Bifurcations of indifference points in discrete time optimal control problems
Mohammadian Moghayer, S.

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This thesis develops new methods to analyse non-convex discrete time optimal control problems. A distinctive feature of such problems is that indifference states may occur: these are initial states at which several optimising trajectories originate. In the thesis, the genesis of such points through indifference-attractor bifurcations is studied as system parameters are varied. This necessitates an analysis of heteroclinic bifurcation scenarios of the state-costate dynamics. In particular, it is found that infinitely many indifference points exist at certain parameter values, or equivalently, that the associated value function is not differentiable at infinitely many points in state space. The results make it possible to analyse the bifurcation structure of the discrete-time lake pollution management problem.

Saeed M Moghayer holds a BSc in Mathematics from Iran University of Science and Technology, a MSc in Mathematics from the University of Kerman (Iran), and a Master Certificate in Dynamical Systems from MRI (Mathematical Research Institute), Utrecht University (Netherlands). In 2006 he joined CeNDEF (Center for Nonlinear Dynamics in Economics and Finance) at the University of Amsterdam to pursue his PhD study. From September 2010, he is working as a researcher at TNO (Netherlands Organisation for Applied Scientific Research) in Delft. He also continues his collaboration with CeNDEF. His current research interests include applications of optimal control methods and bifurcation theory to complex dynamic models, heterogeneous agent modeling, and environmental economics.
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Promotiecommissie:

Promotor: Prof. dr. C.H. Hommes

Co-promotor: Dr. ir. F.O.O. Wagener

Overige Leden: Prof. dr. H.W. Broer
Prof. dr. G. Feichtinger
Prof. dr. A.J. Homburg
Dr. C.G.H. Diks
Dr. D. Grass

Faculteit Economie en Bedrijfskunde
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Chapter 1

Introduction

Dynamic optimisation deals with economic problems that call for an optimal sequential schedule of actions. In a dynamic optimisation problem we are looking for an optimising schedule of a choice variable in each period of time (discrete-time case) or at each point of time (continuous-time case) over a finite or an infinite planning horizon. Regardless of whether the time dependence is discrete or continuous, a typical dynamic optimisation problem contains the following: initial and end point conditions; a set of admissible paths satisfying these conditions; a performance measure such as profit or cost; and a specified objective to maximise or minimise the performance measure by choosing the path optimally.

Historically, the first dynamic optimisation problems were solved using the calculus of variations. In the second half of the 20th century, two new approaches came to prominence: optimal control theory, culminating in the development of the maximum principle by Pontryagin and his co-authors (Pontryagin et al. (1962)), and the Bellman dynamic programming approach based on the value function (Bellman (1952)).
In the control formulation, the set of admissible paths is expressed as the set of solutions to a state evolution equation that depends on one or more admissible control schedules. This seemingly simple reformulation lends itself much better to applications. Dorfman (1969) gives an economic interpretation of optimal control theory by assigning to each element of the maximum principle an economic interpretation. He refers to capital theory, which is used to explain how (and whether) a production factor can be expected to contribute more to the value of output during its lifetime than it costs to produce. In optimal control problems of capital theory, the stock variables are interpreted as physical or environmental capital, controls as some kind of investment in capital, and the main problem is to obtain an optimal investment schedule. Capital accumulation is usually limited by technological constraints on production, which are described by a production function. In addition, for an economy as a whole or even for a corporation, unlike an individual, the assumption of permanent existence is reasonable. Therefore a common specification of such economic problems is the assumption of an infinite planning horizon.

Many models consider situations where technology is convex and the optimal path always converges to a single steady state. Dechert and Nishimura (1983) were among the first to investigate one-sector optimal growth models for which technology is not convex. In this context they showed that the time dependence of the capital stock is necessarily monotonic for an optimal investment schedule, and that depending on the discount rate three situations can occur: the stock converges for all initial values to some positive steady state value, or for all initial values it converges to zero, or finally it depends on the initial state whether the capital stock converges to a positive steady
state or to zero. In the last situation, there can be an initial state for which there are two optimal solutions, one tending to the positive steady state, the other tending to zero.

Intermediate states of this type, which occur in the so-called non-convex class of optimal control problems, subsequently have been called ‘Skiba’, ‘Dechert-Nishimura-Skiba’ (DNS) or ‘Dechert-Nishimura-Sethi-Skiba’ (DNSS) states (see Grass et al., 2008), recognising the contributions of Sethi (1977) and Skiba (1978). In this thesis the designation *indifference state* is preferred for a state from which several different optimal solutions originate, possibly converging to the same long-run steady state or long-run dynamics, and the designation *indifference threshold* for an indifference state for which the originating optimal solutions converge to different long-run steady states or long-run dynamics.

The study of indifference thresholds in dynamic optimisation problems, initiated in the late 1970’s, took off only comparatively recently. Especially in the context of environmental economics, where natural systems often feature non-convexities, indifference thresholds have been studied by several authors; see for instance Tahvonen (1995); Brock and Starrett (2003); Mäler et al. (2003); Wagener (2003); Kiseleva and Wagener (2010); other studies include Grüne and Semmler (2004); Steindl and Feichtinger (2004); Dawid and Deissenberg (2005); and Caulkins et al. (2007).

The following example will recur throughout this thesis. Consider a lake that is affected by some human activities, for instance through pollution that is a by-product of agricultural activities around the lake. The terminology of Brock and Starrett (2003) is used to distinguish two groups of lake users
with conflicting interests: ‘affecters’ of the lake, who indirectly benefit from polluting the lake by using fertilisers; and ‘enjoyers’ of the lake, like fishermen, tourists or water companies, who benefit from high quality of the lake water. There is the problem of assessing the relative interests of these two groups. For this consider a social planner who tries to maximise society’s welfare, which is composed of a weighted sum of benefits that are derived from the lake by both groups. This would be an entirely classical economic problem, if the dynamics of the pollution stock were convex. It being non-convex changes matters, as there may be more than one optimal steady state.

In problems featuring multiple steady states there can be changes in the qualitative (or topological) structure of the set of solutions if parameters are varied, such as the occurrence of single steady states for some sets of parameter values and multiple steady states for others.

In the lake pollution problem qualitative changes can occur in the optimal policy if the weight assigned to benefits of each group of agents is varied. Consequently, in the analysis of this problem there is a need to classify the solutions of the optimisation problem by their qualitative characteristics: are the interests of the producers to be preferred and is the ecosystem to function mainly as a waste dump, or is it better for society to restrict production and enjoy the ecosystem? If the weight of the benefits of enjoyers in society’s welfare is low, the former situation prevails, whereas if the value of this parameter is high, the latter occurs. Therefore, there should be in-between these ranges a critical value of the weight where the solution structure changes qualitatively. To find such critical values bifurcation theory is an appropriate tool, as it provides the mathematics of qualitative changes. Its central aim is to study the dependence of solution structures on model parameters.
Although Dechert and Nishimura (1983) investigated all possible generic configurations of the optimal solution of a non-convex discrete time dynamic optimisation problem where the discount rate is varied, they gave no attention to the precise mechanics of bifurcation. In this thesis the genesis of indifference thresholds is studied for a class of single-state discrete time dynamic optimisation problems as a system parameter changes. The class under consideration contains a range of economic models already treated in the literature, like the optimal growth models studied by Dechert and Nishimura (1983), but also the discrete time version of the lake pollution models introduced by Mäler et al. (2003).

Previously Wagener (2003) showed that in continuous time non-convex optimal control problems, provided that the state variable of the problem is one dimensional, knowledge of the bifurcations of the state-costate dynamical system gives enough information to determine whether or not, for given values of the parameters, there are indifference points in the system: the set of parameters for which indifference points exist was shown to be bounded by heteroclinic bifurcation curves.

In this thesis, the analogous mechanism is investigated for discrete-time problems. Central to this analysis is the state-costate — or phase — dynamics that is associated to optimal state orbits. In the systems considered here these are on the stable manifolds of saddle fixed points of the phase system. The main result of the thesis links the genesis of indifference thresholds to the occurrence of heteroclinic orbits in phase space; these are orbits that are forward asymptotic to one saddle fixed point and backward asymptotic to another.
In particular, it is shown that if the phase system goes through a so-called heteroclinic bifurcation scenario, an indifference threshold and a locally optimal steady state are generated in an indifference-attractor bifurcation. Moreover, during the bifurcation scenario, infinitely many indifference points that are not indifference thresholds are generated. The findings are illustrated by computing the indifference threshold and some of the indifference points in a slightly modified version of the lake pollution problem.

Typically, the phase dynamics depends on several parameters. Imagine the case that the phase dynamics depends on a parameter and that there is a set of values of this parameter for which indifference thresholds exist, and another for which there are none. There is a qualitative difference between these type of systems and going from one set to the other entails a qualitative change, or, in technical terms, a bifurcation. The simplest bifurcations that can occur for the phase dynamics studied in this thesis are saddle-node and heteroclinic bifurcations and, as mentioned before, one of the main results of this thesis is that heteroclinic bifurcations may be connected to the genesis of indifference thresholds. Bifurcation theory determines the boundary of regions in parameter space for which indifference thresholds do or do not exist.

In the lake pollution problem that is considered in this thesis, optimal loading policies are classified according to qualitative characteristics; hence the analysis can be based on bifurcation theory. If the discount rate is positive, there are only four types, given in Figure 1.1. An optimal loading policy gives rise to either one or two attracting steady states; in the first case, and if the natural robustness of the lake is not too large, the steady state can be typified as being either ‘low pollution’ or ‘high pollution’.
If there are however two steady states, and if the social planner is mainly concerned with the short run, the optimal policy can exhibit jumps. In the optimal state dynamics, these jumps are associated with either indifference points or indifference thresholds. In an indifference threshold the social planner is indifferent between two types of policies that steer the lake either to a low or high pollution steady state. In an indifference point that is not an indifference threshold the social planer is indifferent between two distinct optimal policies that however steer the lake to the same long-run steady state.

As shown in the lake model, the characteristic feature of this interest conflict is the possibility of qualitatively different outcomes. Social choice often incorporates a qualitative aspect of decision making. Bifurcation theory is therefore a powerful tool to classify these outcomes and to give a graphical overview of the qualitative characteristics of the solution depending on the parameters of the problem. Using bifurcation analysis a full picture, a bifurcation diagram, of all qualitatively different optimal pollution policies can be constructed, which describes how these policies depend on the type of ecosystem, on the relative economic weights of the interest groups and on the social discount rate. For the discrete-time version of the lake system, the dependence of the different possibilities on the ‘weighting’ and ‘robustness’ parameters are given in Figure 1.1.

In the bifurcation diagram of the lake system boundaries of regions in parameter space that correspond to different types of optimal policies are shown. Each point in the parameter space determines a particular optimisation problem. A certain type of optimal solution corresponds to a particular region, and an intermediate degenerate situation between two types
Figure 1.1: The parameter space of the discrete lake system: ‘economic importance of clean lake’ versus ‘natural robustness of lake’. In the bifurcation diagram, dashed lines are saddle-node bifurcation curves, separating the region of parameters for which there is a unique equilibrium in the state-costate system from the region of multiple equilibria. In the ‘low pollution’ region, solutions tending to the clean equilibrium are optimal; in the ‘high pollution’ region, solutions tending to the polluted equilibrium are optimal. Solid lines indicate heteroclinic bifurcation curves. These curves bound the region for which indifference thresholds exist in the lake optimal control problem. In each of these regions the optimal dynamics are depicted; attractors are marked by a circle, indifference thresholds by a diamond.
of optimal solution corresponds to a boundary. In the lake system there are roughly speaking three types of optimal solution structures: (1) steering a system to a steady state regardless of its initial state; (2) steering a system to either of two steady states depending on its initial state; (3) steering a system to either of the two attracting steady states unless the initial state is exactly at an intermediate repelling steady state. The last two types of optimal solutions are dependent on initial states.

Methodologically, this thesis contributes to the geometrical analysis of discrete time dynamic optimisation problems using phase space methods. In particular, extensive use is made of differential forms and geometric integration. Contrary to the continuous time setting, phase space methods are not particularly popular in the discrete time setting (see e.g., Shone (2002)). There are several probable reasons for this: the omnipresence of the Bellman equation, which is well-understood, easy to generalise to stochastic problems, and which has an elegant theory of existence and uniqueness of solutions. Moreover, some powerful instruments of continuous time theory are not readily available: for instance, in a continuous time problem with a one-dimensional state space, knowledge of the isoclines allows to reconstruct the geometry of the phase trajectories to a great extent. In the discrete time setting, there are backward and forward isoclines, and their knowledge does not allow to reconstruct orbits of the phase system as easily. Also in the continuous time setting the value function can be evaluated in terms of the initial state and costate values of an optimal orbit; this is an immediate corollary to the Hamilton-Jacobi equation. In the discrete time setting, there is no such direct way to find the value function, though in Proposition 2.3.5 a partial replacement has been obtained.
In this thesis a geometrical point of view is used. The basic geometrical objects associated with the phase system are the orbits and the stable manifolds of the phase map. These can almost never be computed analytically, hence numerical methods are needed. Therefore to apply phase space methods, accurate algorithms are required to compute invariant manifolds numerically. For computation of approximations to invariant manifolds of maps there are several approaches (see e.g. Simó (1989), Homburg et al. (1995)). The approach used in this thesis, while less sophisticated than these methods, still provides accurate approximations to the stable manifolds; it is however restricted to two-dimensional systems.

This thesis aims to contribute to the application of bifurcation theory in non-convex discrete-time optimal control problems. While the special case of two-dimensional phase space is considered, similar theoretical results to those obtained are expected to hold in the general case of \( n \)-dimensional phase space. These results are the starting point of a full bifurcation theory of the solution structure of discrete time dynamic optimisation problems, which will make a general analysis of a wide range of non-convex dynamic economic problems accessible.
Chapter 2

Phase space methods in
discrete time optimal control

In this chapter, a general autonomous discrete time optimisation problem is considered. The necessary first order conditions are formulated in terms of a boundary value problem of the state-costate (or phase) map. If the phase map possesses a saddle point, then all orbits on the stable manifold of this point will solve the boundary value problem. Results are derived that allow to compare the values of these orbits.

2.1 First variation and Hamiltonian formalism

Here discrete optimal control theory for systems with \( n \)-dimensional state space is formulated in a way that is suitable for the purpose of this thesis.
**2.1.1 Definitions and problem specification.** To keep notation minimal, for a function $f : \mathbb{R}^n \to \mathbb{R}$ with $f = f(x)$ the following notation is employed for the derivative

$$\frac{df}{dx} = \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right).$$

Likewise, for a function $g = g(x, y), g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, the notation

$$g_x = \left( \frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n} \right)$$

is used etc.

Time $t$ is discrete, and takes values $0, 1, 2, \ldots$. The state space $\mathcal{X}$ and the control set $\mathcal{U}$ are open and convex subsets of $\mathbb{R}^n$. Note that only interior solutions are considered in the following. On the state space the state dynamics is given as

$$x_t = f(x_{t-1}, u_t), \quad (2.1)$$

for $t \geq 1$ where $f : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ is smooth. A function is smooth if it has as many derivatives as necessary; ordinarily, we shall think of $C^\infty$ functions, but the reader can substitute $C^k$ with $k > 0$ sufficiently large.

For technical convenience the special assumption is made that for all $(x, u) \in \mathcal{X} \times \mathcal{U}$ we have

$$\det f_u(x, u) \neq 0.$$ 

Note that this encompasses a large class of optimal control problems, as well as all discrete calculus of variations problems, where $f(x, u) = u$.

If $\mathbf{x} = \{x_t\}_{t=0}^\infty$ and $\mathbf{u} = \{u_t\}_{t=1}^\infty$ are sequences in $\mathcal{X}$ and $\mathcal{U}$ respectively, the pair $(\mathbf{x}, \mathbf{u})$ is called *weakly admissible* if equation (2.1) holds for all $t \geq 1$. 

