilde{E}_{i,t} \sum_{s=t}^{\infty} \delta^{s-t} \left( \tilde{R}_s - \tilde{n}_{s+1} \right). 
\]  

(3.17)

The individual consumption rule (3.17) can be interpreted in the spirit of the canonical consumption model. The first three terms reflect the basic insight that current consumption depends on the expected future discounted wealth, while the last term arises from the assumption of a time-varying real interest rate, and represents deviations from the equilibrium level \( R = \delta^{-1} \) due to either variations in the nominal interest rate or inflation.

### 3.3.2 Firms

The optimization problem for a firm able to reset its price reads

\[
\max \tilde{E}_{j,t} \sum_{s=t}^{\infty} \omega^{s-t} Q_s \left( \frac{P_{j,t}}{P_s} - \frac{W_s}{P_s} \right) \left( \frac{P_{j,t}}{P_s} \right)^{-\eta} c_s,
\]

where \( \tilde{E}_{j,t} \) denotes the subjective expectation of firm \( j \). Since we are in an heterogeneous world and the optimization problem involves subjective expectations which can in principle differ among agents, we will have that in general different firms will set different prices. The average

---

\(^7\)Even if expectations are not rational we are assuming that households are not allowed to believe that they can borrow (and consume) as much as they want and just pay off the interest payments by borrowing more.
price set by firms optimizing at time $t$ will then be

$$P_t^* = \sum_j n_j P_{j,t},$$

where $n_j$ denotes the fraction of firms setting price $P_{j,t}$. The first order condition for firm $j$’s problem is given by

$$\tilde{E}_{j,t} \sum_{s=t}^\infty \omega^{s-t} Q_s \left( (1-\eta) \frac{1}{P_s} + \eta \frac{1}{P_{j,t}} w_s \right) \left( \frac{P_{j,t}}{P_s} \right)^{-\eta} c_s = 0,$$

(3.18)

where again $w_t = \frac{W_t}{P_t}$ denotes real marginal costs. Defining $p_{j,t} \equiv \frac{P_{j,t}}{P_t}$ and $\pi_{t,s} \equiv \frac{P_s}{P_t}$ we can rewrite (3.18) as

$$0 = \tilde{E}_{j,t} \sum_{s=t}^\infty \omega^{s-t} Q_s \left( (1-\eta) \frac{1}{P_s} p_{j,t} + \eta p_{j,t} w_s \right) \left( \frac{P_{j,t}}{P_t} \frac{P_s}{P_t} \right)^{-\eta} c_s$$

$$= \tilde{E}_{j,t} \sum_{s=t}^\infty \omega^{s-t} Q_s \left( (1-\eta) \frac{p_{j,t}}{\pi_{t,s}} + \eta w_s \right) \left( \frac{p_{j,t}}{\pi_{t,s}} \right)^{-\eta} c_s.$$

(3.19)

Moreover we can rewrite the average price set by firms optimizing at time $t$ as

$$p_t^* = \sum_j n_j p_{j,t},$$

so that the aggregate price level equation (3.9), multiplying both sides by $P_t^{\eta-1}$, can be written as

$$\left( \frac{P_t}{P_t} \right)^{1-\eta} = \omega \left( \frac{P_{t-1}}{P_t} \right)^{1-\eta} + (1-\omega) \left( \frac{P_t^*}{P_t} \right)^{1-\eta},$$

or, in terms of the inflation rate

$$1 = \omega \left( \frac{1}{\pi_t} \right)^{1-\eta} + (1-\omega) \left( p_t^* \right)^{1-\eta}.$$
Log-linearizing equation (3.20) gives

$$\hat{\pi}_t = \frac{(1 - \omega)}{\omega} \hat{p}_t.$$  \hfill (3.21)

As shown in Appendix 3.C we can linearize (3.19) to get the following pricing equation:

$$\hat{p}_{j,t} = (1 - \omega \delta) \tilde{E}_{j,t} \sum_{s=t}^{\infty} (\omega \delta)^{s-t} \hat{\psi}_s + \omega \delta \tilde{E}_{j,t} \sum_{s=t}^{\infty} (\omega \delta)^{s-t} \hat{\pi}_{s+1}. \hfill (3.22)$$

Equation (3.22) shows that when a firm sets its price it must be concerned with future marginal costs and future inflation because it may be unable to adjust its price for several periods.

### 3.3.3 Aggregation of individual decision rules

We assume for simplicity that households are running firms (or CEOs are appointed by shareholders so they are discounting profits the same way shareholders would do) so expectations of households and firms will be the same. This means that indexes $i$ and $j$ will coincide. In this way we can also justify the fact that firms are using the stochastic multiplier $Q_s = \delta^{s-t} (c_s/c_t)^{-\sigma}$ to discount future profits. Assume that there are different predictors available to agents in the economy. Agents using the same predictor will have the same expectation, so we can denote the fraction of agents adopting the forecasting rule $h$ at time $t$ as $n_{h,t}$.

Start from households consumption rule (3.17) and integrate over $i$ to get

$$\hat{c}_t = \frac{(1 - \delta) \gamma}{\gamma + (1 - \eta^{-1}) \sigma} \left( \frac{(1 - \eta^{-1}) (1 + \gamma)}{\gamma} \tilde{E}_t \sum_{s=t}^{\infty} \delta^{s-t} \hat{\psi}_s + \frac{1}{\eta} \tilde{E}_t \sum_{s=t}^{\infty} \delta^{s-t} \hat{\psi}_s \right)$$

$$- \frac{\delta}{\sigma} \tilde{E}_t \sum_{s=t}^{\infty} \delta^{s-t} \left( \tilde{R}_s - \hat{\pi}_{s+1} \right), \hfill (3.23)$$

where $\hat{c}_t = \int_0^1 \hat{c}_{i,t} f(i) di$, $\tilde{E}_t = \int_0^1 \tilde{E}_{i,t} f(i) di = \sum_{h=1}^{H} n_{h,t} \tilde{E}_{h,t}$, and we used that $\int_0^1 \tilde{b}_{i,t-1} f(i) di = 0$ by market clearing. We now want to derive an IS curve in terms of output and real interest rate. The optimality condition (3.14) can be rewritten in terms of the real wage, which is taken
as parametric by the individuals, as \( \hat{w}_t = \gamma \hat{h}_{i,t} + \sigma \hat{c}_{i,t} \). Aggregating over individuals we have that 
\( \hat{w}_t = \gamma \hat{h}_t + \sigma \hat{c}_t \). The market clearing condition implies \( \hat{c}_t = \hat{y}_t \), so we can write \( \hat{w}_t = \gamma \hat{h}_t + \sigma \hat{y}_t \).

In order to eliminate the term \( \hat{h}_t \) we can use the production function \( \hat{y}_t = \hat{h}_t \), so that

\[
\hat{w}_t = (\gamma + \sigma) \hat{y}_t. \tag{3.24}
\]

Moreover we can log-linearize the expression for total dividends \( d_t = y_t - w_t h_t \), using relation (3.24) and steady state values, in order to get

\[
\hat{d}_t = (1 - (\eta - 1) (\gamma + \sigma)) \hat{y}_t. \tag{3.25}
\]

Substituting (3.24) and (3.25) into (3.23) and using \( \hat{c}_t = \hat{y}_t \) we get\(^8\)

\[
\hat{y}_t = \frac{(1 - \delta) \gamma}{\gamma + (1 - \eta^{-1}) \sigma} \left( \frac{(1 - \eta^{-1}) \gamma}{\gamma} \tilde{E}_t \sum_{s=t}^{\infty} \delta^{s-t} (\gamma + \sigma) \hat{y}_s + \frac{1}{\eta} \tilde{E}_t \sum_{s=t}^{\infty} \delta^{s-t} (1 - (\eta - 1) (\gamma + \sigma)) \hat{y}_s \right) - \frac{\delta}{\sigma} \tilde{E}_t \sum_{s=t}^{\infty} \delta^{s-t} (\tilde{R}_s - \hat{\pi}_{s+1}) .
\]

Rearranging terms gives:

\[
\hat{y}_t = \frac{(1 - \delta) \gamma}{\gamma + (1 - \eta^{-1}) \sigma} \left( \frac{(1 - \eta^{-1}) (1 + \gamma) (\gamma + \sigma)}{\gamma} + \frac{1 - (\eta - 1) (\gamma + \sigma)}{\eta} \right) \tilde{E}_t \sum_{s=t}^{\infty} \delta^{s-t} \hat{y}_s - \frac{\delta}{\sigma} \tilde{E}_t \sum_{s=t}^{\infty} \delta^{s-t} (\tilde{R}_s - \hat{\pi}_{s+1}) ,
\]

which can be simplified to obtain the *Heterogeneous Expectations IS equation* (HE-IS)

\[
\hat{y}_t = (1 - \delta) \tilde{E}_t \sum_{s=t}^{\infty} \delta^{s-t} \hat{y}_s - \frac{\delta}{\sigma} \tilde{E}_t \sum_{s=t}^{\infty} \delta^{s-t} (\tilde{R}_s - \hat{\pi}_{s+1}) . \tag{3.26}
\]

Consider now firms’ pricing rule (3.22). Using that \( \hat{p}_t = \sum_j n_j \hat{p}_{j,t} \) we obtain from (3.21)

\(^8\)It is consistent to replace \( \hat{w}_t = (\gamma + \sigma) \hat{y}_t \) and \( \hat{d}_t = (1 - (\eta - 1) (\gamma + \sigma)) \hat{y}_t \) in equation (3.23) because we know that (ex post) these equalities will hold in each period \( t \).
that

\[ \hat{\pi}_t = \frac{(1 - \omega)(1 - \omega \delta)}{\omega} \tilde{E}_t \sum_{s=t}^{\infty} (\omega \delta)^{s-t} \hat{w}_s + \frac{(1 - \omega)\omega \delta}{\omega} \tilde{E}_t \sum_{s=t}^{\infty} (\omega \delta)^{s-t} \hat{\pi}_{s+1} \tag{3.27} \]

where again \( \tilde{E}_t = \int_0^1 \tilde{E}_{t,f}(\omega) d\omega = \sum_{h=1}^H \eta_{h,t}\tilde{E}_{h,t} \). Using that \( \hat{w}_t = (\sigma + \gamma) \hat{y}_t \), we can rewrite (3.27) in terms of output as

\[ \hat{\pi}_t = \frac{(1 - \omega)(1 - \omega \delta)}{\omega} (\sigma + \gamma) \tilde{E}_t \sum_{s=t}^{\infty} (\omega \delta)^{s-t} \hat{y}_s + \frac{(1 - \omega)\omega \delta}{\omega} \tilde{E}_t \sum_{s=t}^{\infty} (\omega \delta)^{s-t} \hat{\pi}_{s+1} \]

to obtain the Heterogeneous Expectations New Keynesian Phillips Curve (HE-NKPC)

\[ \hat{\pi}_t = k \tilde{E}_t \sum_{s=t}^{\infty} (\omega \delta)^{s-t} \hat{y}_s + (1 - \omega)\delta \tilde{E}_t \sum_{s=t}^{\infty} (\omega \delta)^{s-t} \hat{\pi}_{s+1}. \tag{3.28} \]

The benchmark RE model as a special case

It is easy to see that, under the hypothesis of homogeneous rational agents, the HE-IS equation (3.26) and the HE-NKPC relation (3.28) can be reduced to the standard IS equation (3.10) and NKPC relation (3.12). Leading (3.28) one period ahead and taking rational expectations gives

\[ E_t \hat{\pi}_{t+1} = k \hat{E}_t \hat{E}_{t+1} \sum_{s=t+1}^{\infty} (\omega \delta)^{s-t-1} \hat{y}_s + (1 - \omega)\delta \hat{E}_t \hat{E}_{t+1} \sum_{s=t+1}^{\infty} (\omega \delta)^{s-t-1} \hat{\pi}_{s+1} \]

\[ E_t \hat{\pi}_{t+1} = k \hat{E}_t \sum_{s=t+1}^{\infty} (\omega \delta)^{s-t-1} \hat{y}_s + (1 - \omega)\delta \hat{E}_t \sum_{s=t+1}^{\infty} (\omega \delta)^{s-t-1} \hat{\pi}_{s+1}, \]

where the second equality makes use of the law of iterated expectations. Rewriting (3.28) as

\[ \hat{\pi}_t = k \hat{y}_t + (1 - \omega)\delta \hat{E}_t \hat{\pi}_{t+1} + k \hat{E}_t \sum_{s=t+1}^{\infty} (\omega \delta)^{s-t} \hat{y}_s + (1 - \omega)\delta \hat{E}_t \sum_{s=t+1}^{\infty} (\omega \delta)^{s-t} \hat{\pi}_{s+1} \]
then gives

\[ \hat{\pi}_t = k\hat{y}_t + (1 - \omega)\delta E_t \hat{\pi}_{t+1} + \omega \delta E_t \hat{\pi}_{t+1} \]

\[ = k\hat{y}_t + \delta E_t \hat{\pi}_{t+1}. \]

For the IS equation we have that (3.26) can be rewritten as

\[ \hat{y}_t = (1 - \delta) \hat{y}_t + (1 - \delta) E_t \sum_{s=t+1}^{\infty} \delta^{s-t} \hat{y}_s - \frac{1}{\sigma} E_t \sum_{s=t}^{\infty} \delta^{s-t} \left( \hat{R}_s - \hat{\pi}_{s+1} \right) \]

\[ = (1 - \delta) E_t \sum_{s=t+1}^{\infty} \delta^{s-t-1} \hat{y}_s - \frac{1}{\sigma} E_t \sum_{s=t}^{\infty} \delta^{s-t} \left( \hat{R}_s - \hat{\pi}_{s+1} \right). \]  

(3.29)

Leading (3.29) one period ahead and taking rational expectations gives

\[ E_t \hat{y}_{t+1} = (1 - \delta) E_t \sum_{s=t+2}^{\infty} \delta^{s-t-2} \hat{y}_s - \frac{1}{\sigma} E_t \sum_{s=t+1}^{\infty} \delta^{s-t-1} \left( \hat{R}_s - \hat{\pi}_{s+1} \right) \]

so that equation (3.29) can be rewritten as

\[ \hat{y}_t = (1 - \delta) E_t \hat{y}_{t+1} + (1 - \delta) E_t \sum_{s=t+1}^{\infty} \delta^{s-t-1} \hat{y}_s - \frac{1}{\sigma} \left( \hat{R}_t - E_t \hat{\pi}_{t+1} \right) \]

\[ - \frac{1}{\sigma} E_t \sum_{s=t+1}^{\infty} \delta^{s-t} \left( \hat{R}_s - \hat{\pi}_{s+1} \right) \]

\[ = (1 - \delta) E_t \hat{y}_{t+1} + \delta E_t \hat{y}_{t+1} - \frac{1}{\sigma} \left( \hat{R}_t - E_t \hat{\pi}_{t+1} \right) \]

\[ = E_t \hat{y}_{t+1} - \frac{1}{\sigma} \left( \hat{R}_t - E_t \hat{\pi}_{t+1} \right). \]

The analysis performed in this section shows that, when the microfoundations underpinning the New Keynesian model are solved under non rational heterogeneous expectations, the aggregate dynamics depend on long horizon forecasts\(^9\) as can be seen in equations (3.26) and

\(^9\)From a behavioral perspective it may seem a little awkward that boundedly rational agents are basing their decisions on long horizon forecasts. However this is a direct consequence of the fact that we are only departing from the benchmark model by relaxing the rationality assumption in the way agents form expectations, but we are keeping the assumption that agents behave optimally given their subjective beliefs.
In fact, boundedly rational agents do not have a complete model of determination of aggregate variables. As noted in Preston (2005), neither the aggregate demand relation (3.26) nor the Phillips curve (3.28) can be simplified as in the rational expectations equilibrium analysis where, since expectations are taken with respect to the correct distribution of future endogenous variables, the law of iterated expectations holds at the aggregate level and therefore only one period ahead expectations matter for aggregate dynamics.

3.4 Monetary policy with heterogeneous expectations

In this section we use the model with heterogeneous expectations developed in Section 3.3 for monetary policy analysis. Contemporary policy discussions argued that a desirable interest rate rule has to involve feedback from endogenous variables such as inflation and/or real activity. Many authors considered simple interest rate rules with endogenous components of the form:

\[ \hat{R}_t = \phi_\pi \hat{\pi}_t + \phi_y \hat{y}_t \]  

(3.30)

and analyzed determinacy properties of the rational expectation equilibrium. Desirable monetary rules should avoid indeterminacy (i.e. multiple bounded equilibria) and sunspot fluctuations as emphasized in the original analysis of Sargent and Wallace (1975). Under rational expectations a necessary and sufficient condition for the equilibrium to be determinate is given by the Taylor principle:

\[ k(\phi_\pi - 1) + (1 - \delta)\phi_y > 0 \]  

(3.31)

stating that the monetary authority should respond to inflation and real activity by adjusting the nominal interest rate with "sufficient strength"\textsuperscript{10}. Recent studies investigated the validity of such a policy recommendation when expectations depart from the rational benchmark. In the context of a New Keynesian monetary model Bullard and Mitra (2002) assume that agents do not initially have rational expectations, and that they instead form forecasts by using recursive

\textsuperscript{10}See Woodford (2003) for a proof.
least squares. Using the \textit{E-stability}\textsuperscript{11} criterion they show that an interest rate rule that satisfies the Taylor principle induces learnability of the RE equilibrium. Preston (2005) studies the learnability of the RE equilibrium in a New Keynesian setting with long horizon forecasts. He shows that under least squares learning dynamics the Taylor principle (3.31) is necessary and sufficient for E-stability. However, most models studying the validity of classical policy recommendation in contexts that depart from the RE assumption assume a representative agent who is learning about the economy.

The focus of this section is to analyze the dynamical consequences of a policy regime as in (3.30) when agents have \textit{heterogeneous beliefs}. The framework developed in the first part of the paper can now be applied to evaluate the desirability of the monetary policy rule (3.30) in the presence of heterogeneous expectations. In particular we want to investigate whether the standard advices to policy makers lead the dynamics to converge to the RE equilibrium in a world with heterogeneous expectations.

3.4.1 Specification of expectations and evolutionary dynamics

In this section we characterize individuals’ expectation schemes and describe the evolution of beliefs over time. There is ample empirical evidence documenting that private sector beliefs, when proxied by surveys, are characterized by heterogeneity. Carroll (2003) and Branch (2004) analyze the Michigan survey data on inflation expectations and find results pointing in the direction of an intermediate degree of rationality. Pfajfar and Santoro (2010) study the same survey and document the fact that agents in different percentiles of the survey seem to be associated with forecasting schemes characterized by different degrees of rationality. Hommes, Sonnemans, Tuinstra, and van de Velden (2005), Adam (2007), Pfajfar and Zakelj (2010), and Assenza, Heemeijer, Hommes, and Massaro (2011) find evidence for pervasive heterogeneity of beliefs in learning to forecast experiments with human subjects. In particular, Pfajfar and Zakelj (2010) find a significant proportion of rational agents (around 40\%) in a monetary policy

\textsuperscript{11}For a detailed presentation of the E-stability concept and its relation with real time learning, see Evans and Honkapohja (2001)
experiment set up in the New Keynesian framework.

Building on the empirical evidence mentioned above, we assume that a fraction of agents are perfectly rational and the remainder are boundedly rational, using heuristics to forecast macroeconomic variables. We thus have in mind an economy in which some agents face cognitive problems in understanding and processing information and hence eventually make mistakes in forecasting macroeconomic variables while other agents, by paying some information gathering and processing costs \( C \geq 0 \) per period, have rational expectations.

Before describing the full model with both perfectly rational and boundedly rational agents, we will characterize the dynamic feedback system in which macroeconomic variables and heterogeneous subjective expectations co-evolve over time along the lines of Brock, Hommes, and Wagener (2005) and Diks and van der Weide (2005). We assume that agents do not fully understand how macroeconomic variables are determined and hence have biased forecasts. One might think about an economy in which subject do not know the target of the monetary authority and have biased beliefs about it. Moreover, as a result of cognitive limitations, there are differences in the use of information and thus heterogeneity in individual forecasts. As argued in De Grauwe (2010), the assumption of a simple biased forecast can be viewed as a parsimonious representation of a world where agents do not know the underlying economic model and have a biased view about this model.

The point predictors used by the agents are represented in the belief space \( \Theta \) and parameterized by the belief parameter \( \theta \). Each value \( \theta_i \in \Theta \) represents a strategy that fully characterizes the behavior of individual \( i \). We will consider the simplest possible case of constant predictors\(^{12}\). Within this class of simple rules we allow for differences in the conclusions that agents draw when processing information, as well as biases and idiosyncracies. Therefore in every period we will have a distribution of point predictions\(^{13}\). This specification is quite general in


\(^{13}\)Anufriev, Assenza, Hommes, and Massaro (2008) use the concept of Large Type Limit (LTL) developed by Brock, Hommes, and Wagener (2005) to analyze inflation dynamics when the number of strategies available to the
the sense that the individual point predictions can be thought of as the outcome of any mental process or estimation technique. Alternatively one could think of agents selecting between various Bayesian forecasting models each differing by their very strong priors. From the modeler’s point of view, the strategy \( \theta_{i,t} \) used by agent \( i \) at time \( t \) is a random variable distributed according to the probability density function \( \psi_t(\theta) \). Given that agents have limited cognitive abilities, their forecasts will typically be biased\(^{14}\). In order to limit the wilderness of bounded rationality and avoid completely irrational behavior we have to introduce discipline in the selection of rules. We will achieve this discipline by subjecting the choice of heuristics to a fitness criterion, and by introducing a selection mechanism that allows agents to learn from their forecasting mistakes.

Predictors will, in fact, be evaluated according to an evolutionary fitness measure. Given the performance of predictors at time \( t-1 \), denoted by \( U_{t-1}(\theta) \), we assume as in Diks and van der Weide (2005) that the distribution of beliefs evolves over time as a function of past performances according to the continuous choice model:

\[
\psi_t(\theta) = \frac{\nu(\theta) e^{\beta U_{t-1}(\theta)}}{Z_{t-1}},
\]

where \( Z_{t-1} \) is a normalization factor independent of \( \theta \) given by

\[
Z_{t-1} = \int_{\Theta} \nu(\vartheta) e^{\beta U_{t-1}(\vartheta)} d\vartheta,
\]

and \( \nu(\theta) \) is a opportunity function that can put different weights on different parts of the beliefs space. The parameter \( \beta \) refers to the intensity of choice and measures how sensitive agents are agents tend to infinity. The concept of Continuous Beliefs System (CBS) developed by Diks and van der Weide (2002) and used in this paper is a generalization of the LTL concept (see Diks and van der Weide (2003) for a discussion).

\(^{14}\)The point predictions of the heuristics can coincide with the perfect foresight point forecast because we did not make any restrictive assumption on the support of the distribution of beliefs.
to differences in performances. Notice that (3.32), which can be rewritten as

\[ \psi_t(\theta) \propto \nu(\theta) e^{\beta U_{t-1}(\theta)}, \]  

(3.33)

is a rule for updating the distribution of beliefs as new information becomes available similar to a Bayesian updating rule. In fact, \( \nu(\theta) \) plays a role similar to a prior, reflecting the a priori faith of individuals in parameters within certain regions of the parameter space, and \( \psi_t(\theta) \) to a posterior. The fitness measure enters the beliefs distribution through the term \( e^{\beta U_{t-1}(\theta)} \), which plays a role similar to the likelihood in Bayesian statistics. In fact, a Bayesian updating rule in the usual form is recovered exactly when \( \beta = 1 \) and the performance measure \( U_{t-1}(\theta) \) is the log-likelihood function of an econometric model, given the available observations. In order to simplify the analysis we will assume that the performance measure is given by the past squared forecast error\(^1\)

\[ U_{t-1}(\theta) = -(\theta - x_{t-1})^2, \]  

(3.34)

for \( x \in \{y, \pi, R\} \). Having observed and compared overall past performance, all agents subsequently adapt their beliefs. The distribution of beliefs at time \( t \) is thus given by means of the continuous choice model (3.32). Co-evolution of the distribution of beliefs with the observed aggregate variables thus emerges through the ongoing evaluation of predictors. As in Diks and van der Weide (2005) we assume a constant opportunity function \( \nu(\theta) = 1 \), meaning that agents assign the same initial weight to all parameter values. Since the utility function is a quadratic function in the belief parameter \( \theta \), it follows that, for all \( t \), the distribution of beliefs is described by a normal distribution:

\[ \psi_t(\theta) = \frac{1}{\sqrt{2\pi} \sigma_t} \exp \left( -\frac{1}{2} \left( \frac{\theta - \mu_t}{\sigma_t} \right)^2 \right). \]  

(3.35)

Branch (2004) finds empirical evidence for dynamic switching depending on the squared errors of the predictors in survey data on individuals’ expectations, while Pfajfar and Zakelj (2010) and Hommes (2011) find empirical evidence for dynamic switching depending on the squared errors of the predictors in experimental data on individuals’ expectations.
Substituting the specified performance measure $U_{t-1}(\theta) = -(\theta - x_{t-1})^2$ and opportunity function $\nu(\theta) = 1$ in (3.33) we get

$$\psi_t(\theta) \propto \exp \left( -\beta (\theta - x_{t-1})^2 \right). \quad (3.36)$$

By comparing the exponents in equations (3.35) and (3.36), the mean $\mu_t$ and the variance $\sigma_t^2$ can be seen to evolve according to

$$\mu_t = x_{t-1} \quad (3.37)$$
$$\sigma_t^2 = \frac{1}{2\beta^t}, \quad (3.38)$$

while the remaining terms independent of $\theta$ are accounted for by the normalization factor $Z_{t-1}$.

The dynamics of $\mu_t$ and $\sigma_t^2$ fully characterize the evolution of $\psi_t(\theta)$ over time. Dividing the continuum of agents interval $[0,1]$ in $n$ parts and taking the limit for $n \to \infty$ we can write

$$\int_0^1 \theta_{i,t} f(i) di = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \theta_{j,t}. \quad (3.39)$$

In the absence of dependence among agents, the law of large number applies, and it follows that the average belief will converge to $\mu_t$, that is

$$\int_0^1 \theta_{i,t} f(i) di = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \theta_{j,t} = \mathbb{E}[\theta_{i,t}] = \int_{\Theta} \theta \psi_t(\theta) d\theta = \mu_t. \quad (3.39)$$

**Rational versus biased beliefs**

We now turn to the description of the heterogeneous expectations model with rational and biased beliefs. Introducing rational agents in the economy we can rewrite the aggregate equations
\[
\hat{y}_t = n_{RE,t}E_t \sum_{s=t}^{\infty} \delta^{s-t} \left( (1 - \delta) \hat{y}_s - \frac{\delta}{\sigma} (\hat{R}_s - \hat{\pi}_{s+1}) \right) + \int_{n_{RE,t}}^{1} \tilde{E}_{i,t} \sum_{s=t}^{\infty} \delta^{s-t} \left( (1 - \delta) \hat{y}_s - \frac{\delta}{\sigma} (\hat{R}_s - \hat{\pi}_{s+1}) \right) f(i) di
\]

\[
\tilde{\pi}_t = n_{RE,t}E_t \sum_{s=t}^{\infty} (\omega \delta)^{s-t} \left( k \hat{y}_s + (1 - \omega) \delta \hat{\pi}_{s+1} \right) + \int_{n_{RE,t}}^{1} \tilde{E}_{i,t} \sum_{s=t}^{\infty} (\omega \delta)^{s-t} \left( k \hat{y}_s + (1 - \omega) \delta \hat{\pi}_{s+1} \right) f(i) di, \tag{3.40}
\]

where \( n_{RE,t} \) denotes the fraction of rational agents at time \( t \) and \( E_t \) is the rational expectation operator.

We follow Kreps (1998) and Sargent (1999) in assuming that agents solve an anticipated utility problem, i.e. when agents solve their optimization problem they hold their expectation operator fixed and assume that it remains fixed for all future periods. Using this assumption we can simplify boundedly rational individual forecasts over the long horizon as

\[
\tilde{E}_{i,t} \sum_{s=t}^{\infty} \delta^{s-t} \tilde{x}_s = \frac{1}{1 - \delta} \theta_{i,x,t}, \tag{3.41}
\]

where \( \theta_{i,x,t} \) denotes the biased forecasts of agent \( i \) in period \( t \) for \( x \in \{y, \pi, R\} \). Using the results in (3.37), (3.38) and (3.39) we have that the average forecast of boundedly rational biased agents is given by

\[
\int_{n_{RE,t}}^{1} \theta_{i,x,t} f(i) di = (1 - n_{RE,t}) \mu_{x,t} = (1 - n_{RE,t}) \tilde{x}_{t-1}, \tag{3.42}
\]

for \( x \in \{y, \pi, R\} \). Therefore, using the results (3.41) and (3.42), we can rewrite system (3.40)
\[
\hat{y}_t = n_{RE,t} E_t \sum_{s=t}^{\infty} \delta^{s-t} \left( (1 - \delta) \hat{y}_s - \frac{\delta}{\sigma}(\hat{R}_s - \hat{\pi}_{s+1}) \right) + (1 - n_{RE,t}) V_y(\hat{y}_{t-1}, \hat{R}_t, \hat{R}_{t-1}, \hat{\pi}_{t-1})
\]

\[
\hat{\pi}_t = n_{RE,t} E_t \sum_{s=t}^{\infty} (\omega \delta)^{s-t} \left( k \hat{y}_s + (1 - \omega) \delta \hat{\pi}_{s+1} \right) + (1 - n_{RE,t}) V_\pi(\hat{y}_{t-1}, \hat{\pi}_{t-1}),
\]

(3.43)

where \(V_x(.)\) denotes a function linear in its arguments, for \(x \in \{y, \pi\}\), described in equations (3.56) and (3.57) in Appendix 3.D.

At the beginning of each period agents have to decide whether to pay an information gathering and processing cost \(C \geq 0\) or use simple heuristics in order to forecast macroeconomic variables. We use the discrete choice model of Brock and Hommes (1997) to model this decision between the costly rational predictor or freely use a simple biased rule. We now recall the discrete choice model in general terms and then we apply the main results to our framework.

Assume that in general agents can form expectations choosing from \(H\) different forecasting strategies. The fraction of individuals using the forecasting rule \(h\) at time \(t\) is denoted by \(n_{h,t}\).