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Let $\mathcal{W}_\alpha = \mathcal{W}$ denote the set of weakly admissible pairs of sequences $(x, u)$ with $x_0 = \alpha$.

Let $\rho > 0$ be a positive real number, and let $g : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ be a smooth real-valued function. For each integer $T \geq 1$, define a functional $J_T : \mathcal{W} \to \mathbb{R}$ by setting
\[
J_T(x, u) = \sum_{t=1}^{T} g(x_{t-1}, u_t) e^{-\rho t}.
\] (2.2)

A sequence $a = \{a_t\}$ of positive real numbers is called *summable* if $\sum_t a_t < \infty$. A pair of weakly admissible sequences $(x, u) \in \mathcal{W}$ is called *admissible*, if there is a positive summable sequence $a$ such that for all $t \geq 1$
\[
|g(x_{t-1}, u_t) e^{-\rho t}| \leq a_t.
\]
The set of admissible pairs $(x, u)$ is denoted by $\mathcal{A}$. Define a functional $J : \mathcal{A} \to \mathbb{R}$ by
\[
J(x, u) = \sum_{t=1}^{\infty} g(x_{t-1}, u_t) e^{-\rho t}.
\]
Note that $J$ is well-defined on $\mathcal{A}$.

The general infinite horizon autonomous discrete time optimisation problem is formulated as follows: maximise an objective
\[
J = \sum_{t=1}^{\infty} g(x_{t-1}, u_t) e^{-\rho t},
\] (2.3)
where $\rho > 0$, under the side condition that
\[
x_t = f(x_{t-1}, u_t),
\] (2.4)
for all $t \geq 1$. Note that as $f$ takes values in $\mathcal{X}$, there are no binding state constraints.
2.1.2 Variations. In this subsection the necessary first order conditions are derived and formulated in terms of a boundary value problem of the state-costate (or phase) map. Given an admissible pair \((x^0, u^0)\), consider the variations

\[
(x(\varepsilon), u(\varepsilon)) = (x^0 + \varepsilon \xi(\varepsilon), u^0 + \varepsilon \upsilon(\varepsilon)).
\]

A variation is called (weakly) admissible, if \((x(\varepsilon), u(\varepsilon))\) is (weakly) admissible for every \(\varepsilon \in [0, 1]\). Throughout, it will be assumed that all variations are weakly admissible for all \(\varepsilon \in [0, 1]\), and that for all \(t\) the functions

\[
\varepsilon \mapsto \xi_t(\varepsilon) \quad \text{and} \quad \varepsilon \mapsto \upsilon_t(\varepsilon)
\]

are smooth. Write

\[
\xi^0_t = \xi_t(0) \quad \text{and} \quad \upsilon^0_t = \upsilon_t(0).
\]

Define a function \(j_T : [0, 1] \to \mathbb{R}\) by setting

\[
 j_T(\varepsilon) = J_T(x(\varepsilon), u(\varepsilon)).
\]

where \(J_T\) is defined in (2.2).

To compute the derivative of \(j_T\) at \(\varepsilon = 0\), note that if the pair \((x(\varepsilon), u(\varepsilon))\) is weakly admissible for every \(\varepsilon \in [0, 1]\), then for all \(t\)

\[
x_t + \varepsilon \xi_t(\varepsilon) = f(x_{t-1} + \varepsilon \xi_{t-1}(\varepsilon), u_t + \varepsilon \upsilon_t(\varepsilon)).
\]

Expanding in \(\varepsilon\) around \((x_t, u_t)\) and solving for \(\upsilon_t\) yields

\[
\upsilon_t(\varepsilon) = f^{-1}_u[\xi_t(\varepsilon) - f_x \xi_{t-1}(\varepsilon)] + \varepsilon r,
\]

where \(|r(x, u, \varepsilon, \xi_1, \xi_2)| \leq C|\xi|^2\), uniformly in \((x, u, \varepsilon)\). Note that in equation (2.5) the arguments \((x_{t-1}, u_t)\) have been omitted; this will be done
whenever there is no chance for confusion. Taking \( \varepsilon \to 0 \) yields
\[
\nu_t^0 = f_u^{-1}\left[\xi_t^0 - f_x\xi_{t-1}^0\right].
\] (2.6)

Moreover
\[
\lim_{\varepsilon \to 0} \frac{j_T(\varepsilon) - j_T(0)}{\varepsilon} = \frac{J_T(x(\varepsilon), u(\varepsilon)) - J_T(x^0, u^0)}{\varepsilon}
= \sum_{t=1}^{T} \left[ g_x\xi_{t-1}^0 + g_u\nu_t^0 \right] e^{-\rho t} + O(\varepsilon)
= \sum_{t=1}^{T} \left[ (g_x - g_u f_u^{-1} f_x) \xi_{t-1}^0 + g_u f_u^{-1} \xi_{t-1}^0 \right] e^{-\rho t} + O(\varepsilon),
\] (2.7)
where in the last equality (2.6) has been used.

Introduce the sequence of costates \( y = \{y_t\}_{t=0}^\infty \) by setting
\[
\begin{cases}
y_t = -g_u(x_{t-1}, u_t) f_u^{-1}(x_{t-1}, u_t) & \text{for } t \geq 1, \text{ and }
y_0 = e^{-\rho} \left( g_x(x_0, u_1) + y_1 f_x(x_0, u_1) \right).
\end{cases}
\] (2.8)

Note that the \( y_t \) are row vectors. Taking in (2.7) the limit \( \varepsilon \to 0 \), the following result is obtained.

**Proposition 2.1.1.** Let \((x(\varepsilon), u(\varepsilon))\) be admissible variations and let
\[
j_T(\varepsilon) = J_T(x(\varepsilon), u(\varepsilon)).
\]

The right derivative \( D_+j_T(0) \) at \( \varepsilon = 0 \) exists and equals
\[
D_+j_T(0) = y_0\xi_0^0 - e^{-\rho} y_T\xi_0^0 + \sum_{t=2}^{T} \left[ (g_x + y_t f_x - e^\rho y_{t-1}) \xi_{t-1}^0 \right] e^{-\rho t}.
\] (2.9)

From this, the necessary conditions for optimality are obtained.
Proposition 2.1.2. If \((x^*, u^*) \in \mathcal{W}\) is such that \(J_T(x^*, u^*) \geq J_T(x, u)\) for all \((x, u) \in \mathcal{W}\) with \(x_0 = \alpha\), then

\[ y_T = 0 \quad (2.10) \]

and

\[ e^\rho y_{t-1} = g_x(x_{t-1}, u_t) + y_t f_x(x_{t-1}, u_t) \quad (2.11) \]

for all \(1 \leq t \leq T\).

Note that (2.11) holds for \(t = 1\) by definition of \(y_0\). Any admissible pair \((x, u)\) that satisfies (2.11) for all \(t \geq 1\) is called extremal.

As \(J_T\) depends only on finitely many variables, there are no technical problems in obtaining the first variation formula (2.9). To find the analogous expression for the infinite horizon functional \(J\), we have to be able to interchange differentiation and infinite summation. This is permitted if the variations are strongly admissible.

Definition 2.1.1. (Strongly admissible variations) An admissible variation

\[(x(\varepsilon), u(\varepsilon))\]

is called strongly admissible, if there is a fixed positive summable sequence \(a = \{a_t\}\) such that for all \(t \geq 1\) and for all \(\varepsilon \in (0,1)\)

\[
\varepsilon^{-1} \left| g(x_{t-1} + \varepsilon \xi_{t-1}(\varepsilon), u_t + \varepsilon v_t(\varepsilon)) - g(x_{t-1}, u_t) \right| e^{-\rho t} \leq a_t. \quad (2.12)
\]

Proposition 2.1.3. Let the variation \((x(\varepsilon), u(\varepsilon))\) be strongly admissible, and let

\[
j(\varepsilon) = J(x(\varepsilon), u(\varepsilon)).
\]

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Then the right-hand derivative $D_+ j(0)$ exists. Moreover, there is a positive summable sequence $\{a_t\}$ and a sequence $\{R_t\}$ such that

$$D_+ j(0) = y_0 \xi_0^0 - e^{-\rho T} y_T \xi_T^0 + \sum_{t=2}^{T} \left[ (g_x + y_t f_x - e^{\rho} y_{t-1}) \xi_{t-1}^0 \right] e^{-\rho t} + R_T$$

and

$$|R_T| \leq \sum_{t=T+1}^{\infty} a_t$$

for every $T \geq 1$.

**Proof.** The conditions of strong admissibility precisely guarantee that the series $j(\varepsilon)$ is uniformly convergent and that we may pass to the limit $\varepsilon \to 0$ under the summation sign; see for instance Knopp (1996).

---

**Definition 2.1.2.** (δ-interior) *The pair of sequences $(x, u)$ is δ-interior, if for every sequence $\xi$ with $\xi_0 = 0$ and $|\xi_t| \leq \delta$ for all $t > 0$, there is a sequence $v(\varepsilon)$ such that the variation $(x^* + \varepsilon \xi, u^* + \varepsilon v(\varepsilon))$ is strongly admissible.*

**Proposition 2.1.4.** Let $(x^*, u^*) \in A$ be such that $J(x^*, u^*) \geq J(x, u)$ for all $(x, u) \in A$ with $x_0 = \alpha$, and let $y^*$ be the associated sequence of costates, given by (2.8). Assume that $(x^*, u^*)$ is δ-interior for $\delta > 0$. Then

$$e^\rho y_{t-1}^* = g_x(x_{t-1}^*, u_{t-1}^*) + y_{t-1}^* f_x(x_{t-1}^*, u_{t-1}^*)$$ (2.13)

for all $t \geq 1$ and

$$\lim_{t \to \infty} e^{-\rho t} y_t^* = 0.$$ (2.14)
Equation (2.14) is commonly referred to as the transversality condition. The rather simple form of (2.14) is a consequence of the fact that variations in all directions are assumed to be admissible.

Proof. Since \((x^*, u^*)\) maximises \(J\), necessarily \(D_+ j(0) \leq 0\). Noting that necessarily \(\xi_0 = 0\), it follows that

\[
0 \geq D_+ j(0) = \sum_{t=2}^{T} \left[ (g_x + y_t f_x - e^\rho y_{t-1}) \xi_{t-1} \right] e^{-\rho t} - e^{-\rho T} y_T \xi_T + R_T.
\]

Let \(\text{sign}(x)\) denote the sign function

\[
\text{sign}(x) = \begin{cases} 
1 & \text{if } x > 0, \\
-1 & \text{if } x < 0, \\
0 & \text{otherwise.}
\end{cases}
\]

Setting

\[
\begin{cases} 
\xi_{t-1} = \delta \text{sign}(g_x + y_t f_x - e^\rho y_{t-1}) & \text{for } 2 \leq t \leq T, \text{ and} \\
\xi_T = -\delta \text{sign}(y_T)
\end{cases}
\]

yields that

\[
\delta \sum_{t=2}^{T} |g_x + y_t f_x - e^\rho y_{t-1}| e^{-\rho t} + \delta e^{-\rho T} |y_T| \leq |R_T| \leq \sum_{t=T+1}^{\infty} a_t.
\]

Since this inequality has to hold for all \(T \geq 2\), and since \(\sum_{t=T+1}^{\infty} a_t \to 0\) as \(T \to \infty\), the result follows.

The necessary first order conditions of the optimisation problem considered in this section are that \((x_t, y_t)\) solves the boundary value problem (2.13) with the initial condition \(x_0 = \alpha\) and the terminal condition (2.14).
2.1.3 The discrete Hamiltonian. The results of the previous subsection can be formulated very elegantly if discrete Pontryagin and Hamilton functions are introduced. The former is given as

\[ P(x, y, u) = g(x, u) + yf(x, u). \]

In terms of \( P \), equations (2.1), (2.8) and (2.13) take the form

\[ 0 = P_u, \quad x_t = P_y, \quad e^\rho y_{t-1} = P_x, \]  

(2.15)

here the argument of the derivatives of \( P \) is \((x_{t-1}, y_t, u_t)\).

It will be assumed that \( P_{uu} \) is negative definite. Then the equation \( P_u = 0 \) can be solved for \( u = U(x, y) \), which yields the discrete present value Hamilton function

\[ H(x, y) = P(x, y, U(x, y)) \]

\[ = g(x, U(x, y)) + yf(x, U(x, y)). \]

(2.16)

Note for later reference that, since \( P_u(x, y, U(x, y)) = 0 \) identically in \((x, y)\), we have

\[ g(x, U(x, y)) = H(x, y) - yH_y(x, y). \]

(2.17)

The necessary equations can be written in the (present-value) Hamiltonian form

\[ x_t = H_y(x_{t-1}, y_t) \quad \text{and} \quad e^\rho y_{t-1} = H_x(x_{t-1}, y_t). \]

(2.18)

By extension, the pair \((x, y)\) is called extremal if equation (2.18) is satisfied for every \( t \geq 1 \). Note that if \((x, y)\) is extremal, and if a control sequence \( u \) is obtained by setting \( u_t = U(x_{t-1}, y_t) \) for \( t \geq 1 \), then the pair \((x, u)\) is extremal in the former sense.
2.1.4 The phase map. The next step is to solve the present-value Hamiltonian equations (2.18) to obtain a phase map $\varphi$ that satisfies

$$(x_t, y_t) = \varphi(x_{t-1}, y_{t-1})$$

for every $t \geq 1$. Sometimes, the coordinate representation of this relation is used:

$$(x_t, y_t) = (\varphi_1(x_{t-1}, y_{t-1}), \varphi_2(x_{t-1}, y_{t-1})).$$

The domain of definition of $\varphi$ is $T^* (\mathcal{X}) = \mathcal{X} \times \mathbb{R}^n$, called the phase space, to distinguish it from the state space $\mathcal{X}$. The notation indicates that $T^* (\mathcal{X})$ is, mathematically speaking, the cotangent bundle of $\mathcal{X}$.