With dynamic predictor selection the fractions are updated in every period according to the well known multinomial logit law of motion (see Manski and McFadden (1981) for details)

\[
n_{h,t} = \frac{e^{\beta U_{h,t-1}}}{\sum_{h=1}^{H} e^{\beta U_{h,t-1}}},
\]

(3.44)

where \(U_{h,t-1}\) is the fitness metric of predictor \(h\) at time \(t - 1\). Note that the higher the past performance of a forecasting rule \(h\), the higher the probability that an agent will select strategy \(h\). The parameter \(\beta\) refers to the intensity of choice and it reflects the sensitivity of the mass of agents to selecting the optimal prediction strategy. The case \(\beta = 0\) corresponds to the situation in which differences in fitness can not be observed, so agents do not switch between strategies and all fractions are constant and equal to \(1/H\). The case \(\beta = \infty\) corresponds to the “neoclassical” limit in which the fitness can be observed perfectly and in every period all agents choose the best predictor.
In our setup the discrete logit model is used to model the choice between two different classes of predictors, namely the RE predictor and the heuristics. Assuming that the fitness measure is given by past squared forecast error minus the cost of the predictor, in order to decide whether to be rational or not, each individual is comparing the cost \( C \) (since the RE predictor corresponds to perfect foresight and thus has zero forecast error) with the average past forecast error of the freely available heuristics, summed over the three variables being forecast, given by\(^{16}\)

\[
\sum_x \left( \int (\theta_{x,t-1} - \hat{x}_{t-1})^2 \psi_{t-1}(\theta_x) d\theta_x \right) = \frac{3}{2\beta},
\]

for \( x \in \{y, \pi, R\} \). We thus have that the fraction \( n_{RE,t} \) is constant over time and given by

\[
n_{RE} = \frac{e^{-\beta_d C}}{e^{-\beta_d C} + e^{-\beta_d \frac{1}{2\beta}C}}, \tag{3.45}
\]

where \( \beta_d \) and \( \beta_c \) denote respectively the intensities of choice of the discrete and continuous choice model.

Using the results derived before and closing the model with the interest rate rule (3.30) we can rewrite system (3.43) in the standard matrix form as\(^{17}\)

\[
\begin{bmatrix}
E_t \hat{y}_{t+1} \\
E_t \hat{\pi}_{t+1} \\
\hat{Y}_{1t+1} \\
\hat{\Pi}_{1t+1}
\end{bmatrix}
= \begin{bmatrix}
\gamma_{yy} & \gamma_{y\pi} & \gamma_{yY1} & \gamma_{y\Pi1} \\
\gamma_{\pi y} & \gamma_{\pi\pi} & \gamma_{\pi Y1} & \gamma_{\pi\Pi1} \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\hat{y}_t \\
\hat{\pi}_t \\
\hat{Y}_{1t} \\
\hat{\Pi}_{1t}
\end{bmatrix},
\tag{3.46}
\]

where \( \hat{Y}_{1t} = \hat{y}_{t-1} \) and \( \hat{\Pi}_{1t} = \hat{\pi}_{t-1} \). We are now ready to investigate the validity of the standard monetary policy recommendations in the context of a New Keynesian model with a rich variety of forecasting rules, including the rational expectations predictor.

\(^{16}\)Note that we are assuming that the intensity of choice governing the evolution of the distributions \( \psi_t(\theta_y) \), \( \psi_t(\theta_\pi) \), and \( \psi_t(\theta_R) \) is the same. Relaxing this assumption does not alter our qualitative results.

\(^{17}\)Description of the coefficients is given in Appendix 3.D.
3.4.2 Numerical analysis of determinacy

Model (3.46) has the form of a rational expectations model with predetermined variables. Techniques for analyzing the determinacy properties of a linear model under rational expectations are well known (see, for example, Blanchard and Kahn (1980)). Define the transition matrix in (3.46) as

\[
\mathbf{M} = \begin{bmatrix}
\gamma_{yy} & \gamma_{y\pi} & \gamma_{yY} & \gamma_{y\Pi} \\
\gamma_{\pi y} & \gamma_{\pi\pi} & \gamma_{\pi Y} & \gamma_{\pi\Pi} \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}.
\]

The determinacy properties of the model depend on the magnitude of the eigenvalues of \(\mathbf{M}\).

In our model with heterogeneous agents we have two predetermined variables, so that determinacy obtains when two eigenvalues are inside the unit circle. Fewer eigenvalues inside the unit circle imply explosiveness and more imply indeterminacy. Given the large number of the model’s parameters we will perform a numerical analysis of determinacy properties. The model’s parameters could be estimated but such an estimation is beyond the scope of this simple monetary policy exercise. We will therefore use calibrated values for the structural parameters as in Woodford (2003), namely \(\delta = 0.99\), \(\sigma = 0.157\), \(k = 0.024\) and \(\omega = 0.66\), and treat the policy parameters \(\phi_\pi\) and \(\phi_y\), together with the fraction of rational agents \(n_{RE}\), as bifurcation parameters\(^{18}\). We will follow Branch and McGough (2009) and use \(0 < \phi_\pi < 2\), \(0 < \phi_y < 2\) as the benchmark policy space.

Fig. 3.1 shows the bifurcation surfaces at which eigenvalues of \(\mathbf{M}\) cross the unit circle in the 3-dimensional \((\phi_\pi, \phi_y, n_{RE})\) parameters space\(^{19}\). By looking at the graphs in Fig. 3.1 it is possible to study how the bifurcations loci shift in the benchmark policy space as a continuous function of the degree of rationality in the economy.

Fig. 3.1a refers to the fold bifurcation, associated with a qualitative change in the behavior

\(^{18}\)The fundamental result of this section, namely that the presence of bounded rationality represents an important source of instability which may alter significantly the determinacy properties of the model, is robust across calibrations.

\(^{19}\)See Kuznetsov (1995) for a detailed mathematical treatment of bifurcation theory.
Figure 3.1: **Left panels:** Bifurcation surfaces. **Right panels:** Topviews.

(a) Fold

(b) Flip

(c) Neimark-Sacker
of the economy due to the appearance of an eigenvalue $\lambda_1 = 1$ of the matrix $M$.

Fig. 3.1b refers to the flip bifurcation, associated with a qualitative change in the behavior of the economy due to the appearance of an eigenvalue $\lambda_1 = -1$ of the matrix $\tilde{M}$.

Fig. 3.1c refers to the Neimark-Sacker bifurcation, associated with a qualitative change in the behavior of the economy due to the appearance of complex eigenvalues $\lambda_{1,2} = e^{\pm i\vartheta}$ of the matrix $\tilde{M}$, where $\vartheta$ is an arbitrary angle. Details about the computation of the bifurcation surfaces are provided in Appendix 3.E.

In order to draw conclusions about the determinacy properties of the model, one would need to overlap Figs. 3.1a, 3.1b, and 3.1c, and study the regions resulting from the intersections of the bifurcation surfaces. Even though this does not present particular problems from the computational point of view, it is easier to visualize the results in the 2-dimensional $(\phi_x, \phi_y)$ parameter space for a given fraction of rational agents $n_{RE}$. The graphs in Fig. 3.1 are then useful to understand how the bifurcation curves shift in the 2-dimensional $(\phi_x, \phi_y)$ space as more and more boundedly rational agents are introduced in the economy.

Fig. 3.2 shows how the determinacy properties of the model change when we allow for small departures from the representative rational agent benchmark. We present determinacy results for some empirically relevant cases. The choice of the fractions of rational agents in the economy reflects the findings of Galí and Gertler (1999) who estimated a reduced form NKPC with a fraction of forward looking rational firms and a fraction of firms with backward looking behavior and found a degree of rationality between 0.6 and 0.8, and the findings of Pfajfar and Zakelj (2010) who provided evidence of a degree of rationality around 0.4 in a monetary policy laboratory experiment.

Fig. 3.2a indicates the outcome under full rationality and it is consistent with the usual prescription of the RE literature: an interest rate rule satisfying the Taylor principle ensures determinacy. However, when we add a fraction of boundedly rational agents in the economy, the boundaries of the determinacy regions are sensibly altered. We indeed observe in Fig. 3.2b that when a small portion of non-rational agents is added to the economy, the region corresponding
Figure 3.2: Determinacy properties for different values of the fraction $n_{RE}$.

D denotes determinacy, I denotes indeterminacy, and E denotes explosiveness.
The labels “fold”, “flip”, and “NS” correspond respectively to fold, flip, and Neimark-Sacker bifurcations.

(a): RE benchmark, $n_{RE} = 1$, (b): Galí and Gertler (1999) (upper bound), $n_{RE} = 0.8$, (c): Galí and Gertler (1999) (lower bound), $n_{RE} = 0.6$, (d): Pfajfar and Zakelj (2010), $n_{RE} = 0.4$.

To determinacy decreases in size, meaning that policy rules that obey the Taylor principle may not enforce determinacy. By looking at Fig. 3.1a we notice that the curve separating the regions of determinacy and indeterminacy in the benchmark RE case rotates counterclockwise and dis-
appears from the relevant policy space as the fraction of rational agents decreases. Moreover, a new fold bifurcation curve emerges and rotates counterclockwise as $n_{RE}$ decreases. This curve corresponds to the curve labeled as “fold” in Fig. 3.2b. In the passage from Fig. 3.2a to Fig. 3.2b we notice that both a Neimark-Sacker and a flip bifurcation curve, respectively labeled as “N-S” and “flip”, emerge. These bifurcation curves separate respectively a region of explosiveness from the region of indeterminacy, and another region of explosiveness from the region of determinacy.

It is important to notice that, in the presence of bounded rationality, the dynamics out of the determinacy region are quite different from the benchmark RE model. In fact, when boundedly rational agents are present in the economy, the model can show instability (i.e., explosive equilibria) instead of indeterminacy (i.e., multiple bounded equilibria).

As the fraction of rational agents decreases further, the determinacy region keeps on shrinking as shown in Figs. 3.2c and 3.2d. In fact, while Fig. 3.1a shows that the fold bifurcation curve approaches an asymptote, Fig. 3.1b shows that the flip bifurcation curve keeps on rotating counterclockwise as $n_{RE}$ decreases, restricting the region of policy actions ensuring determinacy.

Notice also that the indeterminacy region shrinks as $n_{RE}$ decreases. In fact Fig. 3.1c shows that, as the fraction of rational agents decreases, the Neimark-Sacker bifurcation surface associated with values of the policy reaction coefficient $\phi_\pi < 1$ approaches the fold bifurcation surface of Fig. 3.1a.

Of course the results presented in Fig. 3.2 are not exhaustive of all possible determinacy scenarios, but they suffice to make clear the main point of our monetary policy exercise, namely that the presence of bounded rationality may alter significantly the determinacy properties of the model, and therefore that in a world with heterogeneous expectations an interest rate rule that obeys the Taylor principle does not necessarily guarantee a determinate equilibrium. As a concrete example, consider the interest rate rule with $\phi_\pi = 1.5$ and $\phi_y = 0.125$ proposed by
Taylor (1993). This policy rule ensures determinacy in the cases $n_{RE} = 1$ and $n_{RE} = 0.8$ (see top panels Fig. 3.2), however, when the fraction of rational agents decreases to $n_{RE} = 0.6$ and $n_{RE} = 0.4$, the Taylor rule falls in the explosiveness region (see bottom panels Fig. 3.2). The intuition for such result can be found in the logic that produces unique stable dynamics in a New Keynesian model with homogeneous rational expectations. The stabilization mechanism after a shock in the New Keynesian model relies on unstable dynamics in the sense that, by obeying the Taylor principle, the monetary authority induces dynamics that will explode in any equilibrium but one. Ruling out explosive paths guarantees then uniqueness of output and inflation equilibrium paths. However the presence of boundedly rational agents, whose expectations co-evolve with macroeconomic variables in a dynamic feedback system, introduces backward-looking components in the dynamics of the model. In the presence of backward-looking agents in the economy, parameters’ regions that ensured unstable eigenvalues and thus determinate equilibrium in a completely forward-looking model, may now induce unstable dynamics. This result is of a crucial importance for the conduct of a sound monetary policy. In fact the rationale for recommendations advising to conduct policies within the determinacy region is based on the fact that determinacy reduces volatility of inflation and output. It is therefore very important to account for bounded rationality when designing monetary policy since policies constructed to achieve determinacy under homogeneous rational expectations may be destabilizing when expectations are heterogeneous.

### 3.5 Conclusions

Recent papers provided empirical evidence in favor of heterogeneity in individual expectations using survey data as well as experimental data. Building on this evidence, we derived a general micro-founded version of the New Keynesian framework for the analysis of monetary policy

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20 The original coefficient on the output term $\phi_y = 0.5$ has been divided by 4 so that $\phi_y$ corresponds to the output coefficient in a standard Taylor rule written in terms of annualized interest and inflation rates (see Woodford (2003)).

21 See for example Cochrane (2010) for a discussion.
in the presence of heterogeneous expectations. We model individual behavior as being optimal given subjective expectations and derive a law of motion for output and inflation by explicitly aggregating individual decision rules. One advantage of our approach is that it is sufficiently general to consider a rich ecology of forecasting rules, ranging from simple heuristics to the very sophisticated rational expectation predictor. The RE benchmark is indeed a special case of our heterogeneous expectations New Keynesian model.

We designed an economy where agents can decide whether to pay some costs for rationality or use simple heuristics to forecast macroeconomic variables. After having characterized the dynamic feedback system in which aggregate variables and subjective expectations co-evolve over time, we performed a simple monetary policy exercise to illustrate the implications of expectations’ heterogeneity on the determinacy properties of the model. Our central finding is that in a world with heterogeneous agents, the Taylor principle does not necessarily guarantee a unique equilibrium. Therefore policy makers should seriously take into account bounded rationality when designing monetary policies. In fact policy attempts to achieve determinacy under homogeneous rational expectations may destabilize the economy even when only a small fraction of boundedly rational agents is present in the economy.

This paper provides a theoretical DSGE framework for the analysis of monetary policy in the presence of heterogeneous beliefs. Future work should try to estimate the degree of rationality in the economy and investigate further the implications of heterogeneity in the way agents form their beliefs for the global dynamics of the economy and optimal monetary policy design.
Appendix 3.A  Steady state representative rational agent model

We consider a zero inflation steady state ($\pi = 1$) around which we will linearize the equations of the model. The consumption Euler equation (3.3) implies then

$$ R = \delta^{-1}. $$

Using the production function we get

$$ y = h. $$

From the pricing equation (3.7) computed in steady state we get

$$ w = \frac{\eta - 1}{\eta}. $$

Using that $c = y = h$ in the labor supply function (3.4) implies

$$ \chi h^{\gamma+\sigma} = \frac{\eta - 1}{\eta}, $$

so normalizing $\chi = \frac{\eta - 1}{\eta}$ we have

$$ h = c = y = 1. $$

Finally, using the definition of total dividends we get the steady state value

$$ d = 1 - \frac{\eta - 1}{\eta} = \frac{1}{\eta}. $$
Appendix 3.B Derivation of consumption rule (3.17)

Consider household $i$’s first order conditions

$$\hat{c}_{i,t} = \tilde{E}_{i,t}\hat{c}_{i,t+1} - \sigma^{-1}\left(\hat{R}_t - \tilde{E}_{i,t}\hat{\pi}_{t+1}\right) \tag{3.47}$$

and the intertemporal budget constraint

$$\tilde{E}_{i,t}\sum_{s=t}^{\infty} \delta^{s-t}\hat{c}_{i,s} = \tilde{E}_{i,t}\sum_{s=t}^{\infty} \delta^{s-t}\left(\delta^{-1}\hat{b}_{i,t-1} + \left(1 - \eta^{-1}\right)\left(\hat{w}_{s} + \hat{h}_{i,s}\right) + \eta^{-1}\hat{d}_{s}\right). \tag{3.49}$$

By substituting (3.48) into (3.49) we get

$$\tilde{E}_{i,t}\sum_{s=t}^{\infty} \delta^{s-t}\hat{c}_{i,s} = \tilde{E}_{i,t}\sum_{s=t}^{\infty} \delta^{s-t}\left(\delta^{-1}\hat{b}_{i,t-1} + \left(1 - \eta^{-1}\right)\hat{w}_{s} + \frac{1 - \eta^{-1}}{\gamma}\left(\hat{w}_{s} - \sigma\hat{c}_{i,s}\right) + \eta^{-1}\hat{d}_{s}\right),$$

which can be rewritten as

$$\tilde{E}_{i,t}\sum_{s=t}^{\infty} \delta^{s-t}\left(1 + \frac{(1 - \eta^{-1})\sigma}{\gamma}\right)\hat{c}_{i,s} = \tilde{E}_{i,t}\sum_{s=t}^{\infty} \delta^{s-t}\left(\delta^{-1}\hat{b}_{i,t-1} + \frac{(1 - \eta^{-1})(1 + \gamma)}{\gamma}\hat{w}_{s} + \eta^{-1}\hat{d}_{s}\right). \tag{3.50}$$

We can now iterate equation (3.47) to get

$$\tilde{E}_{i,t}\hat{c}_{i,s} = \hat{c}_{i,t} + \sigma^{-1}\tilde{E}_{i,t}\sum_{k=t}^{s} \left(\hat{R}_{k} - \hat{\pi}_{k+1}\right). \tag{3.51}$$

Substituting (3.51) in the LHS of (3.50) gives

$$\left(1 + \frac{(1 - \eta^{-1})\sigma}{\gamma}\right)\tilde{E}_{i,t}\sum_{s=t}^{\infty} \delta^{s-t}\left(\hat{c}_{i,t} + \sigma^{-1}\sum_{k=t}^{s} \left(\hat{R}_{k} - \hat{\pi}_{k+1}\right)\right) = \left(1 + \frac{(1 - \eta^{-1})\sigma}{\gamma}\right)\left(\tilde{E}_{i,t}\sum_{s=t}^{\infty} \delta^{s-t}\hat{c}_{i,t} + \sigma^{-1}\tilde{E}_{i,t}\sum_{k=t}^{\infty} \sum_{k=t}^{s} \delta^{s-t}\left(\hat{R}_{k} - \hat{\pi}_{k+1}\right)\right)$$

$$= \left(1 + \frac{(1 - \eta^{-1})\sigma}{\gamma}\right)\left(\frac{1}{1 - \sigma}\hat{c}_{i,t} + \sigma^{-1}\tilde{E}_{i,t}\sum_{s=t}^{\infty} \sum_{k=t}^{s} \delta^{s-t}\left(\hat{R}_{k} - \hat{\pi}_{k+1}\right)\right). \tag{3.52}$$
Since the term

\[
\tilde{E}_{i,t} \sum_{s=t}^{\infty} \sum_{k=t}^{s} \delta^{s-t} \left( \hat{R}_k - \hat{\pi}_{k+1} \right)
\]

\[
= \tilde{E}_{i,t} \left[ \delta \left( \hat{R}_t - \hat{\pi}_{t+1} \right) + \delta^2 \left( \hat{R}_t - \hat{\pi}_{t+1} \right) + \delta^2 \left( \hat{R}_{t+1} - \hat{\pi}_{t+2} \right) + \delta^3 \left( \hat{R}_t - \hat{\pi}_{t+1} \right) + \delta^3 \left( \hat{R}_{t+1} - \hat{\pi}_{t+2} \right) + \delta^3 \left( \hat{R}_{t+2} - \hat{\pi}_{t+3} \right) + \right.
\]

\[
\left. \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quasi-plaintext
Appendix 3.C Derivation of pricing rule (3.22)

We can rewrite equation (3.19) as

\[ p_{j,t} \eta \tilde{E}_{j,t} \sum_{s=t}^{\infty} \omega^{s-t} Q_s \left( (1 - \eta) \frac{p_{j,t}}{\pi_{t,s}} + \eta w_s \right) \pi_{t,s}^{\eta} c_s = 0 \]

\[ \Leftrightarrow \]

\[ \tilde{E}_{j,t} \sum_{s=t}^{\infty} \omega^{s-t} Q_s (\eta w_s \pi_{t,s}^{\eta} c_s) = \tilde{E}_{j,t} \sum_{s=t}^{\infty} \omega^{s-t} Q_s ((\eta - 1) \pi_{t,s}^{\eta-1} c_s p_{j,t}) \]

\[ \Leftrightarrow \]

\[ p_{j,t} = \eta \frac{\tilde{E}_{j,t} \sum_{s=t}^{\infty} \omega^{s-t} Q_s (\pi_{t,s}^{\eta} w_s c_s)}{\pi_{t,t}^{\eta-1}} \]

\[ \Leftrightarrow \]

\[ p_{j,t} \left( c_t + \tilde{E}_{j,t} \omega Q_{t+1} \pi_{t,t+1}^{\eta-1} c_{t+1} + ... \right) = \eta \frac{\tilde{E}_{j,t} \omega Q_{t+1} c_{t+1}}{\eta - 1} \left( w_t c_t + \tilde{E}_{j,t} \omega Q_{t+1} \pi_{t,t+1}^{\eta} w_{t+1} c_{t+1} + ... \right). \]

Log-linearizing gives

\[ (c + \omega \bar{Q}_{t+1} \pi_{t+1}^{\eta} c + ...) \left( p_{j,t} - p \right) + p (c_t - c) + p \omega \bar{Q}_{t+1} \pi_{t+1}^{\eta-1} \tilde{E}_{j,t} (c_{t+1} - c) + \]

\[ p \omega \pi_{t+1}^{\eta-1} c \tilde{E}_{j,t} (Q_{t+1} - \bar{Q}_{t+1}) + (\eta - 1) p \omega \pi_{t+1}^{\eta-2} c \tilde{E}_{j,t} (\pi_{t,t+1} - \pi) + ... \]

\[ = \frac{\eta}{\eta - 1} \left( c (w_t - w) + w (c_t - c) + \omega \bar{Q}_{t+1} \pi_{t+1}^{\eta} c \tilde{E}_{j,t} (w_{t+1} - w) + \omega \bar{Q}_{t+1} \pi_{t+1}^{\eta} w \tilde{E}_{j,t} (c_{t+1} - c) \right) + \]

\[ \omega \pi_{t+1}^{\eta} w c \tilde{E}_{j,t} (Q_{t+1} - \bar{Q}_{t+1}) + \eta \omega \bar{Q}_{t+1} \pi_{t+1}^{\eta-1} w c \tilde{E}_{j,t} (\pi_{t,t+1} - \pi) + ... \]
Using steady state values $c = 1$, $\pi = 1$, $p = 1$, $w = \frac{n-1}{\eta}$, $\overline{Q}_s = \delta^{s-t}$ we can write

$$(1 + \omega \delta + \omega^2 \delta^2 + \ldots) \bar{p}_{j,t} + (\bar{c}_t + \omega \delta \bar{E}_{j,t} \bar{c}_{t+1} + \ldots) + (\omega \delta \bar{E}_{j,t} \bar{Q}_{t+1} + \ldots) +$$

$$(\eta - 1) \left( \omega \delta \bar{E}_{j,t} \bar{\pi}_{t,t+1} + \omega^2 \delta^2 \bar{E}_{j,t} \bar{\pi}_{t,t+2} + \ldots \right)$$

$$= \left( \bar{w}_t + \omega \delta \bar{E}_{j,t} \bar{w}_{t+1} + \ldots \right) + (\bar{c}_t + \omega \delta \bar{E}_{j,t} \bar{c}_{t+1} + \ldots) + (\omega \delta \bar{E}_{j,t} \bar{Q}_{t+1} + \ldots) +$$

$$\eta \left( \omega \delta \bar{E}_{j,t} \bar{\pi}_{t,t+1} + \omega^2 \delta^2 \bar{E}_{j,t} \bar{\pi}_{t,t+2} + \ldots \right)$$

$$\implies$$

$$\frac{1}{1 - \omega \delta} \bar{p}_{j,t} = \omega \delta \bar{E}_{j,t} \bar{\pi}_{t,s+1} + \omega^2 \delta^2 \bar{E}_{j,t} \bar{\pi}_{t,t+2} + \ldots$$

$$\implies$$

$$\frac{1}{1 - \omega \delta} \bar{p}_{j,t} = \bar{E}_{j,t} \sum_{s=t}^{\infty} (\omega \delta)^{s-t} \bar{\pi}_{t,s+1}$$

Using $\pi_{t,s+1} = \pi_{s+1} \cdot \pi_s \cdot \ldots \cdot \pi_{t+1}$ which means that $\bar{\pi}_{t,s+1} = \bar{\pi}_{s+1} + \bar{\pi}_s + \ldots + \bar{\pi}_{t+1}$ we have that

$$\omega \delta \bar{E}_{j,t} \sum_{s=t}^{\infty} (\omega \delta)^{s-t} \bar{\pi}_{t,s+1}$$

$$= \omega \delta \bar{E}_{j,t} \bar{\pi}_{t,t+1} + \omega^2 \delta^2 \bar{E}_{j,t} \bar{\pi}_{t,t+2} + \ldots$$

$$= \begin{bmatrix} \omega \delta \bar{\pi}_{t+1} + \\ \omega^2 \delta^2 \bar{\pi}_{t+1} + \omega^2 \delta^2 \bar{\pi}_{t+2} + \\ \omega^3 \delta^3 \bar{\pi}_{t+1} + \omega^3 \delta^3 \bar{\pi}_{t+2} + \omega^3 \delta^3 \bar{\pi}_{t+3} + \\ \ldots \end{bmatrix} = \begin{bmatrix} \omega \delta (1 + \omega \delta + \omega^2 \delta^2 + \ldots) \bar{\pi}_{t+1} + \\ \omega^2 \delta^2 (1 + \omega \delta + \omega^2 \delta^2 + \ldots) \bar{\pi}_{t+2} + \\ \omega^3 \delta^3 (1 + \omega \delta + \omega^2 \delta^2 + \ldots) \bar{\pi}_{t+3} + \\ \ldots \end{bmatrix}$$

$$= \frac{\omega \delta}{1 - \omega \delta} \sum_{s=t}^{\infty} (\omega \delta)^{s-t} \bar{\pi}_s.$$

We can thus write

$$\frac{1}{1 - \omega \delta} \bar{p}_{j,t} = \bar{E}_{j,t} \sum_{s=t}^{\infty} (\omega \delta)^{s-t} \bar{w}_{s} + \frac{\omega \delta}{1 - \omega \delta} \sum_{s=t}^{\infty} (\omega \delta)^{s-t} \bar{\pi}_s,$$

which leads to equation (3.22).
Appendix 3.D Derivation of systems (3.43) and (3.46)

Start from system (3.40) which can be rewritten using results (3.41) and (3.42) as

\[ \hat{y}_t = n_{RE,t} E_t \sum_{s=t}^{\infty} \delta^{s-t} \left( 1 - \delta \right) \hat{y}_s - \frac{\delta}{\sigma} (\hat{R}_s - \hat{\pi}_{s+1}) + (1 - n_{RE,t}) \left( \hat{y}_{t-1} - \frac{\delta}{\sigma} \hat{R}_{t-1} - \frac{\delta^2}{(1 - \delta)\sigma} \hat{R}_{t-1} + \frac{\delta}{(1 - \delta)\sigma} \hat{\pi}_{t-1} \right) \]

\[ \hat{\pi}_t = n_{RE,t} E_t \sum_{s=t}^{\infty} (\omega \delta)^{s-t} \left( k \hat{y}_s + (1 - \omega) \delta \hat{\pi}_{s+1} \right) + (1 - n_{RE,t}) \left( k \frac{1}{1 - \omega \delta} \hat{y}_{t-1} + (1 - \omega) \delta \frac{1}{1 - \omega \delta} \hat{\pi}_{t-1} \right). \] (3.55)

Therefore we have that \( V_x(\cdot) \), for \( x \in \{ y, \pi \} \), in (3.43) are defined as

\[ \begin{align*}
V_y(\hat{y}_{t-1}, \hat{R}_t, \hat{\pi}_{t-1}) &= \hat{y}_{t-1} - \frac{\delta}{\sigma} \hat{R}_{t-1} - \frac{\delta^2}{(1 - \delta)\sigma} \hat{R}_{t-1} + \frac{\delta}{(1 - \delta)\sigma} \hat{\pi}_{t-1} \quad (3.56) \\
V_\pi(\hat{y}_{t-1}, \hat{\pi}_{t-1}) &= \frac{k}{1 - \omega \delta} \hat{y}_{t-1} + \frac{(1 - \omega)\delta}{1 - \omega \delta} \hat{\pi}_{t-1} \quad (3.57)
\end{align*} \]

As standard in the learning literature we assumed that the current interest rate is observed by non-rational agents while current output and inflation are not. The term \( \hat{R}_{t-1} \) shows up in the aggregate demand via the performance measure (3.34) in the heuristics selection process. This is due to the assumption that the selection of heuristics takes place at the beginning of period \( t \), before observing \( \hat{R}_t \).