Let $F : T^* (\mathcal{X}) \times T^* (\mathcal{X}) \to T^* (\mathcal{X})$ be given as

$$F(z, \varphi) = \begin{pmatrix} \varphi_1 - H_y(x, \varphi_2) \\ e^\rho y - H_x(x, \varphi_2) \end{pmatrix},$$

where $z = (x, y) \in T^* (\mathcal{X})$. Let the map $\varphi$ be implicitly defined by the equation

$$F(z, \varphi) = 0.$$

**Proposition 2.1.5.** If $H_{xy}$ is invertible, then the equation $F = 0$ can be solved for $\varphi = \varphi(z)$. Moreover,

$$D\varphi = \begin{pmatrix} H_{xy} - H_{yy} H_{xy}^{-1} H_{xx} & e^\rho H_{yy} H_{xy}^{-1} \\ -H_{xy}^{-1} H_{xx} & e^\rho H_{xy}^{-1} \end{pmatrix}$$

and

$$\det D\varphi = e^{\rho \rho}. \quad (2.19)$$

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Proof. Compute

\[ D_\varphi F = \begin{pmatrix} I & -H_{yy} \\ 0 & -H_{xy} \end{pmatrix}. \]

Under the assumption of the lemma, this matrix is invertible at \((z_0, \varphi_0)\), and

\[ (D_\varphi F)^{-1} = \begin{pmatrix} I & -H_{yy}H_{xy}^{-1} \\ 0 & -H_{xy}^{-1} \end{pmatrix}. \]

By the implicit function theorem, the solution \(\varphi = \varphi(z)\) of \(F(z, \varphi) = 0\) satisfies

\[
D\varphi = - (D_\varphi F)^{-1} D_z F \\
= \begin{pmatrix} I & -H_{yy}H_{xy}^{-1} \\ 0 & -H_{xy}^{-1} \end{pmatrix} \begin{pmatrix} H_{xy} & 0 \\ H_{xx} & e^\rho I \end{pmatrix} \\
= \begin{pmatrix} H_{xy} - H_{yy}H_{xy}^{-1}H_{xx} & e^\rho H_{yy}H_{xy}^{-1} \\ -H_{xy}^{-1}H_{xx} & e^\rho H_{xy}^{-1} \end{pmatrix}.
\]

Moreover

\[
\det D\varphi = \det \begin{pmatrix} I & -H_{yy}H_{xy}^{-1} \\ 0 & -H_{xy}^{-1} \end{pmatrix} \det \begin{pmatrix} H_{xy} & 0 \\ H_{xx} & e^\rho I \end{pmatrix} = e^{\rho \rho}.
\]

Summarising, a map \(\varphi\) has been found such that the orbits \(z = \{z_t\} = \{(x_t, y_t)\}\) of \(\varphi\) are extremal and such that \(\varphi\) satisfies

\[
\varphi_1(x, y) = H_y(x, \varphi_2(x, y)) \quad \text{and} \quad e^\rho y = H_x(x, \varphi_2(x, y)). \tag{2.20}
\]

2.1.5 Comparison with continuous time case. Note that the properties of the phase map have well-known analogues in continuous time. They are sketched briefly here.
The continuous time problem asks to maximise a functional

\[ J = \int_{0}^{\infty} g(x, u) e^{-\rho t} dt \]

under the condition that

\[ \dot{x} = f(x, u). \]

The *continuous time present-value Pontryagin function* takes the form

\[ P(x, y, u) = g(x, u) + y f(x, u). \]

An interior optimising orbit satisfies necessarily (cf. equation (2.15))

\[ P_u = 0, \quad \dot{x} = P_y, \quad \rho y - \dot{y} = -\left. \frac{d}{dt} (e^{-\rho t} y) \right|_{t=0} = P_x. \]

If \( P_{uu} < 0 \) everywhere, \( u = U(x, y) \) can be solved from \( P_u = 0 \), yielding the *continuous time present-value Hamilton function*

\[ H(x, y) = P(x, y, U(x, y)). \]

In terms of \( H \), the necessary conditions read as (cf. (2.18))

\[ \dot{x} = H_y \quad \text{and} \quad \rho y - \dot{y} = H_x. \]

If \( X(x, y) = (H_y, \rho y - H_x) \) denotes the vector field defined by these equations,

\[ \text{div} \ X = n\rho. \]

If moreover \( \Phi_t = e^{tX} \) is the phase map defined by the vector field \( X \), then

\[ \det D\Phi_t = e^{t \text{div} X} = e^{n\rho}. \]

This equality can be verified easily by differentiation with respect to \( t \). In particular (cf. equation (2.19))

\[ \det D\Phi_1 = e^{n\rho}. \]
2.2 Local and associated value functions

In the previous section, the necessary first order conditions of the boundary value problem (2.13) were formulated in terms of the phase map $\varphi$: if $(x, y)$ is maximising and $\delta$-interior for $\delta > 0$, then

$$(x_t, y_t) = \varphi(x_{t-1}, y_{t-1}),$$

$$x_0 = \alpha \quad \text{and} \quad \lim_{t \to \infty} e^{-\rho t} y_t = 0.$$  \hspace{1cm} (2.21)

If all optimal state trajectories $x$ starting in some open sub-region of $\mathcal{X}$ converge to some steady state, this state corresponds to a fixed point of the phase map $\varphi$; since

$$\det D\varphi = e^{n\rho} > 1,$$

such a fixed point is necessarily a saddle. In proper coordinates around such a saddle $\bar{z}$, the linear map $D\varphi$ takes the form

$$D\varphi(\bar{z}) = \begin{pmatrix} \Lambda_s & 0 \\ 0 & \Lambda_u \end{pmatrix},$$

where all eigenvalues of $\Lambda_s$ are inside the unit circle, while all eigenvalues of $\Lambda_u$ are outside. The union of phase orbits converging to $\bar{z}$ forms a manifold, the so-called stable manifold $W^s$ of the fixed point, and orbits on stable manifolds of saddles of $\varphi$ thus solve the boundary value problem (2.21).

The stable linear eigenspace $E^s$ of $\bar{z}$ is tangent to $W^s$. The natural projection $\pi : E^s \to \mathcal{X}$ is surjective, which implies that locally the stable manifold can be represented as the graph of a function $\psi$.

In the following two kinds of value functions are introduced: the orbit value function $v$ that associates a value to points on $W^s$ and the value function $\bar{V}$ associated to the stable manifold $W^s$. If the stable manifold of $\bar{z}$ can be
represented as the graph of a function \( y = \psi(x) \), then the value function \( \bar{V} \) of the trajectories in the stable manifold \( W^s \) turns out to satisfy \( \partial \bar{V} / \partial x_i = \psi_i \) for all \( i \), at least locally around \( \bar{z} \). To recover the value function from \( \psi \) requires an integration. The components of \( \psi \) have therefore to satisfy an integrability condition. To formulate this condition, and to demonstrate that it is satisfied, differential forms are introduced.

These forms will be also used in the proof of some propositions that compare values at different points on a stable manifold which in turn are used to construct the global value function \( V \) of the problem.

2.2.1 Integrability and symplectic forms. As mentioned already, a value function \( \bar{V} \) will be associated to the stable manifold \( W^s \); for this, some concepts have to be introduced. On \( T^*(\mathcal{X}) \), the canonical 1-form \( \eta = y \, dx = \sum y_i \, dx_i \) is defined, as well as its derivative \( \omega = d\eta = dy \wedge dx \), the symplectic 2-form.

The symplectic form \( \omega \) is said to vanish on a submanifold \( \mathcal{M} \) of \( T^*(\mathcal{X}) \), if for any point \( z \in \mathcal{M} \) and any tangent vectors \( v_1, v_2 \in T_z(\mathcal{M}) \) to \( \mathcal{M} \) at \( z \) the equality \( \omega_z(v_1, v_2) = 0 \) holds. A \( n \)-dimensional submanifold \( \mathcal{M} \) of \( T^*(\mathcal{X}) \) is called Lagrangian if \( \omega \) vanishes on \( \mathcal{M} \). Being Lagrangian is an integrability condition: to see this assume that \( \mathcal{M} \) can be represented as the graph \( y = Y(x) \) of a function \( Y : \mathcal{N} \subset \mathcal{X} \to \mathbb{R}^n \).

Recall that since \( \mathcal{X} \) is convex, it is topologically trivial. Hence there is a function \( w : \mathcal{X} \to \mathbb{R} \) that satisfies

\[
\frac{\partial w}{\partial x_i} = Y_i
\]
for $i = 1, 2, \cdots, n$ if and only if the integrability conditions
\[
\frac{\partial Y_j}{\partial x_i} - \frac{\partial Y_i}{\partial x_j} = 0
\]
are satisfied for all $i, j$.

Being Lagrangian expresses the same thing. To see this, let $\Psi : \mathcal{N} \subset \mathcal{X} \to T^*(\mathcal{X})$ be given by $\Psi(x) = (x, Y(x))$. Set $\mathcal{M} = \Psi(\mathcal{N})$. The manifold $\mathcal{M}$ is Lagrangian if and only if $\Psi^* \omega = 0$. Compute
\[
0 = \Psi^* \omega = \sum_i dY_i(x) \wedge dx_i = \sum_i \sum_j \frac{\partial Y_i}{\partial x_j} dx_j \wedge dx_i = \sum_i \sum_{j<i} \left( \frac{\partial Y_j}{\partial x_i} - \frac{\partial Y_i}{\partial x_j} \right) dx_i \wedge dx_j;
\]
The classical integrability conditions have been recovered.

The phase map preserves the symplectic form up to a constant factor. Of course, the presence of this factor is an echo of the fact the optimisation problem is formulated in current value variables. Let now $\varphi$ be the phase map defined in (2.20) by
\[
\varphi_1 = H_y(x, \varphi_2(x, y)) \quad \text{and} \quad e^\rho y = H_x(x, \varphi_2(x, y)).
\]

**Proposition 2.2.1.** The equality $\varphi^* \omega = e^\rho \omega$ holds. Moreover, if $\psi$ satisfies $\varphi = e^{\rho/2} \psi$, then $\psi^* \omega = \omega$.

**Proof.** Using that $\varphi_1 = H_y(x, \varphi_2)$ (i.e. equation (2.20)), compute
\[
\varphi^* \omega = d\varphi_2 \wedge d\varphi_1 = d\varphi_2 \wedge (H_{xy} dx + H_{yy} d\varphi_2) = H_{xy} d\varphi_2 \wedge dx.
\]
Analogously, using $e^\rho y = H_x(x, \varphi_2)$, it follows that

$$e^\rho dy \wedge dx = dH_x(x, \varphi_2) \wedge dx = (H_{xx} dx + H_{xy} d\varphi_2) \wedge dx = H_{xy} d\varphi_2 \wedge dx.$$  

The proof for $\psi$ runs similarly, using equation (2.20) in the form

$$\psi_1 = e^{-\rho/2}H_y(x, e^{\rho/2}\psi_2) \quad \text{and} \quad e^\rho y = H_x(x, e^{\rho/2}\psi_2).$$

This proves the proposition. ■

Note in particular that if $\omega_z = 0$, then $\omega = 0$ along the orbit of $\varphi$ through $z$.

**Definition 2.2.1. (Symplectic transformation)** A differential map $\psi$ that preserves the 2-form $\omega$, that is, which is such that $\psi^* \omega = \omega$, is called symplectic.

The fact that $\psi$ is symplectic has implications for the spectrum of the Jacobian matrix $D\psi$.

**Proposition 2.2.2.** (1) If $\psi$ is symplectic, and if $\lambda$ is an eigenvalue of $D\psi$, so is $1/\lambda$.

(2) If $\varphi$ satisfies $\varphi^* \omega = e^\rho \omega$ and if $\lambda$ is an eigenvalue of the phase map $D\varphi$, then so is $e^\rho / \lambda$.

**Proof.** See Abraham and Marsden (1978), Proposition 3.1.12, p. 168. ■

**2.2.2 Invariant manifolds.** Let $\bar{z} = (\bar{x}, \bar{y})$ be a saddle fixed point of $\varphi$.

The linear stable and unstable manifolds are the eigenspaces $E^s$ and $E^u$ associated to the eigenvalues that are respectively lesser and greater than
one in absolute value. The \textit{stable manifold} \( W^s \) and the \textit{unstable manifold} \( W^u \) of \( \bar{z} \) are defined as the set of all points \( z \in T^s(\mathcal{X}) \) such that respectively the forward orbit or the backward orbit of \( \varphi \) through \( z \) tends to \( \bar{z} \):

\[
W^s = \{ z \in \mathcal{X} \times \mathbb{R}^n | \varphi^t(z) \to \bar{z}, \ t \to +\infty \},
\]

\[
W^u = \{ z \in \mathcal{X} \times \mathbb{R}^n | \varphi^t(z) \to \bar{z}, \ t \to -\infty \}.
\]

The basic result about the sets \( W^s \) and \( W^u \) is the invariant manifold theorem (Hirsch et al., 1977), which states that \( W^s \) and \( W^u \) are smooth manifolds, thus justifying the names.

\textbf{Invariant Manifold Theorem.} \textit{Let} \( \varphi : T^s(\mathcal{X}) \to T^s(\mathcal{X}) \) \textit{be a} \( C^k \) \textit{invertible map,} \( k \geq 1 \), \textit{and let} \( \bar{z} \) \textit{be a saddle fixed point. Then the sets} \( W^s \) \textit{and} \( W^u \) \textit{are both} \( C^k \)-\textit{smooth manifolds, tangent to the corresponding eigenspaces.}

In the following it is shown that the invariant manifolds of a saddle point \( \bar{z} \) are Lagrangian.

\textbf{Proposition 2.2.3.} \textit{Let} \( \bar{z} \) \textit{be a saddle fixed point of the phase map} \( \varphi \), \textit{and let} \( W^s \) \textit{and} \( W^u \) \textit{be the associated stable and unstable manifolds. Assume that both} \( W^s \) \textit{and} \( W^u \) \textit{are} \( n \)-\textit{dimensional. Then the symplectic form} \( \omega \) \textit{vanishes on} \( W^s \) \textit{and} \( W^u \).

\textbf{Proof.} \textit{Assume that the theorem does not hold; that is, there are vectors} \( v_1, v_2 \), \textit{tangent to} \( W^s \) \textit{at some point} \( z \in W^s \) \textit{such that} \( |z - \bar{z}| \leq \varepsilon \), \textit{for which} \( \omega(v_1, v_2) \neq 0 \); \textit{after rescaling it may be assumed that} \( \omega(v_1, v_2) = 1 \). \textit{Denote, as above, the restriction of} \( D\varphi \) \textit{to the stable eigenspace} \( E^s \) \textit{by} \( \Lambda_s \), \textit{and let} \( |\Lambda_s| = \lambda_s < 1 \), \textit{where} \( |\Lambda_s| \) \textit{is the matrix norm associated to the Euclidean vector norm} \( |\cdot| \).
Note that for \( v \) tangent to \( W^s \) at \( z \),

\[ |D\varphi^t(z)v_1| < c^t|v_1| \]

for some \( \lambda_s < c < 1 \) and for all \( t \). Consequently

\[ 1 \leq e^{\rho t} = e^{\rho t} \omega(v_1, v_2) = (\varphi^t)^* \omega(v_1, v_2) = \omega(D\varphi^t v_1, D\varphi^t v_2) \leq c^{2t}|v_1||v_2|. \]

But for \( t > 0 \) sufficiently large, this entails a contradiction.

If \( W^s \) is \( n \)-dimensional, there are \( n \) eigenvalues \( \lambda_i \) of \( D\varphi(z) \) such that \( |\lambda_i| < 1 \), \( i = 1, \ldots, n \). Proposition 2.2.2 implies that the other \( n \) eigenvalues then

\[ |\lambda_{n+i}| > e^\rho, \ i = 1, \ldots, n. \]

It follows that \( |\Lambda_u| = \lambda_u > e^\rho \), and that for \( v_1 \) tangent to \( W^u \) at \( z \), the inequality

\[ |D\varphi^{-t}(z)v_1| < c^t|v_1| \]

holds for some \( \lambda_u^{-1} < c < e^{-\rho} \). It follows that for all \( t \)

\[ 1 = \omega(v_1, v_2) = e^{\rho t} (\varphi^{-t})^* \omega(v_1, v_2) = e^{\rho t} \omega(D\varphi^{-t} v_1, D\varphi^{-t} v_2) \leq e^{\rho t} c^{2t}|v_1||v_2| \leq c^t|v_1||v_2|. \]

This also leads to a contradiction. \( \blacksquare \)

Introduce the function \( G : T^*(X) \to \mathbb{R} \) by setting

\[ G(x, y) = g(x, U(x, \varphi_2(x, y))). \]

Note that with this definition

\[ G(z_t) = G(x_t, y_t) = g(x_{t-1}, U(x_{t-1}, y_t)) = g(x_{t-1}, u_t). \]

Let \( y \) be the sequence of costates associated to the extremal pair \( (x, u) \).
Set \( z_t = (x_t, y_t) \) for all \( t \).
Consider a point \( \alpha = (\alpha_1, \alpha_2) \in W^s \) such that in a neighbourhood of \( \alpha \) the manifold \( W^s \) can be represented as the graph of a function \( y : N \subset X \rightarrow T^*(X) \). Then by Proposition 2.2.3 there is a function \( w : N \rightarrow \mathbb{R} \) such that \( dw = y \, dx \). Define \( \tilde{V} : N \rightarrow \mathbb{R} \) as

\[
\tilde{V}(\alpha_1) = \tilde{J}(z) = \sum_{t=1}^{\infty} G(z_{t-1}) e^{-\rho t} = \sum_{t=1}^{\infty} g(x_{t-1}, u_t) e^{-\rho t} = J(x, u), 
\]

(2.22)

where \( z \) is the \( \varphi \)-orbit that originates in \( (x_0, y_0) = (\alpha_1, \alpha_2) \). The next thing to demonstrate is that up to a constant the function \( w(x) \) is actually equal to the value \( \tilde{V}(\alpha_1) \) for orbits of \( \varphi \) starting at \( (\alpha_1, y(\alpha_1)) \). For this it is sufficient to show that the value function \( \tilde{V} \) for orbits on \( W^s \) is differentiable and satisfies

\[
d\tilde{V} = y \, dx.
\]

To formulate this more precisely, choose a smooth parametrisation

\[
z : \mathbb{R}^n \rightarrow T^*(X)
\]

of the stable manifold \( W^s \) of \( \bar{z} \). Write

\[
z(\sigma) = (x(\sigma), y(\sigma))
\]

and assume that \( z(0) = \bar{z} \). Let

\[
\sigma_{t+1} = \psi(\sigma_t)
\]

be the smooth map induced by \( \varphi \) on \( \mathbb{R}^n \). That is, if \( z_t = (x_t, y_t) = (x(\sigma_t), y(\sigma_t)) \) is an orbit of \( \varphi \) on \( W^s \), then

\[
\varphi(z(\sigma_t)) = \varphi(x_t, y_t) = (x_{t+1}, y_{t+1}) = z(\sigma_{t+1}) = z(\psi(\sigma_t)).
\]

Let \( z(\sigma_0) = (x_0, y_0) \in W^s \). If \( dx/\, d\sigma(\sigma_0) \neq 0 \), then on a neighbourhood of \( x(\sigma) \) a function \( y = y(x) \) can be found such that \( y(\sigma) = y(x(\sigma)) \) for \( \sigma \) close to \( \sigma_0 \).
Proposition 2.2.4. If \( \frac{dx}{d\sigma(\sigma_0)} \neq 0 \), then \( d\tilde{V}(x) = ydx \) on a neighbourhood of \( x_0 \). Consequently \( \tilde{V} - w \) is constant.

Proof. Let \( z_0 + \varepsilon \zeta_0(\varepsilon) \) be an arbitrary curve of initial points in \( W^s \); let \( z + \varepsilon \zeta(\varepsilon) \) be the trajectories in \( W^s \) defined by these initial points, and let \( (x + \varepsilon \xi(\varepsilon), u + \varepsilon u(\varepsilon)) \) be the corresponding state-control trajectories.

Set

\[
j(\varepsilon) = J(x + \varepsilon \xi(\varepsilon), u + \varepsilon u(\varepsilon));
\]

then

\[
j(\varepsilon) = \tilde{V}(x_0 + \varepsilon \xi_0(\varepsilon)).
\]

It will be shown below that \( j \) is differentiable for all curves of initial points through \( z_0 \); then

\[
j'(0) = \frac{d\tilde{V}}{dx}(x_0)\xi_0(0).
\]

But it follows from Proposition 2.1.3 that

\[
j'(0) = y_0 \xi_0(0).
\]

Since \( \xi_0(0) \) is arbitrary, the theorem follows.

It remains therefore to show that \( j \) is differentiable at \( \varepsilon = 0 \); this will follow from Proposition 2.1.3.

If \( \Lambda_s \) denotes the stable part of \( D\varphi(\bar{z}) \), then it is possible to choose the parametrisering coordinate \( \sigma \) of the stable manifold \( W^s \) such that

\[
\psi(\sigma) = \Lambda^s \sigma + O(|\sigma|^2),
\]

and

\[
D\psi(\sigma) = \Lambda^s + O(|\sigma|).
\]
Define 
\[ u(\sigma) = U(x(\psi^{-1}(\sigma)), y(\sigma)) \]
and note that with this definition \( u(\sigma_t) = U(x_{t-1}, y_t) = u_t \).

Let \( z = \{z_t\} \) be the orbit in \( W^s \) starting at \( z_0 \), and let \( \sigma = \{\sigma_t\} \) be its associated orbit of parameters \( z_t = z(\sigma_t) \). Let moreover \( c_0 : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n \) be a smooth curve of the form
\[ c_0(\varepsilon) = \sigma_0 + \varepsilon \tau_0, \]
with \( \tau_0 \in \mathbb{R}^n \). Consider the forward iterates \( c_t = \psi^t(c_0) \), parametrised as
\[ c_t(\varepsilon) = \sigma_t + \varepsilon \tau_t(\varepsilon), \]
where \( \sigma_t = \psi^t(\sigma_0) \). Note that for all \( t \geq 1 \)
\[ \psi(\sigma_{t-1}) + \varepsilon \tau_t(\varepsilon) = \sigma_t + \varepsilon \tau_t(\varepsilon) = \psi(\sigma_{t-1} + \varepsilon \tau_{t-1}(\varepsilon)) \]
and hence that
\[ \tau_t(\varepsilon) = D\psi(\sigma_{t-1})\tau_{t-1}(\varepsilon) + \varepsilon \Psi_1(\varepsilon, \sigma_{t-1}, \tau_{t-1}) \]
\[ = \Lambda^s \tau_{t-1}(\varepsilon) + \varepsilon \Psi_1(\varepsilon, \sigma_{t-1}, \tau_{t-1}) + \Psi_2(\sigma_{t-1}, \tau_{t-1}), \]
where \( |\Psi_1(\varepsilon, \sigma, \tau)| \leq C|\tau|^2 \) and \( |\Psi_2(\sigma, \tau)| \leq C|\sigma||\tau| \). Choosing \( T > 0 \) sufficiently large and \( \varepsilon_0 > 0 \) sufficiently small, this implies for \( t > T \) and \( 0 < \varepsilon < \varepsilon_0 \) that
\[ |\tau_t(\varepsilon)| \leq (\lambda^s + \delta)|\tau_{t-1}(\varepsilon)|, \]
where \( 0 < \lambda^s + \delta < 1 \). As a consequence
\[ |\tau_t(\varepsilon)| \to 0, \]
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uniformly in $\varepsilon$, as $t \to \infty$.

It follows that the curves $z(c_t)$ in $W^s$ take the form

$$z(c_t) = z(\sigma_t + \varepsilon \tau_t(\varepsilon)) = \sigma_t + \varepsilon \zeta_t(\varepsilon) = (x_t + \varepsilon \xi_t(\varepsilon), y_t + \varepsilon \eta_t(\varepsilon)),$$

and $|\zeta_t(\varepsilon)| \to 0$ uniformly in $\varepsilon$. Using the control map $U$, it is found that the associated control sequence is also of the form $u_t + \varepsilon \upsilon_t(\varepsilon)$ with $\upsilon_t(\varepsilon) \to 0$ uniformly in $\varepsilon$. As a consequence, the family $(x + \varepsilon \xi(\varepsilon), u + \varepsilon \upsilon(\varepsilon))$ is extremal and strongly admissible. Proposition 2.1.3 can be applied to yield differentiability of $j$ at $\varepsilon = 0$. □

Consider an arbitrary point $\alpha = (\alpha_1, \alpha_2) \in W^s$. A value $v(\alpha)$ is associated to $\alpha \in W^s$ by evaluating the objective functional for the point $\alpha$:

$$v(\alpha) = \bar{J}(z) = \sum_{t=1}^{\infty} G(z_{t-1}) e^{-\rho t} = \sum_{t=1}^{\infty} g(x_{t-1}, u_t) e^{-\rho t} = \bar{J}(x, u), \quad (2.23)$$

for $\varphi$-orbits on $W^s$ originating in $\alpha$. The function $v : W^s \to \mathbb{R}$ is called the orbit value function.

Now the value function $\bar{V} : \mathcal{X} \to \mathbb{R}$ associated to the stable manifold $W^s$ of $\tilde{z}$ can be defined by setting

$$\bar{V}(x) = \sup \left\{ v(\alpha) | \alpha_1 = x \right\} \quad (2.24)$$

(recall that $\sup \emptyset = -\infty$). Note that locally around $\bar{x}$, $\bar{V}(x) = \tilde{V}(x)$. Therefore, by Proposition 2.2.4

$$d\tilde{V} = y \, dx$$

on a neighbourhood of $\bar{x}$. 32
2.3 Value comparison theorems

In this section results are stated that are used to compare values at different points on a stable manifold. These results depend on Stokes’ theorem, which relates the integral of a form over the boundary of region to the integral of the derived form over the region itself.

2.3.1 Regions and area. Here, the concepts of simple regions and oriented simple regions are introduced. Regions will be the domains of the integral $\Omega(A) = \int_A \omega$, which is used extensively in the following.

A simple region in $\mathbb{R}^2$ is any simply connected bounded submanifold $A$ of $\mathbb{R}^2$ which is such that its boundary $\partial A$ is a closed piecewise smooth curve. An oriented simple region is a pair $(A, \mu)$ where $A$ is a simple region and where $\mu = \pm 1$ is the orientation; the orientation is often not given explicitly. The oriented simple region $-A$ is defined by:

$$(-A, \mu) = (A, -\mu).$$

A region $A$ in $\mathbb{R}^2$ is a formal sum of oriented simple regions $A_i$:

$$A = A_1 + A_2 + \cdots + A_k.$$ 

Moreover, set

$$A - A = 0;$$

that is, equal regions with opposite orientations cancel. It is evident how these concepts generalise to subsets of 2-dimensional oriented manifolds that are diffeomorphic to $\mathbb{R}^2$; these will be called surface regions, if there is a need to stress the difference.
If \((A, \mu)\) is an oriented simple region, and \(z\) the boundary curve of \(A\), then \(z\) is oriented \textit{consistently} with \(A\) if the winding number \(n_p(z)\) of \(z\) relative to any point \(p\) in the interior of \(A\) is \(\mu\); recall that if \(\vartheta_p\) is the angle of \(z - p\) with the positive horizontal axis, the winding number is defined as

\[
n_p(z) = \frac{1}{2\pi} \int_z d\vartheta_p.
\]

The boundary curve \(\partial A\) of an oriented region is always chosen consistently with the orientation of \(A\). Inversely, to any closed curve \(z\) without self-intersection, an oriented region \(A\) is associated such that \(\partial A = z\). More generally, if \(z\) is a piecewise smooth curve with a finite number of self-intersections then it divides the plane in a finite number of bounded regions \(A_i\) that are such that the boundary of a \(A_i\) is made up of segments of the curve \(z\). Again, the orientation of \(A_i\) is chosen consistently with that of its boundary arcs. Consider \(A = A_1 + ... + A_k\) and let \(I \subset \{1, 2, ..., k\}\) be the index set of positively oriented simple regions \(A_i\); that is, if \(i \in I\), then the orientation of \(A_i\) is positive, otherwise it is negative. Define then the positively and negatively oriented parts of \(A\) by respectively

\[
A^+ = \sum_{i \in I} A_i, \quad \text{and} \quad A^- = \sum_{i \not\in I} A_i.
\]

It is clear how these definitions extend to surface regions.

If \((A, \mu)\) is an oriented simple region, the area of \(A\) is given as

\[
\text{area} (A) = \mu \int_A dx \wedge dy = -\mu \int_A \omega.
\]

Introduce for simple regions the map \(\Omega\) as

\[
\Omega(A) = \int_A \omega = -\mu \text{area} (A).
\]
If the $A_i$, $i = 1, 2, \cdots$, are simple regions and $A = \sum A_i$, then define

$$\Omega(A) = \sum_i \Omega(A_i).$$

The regions $A_i$ are called the simple components of the region $A$. In particular

$$\Omega(A) = \sum_i -\mu_i \text{area}(A_i).$$

Recall Stokes’ theorem:

**Stokes’ theorem**

*If $\beta$ is an $(n-1)$-form with compact support on $\Omega$ and $\partial \Omega$ denotes the boundary of $\Omega$ with its induced orientation, then*

$$\int_{\Omega} d\beta = \int_{\partial \Omega} \beta. \tag{2.25}$$

*For instance, as $\eta = y \, dx$ implies $d\eta = \omega$, Stokes’ theorem yields, for a simple region $A$, that*

$$\text{area}(A) = -\mu \int_{A} \omega = -\mu \int_{\partial A} \eta.$$  

**2.3.2 The area rule.** In this subsection a result is derived that links the location of discontinuities of the derivative of the value function to the geometry of the manifold $W^s$.

First some more geometrical facts are recalled. Assume that a saddle fixed point $\bar{z}$, a point $\alpha \in W^s$ and a curve $z$ with $z = (x, y) : [0, 1] \rightarrow W^s \subset T^*(\mathcal{X})$ on the stable manifold $W^s$ of $\bar{z}$ are given such that $z(0) = \bar{z}$ and $z(1) = \alpha$.

As seen before, the value $v(\alpha)$ is associated to $\alpha$ by evaluating the objective functional for the phase trajectory starting at $\alpha$ (cf. (2.23)). At the fixed point the value $v(\bar{z})$ can be computed. It is given as

$$v(\bar{z}) = \frac{g(\bar{x}, U(\bar{x}, \bar{y}))}{e^{\nu} - 1}.$$
By Proposition 2.2.4,

\[ v(\alpha) = v(\bar{z}) + \int_0^1 y(s)x'(s) \, ds. \] (2.26)

Note that \( v \) is only defined for points on \( W^s \). Moreover, \( v(\alpha) \) does not depend on the curve \( z \); if \( z_1 \) and \( z_2 \) are two curves on \( W^s \) that connect \( \bar{z} \) to \( \alpha \), then \( z_1 - z_2 \) is a closed curve that encloses an oriented surface region \( A \subset W^s \).

By Stokes’ theorem,

\[ \int_{z_1} y \, dx - \int_{z_2} y \, dx = \int_{z_1 - z_2} y \, dx = \int_A dy \wedge dx = \int_A \omega = 0, \]

since \( W^s \) is Lagrangian.

**Proposition 2.3.1.** Consider a curve \( z_1 : [0, 1] \to W^s \subset T^*(\mathcal{X}) \) without self-intersections, connecting two distinct phase points \( \alpha = z_1(0) \) and \( \beta = z_1(1) \) that have equal state coordinates \( x_1(0) \), and satisfying

\[ x_1(\sigma) \neq x_1(0) = x_1(1) \quad \text{for all} \quad 0 < \sigma < 1. \]

Let moreover \( z_2 \) be the straight line joining \( \beta \) to \( \alpha \), and let \( z = z_1 + z_2 \). The curve \( z \) is closed without self-intersections; let \( A \) be an oriented surface region that is bounded by \( z \). Then

\[ v(\beta) - v(\alpha) = \int_{z_1} y \, dx = \int_z y \, dx = \int_A \omega. \]

Moreover, the value of \( \int_A \omega \) is independent of the choice of \( z_1 \) and \( A \).

**Proof.** Using Stokes’ theorem and \( \int_{z_2} y \, dx = 0 \) the first result follows. To show the second statement let \( z_1, \, \tilde{z}_1, \, A, \) and \( \tilde{A} \) be such that

\[ z_1 + z_2 = \partial A \quad \text{and} \quad \tilde{z}_1 + z_2 = \partial \tilde{A}. \]
Let $D \subset W^s$ be bounded by $z_1 - \tilde{z}_1$. Then there is a two dimensional manifold-with-boundary $S$ such that

$$A - D - \tilde{A} = \partial S.$$  

Using Stokes’s theorem and the fact that $d\omega = 0$

$$0 = \int_S d\omega = \int_A \omega - \int_D \omega - \int_{\tilde{A}} \omega.$$  

Since $D \subset W^s$ and $W^s$ is Lagrangian, it follows that $\int_D \omega = 0$, and consequently that

$$0 = \int_A \omega - \int_{\tilde{A}} \omega.$$  

The proposition is illustrated in Figure 2.3.2. In this figure let $z$ be the curve from $\alpha$ to $\beta$ along $W^s$ and from $\beta$ to $\alpha$ along the straight connecting line. If $z$ surrounds $A$ negatively, then

$$v(\beta) - v(\alpha) = \text{area}(A);$$  

if positively, then

$$v(\beta) - v(\alpha) = -\text{area}(A).$$  

2.3.3 The iterated area rule. In actual optimisation problems, the phase map $\varphi$ may not be a diffeomorphism; in particular, it may not be surjective everywhere. This has consequences for the stable manifold: there may be “holes” in it. If an area rule is applied to compare values of orbits, it has to be ascertained that the regions featuring in the rule are actually defined.
Figure 2.1: *The area rule.*
Proposition 2.2.1 states that the phase map $\varphi$ leaves the symplectic form $\omega = dy \wedge dx$ invariant up to a factor. A directly related result can be derived for the 1-form $\eta = y \, dx$.