Using the result (3.45) to drop the time index from the term \( n_{RE,t} \) and closing the model with the interest rate rule \( \hat{R}_t = \phi_\pi \hat{\pi}_t + \phi_y \hat{y}_t \), after some algebraic manipulations we can rewrite the system (3.55) as

\[ \begin{align*}
\hat{y}_{t+1} &= q_{yy} \hat{y}_t + q_{yE_y} E_t \hat{\pi}_{t+1} + q_{y\pi} \hat{\pi}_t + q_{Y1} \hat{y}_{t-1} + q_{Y11} \hat{\pi}_{t-1} \\
\hat{\pi}_{t+1} &= \gamma_{\pi y} \hat{y}_t + \gamma_{\pi \pi} \hat{\pi}_t + \gamma_{\pi Y1} \hat{y}_{t-1} + \gamma_{\pi Y11} \hat{\pi}_{t-1} \quad (3.58, 3.59)
\end{align*} \]

with

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\[
\begin{align*}
\Omega_y &= \left(1 + \frac{\delta(1 - n_{RE})}{\sigma} \phi_y\right)^{-1} \delta^{-1} \\
q_{yy} &= \Omega_y \left(1 - n_{RE} + \delta - \left(\frac{\delta^3(1 - n_{RE})}{(1 - \delta)\sigma} - \frac{\delta}{\sigma}\right) \phi_y\right) \\
q_{yE\pi} &= \Omega_y \left(-\frac{n_{RE}\delta}{\sigma} - \frac{\delta^2(1 - n_{RE})}{\sigma} \phi_\pi\right) \\
q_{y\pi} &= \Omega_y \left(\frac{\delta^2(1 - n_{RE})}{(1 - \delta)\sigma} - \left(\frac{\delta^3(1 - n_{RE})}{(1 - \delta)\sigma} - \frac{\delta}{\sigma}\right) \phi_\pi\right) \\
q_{yY_1} &= \Omega_y \left(-\frac{1 - n_{RE}}{\delta} \left(1 - \frac{\delta^2}{(1 - \delta)\sigma} \phi_y\right)\right) \\
q_{y\Pi_1} &= \Omega_y \left(-\frac{1 - n_{RE}}{\delta} \left(1 - \frac{\delta^2}{(1 - \delta)\sigma}\right) \phi_\pi\right) \\
\Omega_\pi &= (n_{RE}(1 - \omega)\delta + \omega\delta)^{-1} \\
\gamma_{\pi y} &= \Omega_\pi \left(-n_{RE}k + \omega\delta(1 - n_{RE})\right) \\
\gamma_{\pi\pi} &= \Omega_\pi \left(1 + \frac{(1 - \omega)\omega\delta^2(1 - n_{RE})}{1 - \omega\delta}\right) \\
\gamma_{\pi Y_1} &= \Omega_\pi \left(-\frac{1 - n_{RE}}{1 - \omega\delta}\left(k\right)\right) \\
\gamma_{\pi\Pi_1} &= \Omega_\pi \left(-\frac{1 - n_{RE}}{1 - \omega\delta}\left(1 - \omega\delta\right)\right)
\end{align*}
\]

Plugging (3.59) into (3.58) and rearranging terms we finally get

\[
\begin{align*}
E_t \hat{y}_{t+1} &= \gamma_{yy}\hat{y}_t + \gamma_{y\pi}\hat{\pi}_t + \gamma_{yY_1}\hat{y}_{t-1} + \gamma_{y\Pi_1}\hat{\pi}_{t-1} \\
E_t \hat{y}_{t+1} &= \gamma_{\pi y}\hat{y}_t + \gamma_{\pi\pi}\hat{\pi}_t + \gamma_{\pi Y_1}\hat{y}_{t-1} + \gamma_{\pi\Pi_1}\hat{\pi}_{t-1}
\end{align*}
\]

(3.60)

where

\[
\begin{align*}
\gamma_{yy} &= q_{yy} + q_{yE\pi}\gamma_{\pi y} \\
\gamma_{y\pi} &= q_{y\pi} + q_{yE\pi}\gamma_{\pi\pi} \\
\gamma_{yY_1} &= q_{yY_1} + q_{yE\pi}\gamma_{\pi Y_1} \\
\gamma_{y\Pi_1} &= q_{y\Pi_1} + q_{yE\pi}\gamma_{\pi\Pi_1}.
\end{align*}
\]

System (3.60) can be rewritten in matrix form as in (3.46).
Appendix 3.E  Computation of bifurcation surfaces

The characteristic equation of matrix $M$ is given by

$$P(\lambda) = \lambda^4 + c_3 \lambda^3 + c_2 \lambda^2 + c_1 \lambda + c_0 = 0 \quad (3.61)$$

where

$$
\begin{align*}
c_3 &= -\gamma_{yy} - \gamma_{\pi\pi} \\
c_2 &= \gamma_{yy}\gamma_{\pi\pi} - \gamma_{\pi y}\gamma_{\pi y} - \gamma_{yY1} - \gamma_{\pi\Pi1} \\
c_1 &= -\gamma_{\pi y}\gamma_{y}\gamma_{\Pi} + \gamma_{yy}\gamma_{\pi\Pi1} + \gamma_{\pi\pi}\gamma_{yY1} \\
c_0 &= \gamma_{yY1}\gamma_{\Pi} - \gamma_{y}\gamma_{\Pi1}\gamma_{\pi Y1}.
\end{align*}
$$

The computation of the fold and flip bifurcation surfaces are pretty straightforward. By plugging $\lambda = 1$ and $\lambda = -1$ in (3.61), one finds that the fold and the flip bifurcations loci are given respectively by

$$
\begin{align*}
1 + c_3 + c_2 + c_1 + c_0 &= f_{\text{fold}}(\phi_\pi, \phi_y, n_{RE}) = 0 \\
1 - c_3 - c_2 - c_1 + c_0 &= f_{\text{flip}}(\phi_\pi, \phi_y, n_{RE}) = 0.
\end{align*}
$$

The locus of Neimark-Sacker bifurcations is associated with complex roots of modulus $1$, $\lambda_{1,2} = e^{\pm i\theta}$, where $\theta$ is an arbitrary angle and $i$ is the imaginary unit. Using Vieta’s formulas we find the following relations between the coefficients of $P(\lambda)$ and the roots of (3.61):

$$
\begin{align*}
c_3 &= -\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 \\
c_2 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 \\
c_1 &= \lambda_1\lambda_2\lambda_3 - \lambda_2\lambda_3\lambda_4 - \lambda_1\lambda_3\lambda_4 - \lambda_1\lambda_2\lambda_4 \\
c_0 &= \lambda_1\lambda_2\lambda_3\lambda_4.
\end{align*}
$$
Given that $\lambda_1 \lambda_2 = e^{i\vartheta} e^{-i\vartheta} = 1$, we can rewrite (3.65) as

$$c_0 = \lambda_3 \lambda_4. \quad (3.66)$$

Moreover, using (3.62) we have that the sum of the real roots is given by

$$\lambda_3 + \lambda_4 = -c_3 - e^{i\vartheta} - e^{-i\vartheta}. \quad (3.67)$$

By plugging (3.66) and (3.67) in (3.64) and (3.63) we get

$$c_1 = -c_0(e^{i\vartheta} + e^{-i\vartheta}) + c_3 + e^{i\vartheta} + e^{-i\vartheta} \quad (3.68)$$
$$c_2 = 1 + c_0 + e^{i\vartheta}(-c_3 - e^{i\vartheta} - e^{-i\vartheta}) + e^{-i\vartheta}(-c_3 - e^{i\vartheta} - e^{-i\vartheta}). \quad (3.69)$$

Using Euler’s formula we can rewrite (3.68) and (3.69) as

$$c_1 = (1 - c_0)2\cos(\vartheta) + c_3 \quad (3.70)$$
$$c_2 = 1 + c_0 + 2\cos(\vartheta)(-c_3 - 2\cos(\vartheta)). \quad (3.71)$$

Plugging (3.70) in (3.71) we have that the locus of Neimark-Sacker bifurcation is given by

$$1 + c_0 - c_2 - \frac{c_3(c_1 - c_3)}{1 - c_0} - \left(\frac{c_1 - c_3}{1 - c_0}\right)^2 = f_{NS}(\phi_x, \phi_y, n_{RE}) = 0$$

subject to $-1 \leq (c_1 - c_3)/(2(1 - c_0)) \leq 1$. 

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Chapter 4

Behavioral Heterogeneity in U.S. Inflation Data

4.1 Introduction

In recent years, pricing behavior has been described in the context of models that incorporate both nominal rigidities and optimizing agents with rational expectations.¹ One of the most popular versions of New Keynesian pricing models is derived from Calvo (1983) and it implies a forward-looking inflation equation (a “New Keynesian Phillips curve”, NKPC henceforth) of the form

$$\pi_t = \delta E_t \pi_{t+1} + \gamma mc_t,$$

(4.1)

which relates inflation, $\pi_t$, to next period’s expected inflation rate and to real marginal costs, $mc_t$.² An important implication of this model is that there is no intrinsic inertia in inflation, in the sense that there is no structural dependence of inflation on its own lagged values. As a result, this specification has often been criticized on the grounds that it can not account for the important empirical role played by lagged dependent variables in inflation regressions (see e.g., Rudd and Whelan (2005a,b) for a recent discussion). This critique resulted in various proposals

²Roberts (1995) shows that Eq. (4.1) can be derived from a number of different models of price rigidity.
for so-called “hybrid” variants of the NKPC, which take the form

\[ \pi_t = \theta E_t \pi_{t+1} + (1 - \theta) \pi_{t-1} + \gamma m c_t . \]  

(4.2)

Hybrid models have been theoretically motivated in several ways. Fuhrer and Moore (1995) assume an alternative contracting specification in which workers bargain over relative real wages; Christiano, Eichenbaum, and Evans (2005) use a variant of the Calvo model in which firms that are unable to reoptimize their price instead index it to past inflation; Galí and Gertler (1999) assume the existence of a group of backward looking price setters.\(^3\)

Starting from different theoretical formulations, a significant strand of research focused on assessing the empirical relevance of the forward looking component in Eq. (4.2), generating mixed results. Here we briefly summarize some of the evidence obtained in previous studies.

Galí and Gertler (1999) focus on estimates of \( \theta \) obtained from directly fitting Eq. (4.2) using GMM. Under this procedure \( E_t \pi_{t+1} \) is replaced with \( \pi_{t+1} \) and the model is estimated using instruments for \( \pi_{t+1} \). Employing this technique, Galí and Gertler (1999) estimate large values of \( \theta \) and conclude that rational forward-looking behavior plays an important role in determining U.S. inflation.

Sbordone (2005) follows Christiano, Eichenbaum, and Evans (2005) by assuming that firms that are not allowed to reset prices through the Calvo random drawing are nonetheless allowed to index their current prices to past inflation, and they do so by some fraction \( \rho \in [0, 1] \). She then estimates the closed form solution of the model

\[ \pi_t = \rho \pi_{t-1} + \zeta \sum_{j=0}^{\infty} \delta^j E_t m c_{t+j} , \]

where inflation is a function of lagged inflation and expected future real marginal costs. Using a two-step distance estimator that exploits an auxiliary autoregressive representation of the data as in Campbell and Shiller (1987) to estimate the present value form of the inflation dynam-

\( ^3 \)In the specification of Galí and Gertler (1999) the weights of lagged and expected future inflation are not constrained to sum to unity, unless the time discount factor \( \delta = 1 \).
ics model, Sbordone (2005) finds that the forward-looking component is quantitatively more relevant than the backward-looking component, confirming thus the results of Galí and Gertler (1999).

Lindé (2005) estimates a New-Keynesian sticky price model using a full information maximum likelihood approach and suggests a hybrid version of the NKPC where forward-looking behavior is significant but about equally or less important than backward-looking behavior.

Fuhrer (1997) considers the model developed in Fuhrer and Moore (1995), which extends the staggered contracting framework of Taylor (1980) in a way that imparts persistence to the rate of inflation. Using a maximum likelihood estimation procedure he concludes that forward looking behavior plays essentially no role in observed inflation dynamics.

Rudd and Whelan (2006) estimate closed form solutions of model (4.2) given by

\[
\Delta \pi_t = \lambda_1 \sum_{j=0}^{\infty} \lambda_2 E_t m_{t+j} \quad \text{for } \theta \leq 1/2
\]

\[
\pi_t = \mu_1 \sum_{j=0}^{\infty} E_t m_{t+j} + \mu_2 \pi_{t-1} \quad \text{for } \theta > 1/2
\]

where \( \lambda_1, \lambda_2, \mu_1, \) and \( \mu_2 \) are functions of \( \theta \). Using both VAR-based methods and GMM estimation, they find no significant evidence of rational forward-looking behavior in U.S. data.

One possible explanation for the mixed evidence on the empirical relevance of rational forward-looking behavior stemming from previous tests of sticky price models, may be rooted, as put forward by Rudd and Whelan (2006), in the reliance of these models on a strict form of rational expectations (RE henceforth). Rudd and Whelan (2006) conclude that:

“...further research in this area is probably best aimed toward developing models that deviate from the standard rational expectations framework in favor of alternative descriptions of how agents process information and develop forecasts”.

In this paper we take this criticism seriously and propose a model of inflation dynamics characterized by heterogeneous boundedly rational agents and evolutionary selection of forecasting strategies. Standard New Keynesian models of price setting are based on the assumptions that:
(i) prices are sticky; (ii) agents make optimal decisions given their beliefs about future inflation; (iii) individual expectations are formulated in a rational (i.e., model-consistent) way.

Empirical studies suggest that a significant degree of price stickiness is present in the U.S. economy, providing thus a rationale for firms trying to make predictions about future inflation when setting current prices. In our analysis we keep the assumption of sticky prices and optimizing behavior (given individual beliefs), so that expected future inflation has an important influence on current inflation, but we depart from standard models by assuming heterogeneous subjective expectations.

Heterogeneity in individual expectations has been abundantly documented in the literature. For example, Frankel and Froot (1987, 1990), Allen and Taylor (1990) and Ito (1990) find that financial experts use different forecasting strategies to predict exchange rates. More recently, Carroll (2003), Mankiw, Reis, and Wolfers (2003), Branch (2004) and Pfajfar and Santoro (2010) recently provided empirical evidence in support of heterogeneous expectations using survey data on inflation expectations, while Hommes, Sonnemans, Tuinstra, and van de Velden (2005), Adam (2007), Pfajfar and Zakelj (2010), Assenza, Heemeijer, Hommes, and Massaro (2011), and Hommes (2011) find evidence for heterogeneity in learning to forecast laboratory experiments with human subjects.

We construct a framework with monopolistic competition, staggered price setting and heterogeneous firms. Our stylized model includes two types of price setters. The first type are fundamentalists, or forward-looking, who believe in a present-value relationship between inflation and real marginal costs. The evolution of inflation depends on the discounted sum of expected future values of real marginal costs, and fundamentalists use a VAR methodology to forecast the evolution of the driving variable as in Campbell and Shiller (1987). The second type are naive, or backward-looking agents who use the simplest backward-looking rule of thumb, naive expectations (i.e., their forecast coincides with the last available observation), to forecast future inflation.

All the empirical studies on forward- versus backward-looking behavior in inflation dynam-
ics cited above, take the distribution of heterogeneous firms as fixed and exogenous. However, recent empirical analysis suggest that this assumption is overly restrictive. For example, Zhang, Osborn, and Kim (2008), Kim and Kim (2008) and Hall, Han, and Boldea (2011) find evidence for multiple structural breaks in the relative weights of forward- and backward-looking firms, while Carroll (2003) and Mankiw, Reis, and Wolfers (2003) show that the distribution of heterogeneity evolves over time in response to economic volatility. Furthermore, Frankel and Froot (1991), Bloomfield and Hales (2002), Branch (2004), Assenza, Heemeijer, Hommes, and Massaro (2011) and Hommes (2011), among others, provide evidence that the proportions of heterogeneous forecasters evolve over time as a reaction to past forecast errors using survey data as well as experimental data. On the basis of this empirical evidence, we endogenize the evolution of the distribution of heterogeneous firms by assuming that agents can switch between different forecasting regimes, depending on recent prediction performance of their forecasting rules as in Brock and Hommes (1997).

Obviously we are not the first to introduce a dynamic predictor selection mechanism in macroeconomic models. Recent theoretical papers analyzing inflation dynamics under endogenous selection of expectation rules include, among others, Brock and de Fontnouvelle (2000), Tuinstra and Wagener (2007), Anufriev, Assenza, Hommes, and Massaro (2008), Brazier, Harrison, King, and Yates (2008), Branch and McGough (2010), De Grauwe (2010), Branch and Evans (2010).

The main novelty of our paper consists in the estimation of a NKPC with heterogeneous expectations and endogenous switching between different beliefs using U.S. macroeconomic data. To our knowledge, there are only a few empirical applications that attempt to estimate heterogenous agents models with fully-fledged switching mechanism. Those attempts consider the S&P500 market index (Boswijk, Hommes, and Manzan (2007)), commodity markets (Reitz and Westerhoff (2005, 2007)), the Asian equity markets (De Jong, Verschoor, and Zwinkels (2009)), the DAX30 index options (Frijns, Lehnert, and Zwinkels (2010)), and the U.S. housing market (Kouwenberg and Zwinkels (2010)).
Moreover, our paper contributes to the debate about the empirical relevance of forward-looking behavior in inflation dynamics. In fact our model, although with a different behavioral interpretation, is similar to the hybrid models estimated by Sbordone (2005) and Rudd and Whelan (2006) among others. Our measure of fundamental expectation is constructed in the same way as Sbordone (2005) and Rudd and Whelan (2006) obtain their estimation of the discounted sum of expected future values of real marginal costs in the closed-form solution of the model, i.e., using the Campbell and Shiller methodology, while the expectations of naive firms account for lagged value of inflation in the hybrid specification of the NKPC. The main difference stems from the time-varying weights assigned to fundamentalists and naive price setters, evolving over time according to past forecasting performances.

The results of our analysis provide empirical evidence for behavioral heterogeneity in U.S. inflation dynamics. Moreover, the data support the hypothesis of an endogenous mechanism relating predictors choice to their forecasting performance. In fact, our results suggest that the degree of heterogeneity varies considerably over time, and that the economy can be dominated temporarily by either forward-looking or backward-looking behavior.

These findings have important implications for monetary policy. Standard policy recommendations based on determinacy under RE, may not be a robust criterion for policy advices in the presence of heterogeneous expectations. In fact, recent papers have shown that multiple equilibria, periodic orbits and complex dynamics can arise in presence of dynamic predictor selection, even if the model under RE has a unique stationary solution (see Anufriev, Assenza, Hommes, and Massaro (2008), Branch and McGough (2010), and De Grauwe (2010) among others).

The paper is structured as follows. Section 4.2 derives a NKPC with heterogenous expectations and endogenous switching dynamics. Section 4.3 presents the estimation results and describes the fit of the model. Section 4.4 discusses the robustness of the empirical results to alternative forecasting models for the driving variable in the NKPC and to alternative measures of real marginal costs. Section 4.5 concludes.
4.2 The model

This section derives a NKPC with heterogeneous, potentially nonrational expectations and endogenous switching between forecasting strategies.

4.2.1 The NKPC with heterogeneous expectations

We consider a model with a continuum of monopolistic firms, indexed by $i$, which produce differentiated goods. The demand curve for the product of firm $i$ takes the form:

$$Y_{i,t} = (P_{i,t}/P_t)^{-\eta}Y_t,$$

where $\eta$ is the Dixit-Stiglitz elasticity of substitution among differentiated goods, $Y_t$ is the aggregator function defined as $Y_t = \left[\int_0^1 Y_{i,t}^{(\eta-1)/\eta} di\right]^{\eta/(\eta-1)}$, and $P_t$ is the aggregate price level defined as $P_t = \left[\int_0^1 P_{i,t}^{1-\eta} di\right]^{1/(1-\eta)}$. Nominal price rigidity is modeled by allowing, in every period, only a fraction $(1 - \omega)$ of the firms to set a new price along the lines of Calvo (1983).

Firms that reset prices maximize expected discounted profits, which are given by

$$\max_{P_{i,t}} E_t^\infty \sum_{j=0}^{\infty} \omega^j Q_{t,t+j} \left(\frac{P_{i,t}}{P_{t+j}} - MC_{t+j}\right) \left(\frac{P_{i,t}}{P_{t+j}}\right)^{-\eta} Y_{t+j},$$

where $E_t^i$ denotes the subjective expectation of firm $i$, $Q_{t,t+j}$ is the stochastic discount factor and $MC_t$ are real marginal costs of production. Linearizing the first order conditions of this problem around a zero inflation steady state delivers

$$p_{i,t} = (1 - \omega \delta)E_t^i \sum_{j=0}^{\infty} (\omega \delta)^j mc_{t+j} + \omega \delta E_t^i \sum_{j=0}^{\infty} (\omega \delta)^j \pi_{t+j+1},$$

(4.3)

where $\delta$ is the time discount factor, $\pi_t = p_t - p_{t-1}$, and lower case letter denote log-deviations from steady state.

Optimal pricing decisions involve subjective forecasts of marginal costs and inflation, hence
we will have that firms with different expectations will set different prices. The average price set by optimizing firms is given by

\[ p^*_t = \sum n_{i,t} p_{i,t}, \]

where \( n_{i,t} \) denotes the fraction of firms characterized by the subjective expectation operator \( E^i_t \) at time \( t \). Given the assumed staggered price setting mechanism, the aggregate price level evolves as a convex combination of the lagged price level \( p_{t-1} \) and the average optimal reset price \( p^*_t \) as follows

\[ p_t = \omega p_{t-1} + (1-\omega)p^*_t. \] (4.4)

Under the assumption of a representative firm with rational expectations, Eqs. (4.3) and (4.4) can be used to derive the standard NKPC

\[ \pi_t = \delta E_t \pi_{t+1} + \gamma mc_t, \] (4.5)

where \( \gamma \equiv (1-\omega)(1-\delta\omega)^{-1}. \) Deriving an equation for inflation similar to Eq. (4.5) is not entirely obvious when expectations are heterogeneous. Optimal pricing decisions depend on forecasted marginal costs and inflation. Inflation is determined by other agent’s pricing decisions and their forecasts. As a result, optimal price setting would require forecasting the marginal costs and inflation forecasts of others, see e.g., Woodford (2001a). Denote the average forecast of individuals as \( \bar{E}_t = \sum n_{i,t} E^i_t \) and suppose the following holds:

**Condition 1**

\[ \bar{E}_t \bar{E}_{t+1} x_{t+j} = \bar{E}_t x_{t+j} \text{ for } j \geq 0 \text{ and } x \in \{mc, \pi\}. \]

Condition 1, also known as the “tower property” in probability theory, requires that the average forecast at time \( t \) of the future average forecast of a certain variable \( x \) at time \( t+1 \) is equal to the average forecast of variable \( x \) at time \( t \), that is, agents, on average, do not expect predictable changes in average future expectations of a certain macroeconomic variable \( x \). Stated differently, Condition 1 corresponds to the law of iterated expectations at the aggregate level.
Appendix 4.A shows that whenever Condition 1 is satisfied, it is possible to aggregate individual pricing rules (4.3) and obtain a Phillips curve of the form

$$\pi_t = \delta E_t \pi_{t+1} + \gamma m c_t .$$

(4.6)

In what follows we will assume that Condition 1 is satisfied and we will use a NKPC equation as in (4.6) for our empirical investigation. Condition 1 is similar to the assumptions made by Branch and McGough (2009) and Adam and Padula (2011) in order to derive a NKPC of the form of (4.6) in a context of subjective heterogeneous expectations.

4.2.2 Evolutionary selection of expectations

We assume that agents can form expectations by choosing from \( I \) different forecasting rules, and we denote by \( E_i^t \pi_{t+1} \) the forecast of inflation by rule \( i \). The fraction of individuals using the forecasting rule \( i \) at time \( t \) is denoted by \( n_{i,t} \). Fractions are updated in every period according to an evolutionary fitness measure. At the beginning of every period \( t \) agents compare the realized relative performances of the different strategies and the fractions \( n_{i,t} \) evolve according to a discrete choice model with multinomial logit probabilities (see Manski and McFadden (1981) for details), that is

$$n_{i,t} = \frac{\exp(\beta U_{i,t-1})}{\sum_{i=1}^{I} \exp(\beta U_{i,t-1})},$$

(4.7)

where \( U_{i,t-1} \) is the realized fitness metric of predictor \( i \) at time \( t-1 \), and the parameter \( \beta \geq 0 \) refers to the intensity of choice and it reflects the sensitivity of the mass of agents to selecting the optimal prediction strategy. Brock and Hommes (1997) proposed this model for endogenous selection of expectation rules. The key feature of Eq. (4.7) is that strategies with higher fitness in the recent past attract more followers. The case \( \beta = 0 \) corresponds to the situation in which differences in fitness can not be observed, so agents do not switch between strategies and all fractions are constant and equal to \( 1/I \). The case \( \beta = \infty \) corresponds to the “neoclassical” limit in which the fitness can be observed perfectly and in every period all agents choose the
best predictor.

A strong motivation for switching among forecasting rules can be found in empirical works on individual expectations. Frankel and Froot (1991) find that professional market participants in the foreign exchange markets expect recent price changes to continue in the short term, while they expect mean reversion to fundamental value in the long term. Moreover, Frankel and Froot (1991) report survey evidence showing that professional forecasting services in the foreign exchange markets rely both on technical analysis and fundamental models, but with changing weights over time, and the weights appear to depend strongly on recent forecasting performances. Branch (2004) finds evidence for dynamic switching between alternative forecasting strategies that depends on the relative mean squared errors of the predictors using survey data on inflation expectations. In addition, Bloomfield and Hales (2002), Assenza, Heemeijer, Hommes, and Massaro (2011) and Hommes (2011) document experimental evidence that participants switch between forecasting regimes conditional on recent forecasting performances.

4.2.3 A simple two-type example

We assume that agents can choose between two forecasting rules to predict inflation, namely fundamentalist and naive. The first rule, fundamentalist, is based on a present-value description of the inflation process. When all agents have rational expectations, repeated application of equation Eq. (4.6) gives

\[ \pi_t = \gamma \sum_{k=0}^{\infty} \delta^k E_t mc_{t+k} . \]  

(4.8)

We refer to (4.8) as the fundamental inflation. Fundamentalists use the expression (4.8) to forecast future inflation. In particular, leading (4.8) one-period ahead we get

\[ \pi_{t+1} = \gamma \sum_{k=1}^{\infty} \delta^{k-1} E_{t+1}^f mc_{t+k} , \]  

(4.9)
where $E_{t+1}^f$ denotes fundamentalists forecast. Applying the expectation operator $E_t^f$ on both sides we get
\[
E_t^f \pi_{t+1} = \gamma \sum_{k=1}^{\infty} \delta^{k-1} E_t^f mc_{t+k} .
\]
(4.10)

From a behavioral point of view, fundamentalists can be considered as agents who believe in rational expectations and use the closed form solution of the model to forecast the inflation path. If Eq. (4.6) were the true data generating process and if all agents in the economy were of the same type, then the inflation path implied by fundamental expectations would coincide with the inflation path under rational model-consistent expectations.\(^4\)

In order to characterize the fundamental forecast (4.10) we use the VAR methodology of Campbell and Shiller (1987). Assuming that the forcing variable $mc_t$ is the first variable in the multivariate VAR
\[
Z_t = AZ_{t-1} + \epsilon_t,
\]
(4.11)
we can rewrite the sum of discounted future expectations of marginal costs (4.10) as
\[
E_t^f \pi_{t+1} = \gamma \sum_{k=1}^{\infty} \delta^{k-1} E_t^f mc_{t+k} = \gamma e'_1 (I - \delta A)^{-1} AZ_t,
\]
where $e'_1$ is a suitably defined unit vector.\(^5\)

The second rule, which we call naive, takes advantage of inflation persistence and uses a simple backward-looking forecasting strategy:
\[
E_t^n \pi_{t+1} = \pi_{t-1} .
\]
(4.12)

Notice however that, although being an extremely simple rule, the naive forecasting strategy is optimal when the stochastic process is a random walk; hence for a near unit process, as in the case of inflation, naive expectations are almost optimal.

\(^4\)In fact, substituting the fundamental forecast in Eq. (4.6), we get $\pi_t = \delta \gamma \sum_{k=1}^{\infty} \delta^{k-1} E_t^f mc_{t+k} + \gamma mc_t = \gamma \sum_{k=0}^{\infty} \delta^{k} E_t^f mc_{t+k}$, which corresponds to the inflation path implied by Eq. (4.8) when the discounted sums of current and future expected marginal costs are estimated in the same way.

\(^5\)Technically, because the discounted sum of real marginal costs starts at $k = 1$, we measure it using $(I - \delta A)^{-1} AZ_t$ instead of $(I - \delta A)^{-1} Z_t$. 

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The specific choice of the set of forecasting rules, namely fundamental and naive, will enable us to compare the outcome of our analysis with the results of previous empirical works based on the hybrid Phillips curve specification. In fact, our measure of fundamental expectation is similar to the measure used by Sbordone (2005) and Rudd and Whelan (2006), among others, to estimate the discounted sum of expected marginal costs when fitting the closed-form solution of the hybrid model (4.2) to U.S. data, while the backward-looking component introduced in different ways in RE models is accounted for by the expectations of rule of thumb firms. The main difference between traditional hybrid specifications of the NKPC and our model is the fact that the weights assigned to forward-looking and backward-looking component are endogenously varying over time. We in fact assume that agents can switch between the two predictors based on recent forecasting performance. Defining the absolute forecast error as

\[ F E^i_t = |E^i_{t-1} - \pi_t|, \]

with \( i = f, n \), we can then define the evolutionary fitness measure as

\[ U_{i,t} = -\frac{F E^i_t}{\sum_{i=1}^{I} F E^i_t}. \] (4.13)

The evolution of the weights of different heuristics is then given by Eq. (4.7). Denoting the fraction of fundamentalists as \( n_{f,t} \), we can summarize the full model as

\[ \pi_t = \delta(n_{f,t} E^f_t \pi_{t+1} + (1 - n_{f,t}) E^n_t \pi_{t+1}) + \gamma mc_t + u_t, \] (4.14)

where

\[ E^f_t \pi_{t+1} = \gamma \epsilon_1^f (I - \delta A)^{-1} AZ_t \]
\[ E^n_t \pi_{t+1} = \pi_{t-1} \]

\[ n_{f,t} = \frac{1}{1 + \exp \left( \beta \left( \frac{F E^f_{t-1} - F E^n_{t-1}}{F E^f_{t-1} + F E^n_{t-1}} \right) \right)} \]
\[ F E^i_{t-1} = |E^i_{t-2} - \pi_{t-1}|, \quad \text{with} \quad i = f, n. \]
4.3 Estimation results

This section describes data and methodology used to estimate the nonlinear switching model derived in the previous section.

4.3.1 Data description

We use quarterly U.S. data on the inflation rate, the output gap, unit labor costs, the labor share of income, hours of work and consumption-output ratio, from 1960:Q1 to 2010:Q4. Inflation is measured as log-difference of CPI. Output gap is measured as quadratically detrended log-real GDP. We use unit labor costs, labor share of income, detrended hours of work and detrended consumption-output ratio time series for nonfarm business sector in the construction of the VAR model (4.11). A more detailed description of data sources and variables definition is given in Appendix 4.B.