Recall the function $G : T^*(X) \to \mathbb{R}$:

$$G(x, y) = g\left(x, U\left(x, \varphi_2(x, y)\right)\right).$$

Moreover, recall that with this definition

$$G(z_t) = G(x_t, y_t) = g(x_t, U(x_t, y_{t+1})) = g(x_t, u_{t+1}).$$

Let $y$ be the sequence of costates associated to the extremal pair $(x, u)$.

Set $z_t = (x_t, y_t)$ for all $t$. As seen before,

$$v(z_0) = J(x, u) = \sum_{t=1}^{\infty} g(x_{t-1}, u_t) e^{-\rho t} = \sum_{t=1}^{\infty} G(z_{t-1}) e^{-\rho t}.$$

Using equations (2.17) and (2.20) yields the relation

$$G(z) = G(x, y) = H(x, \varphi_2) - \varphi_1 \varphi_2. \quad (2.27)$$

**Proposition 2.3.2.** Let $z : [0, 1] \to T^*(X)$ be a $C^1$ curve in $T^*(X)$, joining $\alpha = z(0)$ to $\beta = z(1)$, and let $\varphi_\ast z = \varphi \circ z$ be its image under $\varphi$. Then

$$e^\rho \eta - \varphi_\ast \eta = dG. \quad (2.28)$$

**Proof.** This is a simple computation. Deriving equation (2.27) and using (2.20) yields

$$dG = dH - \varphi_1 \, d\varphi_2 - \varphi_2 \, d\varphi_1$$

$$= H_x \, dx + H_y \, d\varphi_2 - \varphi_1 \, d\varphi_2 - \varphi_2 \, d\varphi_1$$

$$= e^\rho y \, dx - \varphi_2 \, d\varphi_1.$$

Since $\eta = y \, dx$ and $\varphi_\ast \eta = \varphi_2 \, d\varphi_1$, this shows the result. \qed
If the curve $z$ is vertical, that is, if $dx = 0$ everywhere along the curve, then the form $\eta = y\,dx$ vanishes on $z$. For such curves proposition 2.3.2 yields the following corollary.

**Proposition 2.3.3.** Let $z: [0, 1] \to T^*(\mathcal{X})$ join $\alpha = z(0)$ to $\beta = z(1)$, and let $dx = 0$ along $z$. Then

$$-\int_{\varphi \circ z} y\,dx = G(\beta) - G(\alpha).$$

The iterated area rule, which is stated and proved next, is used to formulate a value comparison result with respect to a single fixed region.

Consider the following situation: $\alpha$ and $\beta$ are both points on the stable manifold $W^s$ with the same $x$-coordinate and with associated values $v(\alpha)$ and $v(\beta)$, but there is no curve in $W^s$ joining them. There is, however, a curve

$$\tilde{z}_1: [a, b] \to T^*(\mathcal{X})$$

in $W^s$ that joins $\tilde{\alpha} = \phi^T(\alpha) = \tilde{z}_1(a)$ to $\tilde{\beta} = \phi^T(\beta) = \tilde{z}_1(b)$; for this curve

$$\int_{\tilde{z}_1} \eta = v(\phi^T(\beta)) - v(\phi^T(\alpha)) \quad (2.29)$$

From the representations

$$v(\alpha) = \sum_{t=1}^{\infty} G(\varphi^{t-1}(\alpha)) e^{-\rho t}, \quad v(\beta) = \sum_{t=1}^{\infty} G(\varphi^{t-1}(\beta)) e^{-\rho t},$$

it follows easily

$$v(\beta) - v(\alpha) = \sum_{t=1}^{T} [G(\varphi^{t-1}(\beta)) - G(\varphi^{t-1}(\alpha))] e^{-\rho t}$$

$$+ e^{-\rho T} (v(\phi^T(\beta)) - v(\phi^T(\alpha))).$$

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Let now $z_2 : [a, b] \to T^*(\mathcal{X})$ be the vertical curve joining $\beta$ to $\alpha$, and let 
\[ \tilde{z}_2 = \varphi^T \circ z_2. \]

Applying Proposition 2.3.2 repeatedly, the following is obtained
\[
v(\beta) - v(\alpha) = G(\beta) - G(\alpha) + e^{-\rho} \left( G(\varphi(\beta)) - G(\varphi(\alpha)) \right) + \cdots \\
+ e^{-\rho T} \left( v(\varphi^T(\beta)) - v(\varphi^T(\alpha)) \right) \\
= \left( \int_z e^\rho \eta - \int_{\varphi^*z} \eta \right) + e^{-\rho} \left( \int_{\varphi^*z} e^\rho \eta - \int_{\varphi^*z} \eta \right) + \cdots \\
+ e^{-\rho T} \left( v(\varphi^T(\beta)) - v(\varphi^T(\alpha)) \right) \\
= \int_{z_2} \eta - e^{-\rho T} \int_{\varphi^*z_2} \eta + e^{-\rho T} \left( v(\varphi^T(\beta)) - v(\varphi^T(\alpha)) \right). 
\]

Using (2.29), as well as $\int_{z_2} \eta = 0$ and $\tilde{z}_2 = \varphi^*z_2$, leads to
\[ v(\beta) - v(\alpha) = e^{-\rho T} \int_{\tilde{z}_1 + \tilde{z}_2} \eta. \]

The curve $\tilde{z} = \tilde{z}_1 + \tilde{z}_2$ is closed; let $\tilde{A}$ be a surface region that is bounded by this closed curve. Then
\[ v(\beta) - v(\alpha) = \int_{\tilde{A}} \omega. \]

Note that by the same argument as used in Proposition 2.3.1, it can be shown that the value of $\int_{\tilde{A}} \omega$ is independent of the choice of $\tilde{A}$. This discussion is summarised in the following proposition.

**Proposition 2.3.4.** Let $\alpha, \beta \in W^*$ be points with the same $x$-coordinate, $\alpha$, $\beta$ the associated orbits of $\varphi$, and set $v(\alpha) = J(\alpha)$ and $v(\beta) = J(\beta)$. Assume that there is a curve $\tilde{z}_1$ on $W^*$ joining $\varphi^T(\alpha)$ and $\varphi^T(\beta)$. Let $z_2$ be a line connecting $\beta$ to $\alpha$, and set $\tilde{z}_2 = \varphi^T \circ z_2$. Let finally $\tilde{A}$ be a surface region such that $\partial \tilde{A} = \tilde{z} = \tilde{z}_1 + \tilde{z}_2$. Then
\[ v(\beta) - v(\alpha) = e^{-\rho T} \int_{\tilde{A}} \omega. \]
This is independent of \( \tilde{z}_1 \) and \( \tilde{A} \).

The proposition is illustrated in Figure 2.2.

**Figure 2.2:** The iterated area rule: as \( z_1 + z_2 \) is negatively oriented with respect to \( A \), then \( v(\beta) - v(\alpha) = \text{area}(A) \). Even if \( z_1 \) and hence \( A \) were not defined, still \( v(\beta) - v(\alpha) = e^{-\rho T} \text{area}(\tilde{A}) \).

### 2.3.4 Value differences.

Consider now the situation illustrated in Figure 2.3. There are two stable manifolds \( W^- \) and \( W^+ \), associated to the fixed points \( z^- \) and \( z^+ \), and two points \( \alpha \in W^- \) and \( \beta \in W^+ \), such that their \( x \)-coordinates are equal (see Figure 2.3). Let \( z \) be the line segment joining \( \alpha \) to \( \beta \) and let \( A \) be the oriented surface region bounded by the concatenation of \( z \), the part \( w_+ \) of \( W^+ \) joining \( \beta = \varphi(\beta) \), the negative of \( \varphi_*z \) joining \( \tilde{\beta} \) to \( \tilde{\alpha} = \varphi(\alpha) \), and the negative \( -w_- \) of the part of \( W^- \) joining \( \alpha \) to \( \tilde{\alpha} \). Define finally

\[
\Omega(A) = \int_A \omega.
\]
Figure 2.3: Relation between values and area: $v(\beta) - v(\alpha) = \text{area}(A)/(e^\rho - 1)$. The boundary of $A$ is the curve $\alpha \to \beta \to \tilde{\beta} \to \tilde{\alpha} \to \alpha$; it is negatively oriented, consequently the orientation of $A$ is negative as well and $\Omega(A) = \text{area}(A)$. 
Proposition 2.3.5. In the situation sketched above \( \Omega(A) \) is independent of the choice of the curves and

\[
v(\beta) - v(\alpha) = \frac{\Omega(A)}{e^\rho - 1}.
\]

Proof. Again by Stokes’ theorem:

\[
\Omega(A) = \int_A \omega = \int_{\partial A} \eta = \int_z \eta + \int_{c_+} \eta - \int_{\varphi, z} \eta - \int_{c_-} \eta = v(\tilde{\beta}) - v(\beta) - \int_{\tilde{z}} \eta - v(\tilde{\alpha}) + v(\alpha).
\]

Using Proposition 2.3.2 yields that

\[
-\int_{\tilde{z}} \eta = G(\beta) - G(\alpha).
\]

Moreover, using \( v(\alpha) = \sum_1^\infty G(z_i) e^{-\rho t} \) yields

\[
v(\tilde{\alpha}) = e^\rho v(\alpha) - G(\alpha).
\]

Eliminating with these relations the quantities \( \int_{\tilde{z}} y \, dx \) as well as \( v(\tilde{\alpha}) \) and \( v(\tilde{\beta}) \),

\[
\Omega(A) = (e^\rho - 1) \left( v(\beta) - v(\alpha) \right),
\]

as claimed in the proposition. Independence is shown by the same arguments as employed in the proof of Proposition 2.3.1. \( \blacksquare \)
Chapter 3

The indifference-attractor bifurcation

Systems where the phase map \( \varphi \) has a unique saddle fixed point are encountered in many places in the economic literature. Most of the results are about the degree of change of various quantities at that point if parameters are varied (comparative statics). In those systems the unique saddle corresponds to a steady state of the optimal state dynamics which is such that all solutions, regardless of the initial state of the system, tend to this state.

Whenever there is more than one saddle point present, say two, the solution structure is more complicated. It may be that both saddles correspond to an optimal steady state; in that case, Dechert and Nishimura (1983) have showed that there is an intermediate state such that all solutions starting left to this state tend to one steady state and all solutions starting right to it tend to the other. For the intermediate state, there are two possibilities:
either it is a long-term steady state itself, or it is an indifference state\(^1\), i.e. a state that is initial state to two different optimal trajectories.

In analogous continuous time problems, whether or not an indifference state occurs in a system depends on the relative position of the stable and unstable manifolds of the saddle equilibria of the phase flow (see Wagener, 2003; Kiseleva and Wagener, 2011). In particular, the shift from one type of solution to another is characterised by the occurrence of a ‘heteroclinic connection’, where the stable manifold of one saddle equilibrium coincides with the unstable manifold of another saddle. The situation in discrete time problems can be more complex due to the fact that, unlike in continuous time problems, stable and unstable manifolds do not automatically coincide once they have a single point in common. This situation will be analysed in the present chapter.

The chapter investigates indifference-attractor bifurcation of the discrete-time optimisation problems with a single state variable. First a class of optimisation problems that have indifference-attractor bifurcation singularities is defined and assumptions that describe the simplest configuration for which the results hold. Then it is shown that this bifurcation is linked to a heteroclinic bifurcation scenario of the phase map \(\varphi\). The consequences for the optimal solutions are analysed. In particular, the bifurcation value at which indifference thresholds appear is characterised by a geometric condition, which is stated in Section 3.2 together with the required mathematical concepts. Finally, in the last section, the proofs of the main results are given.

\(^1\)Often called Skiba, DNS or DNSS state in reference to the contributions of Skiba (1978), Dechert and Nishimura (1983) and Sethi (1977).
3.1 Heteroclinic orbits and indifference-attractor bifurcations

Here a class of optimisation problems that have indifference-attractor bifurcation singularities is defined. This class is characterised in terms of the phase map, which was introduced in the previous chapter. In particular, attention is restricted to the situation that the phase map is defined on a subset of $T^*\mathbb{R} \cong \mathbb{R}^2$. The main characteristic of the class is that the family of phase maps goes through a heteroclinic bifurcation scenario. Throughout the chapter, it will be found that this abstract mathematical condition has a number of powerful implications for the structure of the set of optimising trajectories.

Consider the problem of optimising (2.3) under the side condition (2.4), where $\mathcal{X}$ and $\mathcal{U}$ are open subsets of $\mathbb{R}$. Recall from Chapter 2 that the necessary first order conditions of this problem can be formulated in terms of the phase map $\varphi$, defined on the phase space $T^*\mathcal{X} = \mathcal{X} \times \mathbb{R}$, as follows: if $(x, y)$ maximises (2.3) subject to (2.4), and if $(x, y)$ is $\delta$-interior for $\delta > 0$, then necessarily

$$(x_t, y_t) = \varphi(x_{t-1}, y_{t-1}), \quad \text{for all} \quad t \geq 1,$$

$$x_0 = \alpha, \quad \text{and} \quad \lim_{t \to \infty} e^{-\rho t} y_t = 0. \quad (3.1)$$

3.1.1 Phase maps. In the following a string of assumptions is made. Their main function is to delineate the simplest configuration for which the results hold; all of them can be checked, at least numerically, for a given system. Moreover, they hold true for a large class of problems of practical
interest. The first assumption implies that the phase maps actually exist.

**Assumption 3.1.1.** The discrete Hamilton function \( H = H(x,y) \), which has been introduced in Section 2.1.3, satisfies \( H_{xy} > 0 \) and \( H_{yy} > 0 \).

The stronger assumption \( H_{xy} = f_x > 0 \) is needed to obtain the results of Theorems 3.3.1 and 3.3.2. This assumption requires \( f \) as a function of \( x \in \mathcal{X} \) to be an orientation-preserving diffeomorphism for each \( u \in \mathcal{U} \).

Phase maps originating from optimisation problems of the type given in this section have, like their continuous time counterparts, special geometrical properties: they are conformally symplectic maps; they are even symplectic in the case \( \rho = 0 \). Symplecticity is an abstract mathematical concept, related to integrability theory; some of its implications have been discussed in Section 2.2.

In the following some concepts from differential geometry are recalled on which the arguments in this chapter are based. For a fuller treatment of this material, especially of differential forms, the reader is referred to the excellent expositions of Spivak (1965) or Arnol’d (1989).