4.3.2 The fit of the model

In this section we discuss the empirical implementation of model (4.14). In the “baseline” specification (the one used in the results reported below) we use the output gap as driving variable, and we use a four-lag, two-variables VAR in output gap and labor share of income to compute fundamental expectations.\(^6\) The selection procedure of the VAR specification is extensively documented in Appendix 4.C. Standard unit root tests reveal that the labor share of income is an I(1) process, therefore the VAR model includes the rate of change of the labor share of income.\(^7\) Denoting by \(Y_t\) the vector of dependent variables, \(Y_t = [y_t, \Delta lsi_t]'\), the vector \(Z_t\) is defined as \(Z_t = [Y_t, Y_{t-1}, Y_{t-2}, Y_{t-3}]'\). Although being parsimonious, our VAR specification captures about 94% of output gap volatility (see Table 4.1). The parameters of the matrix \(A\) are estimated by OLS, and the discount factor \(\delta\) is fixed to the standard value.

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\(^6\)Section 4.4 discusses the sensitivity of the results to the use of labor share as driving variable and to alternative specifications of the forecasting VAR.

\(^7\)Observations in period 1959:4 were taken as initial conditions to build the series in first differences.
Model (4.14) is then estimated using non-linear least squares (NLS). Table 4.1 presents the results and diagnostic checks are reported in Appendix 4.C.

Table 4.1: NLS estimates of model (4.14)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\beta$</th>
<th>$\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimate</td>
<td>3.975***</td>
<td>0.005**</td>
</tr>
<tr>
<td>Std. error</td>
<td>1.100</td>
<td>0.002</td>
</tr>
</tbody>
</table>

$R^2$ from Inflation Equation 0.767

$R^2$ from Output Gap VAR Equation 0.945

Notes: Standard errors are computed using White’s heteroskedasticity-consistent covariance matrix estimator (HC-CME). *, **, *** denote significance at the 10%, 5%, and 1% level.

All coefficients have the correct sign and are significant at least at the 5% level. The positive sign and the significance of the intensity of choice parameter $\beta$ implies that agents switch towards the better performing forecasting rule, based on its past performance.\(^8\) The positive sign and the significance of parameter $\gamma$ is a rather interesting result. It has been quite difficult to obtain parameter estimates with the correct sign and of a plausible magnitude when output gap is used as driving variable for the inflation process. Fuhrer and Moore (1995) and Galí and Gertler (1999), for example, find a negative and insignificant estimate of $\gamma$ when real marginal costs are approximated by detrended output. The results in Table 4.1 show that taking into account non-rational heterogeneous expectations seems to help establishing a plausible link between output and inflation dynamics via the NKPC. Interestingly, Adam and Padula (2011) reach the same conclusion by estimating a NKPC using data from the Survey of Professional Forecasters as proxy for expected inflation.

The series of inflation predicted by (4.14) for the estimated values of $\beta$ and $\gamma$ is plotted in Fig. 4.1, as dashed line, versus the actual series (solid line).\(^9\)

Overall the predicted inflation path tracks the behavior of actual inflation quite well (the $R^2$ from inflation equation (4.14) is about 0.77, see Table 4.1).

Our results are, in some respects, similar to findings obtained in previous empirical works.

\(^8\)The order of magnitude of $\beta$ is more difficult to interpret as it is conditional on the functional form of the performance measure $U$.

\(^9\)The graphs are in deviation from the mean.
In particular, Galí and Gertler (1999) and Sbordone (2005) find that models derived from the assumption of heterogeneous price setting behavior are capable of fitting the level of inflation quite well. However, Rudd and Whelan (2005a) and Rudd and Whelan (2006) show that this good fit reflects the substantial role that these models still allow for lagged inflation, and that forward-looking components play no discernable empirical role in determining inflation.

Our NKPC specification allows for time-varying weights assigned to fundamentalists and naive price setters. Having estimated model (4.14), we are now ready to assess the relative importance over time of forward-looking versus backward-looking components in inflation dynamics. Table 4.2 displays descriptive statistics of the weight of the forward-looking component $n_f$. 

Figure 4.1: Actual vs. predicted inflation
Table 4.2: Descriptive statistics of weight $n_f$

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.338</td>
</tr>
<tr>
<td>Median</td>
<td>0.190</td>
</tr>
<tr>
<td>Maximum</td>
<td>0.976</td>
</tr>
<tr>
<td>Minimum</td>
<td>0.019</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>0.322</td>
</tr>
<tr>
<td>Skewness</td>
<td>0.774</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>2.078</td>
</tr>
<tr>
<td>Auto-corr. Q(-1)</td>
<td>0.383</td>
</tr>
</tbody>
</table>

On average, the majority of agents use the simple backward-looking rule (with mean fraction $1 - 0.34 = 0.66$). However, the spread between the minimum and the maximum indicates that the market can be dominated by either forward-looking or backward-looking agents. Moreover, the autocorrelation of the series $n_f$, about 0.38, indicates that agents do not change quickly their strategy, suggesting a certain degree of inertia in the updating process.

Fig. 4.2 shows the time series of the fraction of fundamentalists, i.e., the forward-looking component in our NKPC specification, the time series of the distance of actual inflation from the fundamental solution, and a scatter plot of the fraction of fundamentalists against the relative forecast error of the naive rule.

It is clear that the fraction of fundamentalists varies considerably over time with periods in which it is close to 0.5 and other periods in which it is close to either one of the extremes 0 or 1. For example, immediately after the oil crisis of 1973, the proportion of fundamentalists drops almost to 0. Soon after the difference between inflation and fundamental value reaches its peak in 1974:Q3/1974:Q4, the estimated weight of the forward-looking component shoots back up to about 0.8. During the second oil crisis inflation was far above the fundamental, causing more and more agents to adopt a simple backward-looking rule to forecast inflation. Fundamentalists dominated the economy in the late-80s, while from 1992 until 2003, inflation stayed continuously well below the fundamental, causing the weight of fundamentalists to fall. From 2004 until the early stages of the recent global financial crisis, the proportion of fundamentalists stayed, on average, around 0.5, reaching peaks of about 0.8. In the aftermath of the crisis we
observe that $n_f$ declines with the fundamentalist rule losing its forecast accuracy when actual inflation falls below its fundamental value in the last quarter of 2008, and then it increases again to around 0.7 in the last two quarters of 2010.

The bottom panel of Fig. 4.2 presents a scatter plot of the relative forecast error of the naive rule, $(FE^n - FE^f)/(FE^n + FE^f)$, versus the fraction of fundamentalist agents, $n_f$. Due to the positive estimated value of $\beta$ this line slopes upwards, such that a more accurate fundamentalist forecast results in a higher weight $n_f$. The S-shape is induced by the logit function in Eq. (4.7).

![Figure 4.2: Top panel: Time series of the fraction of fundamentalists $n_{f,t}$. Middle panel: Distance between actual inflation and fundamental solution. Bottom panel: Scatter plot of the weight $n_{f,t}$ versus the relative forecast error of the naive rule.](image)

The analysis conducted in this section shows that the evolutionary switching model fits the data quite nicely. The positive sign and the significance of the intensity of choice parameter, $\beta$,
implies that the endogenous mechanism that relates predictors choice to their past performance is supported by the data. We also find that the ability of the discounted sum of expected future output gap values to predict the empirical inflation process varies considerably over time. In fact the spread between the minimum and the maximum value of $n_f$, i.e., the fraction of fundamentalists, shows that the economy can be dominated by either forward-looking or backward-looking behavior. Moreover, even though the market is, on average, dominated by agents using a simple heuristic to predict inflation, fundamentalists, or forward-looking components, still have a significant impact on inflation dynamics.

### 4.3.3 Specification tests and out-of-sample forecasting

In order to assess the validity of our baseline model, which will be denoted by $H_1$ for the purposes of this section, we test it against four alternative specifications: a model with heterogeneous agents and exogenous estimated fixed weights ($n_{f,t} \equiv \hat{n}_f$), which is similar to the model estimated by Rudd and Whelan (2006) and Sbordone (2005), and it is denoted by $H_2$; a static model with heterogeneous agents in which we let $\beta = 0$ ($n_{f,t} \equiv 0.5$), which corresponds to the model of Fuhrer and Moore (1995), denoted by $H_3$; a model with homogeneous fundamentalists agents ($n_{f,t} \equiv 1$), which corresponds to the RE closed form solution of the standard NKPC without backward looking component, denoted by $H_4$; a model with homogeneous naive agents ($n_{f,t} \equiv 0$), which recalls the old backward-looking Phillips curve and it is denoted by $H_5$. Given that, with the exception of model $H_3$ which obtains by setting $\beta = 0$, the competing models are nonnested, we will use nonnested hypothesis testing procedures. In particular, we construct a test for the adequacy of our nonlinear specification with endogenous switching in explaining inflation dynamics (null hypothesis) against the alternative specifications mentioned above. Nonnested hypotheses tests are appropriate when rival hypotheses are advanced for the explanation of the same economic phenomenon. We will follow the procedure described in Davidson and MacKinnon (2009) and compute a heteroskedasticity-robust $P$ test of $H_1$, based
on the following Gauss-Newton regression:

$$\hat{u} = \hat{X}b + \hat{Z}c + \text{residuals},$$

(4.15)

where $\hat{u}$ are residuals from the NLS estimates of model $H_1$, $\hat{X} \equiv X(\hat{\theta})$ is the matrix of derivatives of the nonlinear regression function corresponding to the dynamics implied by model $H_1$, evaluated at the NLS estimates of the model, $\hat{\theta} = [\hat{\beta}, \hat{\gamma}]$, and $\hat{Z}$ is a matrix collecting the differences between the fitted values from $H_2$, $H_3$, $H_4$, $H_5$, and the fitted values from $H_1$. The $P$ test is based on the joint significance of the regressors in $\hat{Z}$, i.e., $c = 0$. We report the results of the test in Table 4.3, and refer the reader to Davidson and MacKinnon (2009), p. 284, for details on the construction of the heteroskedasticity-robust test. The results of the nonnested hypotheses test suggest that there is not a statistically significant evidence of departure from the null hypothesis (adequacy of nonlinear switching model $H_1$) in direction of alternative explanation of inflation dynamics (models $H_2$, $H_3$, $H_4$, and $H_5$). Stated differently, the test does not provide evidence in favor of alternative specifications of statistical models when tested against our baseline nonlinear switching model.$^{10}$

As an additional test of the validity of our model with heterogeneous agents, we contrast its forecasting accuracy with the four alternative models. All models are initially estimated over the restricted sample 1960:Q1 - 2001:Q4, and evaluated over the out-of-sample period 2002:Q1 - 2010:Q4. Forecasts are created using an expanding window, meaning that each model is first estimated over the sample 1960:Q1 - 2001:Q4. Subsequently, inflation is forecasted up to one year ahead depending on the forecast horizon, which we vary from 1 to 4 quarters. The models

$^{10}$For completeness, we also compared the switching model to the nested static model without switching ($\beta = 0$) using a likelihood ratio test. We rejected the null of a restricted static model at the 1% level on the basis of the test statistic $2\Delta LL = 91.80^{***}$, where $\Delta LL$ denotes the log-likelihood difference.
are then re-estimated with one extra observation and a new set of forecast is generated. We repeat this process to generate a number of 33 out-of-sample forecasts per horizon. The comparison of forecasting accuracy is assessed using the ratio of the average forecasting accuracy of our benchmark switching model over the average forecasting accuracy of the alternative models. A ratio less than one implies better performance for the benchmark model. Forecasting performance is measured using the mean absolute error and the mean squared error. Table 4.4 presents the forecast performance ratios. The results in Table 4.4 show that, overall, the forecasts of the model with heterogeneous beliefs and evolutionary switching are more accurate than most of the existing models in the literature, with the exception of the model with constant estimated fractions, for which we report a ratio well above one at a horizon of four quarters.

### 4.4 Robustness analysis

The empirical analysis that we presented is conditional upon the assumptions that output gap is well forecast by our baseline VAR specification, and that the output gap itself is a good approximation to real marginal costs. In this section we address the issue of how sensitive our results are to alternative specifications of the VAR forecasting model, and to different measures of real marginal costs.

#### Table 4.4: Comparison of out-of-sample forecast errors

<table>
<thead>
<tr>
<th>Horizon</th>
<th>MAE</th>
<th>MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(n_{f,t} \equiv \hat{n}_f)</td>
<td>(n_{f,t} \equiv 0.5)</td>
</tr>
<tr>
<td>q=1</td>
<td>0.963</td>
<td>0.973</td>
</tr>
<tr>
<td>q=2</td>
<td>1.075</td>
<td>0.968</td>
</tr>
<tr>
<td>q=3</td>
<td>1.096</td>
<td>0.975</td>
</tr>
<tr>
<td>q=4</td>
<td>1.230</td>
<td>0.918</td>
</tr>
</tbody>
</table>

Notes: Ratios of the forecast error of the model with heterogeneous agents and endogenous switching over the forecast error of the model with heterogeneous agents and constant estimated weights (\(n_{f,t} \equiv \hat{n}_f\)), the model with constant equal weights (\(n_{f,t} \equiv 0.5\)), the model with a homogeneous fundamentalists agents (\(n_{f,t} \equiv 1\)), and the model with homogeneous naive agents (\(n_{f,t} \equiv 0\)). MAE is mean absolute error and MSE is mean squared error.
4.4.1 Robustness to the specification of the VAR model

In order to choose the baseline forecasting system, we started from a broad model that recalls the baseline specifications of previous empirical works (see, e.g., Woodford (2001b) and Rudd and Whelan (2005a)) and then restricted the number of variables to include in our VAR as documented in Appendix 4.C. However, one doesn’t necessarily have to exclude from the information set other variables that may help forecasting the output gap beyond the contribution of the rate of change of the labor share of income. Therefore, to investigate how sensitive our results are to the specification of the fundamentalists’ forecasting system, we augmented our baseline VAR model by including hours of work and consumption-output ratio. These variables have been used in the VAR specifications considered by Rudd and Whelan (2005a) and Sbordone (2002). Table 4.5 reports results from alternative VAR forecasting models for the output gap.

Table 4.5: Estimation results using alternative VAR for output gap

<table>
<thead>
<tr>
<th>VAR specification</th>
<th>$y_t$</th>
<th>$\Delta lsi_t$</th>
<th>$h_t$</th>
<th>$c_t/y_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>3.975***</td>
<td>4.037***</td>
<td>4.110***</td>
<td>4.265***</td>
</tr>
<tr>
<td></td>
<td>(1.100)</td>
<td>(1.098)</td>
<td>(1.123)</td>
<td>(1.143)</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.005**</td>
<td>0.005</td>
<td>0.005**</td>
<td>0.006***</td>
</tr>
<tr>
<td></td>
<td>(0.002)</td>
<td>(0.003)</td>
<td>(0.002)</td>
<td>(0.002)</td>
</tr>
<tr>
<td>$R^2$ from Inflation Equation</td>
<td>0.767</td>
<td>0.765</td>
<td>0.769</td>
<td>0.769</td>
</tr>
<tr>
<td>$R^2$ from Output Gap VAR Equation</td>
<td>0.943</td>
<td>0.942</td>
<td>0.951</td>
<td>0.952</td>
</tr>
</tbody>
</table>

Notes: $y_t \equiv$ output gap, $\Delta lsi_t \equiv$ labor share growth, $h_t \equiv$ detrended hours of work, $c_t/y_t \equiv$ detrended consumption-output ratio. Optimal lag length in VAR specifications: $l_1 = l_2 = l_3 = l_4 = 4$. Standard errors are computed using White’s heteroskedasticity-consistent covariance matrix estimator (HCCME). *, **, *** denote significance at the 10%, 5%, and 1% level.

As Table 4.5 shows, the estimates presented in section 4.3.2 are robust to alternative VAR

---

11Hours are quadratically detrended total hours of work in the non-farm business sector, while consumption-output ratio is linearly detrended.
specifications for the output gap. The alternative models provide a good description of the empirical output gap process and the point estimates of coefficients $\beta$ and $\gamma$ do not substantially change. The only relevant difference consists in the insignificance of the coefficient $\gamma$ in the model of column 2, Table 4.5, although the $t$-statistic ($t = 1.52$) is well above one.

### 4.4.2 Robustness to alternative measures of marginal costs

Our benchmark model considers a traditional output gap measure, defined as the deviation of log real GDP from a quadratic trend, as the driving variable in the inflation process. Previous tests of sticky-price models under RE have reported that the NKPC provides a poor description of the actual inflation process when output gap is used as a proxy for real marginal costs (see, e.g., Fuhrer and Moore (1995) and Rudd and Whelan (2005a, 2006) among others). As an alternative to the standard approach, a number of researchers have suggested using the labor’s share of income as driving variable in the NKPC. The motivation for this measure stems from the fact that the micro-foundations underpinning the NKPC imply that the correct driving variable for inflation is actually real marginal cost. Some theoretical restrictions are then required in order for real marginal costs to move with the output gap. Using average unit labor costs (nominal compensation divided by real output) as a proxy for nominal marginal cost results in the labor share of income (nominal compensation divided by nominal output) as a proxy for real marginal cost. Even though empirical implementations of this variant of the NKPC generated mixed evidence,\footnote{Galí and Gertler (1999), Woodford (2001b), Sbordone (2002) and others report that predicted inflation series based on labor share fit actual inflation well, while Rudd and Whelan (2005a,b, 2006) and others show that even the labor share version of the model provides a poor description of the inflation process.} we estimate, as a second robustness exercise, the evolutionary switching model using the (log of) labor’s share of income as driving variable. As noted in section 4.3, the labor share process presents a unit root when considered over the full sample 1960:Q1-2010:Q4. In order to avoid spurious correlations and facilitate comparison with earlier works, we restrict the estimation sample to 1960:Q1-2001:Q4.\footnote{Standard unit root tests motivate this choice.} The estimation results reported in Table 4.6 show that the estimated coefficients are significant and have the correct sign. Moreover, the point
estimates are of the same order of magnitude as in the output gap VAR specification.

Table 4.6: Estimation results using alternative VAR for labor share of income

<table>
<thead>
<tr>
<th>VAR specification</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$R^2$ from Inflation Equation</th>
<th>$R^2$ from Labor Share VAR Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[l_{si_t} \ y_t]$</td>
<td>3.755***</td>
<td>0.009***</td>
<td>0.808</td>
<td>0.823</td>
</tr>
<tr>
<td>$y_t$</td>
<td>3.689***</td>
<td>0.007***</td>
<td>0.811</td>
<td>0.820</td>
</tr>
<tr>
<td>$h_t$</td>
<td>4.037***</td>
<td>0.010***</td>
<td>0.811</td>
<td>0.821</td>
</tr>
<tr>
<td>$c_t/y_t$</td>
<td>3.930***</td>
<td>0.008***</td>
<td>0.811</td>
<td>0.820</td>
</tr>
<tr>
<td>$c_t/y_t$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notes: $l_{si_t}$ is labor share of income, $y_t$ is output gap, $h_t$ is detrended hours of work, $c_t/y_t$ is detrended consumption-output ratio. Optimal lag length in VAR specifications: $l_1 = l_2 = l_3 = l_4 = 2$. Standard errors are computed using White’s heteroskedasticity-consistent covariance matrix estimator (HCCME). *, **, *** denote significance at the 10%, 5%, and 1% level.

Overall, the results presented in this section suggest that our analysis is robust to different VAR forecasting models for the driving variable in the inflation process. Moreover, using the labor share of income as an alternative measure of real marginal costs, does not significantly alter the main results.

4.5 Conclusions

Over the past decade it has become relatively well accepted that the purely forward-looking NKPC cannot account for the degree of inflation inertia observed in the data. In response, the profession has increasingly adopted hybrid models in which lagged inflation is allowed to have an explicit role in pricing behavior. This reformulation of the basic sticky-price model has recently provoked a heated debate as to the extent of forward- versus backward-looking behavior, with little consensus after years of investigation. Most of the empirical studies on the topic take the distribution of heterogeneous pricing behavior as constant and exogenously given. Recent
works on structural stability in short-run inflation dynamics in the U.S. have provided statistical evidence of multiple structural breaks in the relative weights of forward- and backward-looking firms. Moreover, empirical studies based on survey data as well as experimental data, provided evidence that the proportions of heterogeneous forecasters evolve over time as a reaction to past forecast errors. In the light of this empirical evidence, we have proposed a model of monopolistic price setting with nominal rigidities and endogenous evolutionary switching between different forecasting strategies according to their relative past performances. Importantly, heterogeneous firms hold optimizing behavior given their subjective expectations on future inflation. In our stylized framework, fundamentalist firms believe in a present-value relationship between inflation and real marginal costs, as predicted by standard RE models, while naive firms use a simple rule of thumb to forecast future inflation. Although with a different behavioral interpretation, our measure of fundamental expectation mirrors the measure of forward-looking expectations in commonly estimated RE models, while the expectations of naive firms account for the lagged value of inflation in the hybrid specification of the NKPC. The difference with traditional tests of sticky-price models arises from the introduction of time-varying weights and endogenous switching dynamics.

We estimated our behavioral model of inflation dynamics on quarterly U.S. data from 1960:Q1 to 2010:Q4. Our estimation results show statistically significant behavioral heterogeneity and substantial time variation in the weights of forward- and backward-looking price setters. The data gave considerable support for the parameter restrictions implied by the theory. In particular, the intensity of choice was found to be positive, indicating that agents switch towards the better performing rule according to its past performance, and inflation was positively affected by real marginal costs. These results were found to be independent from whether detrended output or the labor share of income were used as a measure of real marginal costs. In addition, the heterogeneous agent model outperforms several well known benchmark models in an assessment of competing out-of-sample forecast.

Our findings have important monetary policy implications. Recent papers have shown that
multiple equilibria, periodic orbits and complex dynamics can arise in New Keynesian models under dynamic predictor selection, even if the model under RE has a unique stationary solution. Given the statistical evidence found in our results for heterogeneous expectations and evolutionary switching, determinacy under RE may not be a robust recommendation and that monetary policy should be designed to account for potentially destabilizing heterogeneous expectations.
Appendix 4.A  NKPC with heterogeneous expectations

Under a Calvo pricing mechanism, the pricing rule of firm $i$ is given by equation (4.3):

$$p_{i,t} = (1 - \omega \delta)E_{i,t} \sum_{j=0}^{\infty} (\omega \delta)^j mc_{t+j} + \omega \delta E_{i,t} \sum_{j=0}^{\infty} (\omega \delta)^j \pi_{t+j+1}.$$  (4.16)

Aggregating (4.16) over different firms and using that $\sum_{i} n_{i,t} p_{i,t} = p^*_t = \omega (1 - \omega)^{-1} \pi_t$, we obtain a NKPC of the form

$$\pi_t = \bar{E}_t \sum_{j=0}^{\infty} (\omega \delta)^j \left[ \frac{(1 - \omega)(1 - \omega \delta)}{\omega} mc_{t+j} + (1 - \omega)\delta \pi_{t+j+1} \right],$$  (4.17)

where the operator $\bar{E}_t$, denoting the average expectation at time $t$, is defined as $\bar{E}_t = \sum_{i} n_{i,t} E_{i,t}$.\footnote{Preston (2005) and Massaro (2011) use a NKPC of the same form as (4.17), where inflation depends on expectations over an infinite horizon, for the analysis of monetary policies.}

Leading (4.17) one-period ahead and applying the operator $\bar{E}_t$ on both sides yields

$$\bar{E}_t \pi_{t+1} = \bar{E}_t \bar{E}_{t+1} \sum_{j=1}^{\infty} (\omega \delta)^{j-1} \left[ \frac{(1 - \omega)(1 - \omega \delta)}{\omega} mc_{t+j} + (1 - \omega)\delta \pi_{t+j+1} \right].$$  (4.18)

Assuming that Condition 1 holds and defining

$$X_{t+1} \equiv \sum_{j=1}^{\infty} (\omega \delta)^{j-1} \left[ \frac{(1 - \omega)(1 - \omega \delta)}{\omega} mc_{t+j} + (1 - \omega)\delta \pi_{t+j+1} \right],$$

we can rewrite

$$\bar{E}_t \pi_{t+1} = \bar{E}_t X_{t+1}$$  (4.19)

$$\pi_t = (1 - \omega)\delta \bar{E}_t \pi_{t+1} \left\{ \frac{(1 - \omega)(1 - \omega \delta)}{\omega} mc_{t} + \omega \delta \bar{E}_t X_{t+1} \right\}. $$  (4.20)

Substituting (4.19) into (4.20) and defining $\gamma \equiv (1 - \omega)(1 - \delta \omega)(\omega)^{-1}$, we finally get

$$\pi_t = \delta \bar{E}_t \pi_{t+1} + \gamma mc_{t}.$$
Appendix 4.B  Data sources

Below we describe the data sources and the data definitions used in the paper.

*Inflation* is constructed using the quarterly Price Indexes for GDP from the March 2011 release of the NIPA Table 1.1.4, 1960:Q1 - 2010:Q4, which can be downloaded at http://www.bea.gov/national/nipaweb/SelectTable.asp.

*Output gap* is constructed using the quarterly real GDP from the March 2011 release of the NIPA Table 1.1.3, 1960:Q1 - 2010:Q4, which can be downloaded at http://www.bea.gov/national/nipaweb/SelectTable.asp.

To construct our measure of the output gap we take logs and quadratically detrend.

*Unit labor costs* are constructed using the Bureau of Labor Statistics quarterly Unit Labor Costs series PRS85006113, 1960:Q1 - 2010:Q4, for the nonfarm business sector. The series can be downloaded at http://data.bls.gov/, under the heading Major Sector Productivity and Costs Index.


*Hours of work* are constructed using the Bureau of Labor Statistics quarterly Hours series PRS85006033, 1960:Q1 - 2010:Q4, for the nonfarm business sector. The series can be downloaded at http://data.bls.gov/, under the heading Major Sector Productivity and Costs Index. To construct our measure of the hours of work we take logs and quadratically detrend.

*Consumption-output ratio* is constructed using the quarterly real GDP from the March 2011 release of the NIPA Table 1.1.3, 1960:Q1 - 2010:Q4, which can be downloaded at http://www.bea.gov/national/nipaweb/SelectTable.asp.

To construct our measure of the consumption-output ratio we take logs and linearly detrend.
Appendix 4.C  Econometric procedure

Here we provide details on the econometric procedure adopted to estimate model (4.14).

Baseline VAR specification

The first step concerns the choice of the baseline VAR specification to estimate the matrix $A$, needed to construct the forecasts of fundamentalists,

$$E_t^f \pi_{t+1} = \gamma e_1' (I - \delta A)^{-1} AZ_t.$$  

We started with a very broad VAR model in the output gap $(\gamma_t)$, unit labor costs $(ulc_t)$, the labor share of income $(lsi_t)$, and the inflation rate $(\pi_t)$.\footnote{This specification extends the baseline specifications of previous empirical works, e.g., Woodford (2001b) and Rudd and Whelan (2005a), by adding lagged inflation in the output gap equation. However, we later eliminate it from the VAR specification together with the rate of change of unit labor costs since we found evidence that none of them Granger cause the output gap.} As shown in Table 4.7, standard unit root tests show that unit labor costs and labor share of income are I(1) processes.\footnote{The presence of a unit root in the labor share time series was not detected in previous empirical works such as Woodford (2001b) and Rudd and Whelan (2005a). This is due to the fact that our dataset incorporates observations until 2010:Q4. Unit root tests performed on the same sample considered by Woodford (2001b) and Rudd and Whelan (2005a) confirms the results found by these authors, i.e., the presence of a unit root in $lsi_t$ is rejected.} Therefore

<table>
<thead>
<tr>
<th>$ulc_t$ has a unit root</th>
<th>0.0860</th>
<th>0.9970</th>
</tr>
</thead>
<tbody>
<tr>
<td>$lsi_t$ has a unit root</td>
<td>-2.6689</td>
<td>0.2508</td>
</tr>
</tbody>
</table>

we estimated VAR models which include the rate of change of unit labor costs ($\Delta ulc_t$) and of labor share ($\Delta lsi_t$). The number of lags was chosen optimally on the basis of the comparison of standard information criteria, namely the sequential modified LR test statistic (LR), the Akaike information criterion (AIC), the Schwarz information criterion (SIC) and the Hannan-Quinn information criterion (HQ). We then performed pairwise Granger causality tests and proceeded iteratively, eliminating insignificant regressors, highest $p$-value first. We found evidence that
neither inflation nor the rate of change of unit labor costs Granger cause the output gap, therefore we excluded the variables $\Delta u_{lt}$ and $\pi_t$ from the VAR and we chose a four-lag bivariate VAR in the output gap and labor share of income growth as our baseline specification.\textsuperscript{17}

The Portmanteau test reports no autocorrelation in the residuals up to the 20th lag ($p$-value $Q(20) = 0.796$) and, although being parsimonious, the baseline VAR captures about 95% of output gap volatility ($R^2 = 0.945$).

Denoting by $Y_t$ the vector of dependent variables, $Y_t = [y_t, \Delta l s_i t]'$, the vector $Z_t$ is defined as $Z_t = [Y_t, Y_{t-1}, Y_{t-2}, Y_{t-3}]'$. The matrix $A$ denotes then the matrix of OLS estimates of the baseline VAR, obtained by regressing $Z_t$ on $Z_{t-1}$.

NLS estimation

Here we describe the estimation of model (4.14) by Non-linear Least Squares (NLS). For notational convenience we rewrite model (4.14) as

$$\pi = x(\theta) + u,$$

where $\theta = [\beta, \gamma]$ is the vector of parameters to be estimated, and the scalar function $x(\theta)$ is a nonlinear regression function corresponding to the dynamics implied by model (4.14). We estimate model (4.21) using NLS, see Table 4.1 in section 4.3.2, and perform diagnostic checks on the residuals. Table 4.8 reports the result of the White test for heteroskedasticity. Given

<table>
<thead>
<tr>
<th>Table 4.8: Heteroskedasticity test</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_0$: homoskedasticity</td>
</tr>
<tr>
<td>$F$-statistic</td>
</tr>
<tr>
<td>Obs$^* R^2$</td>
</tr>
</tbody>
</table>

the presence of heteroskedasticity, we perform the heteroskedasticity-robust test for serial autocorrelation proposed by Davidson and MacKinnon (2009). The procedure is based on the

\textsuperscript{17}The lag order of 4 was selected by 3 out 4 criteria, namely the LR, the AIC, and the HQ.
following Gauss-Newton regression:

\[ \hat{u} = \hat{X}b + \hat{Z}c + \text{residuals}, \]  

(4.22)

where \( \hat{u} = \pi - x(\hat{\theta}) \) denotes the NLS residuals from (4.21), \( X \equiv X(\hat{\theta}) \) denotes the vector of derivatives of \( x(\theta) \) with respect to \( \theta \), computed in \( \hat{\theta} \), and \( \hat{Z} \) is a matrix collecting lagged values of the NLS residuals \( \hat{u} \). In order to test for \( c = 0 \), we use the test statistic suggested by Davidson and MacKinnon (2009), distributed as \( \chi^2(p) \) under the null hypothesis, where \( p \) denotes the order of serial correlation being tested.\(^\text{18}\) The results reported in Table 4.9 show the absence of serial correlation in the residuals up to the 20th lag.