Let

\[
E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

In \( \mathbb{R}^2 \), the *standard symplectic 2-form* is the differential form \( \omega = dy \wedge dx \): on a pair \( v = (v_1, v_2), w = (w_1, w_2) \in \mathbb{R}^2 \), the form \( \omega \) takes the value

\[
\omega(v, w) = \langle v, Ew \rangle = v_2 w_1 - v_1 w_2.
\]

Let \( \psi : \mathbb{R}^2 \to \mathbb{R}^2 \) be of the form \( \psi = (\psi_1(x,y), \psi_2(x,y)) \) and let \( D\psi \) denote
the $2 \times 2$ Jacobi matrix

$$D\psi = \begin{pmatrix}
\frac{\partial \psi_1}{\partial x} & \frac{\partial \psi_1}{\partial y} \\
\frac{\partial \psi_2}{\partial x} & \frac{\partial \psi_2}{\partial y}
\end{pmatrix}.$$  

The pull-back $\psi^*\omega$ of $\omega$ under $\psi$ is defined as

$$\psi^*\omega(v, w) = \omega(D\psi v, D\psi w).$$ (3.2)

The map $\psi$ is called *symplectic*, if

$$\psi^*\omega = \omega.$$

It is called *conformally symplectic*, if there is a function $\lambda : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\psi^*\omega = \lambda \omega.$$

**Proposition 3.1.1.** Let $\mathcal{X} \subset \mathbb{R}$ be an open interval. Then the phase map $\varphi : T^*\mathcal{X} \to T^*\mathcal{X}$, given by (2.20), is conformally symplectic. More precisely, it satisfies

$$\varphi^*\omega = e^\rho \omega.$$  

This implies that

$$\det D\varphi = e^\rho.$$ (3.3)

The proposition is a corollary of the more general Propositions 2.1.5 and 2.2.2, which have been stated and proved in Section 2.

Note that $\det D\varphi = e^\rho$ implies that the map $\varphi$ multiplies phase volume by a factor $e^\rho$. It follows that there are no bounded regions that are invariant under $\varphi$; this implies for instance that $\varphi$ has no invariant circles.
3.1.2 Description of the context. In this subsection a number of assumptions are given (and discussed) which taken together form the definition of a family that has an indifference-attractor singularity. Consider therefore a family of phase maps $\varphi = \varphi_\mu : T^* \mathcal{X} \to T^* \mathcal{X}$ depending on a parameter $\mu \in \mathbb{R}$.

**Assumption 3.1.2.** For all values of the parameter $\mu$ the map $\varphi = \varphi_\mu$ has two saddle fixed points $z_- = (x_-, y_-)$ and $z_+ = (x_+, y_+)$, such that $x_- < x_+$, and such that there is no saddle point $\tilde{z} = (\tilde{x}, \tilde{y})$ of $\varphi_\mu$ with $x_- \leq \tilde{x} \leq x_+$.

As in the continuous time case, orbits on stable manifolds of saddles are candidates for optimal trajectories, as they satisfy the transversality condition automatically (see Subsection 2.1.5). The next assumption postulates that there is an open subinterval $I \subset \mathcal{X}$ with $x_-, x_+ \in I$ such that the optimisation problem has a solution for all initial states, and that orbits on the stable manifolds $W_s^-$ and $W_s^+$ of $z_-$ and $z_+$ respectively are the only candidates for the optimal orbits.

**Assumption 3.1.3.** There is an open interval $I$ such that $x_-, x_+ \in I$ and such that for every $x_0 \in \mathcal{X}$ with $x_- \leq x_0 \leq x_+$, the problem to optimise $J$ subject to equation (2.4) has a solution. Moreover, the state-costate trajectory $z$ of such a solution is either on $W_s^-$ or $W_s^+$.

As mentioned above, the genesis of indifference thresholds will turn out to be intimately connected with the occurrence of heteroclinic orbits in the system. A point $z$ is called heteroclinic to $z_-$ and $z_+$, or a heteroclinic intersection of $W_u^-$ and $W_u^+$, if $z \in W_u^- \cap W_u^+$. Note that if $z$ is heteroclinic, so is $\varphi(z)$ and in fact every iterate $\varphi^k(z)$. The orbit $O(z) = \{\varphi^k(z) \mid k \in \mathbb{Z}\}$ through a heteroclinic point $z$ is called a heteroclinic orbit.
A heteroclinic intersection $z$ is *transversal* if at $z$ the tangent vectors

$W_u^-$ and $W_s^+$ are linearly independent, for instance as in Figure 3.1 or

Figure 3.2(c). Invariant manifolds and their tangent spaces depend continuously on parameters: if for a given parameter value $\mu_0$ there is a transversal heteroclinic intersection, then this is the case for all values of $\mu$ sufficiently close to $\mu_0$. A non-transversal heteroclinic intersection is called a heteroclinic *tangency* (see Figures 3.2(b) and 3.2(d)).

![Figure 3.1: The stable manifold $W_+^s$ of $z_+$ (solid) and the unstable manifold $W_-^u$ of $z_-$ (dashed). The stable manifold $W_+^s$ is composed of all points that are forward asymptotic to $z_+$; likewise, $W_-^u$ is composed of all points backward asymptotic to $z_-$. A heteroclinic point is an intersection of $W_+^s$ and $W_-^u$, hence a point that is forward asymptotic to $z_+$ and backward asymptotic to $z_-$. As both manifolds contain infinitely many orbits, they do not necessarily coincide (unlike in the continuous time case).](image)

The family $\varphi_\mu$ is said to go through a heteroclinic bifurcation scenario, involving for instance $W_-^u$ and $W_+^s$, if there is a parameter interval $[\mu_1, \mu_2]$ such that for $\mu < \mu_1$ and $\mu > \mu_2$, the manifolds $W_-^u$ and $W_+^s$ have no points in common, and if for $\mu \in [\mu_1, \mu_2]$ there is at least one heteroclinic orbit.
Necessarily all heteroclinic orbits are tangencies at $\mu = \mu_1$ and $\mu = \mu_2$. Figure 3.2 illustrates the basic scenario. In general the scenario may be more complex, featuring also tangencies for intermediate values of $\mu$.

**Figure 3.2:** Relative position of $W_u^-$ (dashed) and $W_s^+$ at $z_-$, depending on the parameter $\mu$. At $\mu = \mu_1$ and $\mu = \mu_2$, $W_u^-$ and $W_s^+$ exhibit heteroclinic tangencies; for $\mu_1 < \mu < \mu_2$, the manifolds intersect transversally.

The family $\varphi_\mu$ of phase maps is assumed to go through a heteroclinic bifurcation scenario.

**Assumption 3.1.4.** If $\mu < \mu_1$ or $\mu > \mu_2$, then $W_s^+$ and $W_u^-$ have no points in common. On the other hand, if $\mu_1 \leq \mu \leq \mu_2$, then there are heteroclinic
intersections of $W_u^-$ and $W_s^+$. Moreover, if $\mu > \mu_2$, then $W_s^+$ does not intersect the line $x = x_+$, nor does $W_s^-$ intersect $x = x_-$. If $\mu < \mu_1$, then $W_s^+$ intersects the line $x = x_-$. Assumption 3.1.3 together with the second half of Assumption 3.1.4 implies that if $\mu > \mu_2$, then both $x_-$ and $x_+$ are locally optimal fixed points. For every initial state $x_0 \in I$, the state-costate trajectory $z$ of a solution to the optimisation problem necessarily lies either on $W_s^+$ or $W_s^-$. However, neither of these two stable manifolds cover the whole state space.

For a given heteroclinic intersection $z \in W_u^- \cap W_s^+$, let $c$ be the curve obtained by taking the part of $W_u^-$ that connects $z_-$ to $z$ and the part of $W_s^+$ that connects $z$ to $z_+$. If $c$ is a curve without self-intersections, then $z$ is called a primary heteroclinic intersection.

The next assumption postulates that the map $\varphi$ has some generic properties.

**Assumption 3.1.5.**

1. For $\mu = \mu_1$ and $\mu = \mu_2$, there is a single orbit of heteroclinic tangencies of $W_u^-$ and $W_s^+$.

2. There is a finite set $F \subset [\mu_1, \mu_2]$ such that for each $\mu \in [\mu_1, \mu_2] \setminus F$, the manifolds $W_u^-$ and $W_s^+$ have only transversal primary intersection points. If $\mu \in F$, there is one orbit of primary quadratic heteroclinic tangencies of $W_u^-$ and $W_s^+$, as well as at least two orbits of primary heteroclinic transversal intersections.

Remark that the conditions of the assumptions determine an open set of phase maps $\varphi$. Whether this set is also dense in some suitable function
topology is not immediately clear, due to the indirect definition of $\varphi$. This question is left to a future investigation and instead it is only conjectured that the conditions of Assumption 3.1.5 determine an open and dense set, with respect to the $C^\infty$ topology, of optimisation problems that satisfy Assumptions 3.1.2–3.1.4.

Note that without the restriction to primary intersection points, the conjecture might well be false, as there may be generically infinitely many values of $\mu$ for which there is a heteroclinic tangency (cf. Palis and Takens, 1993, Chapter 6).

The next assumption is required since the inverse of $\varphi$ is not necessarily defined in every part of the phase space $T^*\mathcal{X}$; consequently, the stable manifold need not be connected.

**Assumption 3.1.6.** For each $\mu \in (\mu_1, \mu_2)$ and every orbit $O$ of heteroclinic intersections of $W^s_+$ and $W^u_-$, there exists a point $z \in O$ as well as two smooth curves in $T^*\mathcal{X}$ that connect $z$ to $\varphi(z)$ along $W^u_-$ and $\varphi(z)$ to $z$ along $W^s_+$ respectively.

3.1.3 The main result and its interpretation. A state $\bar{x}$ is called an (optimal) steady state, if the optimal trajectory starting at $\bar{x}$ is given by $x_t = \bar{x}$ for all $t$. An optimal steady state $\bar{x}$ is *globally optimal*, if every optimal trajectory $\{x_t\}$ converges to $\bar{x}$; the steady state $\bar{x}$ is *locally optimal*, if for all initial states $x_0$ in a neighbourhood of $\bar{x}$, the optimal trajectory starting at $x_0$ converges to $\bar{x}$.

Now the main result can be stated.
Theorem 3.1.1. Let the Assumptions 3.1.2–3.1.6 be satisfied. There is a value
\[ \mu_1 < \mu_c < \mu_2 \] such that

1. If \( \mu < \mu_c \), the steady state \( x_+ \) is globally optimal;

2. if \( \mu > \mu_c \), both steady states \( x_- \) and \( x_+ \) are locally optimal, and there is a state \( x_- < x_s < x_+ \) such that \( x_s \) is initial state to two optimal solutions of which one converges to \( x_- \) and the other converges to \( x_+ \);

3. if \( \mu = \mu_c \), the steady state \( x_- \) is semi-stable: optimal solutions starting at \( x_0 \leq x_- \) tend to \( x_- \), whereas optimal solutions starting at \( x_0 > x_- \) tend to \( x_+ \);

4. moreover, if \( \mu = \mu_c \), there is an infinite sequence
\[ x_1^{(1)} > x_1^{(2)} > \cdots > x_- , \]
such that
\[ \lim_{k \to \infty} x_1^{(k)} = x_- \]
and such that each \( x_1^{(k)} \) is initial point to two optimal sequences, both converging to \( x_+ \).

The proof is a direct corollary of Theorems 3.2.1–3.2.3 below.

3.1.4 Optimal maps. Any optimal state-control trajectory \((x, u)\) corresponds one-to-one with a state-costate trajectory \(z\), which in turn is determined by its initial state \(z_0 = (x_0, y_0)\).
By the principle of optimality, if \( z = \{ z_t \}_{t=0}^\infty \) is an optimal state-costate trajectory with initial point \( z_0 \), and if \( n \) is a positive integer, then

\[
\sigma_n z \overset{\text{def}}{=} \{ z_{t+n} \}_{t=0}^\infty
\]

is also an optimal trajectory, but with initial state \( z_n \). Therefore,

\[
J(z) = \sum_{t=1}^n e^{-\rho t} g(x_{t-1}, U(x_{t-1}, y_t)) + e^{-n \rho} J(\sigma_n z),
\]

and it is clear that if \( \sigma_n z \) does not maximise \( J \) over the set of admissible trajectories starting at \( z_n \), then \( z \) does not maximise \( J \) over the set of admissible trajectories starting at \( z_0 \).

The set of optimal state-costate trajectories can be described by a set-valued map

\[
Y^o = Y^o(x) \subseteq \mathbb{R}.
\]

If \( z_0 = (x_0, y_0) \), with \( y_0 \in Y^o(x_0) \), then \( z_0 \) is the initial point of an optimal trajectory. The set-valued map \( Y^o \) is called the optimal costate map. In the present context, it follows from Assumption 3.1.3 that \( Y^o(x) \) is a set containing either one or two elements. Analogously, the optimal state map is defined

\[
\Psi^o(x) = \varphi_1(x, Y^o(x)) = \{ \varphi_1(x, y) \mid y \in Y^o(x) \}.
\]

This is also a set-valued map. The optimal dynamics is given as

\[
x_t \in \Psi^o(x_{t-1}).
\]

Finally, the map

\[
U^o(x) = U(x, Y^o(x)) = \{ u(x, y) \mid y \in Y^o(x) \}
\]
is the policy function. A point $x$ where $U^\alpha(x)$ consists of two elements is a jump point of the policy function.

The complexity of the optimal state dynamics is not as great as would appear at first sight. Indeed, if a state $\xi$ is the first iterate of an initial state $x_0$ under $\Psi^\alpha$, that is, if $\xi \in \Psi^\alpha(x_0)$ for some $x_0$, then $\Psi^\alpha(\xi)$ contains exactly one element; otherwise there would be two optimal state-costate orbits with initial point $(x_0, y_0)$, which contradicts the fact that the phase map $\varphi$ is well-defined (and hence single-valued).

If $\Psi^\alpha(x)$ contains only one element, $\psi^\alpha(x)$ is defined by setting

$$\Psi^\alpha(x) = \{\psi^\alpha(x)\}.$$ 

Note that an optimal steady state as defined above is just a fixed point of the map $\psi^\alpha$. If $\Psi^\alpha(x)$ contains two elements, the state $x$ is an indifference state, as there are two optimal state trajectories starting at $x$. If these two optimal trajectories have different $\omega$-limit sets, then $x$ is an indifference threshold.

The main theorem can be rephrased now in terms of the (parameter-dependent) optimal state dynamics $\Psi^\alpha_\mu$: if the assumptions are satisfied, and if $\mu < \mu_c$, then all orbits of the optimal dynamics tend to $x_+$, and $x_+$ is therefore the global attractor for the optimal dynamics. If $\mu > \mu_c$, there is an indifference threshold $x_s$; all orbits starting at $x_0 < x_s$ tend to $x_-$, whereas all orbits starting at a point $x_0 > x_s$ tend to $x_+$; both $x_-$ and $x_+$ are local attractors of the optimal dynamics. If $\mu = \mu_c$, then the orbit $x = x_-$ is semi-stable: all orbits starting to the left of it converge to $x_-$, while all orbits starting to the right converge to $x_+$. These facts are summarised by saying that at $\mu = \mu_c$, a locally stable attractor and an indifference
threshold of the optimal dynamics are generated through an *indifference-attractor bifurcation*. The last statement of the main theorem is that for $\mu = \mu_c$ the optimal dynamics has an infinity of indifference points that are not indifference thresholds.

### 3.2 Regions and orientations

In this section, three theorems are formulated that give conditions to determine whether the states $x_-$ and $x_+$ are both locally optimal, or whether $x_+$ is globally optimal. These conditions are formulated in terms of the oriented area of a certain region in phase space.

Recall from Subsection 2.3.1 that a region is a collection of oriented open and bounded sets that are simply connected and that have well-behaved boundaries. Assume that the parts of $W^u$ and $W^s_+$ that interact in the heteroclinic bifurcation are parametrised by arclength, starting from the respective fixed points. That is, the parametrisations $\gamma_u(s)$ and $\gamma_s(s)$ satisfy $\gamma_u(0) = z_-$ and $\gamma_s(0) = z_+$ as well as $\|\gamma_u'(s)\| = \|\gamma_s'(s)\| = 1$. Note that this determines an orientation of $W^u$ and $W^s_+$.