Table 4.9: Serial correlation test

<table>
<thead>
<tr>
<th>( H_0: ) no serial correlation ((c = 0))</th>
<th>test-statistic ((p = 20))</th>
<th>Prob. ( \chi^2(p) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>21.51</td>
<td>0.368</td>
</tr>
</tbody>
</table>

\(^\text{18}\)See Davidson and MacKinnon (2009), p. 284, for details about the construction of the test.
Chapter 5

Experiments on Individual Expectations and Aggregate Macro Behavior

5.1 Introduction

Inflation expectations are crucial in the transmission of monetary policy. The way in which individual expectations are formed, therefore, is key in understanding how a change in the interest rate affects output and the actual inflation rate. Since the seminal papers of Muth (1961) and Lucas (1972) the rational expectations hypothesis has become the cornerstone of macroeconomic theory, with representative rational agent models dominating mainstream economics. For monetary policy analysis the most popular model is the New Keynesian (NK) framework which assumes, in its basic formulation, a representative rational agent structure (see e.g. Woodford (2003) and Galí (2008)). The standard NK model with a rational representative agent however has lost most of its appeal in the light of overwhelming empirical evidence: it is clear from the data that this approach is not the most suitable to reproduce stylized facts such as the persistence of fluctuations in real activity and inflation after a shock (see e.g. Chari, Kehoe, and McGrattan (2000) and Nelson (1998)). Economists have therefore proposed a number of extensions to the standard framework by embedding potential sources of endogenous persistence. They
have incorporated features such as habit formation or various adjustment costs to account for the inertia in the data (e.g. Christiano, Eichenbaum, and Evans (2005) and Smets and Wouters (2007)).

In the last two decades adaptive learning has become an interesting alternative to modeling expectations (see e.g. Sargent (1999) and Evans and Honkapohja (2001)). Bullard and Mitra (2002), Preston (2005) among others, introduce adaptive learning in the NK framework and Milani (2007) shows that learning can represent an important source of persistence in the economy and that some extensions which are typically needed under rational expectations to match the observed inertia become redundant under learning. More recently a number of authors have extended the NK model to include heterogeneous expectations, e.g. Galí and Gertler (1999), De Graauwe (2010) and Branch and McGough (2009, 2010).

The empirical literature on expectations in a macro-monetary policy setting can be subdivided in work on survey data and laboratory experiments with human subjects. Mankiw, Reis, and Wolfers (2003) find evidence for heterogeneity in inflation expectations in the Michigan Survey of Consumers and argue that the data are inconsistent with rational or adaptive expectations, but may be consistent with a sticky information model. Branch (2004) estimates a simple switching model with heterogeneous expectations on survey data and provides empirical evidence for dynamic switching that depends on the relative mean squared errors of the predictors. Capistran and Timmermann (2009) show that heterogeneity of inflation expectations of professional forecasters varies over time and depends on the level and the variance of current inflation. Pfajfar and Santoro (2010) measure the degree of heterogeneity in private agents’ inflation forecasts by exploring time series of percentiles from the empirical distribution of survey data. They show that heterogeneity in inflation expectations is persistent and identify three different expectations formation mechanisms: static or highly autoregressive rules, nearly rational expectations and adaptive learning with sticky information. Experiments with human subjects in a controlled laboratory environment to study individual expectations have been carried out by, e.g., Hommes, Sonnemans, Tuinstra, and van de Velden (2005), Adam (2007), Pfajfar and
Zakelj (2010); see Duffy (2008) for an overview of macro experiments, and Hommes (2011) for an overview of learning to forecast experiments to study expectation formation.

In this paper we use laboratory experiments with human subjects to study the individual expectations formation process within a standard New Keynesian setup. We ask subjects to forecast the inflation rate under three different scenarios depending on the underlying assumption on output gap expectations, namely fundamental, naive or forecasts from a group of individuals in the laboratory. An important novel feature of our experiment is that, in one of the treatments, the aggregate variables inflation and output gap depend on individual expectations of two groups of individuals forming expectations on two different variables, inflation and the output gap. In particular, we address the following questions:

• are expectations homogeneous or heterogeneous?
• which forecasting rules do individuals use?
• which theory of expectations and learning fits the aggregate as well as individual experimental data?
• which monetary policy rules can stabilize aggregate outcomes in learning to forecast experiments?

The paper is organized as follows. Section 2 describes the underlying NK-model framework, the different treatments, the experimental design and the experimental results. Section 3 analyzes the individual forecasting rules used by the subjects, while Section 4 proposes a heterogeneous expectations model explaining both individual expectations and aggregate outcomes. Finally, Section 5 concludes.

5.2 The learning to forecast experiment

In 5.2.1 we briefly recall the New Keynesian model and then we give a description of the treatments in the experiment. In 5.2.2 we give an overview of the experimental design and in
5.2.3 we summarize the main results.

5.2.1 The New Keynesian model

In this section we recall the monetary model with nominal rigidities that will be used in the experiment. We adopt the heterogeneous expectations version of the New Keynesian model developed by Branch and McGough (2009), which is described by the following equations:

\[ y_t = \overline{y}_{t+1} - \varphi (i_t - \overline{\pi}_{t+1}) + g_t, \]
\[ \pi_t = \lambda y_t + \rho \pi_{e,t+1}^e + u_t, \]
\[ i_t = \pi + \phi_{\pi}(\pi_t - \pi), \]

where \( y_t \) and \( \overline{y}_{t+1} \) are respectively the actual and average expected output gap, \( i_t \) is the nominal interest rate, \( \pi_t \) and \( \pi_{e,t+1}^e \) are respectively the actual and average expected inflation rates, \( \pi \) is the inflation target, \( \varphi, \lambda, \rho \) and \( \phi_{\pi} \) are positive coefficients and \( g_t \) and \( u_t \) are white noise shocks. The coefficient \( \phi_{\pi} \) measures the response of the nominal interest rate \( i_t \) to deviations of the inflation rate \( \pi_t \) from its target \( \pi \). Equation (5.1) is the aggregate demand in which the output gap \( y_t \) depends on the average expected output gap \( \overline{y}_{t+1} \) and on the real interest rate \( i_t - \overline{\pi}_{t+1} \). Equation (5.2) is the New Keynesian Phillips curve according to which the inflation rate depends on the output gap and on average expected inflation. Equation (5.3) is the monetary policy rule implemented by the monetary authority in order to keep inflation at its target value \( \pi \). The New Keynesian model is widely used in monetary policy analysis and allows us to compare our experimental results with those obtained in the theoretical literature. However the New Keynesian framework requires agents to forecast both inflation and the output gap. Since forecasting two variables at the same time might be a too difficult task for the participants in an experiment we decided to run an experiment using three different treatments. In the first two treatments we make an assumption about output gap expectations (a steady state equilibrium predictor and naive expectations respectively), so that the task of the participants reduces to
forecast only one macroeconomic variable, namely inflation. In the third treatment there are
two groups of individuals, one group forecasting inflation and the other forecasting output gap.
The details of the different treatments are described below.

**Treatment 1: steady state predictor for output gap**

In the first treatment of the experiment we ask subjects to forecast the inflation rate two periods
ahead, given that the expectations on the output gap are fixed at the equilibrium predictor (i.e.
\( \bar{y}_{t+1} = (1 - \rho)\bar{\pi} \lambda^{-1} \)). Given this setup the NK framework (5.1)-(5.3) specializes to:

\[
\begin{align*}
y_t &= (1 - \rho)\bar{\pi} \lambda^{-1} - \varphi(i_t - \bar{\pi}_{t+1}) + g_t, \quad (5.4) \\
\pi_t &= \lambda y_t + \rho \bar{\pi}_{t+1} + u_t, \quad (5.5) \\
i_t &= \varphi \pi_t (\pi_t - \bar{\pi}) + \pi, \quad (5.6)
\end{align*}
\]

where \( \bar{\pi}_{t+1} = \frac{1}{H} \sum_{i=1}^{H} \pi_{i,t+1} \) is the average prediction of the participants in the experiment.
Substituting (5.6) into (5.4) leads to the system

\[
\begin{align*}
y_t &= (1 - \rho)\bar{\pi} \lambda^{-1} + \varphi \pi (\varphi _\pi - 1) - \varphi \phi _\pi \pi_t + \varphi \bar{\pi}_{t+1} + g_t, \quad (5.7) \\
\pi_t &= \lambda y_t + \rho \bar{\pi}_{t+1} + u_t. \quad (5.8)
\end{align*}
\]

The above system can be rewritten in terms of inflation and expected inflation:

\[
\begin{align*}
\pi_t &= A_\pi + \frac{\lambda \varphi + \rho}{1 + \lambda \varphi} \bar{\pi}_{t+1} + \xi_t, \quad (5.9)
\end{align*}
\]

where \( A_\pi = \frac{(1 - \rho)\bar{\pi} + \lambda \varphi \pi (\varphi _\pi - 1)}{1 + \lambda \varphi} \) is a constant and \( \xi_t = \frac{\lambda}{1 + \lambda \varphi} g_t + \frac{1}{1 + \lambda \varphi} u_t \) is a compositive shock. Hence, treatment 1 reduces to a learning to forecast experiment on a single variable, inflation, comparable to the learning to forecast experiments on asset prices in Hommes, Son-
nemans, Tuinstra, and van de Velden (2005) and on inflation in Adam (2007)\(^1\).

**Treatment 2: naive expectations for output gap**

In the second treatment we ask subjects to forecast only the inflation rate (two periods ahead), while expectations on the output gap are represented by naive expectations (i.e. \(\bar{y}_{t+1} = y_{t-1}\)). This treatment is similar to the experiment in Pfajfar and Zakelj (2010) who also implicitly assume naive expectations on output. Given this setup the NK framework (5.1)-(5.3) specializes to:

\[
\begin{align*}
    y_t &= \varphi \pi_t - 1 - \varphi \pi_{t} + \varphi \bar{y}_{t+1} + y_{t-1} + g_t, \\
    \pi_t &= \lambda y_t + \rho \pi_{t+1} + u_t. 
\end{align*}
\]

(5.10) (5.11)

where \(\bar{\pi}_{t+1} = \frac{1}{H} \sum_{i=1}^{H} \pi_{i,t+1}^{e}\) is the average prediction of the participants in the experiment. We can rewrite the above system in matrix form

\[
\begin{bmatrix}
    y_t \\
    \pi_t
\end{bmatrix} = A + \Omega \begin{bmatrix}
    0 & \varphi (1 - \phi_{\pi} \rho) \\
    0 & \lambda \varphi + \rho
\end{bmatrix} \begin{bmatrix}
    \bar{y}_{t+1}^{e} \\
    \pi_{t+1}^{e}
\end{bmatrix} + \Omega \begin{bmatrix}
    1 & 0 \\
    \lambda & 0
\end{bmatrix} \begin{bmatrix}
    y_{t-1} \\
    \pi_{t-1}
\end{bmatrix} + B \begin{bmatrix}
    g_t \\
    u_t
\end{bmatrix}
\]

(5.12)

where \(\Omega = (1 + \lambda \varphi \phi_{\pi})^{-1}\), \(A = \Omega \begin{bmatrix}
    \varphi \pi_t - 1 \\
    \lambda \varphi \pi_t - 1
\end{bmatrix}\) and \(B = \Omega \begin{bmatrix}
    1 & -\varphi \phi_{\pi} \\
    \lambda & 1
\end{bmatrix}\). This setup is more complicated than the learning to forecast experiments in Hommes, Sonnemans, Tuinstra, and van de Velden (2005) and Adam (2007) because inflation is not only driven by expected inflation and exogenous noise, but also by the past output gap \(y_{t-1}\). An important difference with Pfajfar and Zakelj (2010) is that we assume IID noise instead of an AR(1) noise process, so that if fluctuations in inflation will arise in the experiment they must be endogenously driven by expectations.

\(^1\)Given the calibrated values of the structural parameters, described in Section 5.2.3, the coefficient \(\frac{\lambda \varphi + \rho}{1 - \lambda \varphi \phi_{\pi}}\) in (5.9) measuring expectation feedback takes the value of about 0.99 when the policy rule’s reaction coefficient \(\phi_{\pi} = 1\), and of about 0.89 when \(\phi_{\pi} = 1.5\). The corresponding expectation feedback coefficient in Hommes, Sonnemans, Tuinstra, and van de Velden (2005) was 0.95.
**Treatment 3: forecasting inflation and output gap**

In the third treatment there are two groups of participants acting in the same economy but with different tasks: one group forecasts inflation while the other forecasts the output gap. Agents are divided randomly into two groups, one group is asked to form expectations on the inflation rate and another group provides forecasts on the output gap. The aggregate variables inflation and output gap are thus driven by individual expectations feedbacks from two different variables by two different groups. The model describing the experimental economy can be written as

\[
\begin{bmatrix}
y_t \\
\pi_t
\end{bmatrix} = A + \Omega \begin{bmatrix}
1 - \phi_t (1 - \phi \lambda) \\
\lambda
\end{bmatrix} \begin{bmatrix}
y_{t+1} \\
\pi_{t+1}
\end{bmatrix} + B \begin{bmatrix}
g_t \\
u_t
\end{bmatrix},
\]

(5.13)

where \(A, B\) and \(\Omega\) are defined as in treatment 2, while \(\bar{y}_{t+1} = \frac{1}{H} \sum_{i=1}^{H} y_{t+1}^i\) and \(\bar{\pi}_{t+1} = \frac{1}{H} \sum_{i=1}^{H} \pi_{t+1}^i\) are respectively the average output gap and the average inflation predictions of the participants in the experiment. As already pointed out, in treatments 1 and 2 individuals are asked to forecast only the inflation rate two periods ahead, assuming respectively that the expected future output gap is given by the equilibrium predictor \((\overline{y}_{t+1} = (1 - \rho) \pi \lambda^{-1})\) or follows naive expectations \((\overline{y}_{t+1} = y_{t-1})\). An important novel aspect of Treatment 3 is that our experimental economy is driven by individual expectations on two different aggregate variables that interact within a New Keynesian framework.

**Treatments a/b: passive versus active monetary policy**

In order to study the stabilization properties of a monetary policy rule such as (5.3), we ran two experimental sessions for each of the three different treatments described above. In session ”a” the monetary policy responds only weakly to inflation rate fluctuations i.e., the Taylor principle does not hold \((\phi_t = 1)\), while in session ”b” monetary policy responds aggressively to inflation.
i.e., the Taylor principle holds\(^2\) (\(\phi_\pi = 1.5\)).

Table 5.1 summarizes all treatments implemented in the experiments. In total 120 subjects participated in the experiment in 16 experimental economies, 3 for each of the treatments 1a, 1b, 2a, and 2b with 6 subjects each, and 2 experimental economies for treatments 3a and 3b with 12 subjects each. Total average earnings over all subjects were 32 €.

<table>
<thead>
<tr>
<th>Treatment 1a</th>
<th>(\phi_\pi)</th>
<th>(\frac{e^{\pi_{t+1}}}{\pi_{t+1}})</th>
<th>(y_{t+1})</th>
<th># groups</th>
<th>average earnings (\pi(y)) in €</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatment 1b</td>
<td>1.5</td>
<td>(\frac{e^{\pi_{t+1}}}{\pi_{t+1}})</td>
<td>(y_{t-1})</td>
<td>3</td>
<td>37</td>
</tr>
<tr>
<td>Treatment 2a</td>
<td>1</td>
<td>(\frac{e^{\pi_{t+1}}}{\pi_{t+1}})</td>
<td>(y_{t-1})</td>
<td>3</td>
<td>28</td>
</tr>
<tr>
<td>Treatment 2b</td>
<td>1.5</td>
<td>(\frac{e^{\pi_{t+1}}}{\pi_{t+1}})</td>
<td>(y_{t-1})</td>
<td>3</td>
<td>36</td>
</tr>
<tr>
<td>Treatment 3a</td>
<td>1</td>
<td>(\frac{e^{\pi_{t+1}}}{\pi_{t+1}})</td>
<td>(\bar{y}_{t+1})</td>
<td>2</td>
<td>28 (28)</td>
</tr>
<tr>
<td>Treatment 3b</td>
<td>1.5</td>
<td>(\frac{e^{\pi_{t+1}}}{\pi_{t+1}})</td>
<td>(\bar{y}_{t+1})</td>
<td>2</td>
<td>34 (32)</td>
</tr>
</tbody>
</table>

Table 5.1: Treatments summary

5.2.2 Experimental design

The experiment took place in the CREED laboratory at the University of Amsterdam, March-May 2009. For treatments 1 and 2, groups of six (unknown) individuals were formed who had to forecast inflation two periods ahead; for treatment 3 two groups of six individuals were formed, one group forecasting inflation, the other group forecasting the output gap. Most subjects are undergraduate students from Economics, Chemistry and Psychology. At the beginning of the session each subject can read the instructions (see Appendix 5.A) on the screen, and subjects receive also a written copy. Participants are instructed about their role as forecasters and about the experimental economy. They are assumed to be employed in a private firm of professional forecasters for the key variables of the economy under scrutiny i.e. either the inflation rate or the output gap. Subjects have to forecast either inflation or the output gap for 50 periods. We give them some general information about the variables that describe the economy: the output gap (\(y_t\)), the inflation rate (\(\pi_t\)) and the interest rate (\(i_t\)). Subjects are also informed about

\(^2\)Notice that when the policy parameter \(\phi_\pi\) is equal to 1, the system in Treatments 2 and 3 exhibits a continuum of equilibria.
the expectations feedback, that realized inflation and output gap depend on (other) subjects’ expectations about inflation and output gap. They also know that inflation and output gap are affected by small random shocks to the economy. Subjects did not know the equations of the underlying law of motion of the economy nor did they have any information about its steady states. In short, subjects did not have quantitative details, but only qualitative information about the economy.

The payoff function of the subjects describing their score that is later converted into Euros is given by

\[ \text{score} = \frac{100}{1 + f}, \]  

(5.14)

where \( f \) is the absolute value of the forecast error expressed in percentage points. The points earned by the participants depend on how close their predictions are to the realized values of the variable they are forecasting. Information about the payoff function is given graphically as well as in table form to the participants (see Fig. 5.1). Notice that the prediction score increases sharply when the error decreases to 0, so that subjects have a strong incentive to forecast as accurately as they can; see also Adam (2007) and Pfajfar and Zakelj (2010), who used the same payoff function.

![Figure 5.1: Payoff function](image)

<table>
<thead>
<tr>
<th>Absolute forecast error</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Score</td>
<td>100</td>
<td>50</td>
<td>33</td>
<td>25</td>
<td>20</td>
<td>10</td>
</tr>
</tbody>
</table>
In each period individuals can observe on the left side of the screen the time series of realized inflation rate, output gap and interest rate as well as the time series of their own forecasts. The same information is displayed on the right hand side of the screen in table form, together with subjects own predictions scores (see Fig. 5.2). Subjects did not have any information about the forecasts of others.

Figure 5.2: Computer screen for inflation forecasters with time series of inflation forecasts and realizations (top left), output gap and interest rate (bottom left) and table (top right).

5.2.3 Experimental results

This subsection describes the results of the experiment. We fix the parameters at the Clarida, Gali, and Gertler (2000) calibration, i.e. $\rho = 0.99$, $\varphi = 1$, and $\lambda = 0.3$, and we set the inflation target to $\bar{\pi} = 2$.

Figure 5.3 depicts the behavior of the output gap, inflation and individual forecasts in the three different sessions of treatments $1a$ and $1b$ with output expectations given by the steady state predictor. The dotted lines in the figures represent the RE steady states for inflation and output gap that are respectively 2 and 0.07. In treatment $1a$ ($\phi_\pi = 1$) we observe convergence to
Figure 5.3: Time series of Treatment 1, with fundamental predictor for the output gap. **Upper panels:** Treatment 1a \((\phi_\pi = 1)\). **Lower panels:** Treatment 1b \((\phi_\pi = 1.5)\). Blue thick line: realized inflation; yellow thick line: realized output gap; thin lines: individual forecasts for inflation.
a non-fundamental steady state for two groups, while the third group displays highly unstable oscillations\(^3\). In treatment 1b \((\phi_\pi = 1.5)\) we observe convergence to the inflation target for two groups, while the third group exhibits oscillatory behavior which is by far less pronounced than what we observed in treatment 1a, group 2.

We conclude that, under the assumption of a fundamental predictor for expected future output gap, a more aggressive monetary policy that satisfies the Taylor principle \((\phi_\pi > 1)\) stabilizes inflation fluctuations and leads to convergence to the desired inflation target in two of the three groups.

Fig. 5.4 shows the behavior of the output gap, inflation and individual forecasts in three different groups of treatments 2a and 2b with naive output gap expectations. In treatment 2a \((\phi_\pi = 1)\) we observe different types of aggregate dynamics. Group 1 shows convergence to a non-fundamental steady state. Group 2 shows oscillatory behavior with individual expectations coordinating on the oscillatory pattern. In this session the interest rate hits the zero lower bound in period 43 and the experimental economy experiences a phase of decline in output gap but eventually recovers. In group 3 the behavior is even more unstable: inflation oscillates until, in period 27, the interest rate hits the zero lower bound and the economy enters a severe recession and never recovers. In treatment 2b \((\phi_\pi)\) we observe convergence to the fundamental steady state for two groups, while the third group exhibits small oscillations around the fundamental steady state.

We conclude that also under the assumption of naive expectations for the output gap, an interest rate rule that responds more than point to point to deviations of the inflation rate from the target stabilizes the economy.

The upper panels of Fig. 5.5 reproduce the behavior of the output gap, inflation and individual forecasts for both variables in two different sessions of treatment 3a. Recall that in treatment 3 realized inflation and output gap depend on the individual forecasts for both inflation and output gap. In both groups of treatment 3a \((\phi_\pi = 1)\) we observe (almost) convergence

\(^3\)The unstable fluctuations were mainly caused by one participant making very high and very low forecasts.
Figure 5.4: **Upper panels**: Treatment 2a. **Lower panels**: Treatment 2b. Blue thick line: realized inflation; yellow thick line: realized output gap; thin lines: individual forecasts for inflation.
Figure 5.5: **Upper panels:** Treatment 3a. **Lower panels:** Treatment 3b. Blue thick line: realized inflation; yellow thick line: realized output gap; thin lines: individual forecasts for inflation and output gap.

4Note that group 1 ends in period 26 because of a crash of one of the computers in the lab. Moreover realized inflation and output gap in group 2 are plotted until period 49 because of an end effect. In fact, participant 3 predicted an inflation rate of 100% in the last period, causing actual inflation to jump to about 20%.
<table>
<thead>
<tr>
<th>Group</th>
<th>Inflation</th>
<th>Output gap</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a-1</td>
<td>0.3125</td>
<td>0.0052</td>
</tr>
<tr>
<td>1a-2</td>
<td>23.3332</td>
<td>0.0071</td>
</tr>
<tr>
<td>1a-3</td>
<td>0.4554</td>
<td>0.0052</td>
</tr>
<tr>
<td>1a (median)</td>
<td>0.4554</td>
<td>0.0052</td>
</tr>
<tr>
<td>1b-1</td>
<td>0.0715</td>
<td>0.0195</td>
</tr>
<tr>
<td>1b-2</td>
<td>0.0169</td>
<td>0.0115</td>
</tr>
<tr>
<td>1b-3</td>
<td>0.5100</td>
<td>0.0720</td>
</tr>
<tr>
<td>1b (median)</td>
<td>0.0715</td>
<td>0.0195</td>
</tr>
<tr>
<td>2a-1</td>
<td>3.8972</td>
<td>0.0181</td>
</tr>
<tr>
<td>2a-2</td>
<td>3.7661</td>
<td>0.4953</td>
</tr>
<tr>
<td>2a-3</td>
<td>6003.1485</td>
<td>35699.2582</td>
</tr>
<tr>
<td>2a (median)</td>
<td>3.8972</td>
<td>0.4953</td>
</tr>
<tr>
<td>2b-1</td>
<td>0.0160</td>
<td>0.0265</td>
</tr>
<tr>
<td>2b-2</td>
<td>0.0437</td>
<td>0.0400</td>
</tr>
<tr>
<td>2b-3</td>
<td>0.1977</td>
<td>0.1383</td>
</tr>
<tr>
<td>2b (median)</td>
<td>0.0437</td>
<td>0.0400</td>
</tr>
<tr>
<td>3a-1 (excl. t=50)</td>
<td>1.2159</td>
<td>0.1073</td>
</tr>
<tr>
<td>3b-1</td>
<td>0.4804</td>
<td>0.1865</td>
</tr>
<tr>
<td>3b-2</td>
<td>0.4366</td>
<td>0.2256</td>
</tr>
<tr>
<td>3b (median)</td>
<td>0.4585</td>
<td>0.2060</td>
</tr>
</tbody>
</table>

Table 5.2: Average quadratic difference from the REE benchmark for all treatments. The table confirms our earlier graphical observation that a more aggressive Taylor rule stabilizes inflation. Increasing the Taylor coefficient from 1 to 1.5 leads to more stable inflation by a factor around 6 in Treatment 1, a factor of 90 in Treatment 2 and a factor of about 3 in Treatment 3. In contrast to inflation the output gap is not stabilized in our experimental economy where the central bank sets the interest rate responding only to inflation.
5.3 Individual forecasting rules

Estimation of general linear forecasting rules

For each participant we estimated a simple linear prediction rule of the form

\[
\pi_{j,t+1}^e = c + \sum_{i=0}^{2} \alpha_i \pi_{j,t-i}^e + \sum_{i=1}^{3} \beta_i \pi_{t-i} + \sum_{i=1}^{3} \gamma_i y_{t-i} + \mu_t \tag{5.15}
\]

\[
y_{j,t+1}^e = c + \sum_{i=0}^{2} \delta_i y_{j,t-i}^e + \sum_{i=1}^{3} \varepsilon_i y_{t-i} + \sum_{i=1}^{3} \zeta_i \pi_{t-i} + \nu_t \tag{5.16}
\]

in which \(\pi_{j,t+1}^e\) and \(y_{j,t+1}^e\) refer to the inflation or output gap forecast of participant \(j\) for period \(t+1\) (submitted in period \(t\)). Prediction rule (5.15) applies to inflation forecasters and prediction rule (5.16) to output gap forecasters, using both lagged inflation and lagged output.

These prediction rules assume that participants do not use information with a lag of more than three periods; the regression results (see below) show that this is generally a reasonable assumption. We allow for a learning phase, during which participants have not yet fully formed their prediction rules, by leaving the first 11 periods of the experiment out of the regression sample.

Tables 5.8 – 5.14 in Appendix 5.B show the regression results. Of the 102 participants\(^5\) 78% submitted predictions that can be described by a linear rule of the form (5.15) or (5.16) satisfying standard diagnostic tests.\(^6\) In all treatments, the most popular significant regressor is the last available value of the forecasting objective (\(\pi_{t-1}^e\) or \(y_{t-1}^e\)). This is followed in most treatments by either the most recent own prediction (\(\pi_{t}^e\) or \(y_{t}^e\)) or the second last available forecasting objective (\(\pi_{t-2}^e\) or \(y_{t-2}^e\)). Looking at the estimated coefficients, a remarkable property is that all but one of the non-zero coefficients with both the last available forecasting objective and the most recent own prediction are positive suggesting some form of adaptive behavior (see below).

In contrast, a clear majority of the non-zero coefficients of remaining regressors, excluding the

---

\(^5\)The prediction rule specifications (5.15) and (5.16) were applied to all participants except for those in group 3 of Treatment 2a (see Fig. 5.4) and group 1 of Treatment 3a (see Fig. 5.5), which experienced respectively total economic collapse and a computer crash during the experiment.

\(^6\)Estimated prediction rules were tested for autocorrelation (Breusch-Godfrey test, 2 lags), heteroskedasticity (White test, no cross terms) and misspecification (Ramsey RESET test, 1 fitted term). A significance level of 5% was used.
constant, is negative. Averaging over the participants of all treatments, the number of significant regressors, including the constant, in the estimated prediction rules is 3.4.

The estimation results indicate that most participants, largely irrespective of the treatment they are in, use a consistent, linear prediction rule, at least after a learning phase of 11 periods. What is more, there are clear regularities across groups and treatments regarding the variables the prediction rules are composed of and the sign of their coefficients. Specifically, the fact that the two latest observations of the forecasting objective and the latest own prediction are generally the most used prediction rule components, implies that these variables are of particular importance in the prediction rule specification. The relatively low average number of significant regressors, at 3.4 compared to the 10 potential regressors (including the constant) in (5.15) and (5.16), means that the other variables are used very little as input to form predictions. It may therefore be worthwhile to restrict specifications (5.15) and (5.16) by leaving out these infrequently used regressors. The fact that the estimated non-zero coefficients for the most recent values of the forecasting objective and the own prediction are almost all positive, while the non-zero coefficients of the other variables tend to be negative, similarly suggests that the rule specifications (5.15) and (5.16) are too flexible. Restricting (5.15) and (5.16) along the lines of these regularities could increase the efficiency of the estimates, as well as make the estimated rules easier to interpret from a behavioral viewpoint.