With respect to these parametrisations, a transversal heteroclinic intersection

\[ z = \gamma_u(s_1) = \gamma_s(s_2) \]

has intersection number $+1$ (cf. Hirsch, 1976), if

\[ \det \left( \begin{array}{cc} \gamma_u'(s_1) & \gamma_s'(s_2) \end{array} \right) > 0. \]

Intersections of intersection number $-1$ are defined analogously; see Figure 3.3. The intersection number of a quadratic heteroclinic tangency is set to be 0.
As $\det D\varphi > 0$, if $p_t$ is a transversal heteroclinic intersection with intersection number $+1$, then so is $\varphi(p_t)$. Therefore, the intersection number of a heteroclinic orbit $p$ is well-defined as the intersection number of any of its elements. Let $p = \{p_k\}_{k=-\infty}^{\infty}$ be a transversal heteroclinic intersection of $W_u^-$ and $W_s^+$ with intersection number $+1$. Heteroclinic orbits of this type will be called upward orbits.

If $p$ is an upward orbit, assume that $p_0$ is such that smooth curves $c^u$, $c^s$ as postulated in Assumption 3.1.6 exist, connecting $p_0$ to $p_{-1}$. Let $c$ be the closed curve obtained by first following $c^s$ from $p_0$ to $p_{-1}$ and then $c^u$ from $p_{-1}$ to $p_0$. Then $c$ is the boundary of a region $A = A_p$, with positively and negatively oriented components $A^+$ and $A^-$ respectively. This situation is illustrated in Figure 3.4.
Differential forms can be integrated over oriented regions (see e.g. Spivak, 1965): for example, if $A$ is an open connected set that has the standard orientation of $T^*\mathcal{X}$, then

$$\int_A \omega = - \text{area} (A).$$

This equality is used to define a function $\Omega$ taking regions as arguments, a cochain, by setting

$$\Omega(A) = \int_A \omega.$$

Note that

$$\Omega(A^-) = \int_{A^-} \omega = \text{area} (A^-) > 0, \quad \Omega(A^+) = \int_{A^+} \omega = - \text{area} (A^+) < 0,$$

and

$$\Omega(A) = \Omega(A^+) + \Omega(A^-) = \text{area} (A^-) - \text{area} (A^+). \quad (3.4)$$
With these notations Theorems 3.2.1–3.2.3 can be formulated, which imply the main theorem. Theorems 3.2.1 and 3.2.2 describe the generic situations.

**Theorem 3.2.1.** If \( \mu > \mu_2 \) or if \( \Omega(A_p) \geq 0 \) for each upward orbit \( p \), then \( x_- \) is a locally optimal fixed point.

**Theorem 3.2.2.** If \( \mu < \mu_1 \) or if \( \Omega(A_p) < 0 \) for some upward orbit \( p \), then \( x_- \) is not an optimal fixed point and any optimal orbit starting in a neighbourhood of \( x_- \) tends to \( x_+ \).

Theorem 3.2.3 characterises the codimension one situation that separates the two generic cases.

**Theorem 3.2.3.** Let \( \mu_3 \in [\mu_1, \mu_2] \) be such that \( \Omega(A_p) = 0 \) for some upward orbit \( p \). Moreover, assume that \( \Omega(A_q) > 0 \) for all other upward orbits \( q \). Then the fixed point \( x_- \) is optimal. For each \( x_0 > x_- \), the optimal trajectories beginning at \( x_0 \) converge to \( x_+ \). Moreover, there are infinitely many points \( x_i^{(k)} > x_- \) which are initial point to two distinct optimal trajectories.

Remark that though the magnitude of \( \Omega(A_p) \) depends on the choice of \( p_0 \) of the heteroclinic orbit, the sign of \( \Omega(A_p) \) is independent of that choice, since

\[
\Omega(\varphi(A_p)) = e^{\rho} \Omega(A_p).
\]

The idea of the proof is sketched. Consider an upward heteroclinic orbit \( p = \{p_t\}_{t=-\infty}^{\infty} \), and let \( \mathcal{N} \) be a small convex open neighbourhood of the fixed point \( z_- \). If \( p_t \in \mathcal{N} \), introduce the set \( W^s_{+,t} \) as the largest connected component of \( W^s \cap \mathcal{N} \) that contains \( p_t \); otherwise, let \( W^s_{+,t} = \emptyset \). Let moreover \( L_t \) be the line through \( p_t \) and parallel to \( E^- \).
Assume first that \( p_t \) is a transversal heteroclinic intersection of \( W^s_+ \) and \( W^u_- \).

The inclination lemma from the theory of dynamical systems, which is quoted in Section 3.3, implies that there is \( T > 0 \), such that for \( t < -T \), the set \( W^s_{+t} \) is a curve segment that is \( C^1 \)-close to \( L_t \). As \( L_t \) intersects the vertical line

\[
\ell = \{(x,y) : x = x_-\}
\]

through \( z_- \), so does \( W^s_{+t} \) if \( t < 0 \) is sufficiently small. Introduce

\[
q_t = \ell \cap W^s_{+t}.
\]

The situation is illustrated in Figure 3.5. An intersection \( q_t \) arising in this way from an upward orbit shall be called an upward intersection.

To every point on the stable manifold \( W^s_+ \), a value can be associated by evaluating the objective functional for the phase trajectory starting at the
point. It follows from Proposition 2.2.4 that the same result is obtained by
integrating the form \( y \, dx \) along the stable manifold; since the manifold is
Lagrangian, the result of the integration is independent of the integration
path (see Subsection 2.2.2).

Let \( z(s) = (x(s), y(s)) \) be the parametrisation of \( W^s_+ \) by arc length, such
that \( z(0) = z_+ \), and such that heteroclinic points correspond to positive
values of the parameters \( s \). To a point \( \alpha \in W^s_+ \) associate the value \( v(\alpha) \)
given by the phase trajectory \( z \) starting at \( \alpha \) (c.f. Section 2.2):

\[
v(\alpha) = \tilde{J}(z) = \sum_{t=1}^{\infty} G(z_{t-1}) e^{-\rho t} = \sum_{t=1}^{\infty} g(x_{t-1}, u_t) e^{-\rho t} = J(x, u).
\]

This implies for instance that for the fixed point \( z_+ \)

\[
v(z_+) = \frac{g(x_+, U(x_+, y_+))}{e^\rho - 1}.
\]

Given a point \( \alpha \in W^s_+ \), let \( s_\alpha \) be such that \( \alpha = z(s_\alpha) \). Then, by Proposi-
tion 2.2.4,

\[
v(\alpha) = v(z_+) + \int_0^{s_\alpha} y(s) x'(s) \, ds. \tag{3.5}
\]

Note that \( v \) is only defined for points on \( W^s_+ \).

It will be established that for every \( t < 0 \) such that \( W^s_{+,t} \) intersects the
line \( \ell \), there is a region \( C_t \) such that

\[
v(q_{t-1}) - v(q_t) = e^{\rho t} \Omega(A) + e^{\rho t} \text{area } (C_t). \tag{3.6}
\]

Consider first the situation that \( \Omega(A) > 0 \). Since \( \text{area } (C_t) > 0 \), this implies
that \( v(q_t) \) increases as \( t \) decreases towards minus infinity. It follows from
Proposition 6.0.1 in Appendix A that \( v(q_t) \uparrow V_-(x_-) \) as \( t \to -\infty \). If equa-
tion (3.6) holds for every upward orbit \( p \), it follows that \( V_-(x_-) \) is larger than
any value \( v(z) \) for \( z \in W^+_s \cap \ell \), and consequently, that it is optimal to remain in \( z_- \). A similar argument holds if \( \ell \) is replaced by any vertical line through a point sufficiently close to \( z_- \), demonstrating local optimality of \( x_- \). This is Theorem 3.2.1.

Equation (3.6) is also helpful for analysing the case that \( \Omega(A) < 0 \) for some upward orbit \( p \), for it can be shown that
\[
\frac{\text{area}(C_t)}{\Omega(A)} \to 0
\]
as \( t \to -\infty \). This implies that the sequence \( v(q_t), v(q_{t-1}), \ldots \) is eventually decreasing. Note that the limit of the sequence is still \( V_-(x_-) \); therefore, there is some \( T \) such that
\[
v(q_T) > V_-(x_-),
\]
and the steady state \( x_- \) cannot be optimal in this case, implying the result of Theorem 3.2.2.

### 3.3 Proofs of the theorems

In this section, the proofs of Theorems 3.2.1, 3.2.2 and 3.2.3 are given.

#### 3.3.1 Local preliminaries.

**Proposition 3.3.1.** If \( v = (v_1, v_2) \) is a nonzero eigenvector of \( D\varphi \), then \( v_1 \neq 0 \).

**Proof.** The derivative \( D\varphi \) takes the form
\[
D\varphi = \begin{pmatrix}
\frac{H_{xy}}{H_{xy}} & e^\rho \frac{H_{yy}}{H_{xy}} \\
\frac{H_{yy}}{H_{xy}} & \frac{H_{xx}}{H_{xy}} \\
\end{pmatrix}.
\]

(3.7)
Assume that $v_1 = 0$. Then by Proposition 2.1.5 the eigenvalue equation $\lambda v = (D\varphi)v$ reads as

$$
\begin{pmatrix}
0 \\
\lambda v_2
\end{pmatrix}
= 
\begin{pmatrix}
e^\rho H_{yy} H_{xy}^{-1} v_2 \\
e^\rho H_{xy}^{-1} v_2
\end{pmatrix}.
$$

If $\lambda = 0$, then $H_{xy}^{-1} v_2 = 0$ and consequently $v_2 = 0$; but then $v$ would be trivial. If $\lambda \neq 0$, then $v_2 = (\lambda/e^\rho) H_{xy} v_2$. Substituting into the first equation yields that

$$
0 = \lambda H_{yy} v_2.
$$

As $H_{yy}$ is positive by Assumption 3.1.1, it follows that $v_2 = 0$, again implying that $v$ is trivial.

This proposition implies that there is a neighbourhood $\mathcal{N}$ of $z_-$ such that $W_s^\pm$ and $W_u^\pm$ restricted to $\mathcal{N}$ can be represented as the graphs of functions $w^s$ and $w^u$ respectively.

**Proposition 3.3.2.** If $v^u = (1, v_2^u)$ and $v^s = (1, v_2^s)$ are stable and unstable eigenvectors of $D\varphi$, then $v_2^u > v_2^s$.

**Proof.** If $v = (1, v_2)$ is an eigenvector with eigenvalue $\lambda$,

$$
(H_{xy}^2 - H_{xx} H_{yy}) - \lambda H_{xy} + e^\rho H_{yy} v_2 = 0.
$$

This can be written as

$$
v_2 = \frac{H_{xx} H_{yy} - H_{xy}^2}{e^\rho H_{yy}} + \frac{\lambda}{e^\rho} H_{xy}.
$$

The result now follows from Assumption 3.1.1 that $H_{xy} > 0$ and $H_{yy} > 0$. ■

From Proposition 3.3.2, the following corollary is obtained.
Proposition 3.3.3. Let $\Delta$ be the triangle bounded by the line connecting $(0,0)$ to $v^u$, followed by the line connecting $v^u$ to $v^u+v^s=(0,v_2^u+v_2^s)$ and the line connecting $v^u+v^s$ to 0. Then $\Delta$ is positively oriented.

Recall that a map $\Phi$ is symplectic if $\Phi^*\omega=\omega$.

Proposition 3.3.4. There is an open neighbourhood $\mathcal{N} \subset T^*\mathcal{X}$ of $z_-$, an open neighbourhood $\tilde{\mathcal{N}} \subset \mathbb{R}^2$ of $(0,0)$ and a symplectic coordinate transformation $\Phi: \mathcal{N} \to \tilde{\mathcal{N}}$ of the form
\[
\zeta = (\xi, \eta) = \Phi(x, y) = \Phi(z),
\]
such that in the new coordinates the map $\varphi$ has the form
\[
\varphi(\zeta) = \begin{pmatrix} \lambda^s & 0 \\ 0 & \lambda^u \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} \xi \psi_1 \\ \eta \psi_2 \end{pmatrix},
\]
where for some $K > 0$, $|\psi_i(\zeta)| \leq K|\zeta|$, $i = 1, 2$ for $\zeta \in \tilde{\mathcal{N}}$.

Proof. The transformation $\Phi$ is the composition of the two successive symplectic coordinate transformations. For the first transformation, the following is taken
\[
\tilde{x} = x - x_-, \quad \tilde{y} = y - w^s(x).
\]
This transformation is symplectic, as
\[
\varphi^*\omega = d\tilde{y} \wedge d\tilde{x} = (dy - (w^s)'(x)\,dx) \wedge dx = dy \wedge dx = \omega.
\]
In $(\tilde{x}, \tilde{y})$ coordinates, the fixed point $z_-$ is given by $(\tilde{x}, \tilde{y}) = (0,0)$, the stable manifold $W^s_-$ by the equation $\tilde{y} = 0$, and the unstable manifold $W^u_-$ by
\[
\tilde{y} = w^u_1(\tilde{x}) = w^u(x_- + \tilde{x}) - w^s(x_- + \tilde{x}).
\]
The function $w^u_1$ defined by this equation satisfies
\[(w^u_1)'(0) = (w^u)'(\tilde{x}) - (w^s)'(\tilde{x}) \neq 0.\]

For the second transformation, take
\[\xi = \tilde{x} - (w^u_1)^{-1}(\tilde{y}), \quad \eta = \tilde{y}.\]

This transformation is well-defined on $\mathcal{N}$ — possibly the neighbourhood has to be taken smaller to ensure the invertibility of $w^u_1$ — it is symplectic, it preserves the location $\eta = 0$ of the stable manifold, and it maps the unstable manifold to $\xi = 0$. In the new coordinates the map $\varphi$ has then the form given in the proposition.

Recall the inclination lemma or $\lambda$-lemma (see Palis and Takens, 1993, p. 155).

**Inclination lemma.** Let $M$ be a $n$-dimensional manifold and let $\varphi : M \to M$ be a $C^k$ diffeomorphism, $k \geq 1$, with a hyperbolic fixed point $z$. Let $W \subset M$ be a $C^k$ submanifold such that $\dim(W) = \dim(W^s(z))$, and such that $W$ has a point $p$ of transversal intersection with $W^u(z)$.

Then for each $t$, one can choose a disk $D_t \subset \varphi^{-t}(W)$, which is a neighbourhood of $\varphi^{-t}(p)$ in $\varphi^{-t}(W)$, such that
\[\lim_{t \to \infty} D_t = D,\]
where $D$ is a disk-neighbourhood of $p$ in $W^s(z)$. Convergence means here that for $t$ sufficiently large $D_t$ and $D$ are $C^k$-near embedded disks.

### 3.3.2 Estimating value differences using the area rule

In this subsection, equation (3.6) is stated, derived, and an estimate of the term $\text{area } (C_t)$
is given. Moreover a variant of equation (3.6) needed to prove Theorem 3.2.1 is derived as well.

Take $\xi$ close to $x_-$ and consider the intersection of $W^s_+$ with

$$\ell_\xi = \{(x, y) : x = \xi\}$$

as in Figure 3.6. The values at these intersection points can be compared using the area rule. To do this, the following definitions are made.

Let $p$ be an upward heteroclinic orbit. Consider the curve $c_1$ given by the part of $W^s_+$ connecting $p_0$ to $p_{-1}$, followed by the part of $W^u_-$ connecting $p_{-1}$ to $p_0$. Then $c_1$ is the boundary of a region $A$ with positively and negatively oriented components $A^+$ and $A^-$ respectively (see Figure 3.6).

![Figure 3.6: The regions $A^+$ and $A^-$ are respectively positively and negatively oriented.](image)

Let moreover $T > 0$ be such that if $t < -T$, then the part of $W^s_+$ con-
necting \( p_t \) to \( p_{t-1} \) intersects the line \( \ell_\xi \). Let \( q_t^0 \) be the first intersection of \( W^s_+ \) with \( \ell_{x_-} \) following \( p_t \) with respect to parametrisation \( z(s) \), that is, let \( q_t^0 \) be such that the segment of \( W^s_+ \) connecting \( p_t \) to \( q_t^0 \) has no other points in common with \( \ell_{x_-} \). Let moreover \( q_t(\xi) \) be a continuous function such that \( q_t(z_-) = q_t^0 \) and \( q_t(\xi) \in \ell_\xi \cap W^s_+ \). This situation is illustrated in Figure 3.7.