**Estimation of an anchoring-and-adjustment heuristic**

The estimation results of the previous section indicate that the general prediction rules (5.15) and (5.16) can for most participants be restricted without losing much explanatory power. One way of strongly reducing the number of parameters while preserving much of the specifications’ flexibility is by fitting an anchoring-and-adjustment heuristic, (Tversky and Kahneman (1974)), named *First-Order Heuristic* (FOH). In the context of our experiment, the FOH has

---

For other applications of the FOH in modeling expectation formation, see Heemeijer, Hommes, Sonnemans, and Tuinstra (2009) and Heemeijer (2009).
the following form:

\[
\pi_{h,t+1}^e = \alpha_1 \pi_{t-1} + \alpha_2 \pi_{h,t} + (1 - \alpha_1 - \alpha_2) \frac{1}{39} \sum_{t=12}^{50} \pi_t + \alpha_3 (\pi_{t-1} - \pi_{t-2}) + \mu_t \tag{5.17}
\]

\[
y_{h,t+1}^e = \gamma_1 y_{t-1} + \gamma_2 y_{h,t} + (1 - \gamma_1 - \gamma_2) \frac{1}{39} \sum_{t=12}^{50} y_t + \gamma_3 (y_{t-1} - y_{t-2}) + \nu_t \tag{5.18}
\]

The first three terms in (5.17) and (5.18) are a weighted average of the latest realization of the forecasting objective, the latest own prediction and the forecasting objective’s sample mean (excluding the learning phase). This weighted average is the (time varying) “anchor” of the prediction, which is a zeroth order extrapolation from the available data at period \(t\). The fourth term in (5.17) and (5.18) is a simple linear, i.e. first order, extrapolation from the two most recent realizations of the forecasting objective; this term is the “adjustment” or trend extrapolation part of the heuristic. An advantages of the FOH rule is that it simplifies to well-known rules-of-thumb for different boundary values of the parameter space. For example, the inflation prediction rule (5.17) reduces to Naive Expectations if \(\alpha_1 = 1, \alpha_2 = 0\); it reduces to Adaptive Expectations if \(\alpha_1 + \alpha_2 = 1, \alpha_3 = 0\). Another special case occurs when \(\alpha_1 = \alpha_2 = 0\), so that the anchor reduces to the sample average; we will refer to this case as Fundamentalists, as the sample average is a proxy of the steady state equilibrium level of inflation or output. In the more flexible case \(\alpha_1 + \alpha_3 = 1, \alpha_2 = 0\) the anchor is time varying; we will refer to this case as a learning anchor and adjustment (LAA) rule.

We estimated the FOH rules (5.17) and (5.18) for participants that have a prediction rule of type (5.15) or (5.16) satisfying standard diagnostic tests (see previous Section), and that are not significantly worse described by a FOH rule than by a general linear rule. The second criterion was verified by a Wald test on joint parameter restriction. It turns out that 59 out of the 102 participants used in the regression analysis, that is, 58%, pass both criteria. These results are in line with the findings in Adam (2007) where simple forecast functions that condition on a

\[8\]In the estimation of (5.17) and (5.18) we included the sample mean of inflation resp. output, which is of course not available to the subjects at the moment of the prediction but acts as a proxy of the equilibrium level. In the heuristic switching model of section 5.4 we will use the sample average of all previous realizations at every point in time, which generally converges quickly to the sample mean, as an anchor.

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single explanatory variable capture subjects’ expectations fairly well. For the participants whose forecasting rules can be successfully restricted to a FOH rule, the estimation results are shown in Tables 5.15 – 5.17 in Appendix 5.B. Also indicated in Tables 5.15 – 5.17 in Appendix 5.B are the rules-of-thumb, if any, that the estimated FOH rules are equivalent to (again according to Wald tests). Looking across treatments, the most frequent classifications for the anchors of the prediction rules are Naive and Adaptive Expectations. However, the anchors of almost half of the participants with a FOH rule (26 out of 59) are not equivalent to a well-known rule-of-thumb and are therefore described as “other.” Regarding the adjustment part of the estimated FOH rules, it is interesting to see that all participants with a trend extrapolating term in their FOH rules are trend followers, i.e. the coefficient $\alpha_3 > 0$, ranging from 0.3 to 1.4. More than half of the estimated FOH rules (31 out of 59) have a trend-following adjustment term.

Figs. 5.6a – 5.6d illustrate the estimation results in the three-dimensional space $(\alpha_1, \alpha_2, \alpha_3)$. The individual FOH rules are represented by dots in the FOH rule’s prismatic parameter space. The prisms show concentrations of dots at the regions corresponding to Naive Expectations $(\alpha_1 = 1, \alpha_2 = \alpha_3 = 0)$, Adaptive Expectations $(\alpha_1 + \alpha_2 = 1, \alpha_3 = 0)$, or Trend-Following Expectations $(\alpha_3 > 0)$, confirming the classification results above. At the same time, there are substantial differences between the prisms. In particular, Figs. 5.6b and 5.6d show that almost all participants using an FOH rule in Treatments 2a, 2b and 3b have a trend-following adjustment term, while this is much less frequently the case in Treatments 1a, 1b and 3a (see Figs. 5.6a and 5.6c). Comparing with the experimental results (Figs. 5.3 – 5.5), the presence of trend-following forecasting rules is clearly related to oscillations in the forecasting objective, which occur more often in Treatments 2a, 3a and 3b than in the rest of the experiment. Also, an interesting difference between Figs. 5.6c and 5.6d is that Fig. 5.6c contains a cluster of dots close to Naive Expectations, while Fig. 5.6d contains a cluster close to Fundamentalist Expectations (i.e. predictions equal to the forecasting objective’s sample mean). Comparing with Fig. 5.5, it is apparent that the reason for this difference is that inflation and, to a lesser extent, output gap, do not fully converge in Treatment 3a, while they do converge in Treatment 3b. This makes
Figure 5.6: Estimated coefficient vectors of First-Order Heuristics (FOH) prediction rules for the participants. The graph on the right of each prism presents a top-down view of the prism. **Top left:** dark dots correspond to participants of Treatment 1a; light dots to participants of Treatment 1b. **Top right:** dark dots correspond to participants of Treatment 2a; light dots to participants of Treatment 2b. **Bottom left:** dark dots correspond to inflation forecasters of Treatment 3a; light dots to output gap forecasters of Treatment 3a. **Bottom right:** Dark dots correspond to inflation forecasters of Treatment 3b; light dots to output gap forecasters of Treatment 3b.
a constant anchor such as in Fundamentalist Expectations more useful in Treatment 3b, and a flexible anchor such as in Naive Expectations more useful in Treatment 3a.

**Graphical evidence for strategy switching**

The estimated forecasting rules (5.17) and (5.18) assume a fixed individual prediction rule over the last 40 periods of the experiment. This should be viewed as an approximate forecasting rule; in reality agents may learn and switch to a different forecasting heuristic. This section presents graphical evidence of switching behavior. Fig. 5.7 shows the time series of some individual forecasts together with the realizations of the variable being forecasted. For every period $t$ we plot the realized inflation or output gap together with the two period ahead forecast of the individual. In this way we can graphically infer how the individual prediction uses the last available observation. For example, if the time series coincide, the subject is using a naive forecasting strategy.

In Fig. 5.7a (group 2, treatment 3a), subject 2 strongly extrapolates changes in the output gap in the early stage of the experiment, but starting from period $t = 18$ he switches to a much weaker form of trend extrapolation.

In Fig. 5.7b (group 1, treatment 3b), subject 4 switches between various constant predictors for inflation in the first 23 periods of the experimental session. She is in fact initially experimenting with three predictors, 2% 3% and 5%, and then switches to a naive forecasting strategy after period 23. In the same experimental session, Fig. 5.7c, participant 6 predicting the output gap is using different trend extrapolation strategies and, in the time interval $t = 19, ..., 30$, he uses a constant predictor for the output gap. This group illustrates an important point: in the same economy individuals forecasting different variables may use different forecasting strategies.

In Fig. 5.7d group 2, treatment 3b, subject 1 uses a trend following rule in the initial part of the experiment, i.e. when inflation fluctuates more. However, when oscillations dampen and inflation converges to the equilibrium level, he uses a forecasting strategy very close to naive.
Figure 5.7: **Individual learning as switching between heuristics.** For every period the subject’s forecast $x_{t+2}$ (green) and the variable being forecast $x_t$, with $x = \pi, y$, are reproduced.

A stylized fact that emerges from the investigation of individual experimental data is that individual learning has the form of switching from one heuristic to another$^9$. Anufriev and Hommes (2009) found a similar result analyzing individual forecasting time series from the asset pricing experiments of Hommes, Sonnemans, Tuinstra, and van de Velden (2005). Moreover, the fact that different types of aggregate behavior, namely convergence to different (non-fundamental) steady states, oscillations and dampening oscillations arise, suggest that heteroge-
neous expectations play an important role in determining the aggregate outcomes. In the light of the empirical evidence for heterogeneous expectations and individual switching behavior, we introduce in the next section a simple model with evolutionary selection between different forecasting heuristics in order to reproduce individual as well as aggregate experimental data.

5.4 A heterogeneous expectations model

Anufriev and Hommes (2009) developed a heuristics switching model along the lines of Brock and Hommes (1997), to explain different price fluctuations in the asset pricing experiment of Hommes, Sonnemans, Tuinstra, and van de Velden (2005). The key idea of the model is that the subjects chose between simple heuristics depending upon their relative past performance. The performance measure of a forecasting heuristic is based on its absolute forecasting error and it has the same functional form as the payoff function used in the experiments. More precisely, the performance measure of heuristic $h$ up to (and including) time $t - 1$ is given by

$$U_{h,t-1} = \frac{100}{1 + |x_{t-1} - x_{e,h,t-1}|} + \eta U_{h,t-2},$$

with $x = \pi, y$. The parameter $0 \leq \eta \leq 1$ represents the memory, measuring the relative weight agents give to past errors of heuristic $h$.

Given the performance measure, the impact of rule $h$ is updated according to a discrete choice model with asynchronous updating

$$n_{h,t} = \delta n_{h,t-1} + (1 - \delta) \frac{\exp(\beta U_{h,t-1})}{Z_{t-1}}$$

where $Z_{t-1} = \sum_{h=1}^{H} \exp(\beta U_{h,t-1})$ is a normalization factor. The asynchronous updating parameter $0 \leq \delta \leq 1$ measures the inertia in the impact of rule $h$, reflecting the fact that not all the participants update their rule in every period or at the same time. The parameter $\beta \geq 0$ represents the intensity of choice measuring how sensitive individuals are to differences in heuristics.
performances.

Our goal is to explain three different observed patterns of inflation and output in the experiment: convergence to (some) equilibrium level, permanent oscillations and oscillatory convergence. In order to keep the number of heuristics small, we use a heterogeneous expectation model with only four forecasting rules. These rules, summarized in Table 5.3, were obtained as heuristics describing typical individual forecasting behavior observed and estimated in our macro experiments. In order to check the robustness of a heterogeneous expectations model across different settings, we fixed the coefficient values to match the set of heuristics used in Anufriev and Hommes (2009) to explain asset pricing experiments. In treatment 3 we apply the same heuristics switching model to both inflation and output forecasting.

Table 5.3: Set of heuristics

<table>
<thead>
<tr>
<th>ADA</th>
<th>adaptive rule</th>
<th>$x_{1,t+1} = 0.65x_{1,t} + 0.35x_{1,t-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>WTR</td>
<td>weak trend-following rule</td>
<td>$x_{2,t+1} = x_{t-1} + 0.4(x_{t-1} - x_{t-2})$</td>
</tr>
<tr>
<td>STR</td>
<td>strong trend-following rule</td>
<td>$x_{3,t+1} = x_{t-1} + 1.3(x_{t-1} - x_{t-2})$</td>
</tr>
<tr>
<td>LAA</td>
<td>anchoring and adjustment rule</td>
<td>$x_{4,t+1} = 0.5(x_{av,t-1} + x_{t-1}) + (x_{t-1} - x_{t-2})$</td>
</tr>
</tbody>
</table>

5.4.1 50-periods ahead simulations

The model is initialized by two initial values for inflation and output gap, $\pi_1$, $y_1$, $\pi_2$ and $y_2$, and initial weights $n_{h,in}$, $1 \leq h \leq 4$. Given the values of inflation and output gap for periods 1 and 2, the heuristics forecasts can be computed and, using the initial weights of the heuristics, inflation and output gap for period 3, $\pi_3$ and $y_3$, can be computed. Starting from period 4 the evolution according to the model’s equations is well defined. Once we fix the four forecasting heuristics, there are three free “learning” parameters left in the model: $\beta$, $\eta$, and $\delta$. We used the same set of learning parameters as in Anufriev and Hommes (2009), namely $\beta = 0.4$, $\eta = 0.7$, $\delta = 0.9$, and we chose the initial shares of heuristics in such a way to match the patterns observed in the first few periods of the experiment. We also experimented with initial values of inflation and output gap close to the values observed in the first two rounds of the
corresponding experimental session. After some trial-and-error experimentation with different initial conditions we were able to replicate all three different qualitative patterns observed in the experiment. For the simulations shown in Fig. 5.8 we used the same realizations for demand and supply shocks as in the experiment and we chose the initial conditions as follows:

- treatment 1b, group 1, with convergence to fundamental equilibrium level
  initial inflation rates: $\pi_1 = 2.5, \pi_2 = 2.5$;
  initial fractions: $n_{1,in} = n_{4,in} = 0.40, n_{2,in} = n_{3,in} = 0.10$;

- treatment 2b, group 3, with permanent oscillations
  initial inflation: $\pi_1 = 2.64, \pi_2 = 2.70$ (experimental data);
  initial output gap: $y_1 = -0.20, y_2 = -0.42$ (experimental data);
  initial fractions: $n_{1,in} = 0, n_{2,in} = n_{3,in} = 0.20, n_{4,in} = 0.60$;

- treatment 3a, group 2, with convergence to a non-fundamental steady state
  initial inflation: $\pi_1 = 2.4, \pi_2 = 2.0$;
  initial output gap: $y_1 = 1.8, y_2 = 2$;
  initial fractions inflation: $n_{1,in} = 0.60, n_{2,in} = 0.05, n_{3,in} = 0.10, n_{4,in} = 0.25$
  initial fractions output gap: $n_{1,in} = 0.6, n_{2,in} = 0.05, n_{3,in} = 0.15, n_{4,in} = 0.20$.

- treatment 3b, group 2, with oscillatory convergence
  initial inflation: $\pi_1 = 3.98, \pi_2 = 3.72$ (experimental data);
  initial output gap: $y_1 = 0.28, y_2 = -0.05$ (experimental data);
  initial fractions inflation: $n_{1,in} = 0, n_{2,in} = 0.10, n_{3,in} = 0.40, n_{4,in} = 0.50$
  initial fractions output gap: $n_{1,in} = 0.15, n_{2,in} = 0.20, n_{3,in} = 0.50, n_{4,in} = 0.15$.

Fig. 5.8 shows realizations of inflation and output gap in the experiment together with the simulated paths using the heuristics switching model\(^{10}\). The model is able to reproduce qualitatively all three different patterns observed in the experiment, which are, convergence to (some)\footnote{Treatment 3a group 2 has been simulated for 49 periods due to a clear ending effect, see footnote 4.}
Figure 5.8: Experimental data (blue points) and **50-periods ahead** heuristics switching model simulations (red lines)
equilibrium, permanent oscillations and oscillatory convergence\textsuperscript{11}. As shown in Table 5.4, the model is also capable to match some quantitative features of the experimental data, such as the mean and the variance\textsuperscript{12}.

Table 5.4: Observed vs simulated moments (50-periods ahead)

<table>
<thead>
<tr>
<th>Treatment</th>
<th>1b</th>
<th>2b</th>
<th>3a ((\pi))</th>
<th>3a ((y))</th>
<th>3b ((\pi))</th>
<th>3b ((y))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(\mu)</td>
<td>(\sigma^2)</td>
<td>(\mu)</td>
<td>(\sigma^2)</td>
<td>(\mu)</td>
<td>(\sigma^2)</td>
</tr>
<tr>
<td>Obs.</td>
<td>2.19</td>
<td>0.01</td>
<td>2.05</td>
<td>0.18</td>
<td>3.06</td>
<td>0.14</td>
</tr>
<tr>
<td>Sim.</td>
<td>2.15</td>
<td>0.02</td>
<td>2.03</td>
<td>0.17</td>
<td>3.13</td>
<td>0.05</td>
</tr>
<tr>
<td>(p)</td>
<td>0.06</td>
<td>0.03</td>
<td>0.74</td>
<td>0.71</td>
<td>*</td>
<td>*</td>
</tr>
</tbody>
</table>

\(* = \) Non stationarity.
The row corresponding to \(p\) reports p-values of tests on the equality of observed and simulated mean and on the equality of observed and simulated variance (HAC Consistent covariance estimators (Newey-West) have been used to compute standard errors).

5.4.2 One-period ahead simulations

The 50-period ahead simulations fix initial states and then predicts inflation and output patterns 50-periods ahead. We now report the results of one-step ahead simulations of the nonlinear switching model. At each time step, the simulated path uses experimental data as inputs to compute the heuristics’ forecasts and update their impacts. Hence, the one-period ahead simulations use exactly the same information as the subjects in the experiments. The one-period ahead simulations match the different patterns in the experimental data quite nicely. Fig. 5.9 compares the experimental data with the one-step ahead predictions made by our model, using the benchmark parameter values \(\beta = 0.4\), \(\eta = 0.7\), \(\delta = 0.9\).

In these simulations initial inflation and output gap initial inflation and output gap in the first two periods are taken from the corresponding experimental group, while the initial impacts

\textsuperscript{11}We reported only simulations for some representative experimental economies which account for the three different aggregate behaviors observed in the experiment. Results for experimental economies with analogous qualitative behavior are similar.

\textsuperscript{12}We performed the tests on the equality of observed and simulated mean and variance on a sample that goes from period 4 to the end of the experimental session in order to minimize the impact of the initial conditions.
Figure 5.9: Experimental data (blue points) and one-period ahead heuristics switching model simulations (red lines)
of all heuristics are equal to 0.25.

Fig. 5.10 shows how in different groups different heuristics are taking the lead after starting from a uniform distribution.

In treatment 1b group 1 (Fig. 5.10a), the initial drop in inflation, from 3.1 to 1.9 respectively in periods 1 and 2, causes an overshooting in the predictions of the trend extrapolating rules, i.e. WTF, STF and LAA, for inflation in period 3. Therefore the relative impacts of these rules starts to drop, while the relative share of adaptive expectations ADA increases to about 70% in the first 14 periods. From period 14 on, the share of the WTF rule increases due to some slow oscillation, and it reaches a peak of about 48% in period 33. During this time span of slow oscillations the fraction of the ADA rule decreases to about 30%. However, in the last part of the experiment inflation stabilizes and the ADA rule dominates the other rules. In group 3 treatment 2b (Fig. 5.10b) we clearly observe that the ADA rule is not able to match the oscillatory pattern and its impact declines monotonically in the simulation. The STF rule can follow the oscillatory pattern and initially dominates (almost 40% in period 8) but its predictions overshoot the trend in realized inflation reverses, and its relative share declines monotonically from period 9 on.

Both the WTF and the LAA rule can follow closely the observed oscillations, but in the last part of the experiment the LAA rule dominates the other rules. As in the quite different setting of the asset pricing experiments in Anufriev and Hommes (2009), our simulation explains oscillatory behavior by coordination on the LAA rule by most subjects. In the early stage of treatment 3a, group 2 (Fig. 5.10c), the oscillations in inflation are relatively small and therefore the WTF rule is able to match the oscillatory pattern; also the ADA rule performs reasonably well, while both the STF and LAA rules overshoot too often. Then inflation undergoes a more turbulent phase with stronger oscillations starting in period 24 and the impact of the strong trend following rule increases and reaches a peak of about 30% in period 35. At the same time, when inflation fluctuates the share of the ADA rule declines. In the last part of the experiment inflation more or less stabilizes and the impact of the WTF rule declines monotonically, while, the impact of the ADA rule rises from less than 10% to about 50% in the last 10 periods of the experiment.
Figure 5.10: Evolution of fractions of 4 heuristics corresponding to one-period ahead simulations in Fig. 5.9: adaptive expectations (ADA, blue), weak trend follower (WTF, red), strong trend follower (STF, black), anchoring and adjustment heuristics (LAA, green).
Interestingly, in the same economy the story is different for the output gap (5.10d). In fact the dynamics are characterized by oscillations in the early stage of the experiment which are less pronounced than the oscillations in the inflation rate. The model then explains the convergence pattern of output gap with small oscillation by coordination of most individuals on the ADA rule and a share of WTF that varies between 7% and 25% throughout the experiment. A novel feature of our heuristics switching model is that it allows for coordination on different forecasting rules for different aggregate variable of the same economy. Inflation expectations are dominated by weak trend followers, causing inflation to slowly drift away to the “wrong” non-fundamental steady state, while output expectations are dominated by adaptive expectations, causing output to converge (slowly) to its fundamental steady state level.

For treatment 3b group 2 (Fig. 5.10e), the one step ahead forecast exercise produces a rich evolutionary competition among heuristics. In the initial part of the experiment, the STF is the only rule able to match the strong decline in the inflation rate and its share increases to 50% in period 8. However the impact of the STF rule starts to decrease after it misses the first turning point. After the initial phase of strong trend in inflation, the LAA rule does a better job in predicting the trend reversal and its impact starts to increase, reaching a share of about 70% in period 18. However oscillations slowly dampen and therefore the impacts of the ADA rule and the WTF rule starts to rise. Towards the end of the simulation, when inflation has converged, the ADA rule dominates the other heuristics. The evolutionary selection dynamics are somewhat different for the output gap predictors (Fig. 5.10f). In fact, oscillations of the output gap are more frequent and this implies a relatively bad forecasting performance of the STF rule that tends to overshoot more often. The switching model explains the oscillatory behavior of output in the initial phase by coordination on the LAA rule by most subjects. However, with dampening oscillations the impact of the LAA rule gradually decreases and the ADA rule starts increasing after period 25 and dominates in the last 10 periods. Fig. 5.11 reports the predictions of the participants in the experiments together with the predictions generated by the four heuristics,
while Table 5.5 compares observed and simulated moments\textsuperscript{13}.

Table 5.5: Observed vs simulated moments (one-period ahead)

<table>
<thead>
<tr>
<th>Treatment</th>
<th>1b</th>
<th>2b</th>
<th>3a (π)</th>
<th>3a (y)</th>
<th>3b (π)</th>
<th>3b (y)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\mu$</td>
<td>$\sigma^2$</td>
<td>$\mu$</td>
<td>$\sigma^2$</td>
<td>$\mu$</td>
<td>$\sigma^2$</td>
</tr>
<tr>
<td>Obs.</td>
<td>2.19</td>
<td>0.01</td>
<td>2.05</td>
<td>0.18</td>
<td>3.06</td>
<td>0.14</td>
</tr>
<tr>
<td>Sim.</td>
<td>2.17</td>
<td>0.01</td>
<td>2.05</td>
<td>0.16</td>
<td>3.08</td>
<td>0.14</td>
</tr>
<tr>
<td>$p$</td>
<td>0.02</td>
<td>0.86</td>
<td>0.83</td>
<td>0.16</td>
<td>0.24</td>
<td>0.78</td>
</tr>
</tbody>
</table>

The row corresponding to $p$ reports p-values of tests on the equality of observed and simulated mean and on the equality of observed and simulated variance (HAC Consistent covariance estimators (Newey-West) have been used to compute standard errors).

**Forecasting performance**

Table 5.6 compares the MSE of the one-step ahead prediction in 10 experimental groups\textsuperscript{14} for 9 different models: the rational expectation prediction (RE), six homogeneous expectations models (naive expectations, fixed anchor and adjustment (AA) rule\textsuperscript{15}, and each of the four heuristics of the switching model), the switching model with benchmark parameters $\beta = 0.4$, $\eta = 0.7$, and $\delta = 0.9$, and the “best” switching model fitted by means of a grid search in the parameters space. The MSEs for the benchmark switching model are shown in bold and, for comparison, for each group the MSEs for the best among the four heuristics are also shown in bold. The best among all models for each group is shown in italic\textsuperscript{16}. We notice immediately that the RE prediction is (almost) always the worst. It also appears that the evolutionary learning model is able to make the best out of different heuristics. In fact, none of the homogeneous expectations models fits all different observed patterns, while the best fit switching model yields the lowest

\textsuperscript{13}We performed tests on the equality of observed and simulated mean and variance on a sample that goes from period 4 to the end of the experimental session in order to minimize the impact of initial conditions.

\textsuperscript{14}The MSE of the one-step ahead prediction for the remaining groups is reported in Appendix 5.B, Table 5.18

\textsuperscript{15}In the AA rule we consider the full sample mean, which is a proxy of the equilibrium level, as an anchor. In the LAA rule instead we use the sample average of all the previous realizations that are available at every point in time as an anchor.

\textsuperscript{16}We evaluate the MSE over 47 periods, for $t = 4, ..., 50$. This minimizes the impact of initial conditions for the switching model in the sense that $t = 4$ is the first period when the prediction is computed with both the heuristics forecasts and the heuristics impacts being updated on the basis of experimental data.
Figure 5.11: **Left panels:** predictions of the participants in the experiment. **Right panels:** predictions of the four heuristics.
MSE in $9/15^{17}$ cases, being the second best, with only a slightly larger MSE compared to the best model, in the other cases (with the exceptions of group 1 in treatment 2a and groups 2 and 3 in treatment 3a). Notice also that the benchmark switching model typically is almost as good as the best switching model, indicating that the results are not very sensitive to the learning parameters.

**Out-of-sample forecasting**

In order to evaluate the out-of-sample forecasting performance of the model, we first perform a grid search to find the parameters of the model minimizing the MSE for a restricted sample, i.e. for periods $t = 4, \ldots, 43$. Then, the squared forecasting errors are computed for the next 7 periods. The results are shown in Table 5.7 and in Table 5.19, Appendix 5.B. Finally, we compare the out-of-sample forecasting performance of the structural heuristics switching model (both the best fit and the model with benchmark parameters) with a simple non-structural AR(2) model with three parameters. Notice that, for treatment 3 we use different AR(2) models for inflation and output gap, so that we have in fact 6 parameters for the AR(2) models in treatment 3.

For the converging groups (treatment 1a groups 1 and 3, treatment 1b groups 1 and 2, treatment 2a group 1, treatment 2b groups 1 and 2, treatment 3a groups 1 and 2, treatment 3b group 1) we typically observe that the squared prediction errors remain very low and comparable with the MSEs computed in-sample. This is due to the fact that the qualitative behavior of the data does not change in the last periods. For the groups that exhibit oscillatory behavior (treatment 1b group 3, treatment 2a, groups 2 and 3, treatment 2b group 3) the out-of-sample errors are larger than the in-sample MSEs, and they typically increase with the time horizon of the prediction. When we instead observe dampening oscillations (treatment 3b, group 2), the out-of-sample prediction errors are smaller than the in-sample MSEs. This is due to the fact that, towards the end of the experimental session, convergence is observed. Comparing the out-of-sample

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$^{17}$We excluded treatment 1a, group 2 and did not fit the heuristics switching model because of the anomalous observed behavior.
Table 5.6: MSE over periods 4 – 50 of the one-period ahead forecast.

<table>
<thead>
<tr>
<th>Model</th>
<th>Tr1a gr1</th>
<th>Tr1b gr1</th>
<th>Tr1b gr3</th>
<th>Tr2a gr1</th>
<th>Tr2a gr2</th>
<th>Tr2b gr1</th>
<th>Tr2b gr3</th>
<th>Tr3a gr2</th>
<th>Tr3b gr1</th>
<th>Tr3b gr2</th>
</tr>
</thead>
<tbody>
<tr>
<td>RE</td>
<td>0.3366</td>
<td>0.0438</td>
<td>0.5425</td>
<td>0.0122</td>
<td>0.1822</td>
<td>Indeterminacy</td>
<td>0.5816</td>
<td>0.4851</td>
<td></td>
<td></td>
</tr>
<tr>
<td>naive</td>
<td>0.0058</td>
<td>0.0016</td>
<td>0.0454</td>
<td>0.1253</td>
<td>0.5024</td>
<td>0.0032</td>
<td>0.0552</td>
<td>0.0579</td>
<td>0.1126</td>
<td>0.2855</td>
</tr>
<tr>
<td>AA</td>
<td>0.0181</td>
<td>0.0066</td>
<td>0.1273</td>
<td>0.0487</td>
<td>0.4099</td>
<td>0.0066</td>
<td>0.0185</td>
<td>0.1170</td>
<td>0.1533</td>
<td>0.1746</td>
</tr>
<tr>
<td>ADA</td>
<td>0.0098</td>
<td>0.0021</td>
<td>0.0908</td>
<td>0.3170</td>
<td>0.9893</td>
<td>0.0110</td>
<td>0.1113</td>
<td>0.0531</td>
<td>0.1536</td>
<td>0.3881</td>
</tr>
<tr>
<td>WTF</td>
<td>0.0045</td>
<td>0.0026</td>
<td>0.0209</td>
<td>0.0840</td>
<td>0.2652</td>
<td>0.0035</td>
<td>0.0273</td>
<td>0.0705</td>
<td>0.1060</td>
<td>0.2215</td>
</tr>
<tr>
<td>STF</td>
<td>0.0137</td>
<td>0.0106</td>
<td>0.0084</td>
<td>0.1359</td>
<td>0.1749</td>
<td>0.0131</td>
<td>0.0474</td>
<td>0.1383</td>
<td>0.2266</td>
<td>0.4329</td>
</tr>
<tr>
<td>LAA</td>
<td>0.0191</td>
<td>0.0066</td>
<td>0.1243</td>
<td>0.0606</td>
<td>0.3931</td>
<td>0.0064</td>
<td>0.0147</td>
<td>0.0985</td>
<td>0.1302</td>
<td>0.1870</td>
</tr>
</tbody>
</table>

| 4 rules (benchmark) | 0.0048 | 0.0017 | 0.0117 | 0.0692 | 0.0934 | 0.0024 | 0.0092 | 0.0632 | 0.0958 | 0.1898 |
| 4 rules (best fit)  | 0.0044 | 0.0016 | 0.0089 | 0.0665 | 0.0897 | 0.0022 | 0.0093 | 0.0609 | 0.0936 | 0.1840 |

Note that the MSE reported for treatment 3 refers to the sum of the MSE relative to inflation and the MSE relative to output gap. In treatment 3a, group 2, the MSE has been computed for periods 4 – 49 due to the observed ending effect. Moreover the fact that in treatment 2b, group 3, the MSE reported for the benchmark model is lower than the MSE of the best fit model is due to the fact that the grid search for parameter $\beta$ takes a step of 1 and thus excludes $\beta = 0.4$. 

<table>
<thead>
<tr>
<th></th>
<th>$\beta$</th>
<th>1</th>
<th>10</th>
<th>2</th>
<th>0</th>
<th>10</th>
<th>7</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta$</td>
<td>0.9</td>
<td>0.7</td>
<td>0.7</td>
<td>0</td>
<td>0.4</td>
<td>0.1</td>
<td>0.7</td>
<td>0.9</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.5</td>
<td>0.8</td>
<td>0.7</td>
<td>0</td>
<td>0.9</td>
<td>0.8</td>
<td>0.9</td>
<td>0.8</td>
</tr>
</tbody>
</table>
forecasting performance, we conclude that the benchmark switching model generally does not perform worse (sometimes even better) than the best in-sample fitted switching model. Compared to the non-structural AR(2) model, the switching model on average performs better. In particular, for treatment 3 the benchmark switching model as well as the 3-parameter best-fit switching model perform better than the AR(2) models with 6 parameters.

5.5 Conclusions

In this paper we use laboratory experiments with human subjects to study individual expectations, their interactions and the aggregate behavior they co-create within a New Keynesian macroeconomic setup. A novel feature of our experimental design is that realizations of aggregate variables depend on individual forecasts of two different variables, output gap and inflation. We find that individuals tend to base their predictions on past observations, following simple forecasting heuristics, and individual learning takes the form of switching from one heuristic to another. We propose a simple model of evolutionary selection among forecasting rules based on past performance in order to explain the different aggregate outcomes observed in the laboratory experiments, namely convergence to some equilibrium level, persistent oscillatory behavior and oscillatory convergence. Our model is the first to describe aggregate behavior in an economy with heterogeneous individual expectations on two different variables. Simulations of the heuristics switching model show that the model is able to match individual forecasting behavior and nicely reproduce the different observed patterns of aggregate variables. A distinguishing feature of our heterogeneous expectations model is that evolutionary selection may lead to different dominating forecasting rules for different variables within the same economy (see Figs. 5.10c and 5.10d where a weak trend following rule dominates inflation forecasting while adaptive expectations dominate output forecasting). We also perform an exercise of empirical validation on the experimental data to test the model’s performance in terms of in-sample forecasting as well as out-of-sample predicting power. Our results show that the heterogeneous
Table 5.7: Out-of-sample performance.

<table>
<thead>
<tr>
<th>Best Fit Switching Model</th>
<th>Tr1a gr1</th>
<th>Tr1b gr1</th>
<th>Tr1b gr2</th>
<th>Tr2a gr1</th>
<th>Tr2a gr2</th>
<th>Tr2b gr1</th>
<th>Tr2b gr2</th>
<th>Tr3a gr2</th>
<th>Tr3b gr1</th>
<th>Tr3b gr2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(β, η, δ)</td>
<td>(1.0, 0.5)</td>
<td>(10.0, 7.0, 8)</td>
<td>(3.0, 7.0, 5)</td>
<td>(0.0)</td>
<td>(10.0, 4.0, 9)</td>
<td>(10.1, 0.8)</td>
<td>(10.7, 0.9)</td>
<td>(10.0, 9.0, 8)</td>
<td>(5.0, 1.0, 8)</td>
<td>(2.0, 5.0, 9)</td>
</tr>
<tr>
<td>1 p ahead</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0287</td>
<td>0.0006</td>
<td>0.1201</td>
<td>0.0000</td>
<td>0.0007</td>
<td>0.0060</td>
<td>0.0113</td>
<td>0.0245</td>
</tr>
<tr>
<td>2 p ahead</td>
<td>0.0020</td>
<td>0.0002</td>
<td>0.0046</td>
<td>0.0003</td>
<td>0.0511</td>
<td>0.0037</td>
<td>0.1076</td>
<td>0.0085</td>
<td>0.0170</td>
<td>0.0501</td>
</tr>
<tr>
<td>3 p ahead</td>
<td>0.0011</td>
<td>0.0000</td>
<td>0.0256</td>
<td>0.0006</td>
<td>0.0931</td>
<td>0.0014</td>
<td>0.2364</td>
<td>0.0033</td>
<td>0.0059</td>
<td>0.0721</td>
</tr>
<tr>
<td>4 p ahead</td>
<td>0.0016</td>
<td>0.0000</td>
<td>0.1930</td>
<td>0.0006</td>
<td>0.0347</td>
<td>0.0038</td>
<td>0.1902</td>
<td>0.0224</td>
<td>0.0344</td>
<td>0.0663</td>
</tr>
<tr>
<td>5 p ahead</td>
<td>0.0083</td>
<td>0.0005</td>
<td>0.4156</td>
<td>0.0070</td>
<td>0.0131</td>
<td>0.0022</td>
<td>0.0056</td>
<td>0.0611</td>
<td>0.0669</td>
<td>0.0131</td>
</tr>
<tr>
<td>6 p ahead</td>
<td>0.0004</td>
<td>0.0020</td>
<td>0.6021</td>
<td>0.0690</td>
<td>0.4067</td>
<td>0.0023</td>
<td>0.1732</td>
<td>0.0259</td>
<td>0.0125</td>
<td>0.0491</td>
</tr>
<tr>
<td>7 p ahead</td>
<td>0.0003</td>
<td>0.0041</td>
<td>0.6235</td>
<td>0.0271</td>
<td>1.5804</td>
<td>0.0007</td>
<td>0.3218</td>
<td>0.3048</td>
<td>0.0061</td>
<td>0.0727</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Benchmark Switching Model</th>
<th>(β, η, δ)</th>
<th>(0.4, 0.7, 0.9)</th>
<th>(0.4, 0.7, 0.9)</th>
<th>(0.4, 0.7, 0.9)</th>
<th>(0.4, 0.7, 0.9)</th>
<th>(0.4, 0.7, 0.9)</th>
<th>(0.4, 0.7, 0.9)</th>
<th>(0.4, 0.7, 0.9)</th>
<th>(0.4, 0.7, 0.9)</th>
<th>(0.4, 0.7, 0.9)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 p ahead</td>
<td>0.0000</td>
<td>0.0004</td>
<td>0.0388</td>
<td>0.0015</td>
<td>0.1087</td>
<td>0.0005</td>
<td>0.0042</td>
<td>0.0062</td>
<td>0.0229</td>
<td>0.0386</td>
</tr>
<tr>
<td>2 p ahead</td>
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<td>0.0003</td>
<td>0.0382</td>
<td>0.0016</td>
<td>0.1442</td>
<td>0.0004</td>
<td>0.0957</td>
<td>0.0045</td>
<td>0.0062</td>
<td>0.0442</td>
</tr>
<tr>
<td>3 p ahead</td>
<td>0.0004</td>
<td>0.0008</td>
<td>0.0190</td>
<td>0.0028</td>
<td>0.5843</td>
<td>0.0002</td>
<td>0.1404</td>
<td>0.0051</td>
<td>0.0085</td>
<td>0.0343</td>
</tr>
<tr>
<td>4 p ahead</td>
<td>0.0000</td>
<td>0.0005</td>
<td>0.0039</td>
<td>0.0026</td>
<td>1.1437</td>
<td>0.0014</td>
<td>0.0659</td>
<td>0.0134</td>
<td>0.0238</td>
<td>0.0337</td>
</tr>
<tr>
<td>5 p ahead</td>
<td>0.0004</td>
<td>0.0017</td>
<td>0.0028</td>
<td>0.0042</td>
<td>1.4969</td>
<td>0.0008</td>
<td>0.0800</td>
<td>0.0464</td>
<td>0.0363</td>
<td>0.0284</td>
</tr>
<tr>
<td>6 p ahead</td>
<td>0.0000</td>
<td>0.0049</td>
<td>0.0019</td>
<td>0.0589</td>
<td>0.9876</td>
<td>0.0021</td>
<td>0.2232</td>
<td>0.0271</td>
<td>0.0393</td>
<td>0.0361</td>
</tr>
<tr>
<td>7 p ahead</td>
<td>0.0000</td>
<td>0.0075</td>
<td>0.0198</td>
<td>0.0235</td>
<td>0.1579</td>
<td>0.0007</td>
<td>0.1997</td>
<td>0.2575</td>
<td>0.0181</td>
<td>0.0240</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>AR(2) Model</th>
<th>(βπ, 0, βπ, 1, βπ, 2)</th>
<th>(0.3, 1.1, -0.2)(0.4, 0.7, 0.1)(0.2, 1.8, -0.9)(1.8, 1.3,-0.7)(0.6,1.8,-1.0)(0.7,0.9,-0.2)(0.9,1.4,-0.9)(0.3,1.1,-0.2)(0.5,1.3,-0.5)(1.3,1.1,-0.6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 p ahead</td>
<td>0.0034</td>
<td>0.0007</td>
</tr>
<tr>
<td>2 p ahead</td>
<td>0.0005</td>
<td>0.0007</td>
</tr>
<tr>
<td>3 p ahead</td>
<td>0.0146</td>
<td>0.0007</td>
</tr>
<tr>
<td>4 p ahead</td>
<td>0.0347</td>
<td>0.0005</td>
</tr>
<tr>
<td>5 p ahead</td>
<td>0.0000</td>
<td>0.0202</td>
</tr>
<tr>
<td>6 p ahead</td>
<td>0.0007</td>
<td>0.0105</td>
</tr>
<tr>
<td>7 p ahead</td>
<td>0.0009</td>
<td>0.0085</td>
</tr>
</tbody>
</table>

Note that the squared prediction errors for treatment 3 refer to the sum of the errors relative to inflation and to output gap. In treatment 3a, group 2, the switching models and the AR(2) model have been estimated on a restricted sample of 42 periods due to the observed ending effect.
expectations model outperforms models with homogeneous expectations, including the rational expectations benchmark. On the policy side we find that the implementation of a monetary policy that reacts aggressively to deviations of inflation from the target leads the economy to the desired target, at least in the long run. In the short run, however, oscillations in inflation and output may arise due to coordination of individual expectations on trend following rules.
Appendix 5.A  (Translation of Dutch) Instructions for participants

**Inflation forecasters**

**Set-up of the experiment**

You are participating in an experiment on economic decision-making. You will be rewarded based on the decisions you make during the experiment. The experiment will be preceded by several pages of instructions that will explain how it works. When the experiment has ended, you will be asked to answer some questions about how it went.

- The whole experiment, including the instructions and the questionnaire, is computerized. Therefore you do not have to submit the paper on your desk. Instead, you can use it to make notes.

- There is a calculator on your desk. If necessary, you can use it during the experiment.

- If you have any question during the experiment, please raise your hand, then someone will come to assist you.

**General information about the experiment**

In the experiment, statistical research bureaus make predictions about the inflation and the so-called "output gap" in the economy. A limited amount of research bureaus is active in the economy. You are a research bureau that makes predictions about inflation. This experiment consists of 50 periods in total. In each period you will be asked to predict the inflation; your reward after the experiment has ended is based on the accuracy of your predictions.

In the following instructions you will get more information about the economy you are in, about the way in which making predictions works during the experiment, and about the way in
which your reward is calculated. Also, the computer program used during the experiment will be explained.

**Information about the economy (part 1 of 2)**

The economy you are participating in is described by three variables: the inflation $\pi_t$, the output gap $y_t$ and the interest rate $i_t$. The subscript $t$ indicates the period the experiment is in. In total there are 50 periods, so $t$ increases during the experiment from 1 through 50.

The inflation measures the percentage change in the price level of the economy. In each period, inflation depends on the inflation predictions and output gap predictions of the statistical research bureaus, and on minor price shocks. There is a positive relation between the actual inflation and both the inflation predictions and output gap predictions of the research bureaus. This means for example that if the inflation prediction of a research bureaus increases, then actual inflation will also increase (assuming that the other predictions and the price shock remain equal). The minor price shocks have an equal chance of influencing inflation positively or negatively.

**Information about the economy (part 2 of 2)**

The output gap measures the percentage difference between the Gross Domestic Product (GDP) and the natural GDP. The GDP is the value of all goods produced during a period in the economy. The natural GDP is the value the total production would have if prices in the economy would be fully flexible. If the output gap is positive (negative), the economy therefore produces more (less) than the natural GDP. In each period the output gap depends on the inflation predictions and output gap predictions of the statistical bureaus, on the interest rate and on minor economic shocks. There is a positive relation between the output gap and the inflation predictions and output gap predictions, and a negative relation between the output gap and the interest rate. The minor economic shocks have an equal chance of influencing the output gap positively or negatively.
The interest rate measures the price of borrowing money and is determined by the central bank. There is a positive relation between the interest rate and the inflation.

Information about making predictions

Your task, in each period of the experiment, consists in predicting the inflation in the next period. Inflation has been historically between $-5\%$ and $15\%$. When the experiment starts, you have to predict the inflation for the first two periods, i.e. $\pi^e_1$ and $\pi^e_2$. The superscript $e$ indicates that these are predictions. When all participants have made their predictions for the first two periods, the actual inflation ($\pi_1$), the output gap ($y_1$) and the interest rate ($i_1$) for period 1 are announced. Then period 2 of the experiment begins.

In period 2 you make an inflation prediction for period 3 ($\pi^e_3$). When all participants have made their predictions for period 3, the inflation ($\pi_2$), the output gap ($y_2$) and the interest rate ($i_2$) for period 2 are announced. This process repeats for 50 periods. Therefore, when at a certain period $t$ you make a prediction of the inflation in period $t + 1$ ($\pi^e_{t+1}$), the following information is available:

- Values of the actual inflation, output gap and interest rate up to and including period $t - 1$;
- Your predictions up to and including period $t$;
- Your prediction scores up to and including period $t - 1$.

Information about your reward (part 1 of 2)

Your reward after the experiment has ended increases with the accuracy of your predictions. Your accuracy is measured by the absolute error between your inflation predictions and the true inflation. For each period this absolute error is calculated as soon as the true value of inflation is known; you subsequently get a prediction score that decreases as the absolute error increases. The table below gives the relation between the absolute predictions error and the prediction score. If at a certain period you predict for example an inflation of $2\%$, and the true inflation
turns out to be 3%, then you make an absolute error of $3\% - 2\% = 1\%$. Therefore you get a prediction score of 50. If you predict an inflation of 1%, and the realized inflation turns out to be $-2\%$, you make a prediction error of $1\% - (-2\%) = 3\%$. Then you get a prediction score of 25. For a perfect prediction, with a prediction error of zero, you get a prediction score of 100.

<table>
<thead>
<tr>
<th>Absolute prediction error</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Score</td>
<td>100</td>
<td>50</td>
<td>33/3</td>
<td>25</td>
<td>20</td>
<td>10</td>
</tr>
</tbody>
</table>

**Information about your reward (part 2 of 2)**

The figure below shows the relation between your prediction score (vertical axis) and your prediction error (horizontal axis). Notice that your prediction score decreases more slowly as your prediction error increases. Points in the graph correspond to the prediction scores in the previous table.

Your total score at the end of the experiment consists simply of the sum of all prediction scores you got during the experiment. During the experiment, your scores are shown on your computer screen. When the experiment has ended, you are shown an overview of your prediction scores, followed by the resulting total score. Your final reward consists of 0.75 euro-cent
for each point in your total score (200 points therefore equals 1.50 euro). Additionally, you will receive a show up fee of 5 euro.

**Information about the computer program (part 1 of 3)**

Below you see an example of the left upper part of the computer screen during the experiment. It consists of a graphical representation of the inflation (red series) and your predictions of it (yellow series). On the horizontal axis are the time periods; the vertical axis is in percentages. In the imaginary situation depicted in the graph, the experiment is in period 30 and you predict the inflation in period 31 (the experiment lasts for 50 periods). Notice that the graph only shows results of at most the last 25 periods and that the next period is always on the right hand side.

The left bottom part of the computer screen also contains a graph. In this graph the output gap and the interest rate are shown in the same way as in the above graph.

**Information about the computer program (part 2 of 3)**

Below you see an example of the right upper part of the computer screen during the experiment. It consists of a table containing information about the results of the experiment in at most the last 25 periods. This information is supplemental to the graphs in the left part of the screen. The
first column of the table shows the time period (the next period, 31 in the example, is always at the top). The second and third columns respectively show the inflation and your predictions of it. The fourth column gives the output gap and the fifth column the interest rate. Finally, the sixth column gives your prediction score for each period separately. Notice that you can use the sheet of paper on your desk to save data longer than 25 periods.

| Tijds- | Inflatie | Stijging | Output gap | Basisc | Voorspel-score |
| periode | (%) | (%) | (%) | (%) |
| 31 | 0.32 | -0.32 | | | |
| 30 | 2.54 | 0.63 | 2.15 | 1.42 | 56 |
| 29 | 0.63 | 2.04 | -0.31 | 1.71 | 75 |
| 28 | 1.61 | 1.26 | 1.06 | 1.35 | 60 |
| 27 | 2.34 | -1.09 | -2.12 | 2.09 | 24 |
| 26 | 1.95 | 0.99 | -1.33 | 1.99 | 50 |
| 25 | 1.41 | 1.9 | 4.1 | 1.85 | 96 |
| 24 | 1.24 | 5.21 | 1.39 | 2.06 | 24 |
| 23 | 1.07 | 0.06 | -3.92 | 1.77 | 55 |
| 22 | 3.58 | 5.44 | 5.07 | 2.39 | 25 |
| 21 | -0.0 | 1.7 | -1.69 | 1.42 | 76 |
| 20 | 2.29 | 4.33 | 1.31 | 2.07 | 31 |
| 19 | -2.49 | -5.55 | -0.85 | 0.58 | 13 |
| 18 | 4.03 | 5.02 | -0.76 | 2.71 | 24 |
| 17 | 1.2 | 0.04 | -0.59 | 2.8 | 36 |
| 16 | 0.68 | 0.29 | -0.34 | 1.67 | 60 |
| 15 | -0.57 | 0.43 | -0.18 | 1.76 | 27 |
| 14 | -1.13 | -0.91 | 3.58 | 2.3 | 60 |
| 13 | 0.65 | 3.81 | -1.24 | 4.4 | 40 |
| 12 | 5.97 | 6.13 | -4.45 | 2.99 | 24 |
| 11 | 1.74 | 2.71 | 1.88 | 1.93 | 56 |
| 10 | 0.23 | -0.14 | 3.28 | 2.66 | 37 |
| 9 | -2.98 | -4.58 | -0.07 | 0.75 | 16 |
| 8 | -2.18 | 0.43 | 2.57 | 0.23 | 67 |

Information about the computer program (part 3 of 3)

Below you see an example of the bottom part of the computer screen during the experiment. In each period you are asked to submit your inflation prediction in the next period (below Submit you prediction). When submitting your prediction, use the decimal point if necessary. For example, if you want to submit a prediction of 2.5%, type ”2.5”; for a prediction of –1.75%, type ”–1.75”. Notice that your predictions and the true inflation in the experiment are rounded to two decimals. Moreover, prediction scores are rounded to integers.
Output gap forecasters

Set-up of the experiment

You are participating in an experiment on economic decision-making. You will be rewarded based on the decisions you make during the experiment. The experiment will be preceded by several pages of instructions that will explain how it works. When the experiment has ended, you will be asked to answer some questions about how it went.

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- There is a calculator on your desk. If necessary, you can use it during the experiment.

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**Information about the economy (part 2 of 2)**

The output gap measures the percentage difference between the Gross Domestic Product (GDP) and the natural GDP. The GDP is the value of all goods produced during a period in the economy. The natural GDP is the value the total production would have if prices in the economy would be fully flexible. If the output gap is positive (negative), the economy therefore produces more (less) than the natural GDP. In each period the output gap depends on the inflation predictions and output gap predictions of the statistical bureaus, on the interest rate and on minor
economic shocks. There is a positive relation between the output gap and the inflation predictions and output gap predictions, and a negative relation between the output gap and the interest rate. The minor economic shocks have an equal chance of influencing the output gap positively or negatively.

The interest rate measures the price of borrowing money and is determined by the central bank. There is a positive relation between the interest rate and the inflation.

**Information about making predictions**

Your task, in each period of the experiment, consists in predicting the output gap in the next period. Inflation has been historically between −5% and 5%. When the experiment starts, you have to predict the output gap for the first two periods, i.e. $y_{e1}$ and $y_{e2}$. The superscript $e$ indicates that these are predictions. When all participants have made their predictions for the first two periods, the actual inflation ($\pi_1$), the output gap ($y_1$) and the interest rate ($i_1$) for period 1 are announced. Then period 2 of the experiment begins.

In period 2 you make an output gap prediction for period 3 ($y_{e3}$). When all participants have made their predictions for period 3, the inflation ($\pi_2$), the output gap ($y_2$) and the interest rate ($i_2$) for period 2 are announced. This process repeats for 50 periods. Therefore, when at a certain period $t$ you make a prediction of the inflation in period $t + 1$ ($y_{e,t+1}$), the following information is available:

- Values of the actual inflation, output gap and interest rate up to and including period $t - 1$;
- Your predictions up to and including period $t$;
- Your prediction scores up to and including period $t - 1$.

**Information about your reward (part 1 of 2)**

Your reward after the experiment has ended increases with the accuracy of your predictions. Your accuracy is measured by the absolute error between your inflation predictions and the true
inflation. For each period this absolute error is calculated as soon as the true value of inflation is known; you subsequently get a prediction score that decreases as the absolute error increases. The table below gives the relation between the absolute predictions error and the prediction score. If at a certain period you predict for example an inflation of 2%, and the true inflation turns out to be 3%, then you make an absolute error of $3\% - 2\% = 1\%$. Therefore you get a prediction score of 50. If you predict an inflation of 1%, and the realized inflation turns out to be $-2\%$, you make a prediction error of $1\% - (-2\%) = 3\%$. Then you get a prediction score of 25. For a perfect prediction, with a prediction error of zero, you get a prediction score of 100.

<table>
<thead>
<tr>
<th>Absolute prediction error</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Score</td>
<td>100</td>
<td>50</td>
<td>33/3</td>
<td>25</td>
<td>20</td>
<td>10</td>
</tr>
</tbody>
</table>

Information about your reward (part 2 of 2)

The figure below shows the relation between your prediction score (vertical axis) and your prediction error (horizontal axis). Notice that your prediction score decreases more slowly as your prediction error increases. Points in the graph correspond to the prediction scores in the previous table.

Your total score at the end of the experiment consists simply of the sum of all prediction scores you got during the experiment. During the experiment, your scores are shown on your computer screen. When the experiment has ended, you are shown an overview of your prediction scores, followed by the resulting total score. Your final reward consists of 0.75 euro-cent for each point in your total score (200 points therefore equals 1.50 euro). Additionally, you will receive a show up fee of 5 euro.

Information about the computer program (part 1 of 3)

Below you see an example of the left upper part of the computer screen during the experiment. It consists of a graphical representation of the output gap (red series) and your predictions of it (yellow series). On the horizontal axis are the time periods; the vertical axis is in percentages.
In the imaginary situation depicted in the graph, the experiment is in period 30 and you predict the output gap in period 31 (the experiment lasts for 50 periods). Notice that the graph only shows results of at most the last 25 periods and that the next period is always on the right hand side.

The left bottom part of the computer screen also contains a graph. In this graph the inflation and the interest rate are shown in the same way as in the above graph.
Information about the computer program (part 2 of 3)

Below you see an example of the right upper part of the computer screen during the experiment. It consists of a table containing information about the results of the experiment in at most the last 25 periods. This information is supplemental to the graphs in the left part of the screen. The first column of the table shows the time period (the next period, 31 in the example, is always at the top). The second and third columns respectively show the output gap and your predictions of it. The fourth column gives the inflation and the fifth column the interest rate. Finally, the sixth column gives your prediction score for each period separately. Notice that you can use the sheet of paper on your desk to save data longer than 25 periods.

<table>
<thead>
<tr>
<th>Tijdsperiode</th>
<th>Output gap</th>
<th>Inflatie</th>
<th>Reële (%)</th>
<th>Voorstel. score</th>
</tr>
</thead>
<tbody>
<tr>
<td>31</td>
<td>3.59</td>
<td>3.43</td>
<td>3.59</td>
<td>55</td>
</tr>
<tr>
<td>30</td>
<td>2.59</td>
<td>2.60</td>
<td>2.59</td>
<td>35</td>
</tr>
<tr>
<td>29</td>
<td>2.08</td>
<td>2.42</td>
<td>2.08</td>
<td>35</td>
</tr>
<tr>
<td>28</td>
<td>1.88</td>
<td>1.64</td>
<td>1.88</td>
<td>35</td>
</tr>
<tr>
<td>27</td>
<td>-1.11</td>
<td>1.27</td>
<td>-1.11</td>
<td>35</td>
</tr>
<tr>
<td>26</td>
<td>-2.99</td>
<td>1.75</td>
<td>-2.99</td>
<td>35</td>
</tr>
<tr>
<td>25</td>
<td>-2.99</td>
<td>1.75</td>
<td>-2.99</td>
<td>35</td>
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<td>24</td>
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<td>23</td>
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<td>-2.99</td>
<td>35</td>
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<td>-2.99</td>
<td>35</td>
</tr>
<tr>
<td>17</td>
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<td>1.75</td>
<td>-2.99</td>
<td>35</td>
</tr>
<tr>
<td>16</td>
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<td>1.75</td>
<td>-2.99</td>
<td>35</td>
</tr>
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</tr>
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<td>-2.99</td>
<td>1.75</td>
<td>-2.99</td>
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</tr>
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<td>1.75</td>
<td>-2.99</td>
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<tr>
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<td>1.75</td>
<td>-2.99</td>
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<td>6</td>
<td>-2.99</td>
<td>1.75</td>
<td>-2.99</td>
<td>35</td>
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<tr>
<td>5</td>
<td>-2.99</td>
<td>1.75</td>
<td>-2.99</td>
<td>35</td>
</tr>
</tbody>
</table>

Information about the computer program (part 3 of 3)

Below you see an example of the bottom part of the computer screen during the experiment. In each period you are asked to submit your output gap prediction in the next period (below
Submit your prediction). When submitting your prediction, use the decimal point if necessary. For example, if you want to submit a prediction of 2.5%, type "2.5"; for a prediction of −1.75%, type " −1.75". Notice that your predictions and the true inflation in the experiment are rounded to two decimals. Moreover, prediction scores are rounded to integers.

Appendix 5.B  List of tables

- Estimation of general linear forecasting rules
  - Tables 5.8-5.14

- Estimation of First Order Heuristics
  - Tables 5.15-5.17

- MSE of the one-period ahead forecast
  - Table 5.18

- Out of sample forecasting performance of the switching model
  - Table 5.19
Table 5.8: Estimated coefficients of general linear prediction rules for the participants of Treatment 1a. Insignificant coefficients have been removed iteratively, highest p-value first. Bold values for the diagnostics tests indicate the presence of autocorrelation of order one or two (Breusch-Godfrey (B-G)), heteroskedasticity (White, no cross terms (W)) or misspecification (Ramsey RESET). For two participants, outliers in the expectation series have been removed by linear interpolation (see Remarks).

<table>
<thead>
<tr>
<th>Subject</th>
<th>B-G W RR</th>
<th>Diagnostics (p values)</th>
<th>Estimated coefficients (OLS regression, 5% significance, learning phase = 11 periods)</th>
</tr>
</thead>
<tbody>
<tr>
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<td></td>
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<td>Diagnostics (p values)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>p values</td>
</tr>
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<td></td>
<td>6</td>
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</table>

Interpolation due to outlier(s)
Table 5.9: Estimated coefficients of general linear prediction rules for the participants of Treatment 1b. Insignificant coefficients have been removed iteratively, highest \( p \) value first. Bold \( p \) values for the diagnostics tests indicate the presence of autocorrelation of order one or two (Breusch-Godfrey (B-G)), heteroskedasticity (White, no cross terms (W)) or misspecification (Ramsey RESET, 1 fitted term (RR)). For one participant, an insignificant regressor was not removed in order to prevent autocorrelation (see Remarks).
Dependent variable:

Estimated coefficients (OLS regression, 5% significance, learning phase = 11 periods) Diagnostics (p-values)

Table 5.10: Estimated coefficients of general linear prediction rules for the participants of Treatment 2a. Insignificant coefficients have been removed iteratively, highest p-value first. Bold values for the diagnostics tests indicate the presence of autocorrelation of order one or two (Breusch-Godfrey (B-G)), heteroskedasticity (White, no cross terms (W)) or misspecification (Ramsey RESET, 1 fitted term). For one participant, outliers in the expectation series have been removed linearly, by linear interpolation (see Remarks).
Dependent variable: $\pi_{t+1}$
Estimated coefficients (OLS regression, 5% significance, learning phase = 11 periods) Diagnostics ($p$ values)

<table>
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<tr>
<th>Subject</th>
<th>$c$</th>
<th>$\pi_t$</th>
<th>$\pi_{t-1}$</th>
<th>$\pi_{t-2}$</th>
<th>$\pi_{t-3}$</th>
<th>$\pi_{t-4}$</th>
<th>$\pi_{t-5}$</th>
<th>$\pi_{t-6}$</th>
<th>$y_{t-1}$</th>
<th>$y_{t-2}$</th>
<th>$y_{t-3}$</th>
<th>$R^2$</th>
<th>B-G</th>
<th>W</th>
<th>RR</th>
<th>Remarks</th>
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<td>0.000</td>
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<td>0.000</td>
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<td>0.000</td>
<td>0.000</td>
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<td>0.180</td>
<td>0.406</td>
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<th>Subject</th>
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<th>$\pi_{t-1}$</th>
<th>$\pi_{t-2}$</th>
<th>$\pi_{t-3}$</th>
<th>$\pi_{t-4}$</th>
<th>$\pi_{t-5}$</th>
<th>$\pi_{t-6}$</th>
<th>$y_{t-1}$</th>
<th>$y_{t-2}$</th>
<th>$y_{t-3}$</th>
<th>$R^2$</th>
<th>B-G</th>
<th>W</th>
<th>RR</th>
<th>Remarks</th>
</tr>
</thead>
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<tr>
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<td>0.000</td>
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<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
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Table 5.11: Estimated coefficients of general linear prediction rules for the participants of Treatment 2b. Insignificant coefficients have been removed iteratively, highest $p$ value first. Bold $p$ values for the diagnostics tests indicate the presence of autocorrelation of order one or two (Breusch-Godfrey ($B$-$G$)), heteroskedasticity (White, no cross terms ($W$)) or misspecification (Ramsey RESET, 1 fitted term ($RR$)). For one participant, a significant regressor was removed in order to prevent autocorrelation (see Remarks).
### Table 5.12: Estimated coefficients of general linear prediction rules for the participants of Treatment 3a.

Insignificant coefficients have been removed iteratively, highest p-value first. Bold p-values for the diagnostics tests indicate the presence of autocorrelation of order one or two (Breusch-Godfrey B-G), heteroskedasticity (White, no cross terms W), or misspecification (Ramsey RESET, 1 fitted term RR). For one participant, an apparently significant regressor was removed in order to prevent autocorrelation. For two participants, interpolated values in the expectation series have been removed iteratively; highest value first. For one participant, an apparently significant regressor was removed in order to prevent autocorrelation.

| Subject | c | y_{t-1} | y_{t-2} | y_{t-3} | \pi_{t-1} | \pi_{t-2} | \pi_{t-3} | \pi_{et-1} | \pi_{et-2} | \pi_{et-3} | B-G | W | RR Remarks |
|---------|---|---------|---------|---------|----------|----------|----------|----------|----------|----------|-----|---|--------|---|
| 1       | -0.102 | 0.864 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | Interpolation due to outlier(s) |
| 2       | -0.019 | 1.064 | -0.255 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | — |
| 3       | 0.503 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | — |
| 4       | 0.317 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | — |
| 5       | 0.516 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | — |
| 6       | 0.277 | 0.266 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | — |
| 7       | 0.415 | 0.646 | -0.499 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | — |
| 8       | 0.020 | 0.492 | -0.181 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | — |
| 9       | 0.067 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | — |
| 10      | 0.018 | 0.468 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | — |
| 11      | 0.029 | 0.507 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | — |
Table 5.13: Estimated coefficients of general linear prediction rules for the participants of Treatment 3b, Group 1. Insignificant coefficients have been removed iteratively, highest \( p \) value first. Bold \( p \) values for the diagnostics tests indicate the presence of autocorrelation of order one or two (Breusch-Godfrey (\textit{B-G})), heteroskedasticity (White, no cross terms (\textit{W})) or misspecification (Ramsey RESET, 1 fitted term (\textit{RR})).

\[
\text{Dependent variable: } \pi_{t+1}^c
\]

Estimated coefficients (OLS regression, 5\% significance, learning phase = 11 periods)  

<table>
<thead>
<tr>
<th>Subject</th>
<th>( c )</th>
<th>( \pi_t^c )</th>
<th>( \pi_{t-1}^c )</th>
<th>( \pi_{t-2}^c )</th>
<th>( \pi_{t-3} )</th>
<th>( \pi_{t-4} )</th>
<th>( y_{t-1} )</th>
<th>( y_{t-2} )</th>
<th>( y_{t-3} )</th>
<th>( R^2 )</th>
<th>\textit{B-G}</th>
<th>\textit{W}</th>
<th>\textit{RR}</th>
<th>Remarks</th>
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<tbody>
<tr>
<td>1</td>
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<td>0.411</td>
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</tr>
<tr>
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<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
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<td>0.581</td>
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</tr>
<tr>
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<td>0.000</td>
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<td>0.841</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
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<td>0.302</td>
<td>0.015</td>
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<tr>
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<td>0.000</td>
<td>0.000</td>
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<td>0.250</td>
<td>0.000</td>
<td>0.000</td>
<td>0.896</td>
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<td>0.427</td>
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</table>

\[
\text{Dependent variable: } y_{t+1}^c
\]

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<tr>
<th>Subject</th>
<th>( c )</th>
<th>( y_t^c )</th>
<th>( y_{t-1}^c )</th>
<th>( y_{t-2}^c )</th>
<th>( y_{t-3} )</th>
<th>( y_{t-4} )</th>
<th>( y_{t-5} )</th>
<th>( y_{t-6} )</th>
<th>( y_{t-7} )</th>
<th>( R^2 )</th>
<th>\textit{B-G}</th>
<th>\textit{W}</th>
<th>\textit{RR}</th>
<th>Remarks</th>
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<td>0.000</td>
<td>0.541</td>
<td>0.000</td>
<td>0.857</td>
<td>0.000</td>
<td>0.000</td>
<td>0.507</td>
<td>0.612</td>
<td>0.657</td>
<td>0.335</td>
</tr>
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<td>-0.449</td>
<td>-0.357</td>
<td>0.000</td>
<td>0.000</td>
<td>0.571</td>
<td>0.675</td>
<td>-0.415</td>
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<td>0.714</td>
<td>0.724</td>
<td>0.247</td>
<td>0.068</td>
</tr>
<tr>
<td>9</td>
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<td>0.000</td>
<td>-0.306</td>
<td>0.000</td>
<td>0.000</td>
<td>0.443</td>
<td>0.839</td>
<td>0.000</td>
<td>0.000</td>
<td>0.802</td>
<td>0.300</td>
<td>0.637</td>
<td>0.308</td>
</tr>
<tr>
<td>10</td>
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<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.663</td>
<td>0.000</td>
<td>0.794</td>
<td>0.000</td>
<td>0.188</td>
<td>0.673</td>
<td>0.179</td>
<td>0.034</td>
</tr>
<tr>
<td>11</td>
<td>0.051</td>
<td>0.000</td>
<td>0.322</td>
<td>0.194</td>
<td>-0.824</td>
<td>0.000</td>
<td>0.792</td>
<td>0.570</td>
<td>-0.345</td>
<td>0.000</td>
<td>0.804</td>
<td>0.231</td>
<td>0.110</td>
<td>0.275</td>
</tr>
<tr>
<td>12</td>
<td>-0.010</td>
<td>0.000</td>
<td>-0.256</td>
<td>-0.228</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.273</td>
<td>0.000</td>
<td>0.389</td>
<td>0.303</td>
<td>0.233</td>
<td>0.966</td>
</tr>
</tbody>
</table>
### Table 5.14: Estimated coefficients of general linear prediction rules for the participants of Treatment 3b, Group 2.

Insignificant coefficients have been removed iteratively, highest p-value first. Bold values for the diagnostics tests indicate the presence of autocorrelation of order one or two (Breusch-Godfrey (B-G)), heteroskedasticity (White, no cross terms (W)), or misspecification (Ramsey RESET). Significant variable(s) removed.

<table>
<thead>
<tr>
<th>Subject</th>
<th>B-G W RR Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.498 0.000 0.000 0.000 1.260 0.000 0.000 1.299 0.000 0.000 1.107 0.000 0.000 0.000 0.000 0.617 0.224 0.755 0.235 —</td>
</tr>
<tr>
<td>2</td>
<td>0.649 0.000 0.000 0.382 1.618 -1.288 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000</td>
</tr>
<tr>
<td>3</td>
<td>0.531 0.386 0.000 0.309 1.041 -0.968 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000</td>
</tr>
<tr>
<td>4</td>
<td>1.758 0.000 0.000 1.826 -1.659 0.000 0.000 -0.481 0.000 0.000 0.847 0.772 0.271 0.949 —</td>
</tr>
<tr>
<td>5</td>
<td>1.504 0.000 0.000 0.276 0.841 -0.834 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000</td>
</tr>
<tr>
<td>6</td>
<td>-0.070 0.000 0.000 1.037 0.000 0.000 1.058 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000</td>
</tr>
<tr>
<td>7</td>
<td>1.060 0.000 0.000 -0.502 0.000 -0.475 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000</td>
</tr>
<tr>
<td>8</td>
<td>4.319 0.000 0.000 -0.454 -2.064 1.316 -1.292 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000</td>
</tr>
<tr>
<td>9</td>
<td>2.974 0.000 0.000 -1.367 0.000 0.908 -1.392 0.581 0.753 0.714 0.229 0.259 —</td>
</tr>
<tr>
<td>10</td>
<td>1.585 0.000 0.000 -0.740 0.000 0.786 -1.242 0.427 0.796 0.574 0.271 0.257 —</td>
</tr>
<tr>
<td>11</td>
<td>4.033 0.000 0.000 -1.829 0.000 -1.633 0.732 0.464 0.715 0.338 0.510 —</td>
</tr>
<tr>
<td>12</td>
<td>0.760 0.000 0.000 -0.949 1.348 0.000 1.000 0.630 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000</td>
</tr>
</tbody>
</table>

Note: Estimated coefficients (OLS regression, 5% significance, learning phase = 11 periods).
Table 5.15: Estimated coefficients of First-Order Heuristics (FOH) prediction rules for the participants of Treatments 1a and 1b. A Wald test indicates whether the general linear specification (see Table 3) can be restricted to the FOH specification. Insignificant regressors have been removed iteratively, highest p value first. The columns “Anchor” and “Adjustment” classify the estimated rules.
Dependent variable: $\pi_{t+1}^e$

Estimated coefficients (OLS regression, 5% significance, learning phase = 11 periods)

<table>
<thead>
<tr>
<th>Subject</th>
<th>Wald test $(p \text{ value})$</th>
<th>Sample mean $\pi$</th>
<th>$\pi_t$</th>
<th>$\pi_{t-1}$</th>
<th>$(\pi_{t-1} - \pi_{t-2})$</th>
<th>$R^2$</th>
<th>Anchor</th>
<th>Adjustment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatment 2A</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
1 | 0.014 | — | — | — | — | — | — | — |
2 | 0.000 | — | — | — | — | — | — | — |
3 | 0.021 | — | — | — | — | — | — | — |
4 | 0.829 | 0.000 | 0.711 | 1.366 | 0.865 | Other | Trend following |
5 | 0.304 | 0.016 | 0.835 | 0.858 | 0.704 | Naive | Trend following |
6 | 0.648 | 0.465 | 0.392 | 0.143 | 0.995 | 0.767 | Other | Trend following |
1 | 0.009 | — | — | — | — | — | — | — |
2 | 0.092 | 0.178 | 0.637 | 0.902 | 0.952 | Other | Trend following |
3 | 0.257 | 0.022 | 0.978 | 0.796 | 0.915 | Naive | Trend following |
4 | 0.017 | — | — | — | — | — | — | — |
5 | 0.019 | — | — | — | — | — | — | — |
6 | 0.212 | 0.187 | 0.444 | 0.369 | 0.989 | 0.953 | Other | Trend following |
| Treatment 2B |
1 | 0.137 | 0.337 | 0.457 | 0.206 | 0.415 | 0.478 | Other | Trend following |
2 | 0.002 | — | — | — | — | — | — | — |
3 | 0.009 | — | — | — | — | — | — | — |
4 | 0.009 | — | — | — | — | — | — | — |
5 | 0.422 | 0.216 | 0.784 | 0.000 | 0.779 | Naive | None |
6 | 0.180 | 1.000 | 0.000 | 0.000 | 0.000 | NR | Fundamentalist | None |
1 | 0.720 | -0.265 | 0.000 | 1.265 | 0.698 | 0.671 | Naive | None |
2 | 0.303 | 0.431 | 0.348 | 0.220 | 0.733 | 0.746 | Other | Trend following |
3 | 0.118 | 0.170 | 0.000 | 0.830 | 0.812 | 0.912 | Other | Trend following |
4 | 0.027 | — | — | — | — | — | — | — |
5 | 0.838 | -0.018 | 0.000 | 1.018 | 0.613 | 0.671 | Naive | Trend following |
6 | 0.815 | 0.437 | 0.462 | 0.100 | 1.048 | 0.761 | Other | Trend following |
1 | 0.021 | — | — | — | — | — | — | — |
2 | 0.130 | 0.206 | 0.688 | 0.106 | 0.742 | 0.858 | Other | Trend following |
3 | 0.234 | 0.454 | 0.479 | 0.067 | 0.940 | 0.777 | Other | Trend following |
4 | 0.041 | — | — | — | — | — | — | — |
5 | 0.063 | 0.295 | 0.000 | 0.705 | 0.401 | 0.846 | Other | Trend following |
6 | 0.085 | 0.420 | 0.000 | 0.580 | 1.028 | 0.835 | Other | Trend following |

Table 5.16: Estimated coefficients of First-Order Heuristics (FOH) prediction rules for the participants of Treatments 2a and 2b. A Wald test indicates whether the general linear specification (see Table 3) can be restricted to the FOH specification. Insignificant regressors have been removed iteratively, highest $p$ value first. The columns “Anchor” and “Adjustment” classify the estimated rules.
Table 5.17: Estimated coefficients of First-Order Heuristics (FOH) prediction rules for the participants of Treatments 3a and 3b. A Wald test indicates whether the general linear specification (see Table 3) can be restricted to the FOH specification. Insignificant regressors have been removed iteratively, highest p value first. The columns “Anchor” and “Adjustment” classify the estimated rules.
Moreover, the MSE for treatment \( a \) group 1 has been computed for periods \( 4 - 45 \) due to the crash of the session. In treatments 2 and 3 the interest rate hits the zero lower bound respectively in period 27 and period 77. Therefore the MSE for treatment 2 groups 2 and 3 the interest rate hits the zero lower bound respectively in period 48 and period 77. In treatment 3 group 1, the MSE for treatment 3 refers to the sum of the MSE relative to inflation and the MSE relative to output gap.

<table>
<thead>
<tr>
<th>Model</th>
<th>( \beta )</th>
<th>( \alpha )</th>
<th>( \eta )</th>
<th>( \delta )</th>
<th>( \omega )</th>
</tr>
</thead>
<tbody>
<tr>
<td>LAA</td>
<td>0.321</td>
<td>0.368</td>
<td>0.321</td>
<td>0.368</td>
<td>0.321</td>
</tr>
<tr>
<td>ADA</td>
<td>0.210</td>
<td>0.234</td>
<td>0.210</td>
<td>0.234</td>
<td>0.210</td>
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<tr>
<td>LIF</td>
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<td>0.039</td>
<td>0.034</td>
<td>0.039</td>
<td>0.034</td>
</tr>
<tr>
<td>RE</td>
<td>0.521</td>
<td>0.539</td>
<td>0.521</td>
<td>0.539</td>
<td>0.521</td>
</tr>
<tr>
<td>TR3A</td>
<td>0.421</td>
<td>0.427</td>
<td>0.421</td>
<td>0.427</td>
<td>0.421</td>
</tr>
<tr>
<td>TR3B</td>
<td>0.321</td>
<td>0.368</td>
<td>0.321</td>
<td>0.368</td>
<td>0.321</td>
</tr>
<tr>
<td>TR4A</td>
<td>0.210</td>
<td>0.234</td>
<td>0.210</td>
<td>0.234</td>
<td>0.210</td>
</tr>
<tr>
<td>TR4B</td>
<td>0.034</td>
<td>0.039</td>
<td>0.034</td>
<td>0.039</td>
<td>0.034</td>
</tr>
<tr>
<td>TR4C</td>
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<td>0.521</td>
<td>0.539</td>
<td>0.521</td>
</tr>
<tr>
<td>TR5A</td>
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<td>0.421</td>
<td>0.427</td>
<td>0.421</td>
</tr>
<tr>
<td>TR6A</td>
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<td>0.368</td>
<td>0.321</td>
<td>0.368</td>
<td>0.321</td>
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<tr>
<td>TR6B</td>
<td>0.210</td>
<td>0.234</td>
<td>0.210</td>
<td>0.234</td>
<td>0.210</td>
</tr>
<tr>
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<td>0.039</td>
<td>0.034</td>
<td>0.039</td>
<td>0.034</td>
</tr>
<tr>
<td>TR7A</td>
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<td>0.539</td>
<td>0.521</td>
<td>0.539</td>
<td>0.521</td>
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<tr>
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<td>0.421</td>
<td>0.427</td>
<td>0.421</td>
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<td>0.321</td>
<td>0.368</td>
<td>0.321</td>
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</table>

Table 5.18: MSE over periods \( 4 - 70 \) of the one-period ahead forecast.
Table 5.19: Out-of-sample performance.

<table>
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<tr>
<th>Model</th>
<th>$Tr_{1a , gr2}$</th>
<th>$Tr_{1a , gr3}$</th>
<th>$Tr_{1b , gr2}$</th>
<th>$Tr_{2a , gr3}$</th>
<th>$Tr_{2a , gr2}^*$</th>
<th>$Tr_{2a , gr3}^*$</th>
<th>$Tr_{2b , gr2}$</th>
<th>$Tr_{3a , gr1}$</th>
</tr>
</thead>
<tbody>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(\beta, \eta, \delta)$</td>
<td>$(1,1,0.8)$</td>
<td>$(3,1,0.6)$</td>
<td>$(3,1,0.2)$</td>
<td>$(0,0)$</td>
<td>$(10,0.7,0.9)$</td>
<td>$(10,0.3,0.7)$</td>
<td>$(1,0.9,0.1)$</td>
<td>$(1,1,0.2)$</td>
</tr>
<tr>
<td>1 p ahead</td>
<td>0.9794</td>
<td>0.0000</td>
<td>0.0003</td>
<td>188.78</td>
<td>0.0023</td>
<td>0.0014</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>2 p ahead</td>
<td>22.501</td>
<td>0.0002</td>
<td>0.0002</td>
<td>10602</td>
<td>0.0749</td>
<td>0.1557</td>
<td>0.0007</td>
<td>0.0227</td>
</tr>
<tr>
<td>3 p ahead</td>
<td>93.957</td>
<td>0.0000</td>
<td>0.0000</td>
<td>805.38</td>
<td>0.8689</td>
<td>0.7025</td>
<td>0.0000</td>
<td>0.1008</td>
</tr>
<tr>
<td>4 p ahead</td>
<td>115.49</td>
<td>0.0010</td>
<td>0.0003</td>
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<td>3.5035</td>
<td>1.5682</td>
<td>0.0077</td>
<td>0.0522</td>
</tr>
<tr>
<td>5 p ahead</td>
<td>84.702</td>
<td>0.0041</td>
<td>0.0000</td>
<td>6297.1</td>
<td>7.7714</td>
<td>3.1128</td>
<td>0.0250</td>
<td>0.0376</td>
</tr>
<tr>
<td>6 p ahead</td>
<td>20.159</td>
<td>0.0037</td>
<td>0.0000</td>
<td>13423</td>
<td>11.831</td>
<td>5.0313</td>
<td>0.0413</td>
<td>0.1475</td>
</tr>
<tr>
<td>7 p ahead</td>
<td>62.590</td>
<td>0.0060</td>
<td>0.0011</td>
<td>1670.9</td>
<td>16.452</td>
<td>8.4157</td>
<td>0.0287</td>
<td>0.0657</td>
</tr>
<tr>
<td><strong>Benchmark Switching Model</strong></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(\beta, \eta, \delta)$</td>
<td>$(0.4,0.7,0.9)$</td>
<td>$(0.4,0.7,0.9)$</td>
<td>$(0.4,0.7,0.9)$</td>
<td>$(0.4,0.7,0.9)$</td>
<td>$(0.4,0.7,0.9)$</td>
<td>$(0.4,0.7,0.9)$</td>
<td>$(0.4,0.7,0.9)$</td>
<td>$(0.4,0.7,0.9)$</td>
</tr>
<tr>
<td>1 p ahead</td>
<td>1.6998</td>
<td>0.0000</td>
<td>0.0002</td>
<td>3458.5</td>
<td>0.0501</td>
<td>0.0103</td>
<td>0.0037</td>
<td>0.0033</td>
</tr>
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Note that the squared prediction errors for treatment 3 refer to the sum of the errors relative to inflation and to output gap.
In treatment 2a groups 2 and 3 the interest rate hits the zero lower bound respectively in period 43 and period 27, therefore the switching models and the AR(2) model for $Tr_{2a \, gr2}^*$ and $Tr_{2a \, gr3}^*$ have been estimated on restricted samples of 36 and 20 periods respectively. Moreover in treatment 3a group 1 the switching models and the AR(2) model have been estimated on a sample of 18 periods due to the crash of the session.
Chapter 6

Summary

This thesis studies the effect of individual bounded rationality on aggregate macroeconomic dynamics. Boundedly rational agents are specified as using simple heuristics in their decision making. An important aspect of the type of bounded rationality described in this thesis is that the population of agents is heterogeneous, that is, actors can choose from different rules to solve the same economic problem. The set of rules is disciplined by an evolutionary selection mechanism where the best performing rule, measured according to some fitness metric, attracts the higher number of agents. An important role in triggering switching between rules is played by the dynamic feedback between individual expectations of macroeconomic variables and their aggregate realizations. This expectational feedback mechanism translates agents’ interaction into a mutual dependence between the individual choices of economic actors and the macro environment against which these choices are evaluated.

The contributions of the thesis are threefold. First, we develop theoretical New Keynesian models with monopolistic competition, nominal rigidities, heterogeneous expectations and evolutionary selection among forecasting strategies and study aggregate macro behavior under heterogeneous expectations among boundedly rational agents. Second, in departing from the traditional assumption of a representative rational agent, we investigate the validity of a frequently used argument in support of rationality, namely that rationality is the outcome of the repeated interaction of heterogeneous boundedly rational agents. In particular, we test the valid-
ity of standard monetary policy advices such as the Taylor principle, derived to ensure a unique stable equilibrium under the assumption of RE, in the presence of heterogeneous agents whose beliefs are updated based on past performance. Third, the thesis contributes to the empirical validation of heterogeneous agents models. In particular, the heuristics switching model is used to explain aggregate time series data as well as experimental data on individual expectations and aggregate macro behavior.

Chapter 2 presents a simple frictionless DSGE model to study the role of heterogeneous expectations about future inflation with evolutionary selection of strategies and the potential (de-)stabilizing effect of different interest rate rules. Macroeconomic stability and inflation dynamics depend in interesting ways on the set of forecasting strategies and the reaction coefficient to inflation of a Taylor-type interest rate rule. In particular, the standard policy recommendation, i.e., to adjust the interest rate more than point for point in reaction to inflation (Taylor principle), is sufficient to guarantee convergence to the RE equilibrium in the case of a continuum of beliefs. However, the Taylor principle is no longer sufficient to guarantee uniqueness and global stability of the RE equilibrium in the (more realistic) case of finitely many belief types. Non-rational beliefs and multiple equilibria may then survive evolutionary competition.

Chapter 3 derives a general New Keynesian framework consistent with heterogeneous expectations by explicitly solving the micro-foundations underpinning the model. We design an economy where agents can decide whether to pay some information gathering and processing costs for rationality or use simple heuristics to forecast macroeconomic variables. In such an environment, we address determinacy issues, i.e., uniqueness and stability of the RE equilibrium, related to the use of different interest rate rules. The central finding is that in a world with heterogeneous agents, standard policy advices do not necessarily guarantee uniqueness and stability of the RE equilibrium. In fact, policy attempts to achieve determinacy under RE, may destabilize the economy even when only a small fraction of boundedly rational agents is present in the economic environment.

Chapter 4 estimates a behavioral model of inflation dynamics with monopolistic price set-
ting, nominal rigidities and endogenous evolutionary switching between different forecasting strategies according to some fitness measure. In the stylized framework presented in the chapter there are two groups of price setters, fundamentalists and naive. Fundamentalists are forward-looking and believe in a present value relationship between inflation and marginal costs, while naive are backward-looking, using the simplest rule of thumb, naive expectations, to forecast future inflation. The estimation results show statistically significant behavioral heterogeneity and substantial time variation in the weights of forward-looking and backward-looking agents.

Chapter 5 reports the results of laboratory experiments with human subjects, designed to study individual expectations, their interactions and the aggregate behavior they co-create within a New Keynesian setup. Experimental data show that individuals tend to base their predictions on past observations, following simple forecasting heuristics, and individual learning takes the form of switching from one heuristic to another. We then use a simple model of individual learning with a performance-based evolutionary selection among different forecasting rules to explain coordination of individual expectations and aggregate macro behavior observed in the laboratory experiments. The analysis shows the simple heterogeneous expectations switching model fits individual learning as well as aggregate outcomes and outperforms homogeneous expectations benchmarks.

A general conclusion following from the theoretical and empirical results of this thesis is that non-rational beliefs may survive evolutionary competition among heterogeneous forecasting strategies. Therefore, policy makers should seriously take into account bounded rationality when designing monetary policy, since policies constructed under the assumption of homogeneous RE may be destabilizing when expectations are heterogeneous.


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Samenvatting (Summary in Dutch)

Dit proefschrift bestudeert het effect van de begrensde rationaliteit van individuen op geaggregeerde macro-economische dynamica. Begrend rationale agenten worden gekenmerkt door het gebruik van eenvoudige vuistregels (heuristieken) in het beslissingsproces. Een belangrijk kenmerk van het type begrensde rationaliteit dat wordt besproken in dit proefschrift is dat de groep agenten heterogeen is. Dit houdt in dat individuen uit verschillende regels kunnen kiezen om hetzelfde economische probleem op te lossen. De verzameling regels wordt gedisciplineerd door een evolutionair selectiemechanisme waarbij de best presterende regel, gemeten aan de hand van een bepaalde fitnessmaat, het hoogste aantal agenten aantrekt. Een belangrijke rol in macro-economische dynamische modellen is weggelegd voor de feedback tussen individuele verwachtingen van macro-economische variabelen en hun geaggregeerde resultaten. Dit feedbackmechanisme gebaseerd op verwachtingspatronen vertaalt de interactie tussen agenten in een wederzijdse afhankelijkheid van de individuele keuzes van economische actoren en de macro omgeving waartegen deze keuzes worden geëvalueerd.

De bijdrage van het proefschrift is drievoudig. Ten eerste ontwikkelen we Nieuw-Keynesiaanse modellen met monopolistische concurrentie, nominale rigiditeiten, heterogene verwachtingen en evolutionaire selectie binnen voorspellende strategieën en bestuderen we geaggregeerd macrogedrag onder heterogene verwachtingen van beperkt rationale agenten. Ten tweede onderzoeken we door de traditionele aanname van een representatieve rationale agent los te laten de validiteit van een vaak gebruikt argument ter ondersteuning van rationaliteit, namelijk dat rationaliteit de uitkomst is van de herhaalde interactie van heterogene, beperkt rationele agenten. In het bijzon-
der testen we de validiteit van standaard monetaire beleidsadviezen, zoals het Taylorprincipe
dat een uniek stabiel evenwicht verzekert onder de veronderstelling van rationele verwachtingen (RE), in de aanwezigheid van heterogene agenten wiens beslissingsregels worden aangepast
op basis van prestaties uit het verleden. Ten derde draagt het proefschrift bij aan de empirische
validatie van modellen gebaseerd op heterogene agenten. In het bijzonder wordt het heuristieke
switching model gebruikt om zowel de geaggregeerde tijdreeksen als experimentele data van
individuele verwachtingen en geaggregeerd macrogedrag te verklaren.

Hoofdstuk 2 presenteert een eenvoudig DSGE-model zonder fricties voor het bestuderen
van de rol van heterogene verwachtingen van toekomstige inflatie met evolutionaire selectie
van strategieën en het potentiële destabiliserende effect van verschillende regels voor rentetarieven. Macro-economische stabiliteit en inflatiedynamica hangen op interessante wijze af
van de verzameling voorspellende strategieën en de reactiecoëfficiënt van inflatie van een regel
à la Taylor. In het bijzonder is de aanbeveling binnen standaardbeleid, het meer dan punt-
voor-punt aanpassen van het rentetarief in reactie op inflatie (het Taylorprincipe), voldoende
om convergentie te garanderen van het RE-evenwicht in geval van een continuüm van voorspelregels. Echter, het Taylorprincipe is onvoldoende om de uniciteit en globale stabiliteit van
het RE-evenwicht te garanderen in het (meer realistische) geval van eindig veel voorspelregels.
Niet-rationele voorspel strategieën en meerdere stabiele evenwichten kunnen in dat geval de
evolutionaire competitie overleven.

In hoofdstuk 3 wordt een Nieuw-Keynesiaans raamwerk afgeleid dat in overeenstemming is
met heterogene verwachtingen door het expliciet oplossen van de micro-economische grondsla-
gen die het model ondersteunen. We ontwerpen een economie waarin agenten kunnen beslissen
of ze kosten willen afdragen voor het vergaren en verwerken van informatie over rationaliteit, of
dat ze gebruik maken van een eenvoudige vuistregel om macro-economische variabelen te voor-
spellen. Op deze manier bespreken we belangrijkse kwesties, te weten uniciteit en stabiliteit van
het RE-evenwicht, onder verschillende renteregels. De centrale bevinding is dat gebruikelijke
beleidsadviezen in een wereld met heterogene agents niet noodzakelijkerwijs uniciteit en sta-
biliteit van het RE evenwicht garanderen. Het is zelfs zo dat sommige beleidsmaatregelen, bepaald onder RE, de economie kunnen destabiliseren, zelfs wanneer slechts een fractie van beperkt rationale agenten aanwezig is in de economie.


Hoofdstuk 5 toont de resultaten van laboratoriumexperimenten met menselijke subjecten, ontworpen om individuele verwachtingen te bestuderen, evenals de interacties en het geaggregeerd gedrag dat zij creren binnen een Nieuw-Keynesiaanse opzet. De experimentele data tonen aan dat individuen geneigd zijn zich te richten op waarnemingen uit het verleden, eenvoudige voorspellende vuistregels volgen, en leren te voorspellen door te switchen tussen de vuistregels. We gebruiken vervolgens een eenvoudig model waarin individuen leren door evolutionaire selectie, gebaseerd op prestaties van verschillende regels in het recente verleden, om de coördinatie van individuele verwachtingen en geaggregeerd macrogedrag geobserveerd in de laboratoriumexperimenten te verklaren. De analyse toont dat het switching model gebaseerd op eenvoudige heterogene verwachtingen zowel overeenkomt met het individuele leerproces als met geaggregeerde uitkomsten en beter voorspelt dan modellen met homogene verwachtingen.

Een algemene conclusie die volgt uit de theoretische en empirische resultaten van dit proefschrift is dat niet-rationele regels de evolutionaire competitie tussen heterogene voorspellende strategieën kunnen overleven. Beleidsmakers zouden daarom serieus rekening moeten houden met beperkte rationaliteit bij het ontwerpen van monetair beleid, omdat beleid onder de veron-
derstelling van homogene RE wellicht juist destabiliseert wanneer de verwachtingen in werkelijkheid heterogeneen zijn.
The Tinbergen Institute is the Institute for Economic Research, which was founded in 1987 by the Faculties of Economics and Econometrics of the Erasmus University Rotterdam, University of Amsterdam and VU University Amsterdam. The Institute is named after the late Professor Jan Tinbergen, Dutch Nobel Prize laureate in economics in 1969. The Tinbergen Institute is located in Amsterdam and Rotterdam. The following books recently appeared in the Tinbergen Institute Research Series:

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