![Figure 3.7: The images of the regions \( A^+ \) and \( A^- \) under \( \varphi^t \), i.e. \( \varphi^t(A^+) \) and \( \varphi^t(A^-) \).](image)

For the sake of notational simplicity, write \( q_t = q_t(\xi) \) for \( \xi \) close to \( x_- \). Define \( \tilde{\ell}_\xi \) as the segment of \( \ell_\xi \) connecting \( q_{t-1} \) to \( q_t \). Then \( \varphi^{-t}\tilde{\ell}_\xi \) is a curve connecting \( \varphi^{-t}(q_{t-1}) \) to \( \varphi^{-t}(q_t) \), which are both located on \( W^s_+ \). Consider the curve \( c_2 \) given by the part of \( W^s_+ \) connecting \( \varphi^{-t}(q_t) \) to \( \varphi^{-t}(q_{t-1}) \), followed by the curve \( \varphi^{-t}\tilde{\ell}_\xi \). Then \( c_2 \) is the boundary of a region \( B_t \) with positively and negatively oriented components \( B^+_t \) and \( B^-_t \) respectively.
Define

\[ C_t^+ = A^+ - B_t^+, \quad C_t^- = A^- - B_t^- , \]

and

\[ C_t = A - B_t = C_t^+ + C_t^- . \]

See Figure 3.8.

**Proposition 3.3.5.** Let \( v \) be as in equation (3.5). If \( t < -T \), then

\[ v(q_{t-1}) - v(q_t) = e^{\rho t} \Omega(\varphi^t(A)) - e^{\rho t} \Omega(\varphi^t(C_t)). \]

In particular, if all simple components of \( C_t \) are positively oriented, then

\[ v(q_{t-1}) - v(q_t) = e^{\rho t} \Omega(\varphi^t(A)) + e^{\rho t} \text{area} \left( \varphi^t(C_t) \right), \]

whereas if all simple components of \( C_t \) are negatively oriented, then

\[ v(q_{t-1}) - v(q_t) = e^{\rho t} \Omega(\varphi^t(A)) - e^{\rho t} \text{area} \left( \varphi^t(C_t) \right). \]

**Proof.** Recall that \( B_t = A - C_t \). By the iterated area rule

\[ e^{-\rho t} \left( v(q_{t-1}) - v(q_t) \right) = \Omega(B_t) = \Omega(A) - \Omega(C_t). \]

The result follows. \( \square \)

We now consider the case that \( \ell \) is the line \( x = x_- \). The following proposition states that in that case, all simple components of \( C_t \) are positively oriented (see Figure 3.8), and gives an estimate of \( \Omega(C_t) \).

**Proposition 3.3.6.** If \( \ell \) is the line \( x = x_- \), then all simple components of \( C_t^+ \) and \( C_t^- \) are positively oriented. Moreover, there are constants \( T_0 > 0 \) and \( K > 0 \) such that for all \( t < -T_0 \), the inequality

\[ -K \lambda_u^{2t} \leq e^{\rho t} \Omega(C_t) \leq 0 \]

holds.
Figure 3.8: The regions $B_t^+$ and $C_t^\pm$. The regions $B_t^+$ and $B_t^-$ are respectively positively and negatively oriented by definition. In the situation depicted in the upper figure, $C_t^+$ and $C_t^-$ are both positively oriented, whereas in the lower figure, only $C_t^-$ is negatively oriented, while $C_t^+$ has both a positively and a negatively oriented component.
Proof. Since $\det D\varphi = e^\rho > 0$, the phase map $\varphi$ preserves orientation. Note that for $T_0 > 0$ sufficiently large, if $t < -T_0$, then the regions $\varphi^t(C_t^\pm)$ are contained in the curvilinear triangle $\tilde{\Delta}$ formed by the part $W_{s,t}^+$ of $W_s^+$ connecting $p_t$ to $q_t$, the part of $\ell$ connecting $q_t$ to $z_-$, and the part of $W_u^-$ connecting $z_-$ to $p_t$ (cf. Figure 3.7). By the $\lambda$-lemma, for large values of $-t$ the curve segment $W_{s,t}^+$ is $C^1$-close to $W_s^+$. Therefore the curvilinear triangle $\tilde{\Delta}$ has the same orientation as the triangle $\Delta$ introduced in Proposition 3.3.3. But that proposition states that $\Delta$ is positively oriented.

Let $m_0$ and $m_{-1}$ be curves through $p_0$ and $p_{-1}$ respectively that intersect $W_u^-$ transversally, and which are such that the region bounded by $m_0$, $\varphi^{-t}\ell$, $m_{-1}$ and $W_u^-$ contains $C_t$. Moreover, in local coordinates introduced in Proposition 3.3.4, let

$$p_t = (0, \eta_t).$$

By the $\lambda$-lemma, the iterates of the $m_i$ have the property that the intersections $\varphi^i m_i \cap \mathcal{N}$ tend to $W_s^+ \cap \mathcal{N}$ in the $C^1$-norm as $t \to -\infty$. That is, given $\varepsilon > 0$, there is a $T > 0$ such that for $t < -T$ the intersections in local $(\xi, \eta)$-coordinates, introduced in Proposition 3.3.4, take the form

$$\varphi^i m_i \cap \mathcal{N} : \eta = \chi_i^t(\xi), \quad i = 0, -1, ...$$

with $\chi_i^t(0) = \eta_{t+i}$ and

$$\max_{W_s^+ \cap \mathcal{N}} |(\chi_i^t)'(\xi)| < \varepsilon.$$

Still in local coordinates, the curve $\ell$ is represented as the graph of the function

$$\xi = \kappa(\eta) = -(w_u^a)^{-1}(\eta)$$

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with \( \kappa(0) = 0 \). Moreover, there is \( M > 0 \) with \( -M < \kappa'(0) < 0 \) for all \( \eta \) such that \( (\eta, \kappa(\eta)) \in \mathcal{N} \). The area \( R_t \) bounded by \( W^{-}_u, \varphi^t m_0 \) and \( \ell \) contains \( \varphi^t C_t \). Consequently

\[
e^{pt} \text{area} (C_t) = \text{area} (\varphi^t C_t) \leq \text{area} (R_t).
\]

The region \( R_t \) itself is contained in the triangle formed by the lines \( \xi = -M\eta, \eta = \eta_t - \varepsilon \xi \) and \( \xi = 0 \); it follows that

\[
\text{area} (R_t) \leq \frac{M}{2(1 - \varepsilon M)\eta_t^2} = M'\eta_t^2.
\]

The fact that \( \eta_t = M\lambda_u + O(\lambda_u^2) \), uniformly in \( t \), proves the proposition.

As noted in the sketch of the proof, Proposition 3.3.5 shows that if \( \Omega(A) > 0 \), then \( v(q_t) \) increases towards \( V^-(x_-) \) as \( t \to -\infty \). However, not all intersections of \( W^+_s \) with \( \varphi^{-t} \ell_t \) follow directly on an upward intersection of \( W^+_s \) with \( W^-_u \); there may be a configuration as the one depicted in Figure 3.9.

Define \( r_{t,1} = \varphi^{-t}(q_t), r_{t,2}, \cdots, r_{t,K_t}, r_{t,K_t+1} = \varphi^{-t}(q_{t-1}) \) as the consecutive positive intersections of \( W^+_s \) with \( \varphi^{-t} \ell_t \) that follow \( p_t \). Set

\[
q_{t,i} = \varphi^t(r_{t,i}).
\]

Denote moreover by \( B_{t,1}^+, B_{t,2}^+ \) the components of the region \( B_t^+ \) that are such that for \( 1 \leq i \leq K_t \), the point \( r_{t,i} \) is contained in the boundary of \( B_{t,i}^+ \). Likewise, denote by \( C_{t,1}^-, C_{t,2}^- \) the components of \( C_t^- \) that are such that for \( 1 \leq i \leq K \) the point \( r_{t,i+1} \) is contained in \( C_{t,i}^- \).

**Proposition 3.3.7.** Let \( 1 \leq k \leq K_t \) be such that \( r_{t,k} = \varphi^{-t}(q_{t,k}) \) follows \( p_0 \), but precedes any other upward intersection of \( W^+_s \) with \( W^u \). Then

\[
v(q_{t-1}) - v(q_{t,k}) \geq \Omega (\varphi^t(A)).
\]
Figure 3.9: Several intersections of $W^s_+$ and $\varphi^{-i}l_i$ following an upward intersection.

Proof. The area rule implies that

$$v(q_{t,k}) - v(q_{t,1}) = \sum_{i=1}^{k-1} \left\{ \Omega\left(\varphi^i(B_{i,i}^+)\right) + \Omega\left(\varphi^i(C_{i,i}^-)\right) \right\}$$

$$= \sum_{i=1}^{k-1} \left\{ - \text{area}\left(\varphi^i(B_{i,i}^+)\right) + \text{area}\left(\varphi^i(C_{i,i}^-)\right) \right\}.$$

Using Proposition 3.3.5 and the equality

$$v(q_{t-1}) - v(q_{t,k}) = v(q_{t-1}) - v(q_t) + v(q_t) - v(q_{t,k})$$

$$= v(q_{t-1}) - v(q_t) + \sum_{i=1}^{k-1} \left\{ \text{area}\left(\varphi^i(B_{i,i}^+)\right) - \text{area}\left(\varphi^i(C_{i,i}^-)\right) \right\}$$

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then yields that
\[ v(q_{t-1}) - v(q_{t,k}) = \Omega(\varphi'(A)) - \Omega(\varphi'(C_t)) - \sum_{i=1}^{k-1} \text{area}(\varphi'(C_{t,i}^-)) + \sum_{i=1}^{k-1} \text{area}(\varphi'(B_{t,i}^+)) . \]

As for \( \ell = \ell_{x_-} \) all regions \( C_{t,i}^- \) and \( C_{t,i}^+ \) are positively oriented,
\[ v(q_{t-1}) - v(q_{t,k}) = \Omega(\varphi'(A)) + \sum_{i=1}^{k-1} \text{area}(\varphi'(B_{t,i}^+)) + \text{area}(\varphi'(C_t)) - \sum_{i=1}^{k-1} \text{area}(\varphi'(C_{t,i}^-)) \geq \Omega(\varphi'(A)) \]

as \( \bigcup_{i=1}^{k-1} C_{t,i}^- \subset C_t \). The result follows.

3.3.3 Proof of Theorem 3.2.1.

Proof. The first part of the Proposition is immediate: if \( \mu > \mu_2 \), then by Assumption 3.1.4, there are open neighbourhoods \( \mathcal{N}_- \), \( \mathcal{N}_+ \) of \( x_- \) and \( x_+ \) respectively, such that \( W_s^+ \cap \mathcal{N}_- \times \mathbb{R} = \emptyset \) and \( W_s^- \cap \mathcal{N}_+ \times \mathbb{R} = \emptyset \). But by Assumption 3.1.3, optimal solutions correspond to trajectories on either \( W_s^- \) or \( W_s^+ \). It follows that all optimal state trajectories starting in \( \mathcal{N}_- \) tend to \( x_- \), and those starting in \( \mathcal{N}_+ \) tend to \( x_+ \).

To prove the second part of the proposition, let as before \( p \) be an upward heteroclinic intersection of \( W_s^+ \) and \( W_u^- \) such that \( p_0 \) and \( A \) be as stated before in this chapter. Let moreover \( \ell \) be the line \( x = x_- \) and as stated in Proposition 3.3.7 let
\[ q_{t,i} = \varphi'(r_{t,i}) , \quad i = 1, \cdots , K_t \]
be the positive intersections of \( W_+^s \) with \( \ell \) that follow \( p_t \) and that precede the next upward intersection of \( W_+^s \) and \( W_-^u \). Set \( q_t = q_{t,1} \).

Using \( \Omega(A) \geq 0 \) together with Proposition 3.3.5, it is obtained that

\[
v(q_{t-1}) - v(q_t) > 0.
\]

and therefore \( v(q_t) \) is an increasing sequence. Since \( q_t \to z_- \), it follows from Proposition 6.0.1 that \( v(q_t) \to V_-(x_-) \). We conclude that

\[
\cdots < v(q_t) < v(q_{t-1}) < v(q_{t-2}) < \cdots < V_-(x_-).
\]

Moreover, from Proposition 3.3.7, it follows that for \( 1 \leq i \leq K_t \), then

\[
v(q_{t,i}) < v(q_{t-1}).
\]

It is immediate that the remaining intersections of \( W_+^s \) and \( \ell \) yield even smaller values. But then no orbit on \( W_+^s \) yields a value that is as high as \( V_-(x_-) \), and the proposition follows.

**3.3.4 Proof of Theorem 3.2.2.**

*Proof.* If \( \mu < \mu_1 \), take \( \alpha = z_- \) and \( \beta \in W_+^s \) such that \( x_\beta = x_- \). Let \( E \) be the region bounded by the segment \( \bar{\ell} \) of the straight line connecting \( \alpha \) to \( \beta \), the segment \( \Sigma \) of \( W_+^s \) connecting \( \beta \) to \( \varphi(\beta) \) and the image \( \varphi(\bar{\ell}) \) of \( \bar{\ell} \). As \( \Sigma \) does not intersect \( W_-^s \) or \( W_-^u \), it follows that \( E \) is negatively oriented (see Figure 3.10). Proposition 2.3.5 then implies that

\[
V_+(x_-) > V_-(x_-).
\]

But then \( x_- \) cannot even be a locally optimal fixed point.
Figure 3.10: Region $E$ is negatively oriented, $\Omega(E) > 0$.

The second part of Theorem 3.2.2, the case that $\mu_s > \mu > \mu_1$ and $\Omega(A) < 0$, follows from Proposition 6.0.1 in Appendix A and Proposition 3.3.5, as a sequence of points $q_t \in W^+_s \cap \ell_{x_-}$ is found which is such that $q_t \to z_-$ as $t \to -\infty$, implying $v(q_t) \to V_-(x_-)$, and which satisfies for all $t < -T_0$ the inequalities

$$a_t = e^{\rho t} \Omega(A) - C'' \lambda^2 u \leq v(q_{t-1}) - v(q_t) \leq e^{\rho t} \Omega(A) + C'' \lambda^2 u = b_t.$$

Using $\Omega(A) < 0$ and the fact that $e^\rho = \det D\varphi(z_-) = \lambda_u \lambda_s$, it follows that

$$e^{-\rho t} b_t = \Omega(A) + C'' \left( \frac{\lambda_u}{\lambda_s} \right)^t,$$

and there is some $T' > 0$ such that $b_t < 0$ for all $t < -T'$. But then the sequence $v(q_t)$ is eventually decreasing as $t \to -\infty$. Therefore there
is some \( t_0 \) such that \( v(q_{t_0}) > V_-(x_-) \), and the state trajectory remaining at \( x = x_- \) cannot be optimal.

This implies that the optimal solution starting at \( x = x_- \) converges to \( x = x_+ \). Consequently, no solution on \( W_-^s \) can be optimal, and therefore, by Assumption 3.1.3, every optimal solution converges to \( x = x_+ \).

3.3.5 Proof of Theorem 3.2.3.

Proof. The optimality of the trajectory \( x_t = x_- \) follows from Theorem 3.2.1.

Fix a small neighbourhood of \( x_- \), and take \( \xi \) in this neighbourhood such that \( \xi > x_- \). Let \( \ell_\xi \) be the vertical line \( x = \xi \), and denote by \( q_t(\xi) \) the intersection of \( W_+^s \) with \( \ell_\xi \) defined in Subsection 3.3.2. Moreover, let \( t_0 = t_0(\xi) \) be such that for \( t \geq t_0 \), the curve segment from \( p_t \) to \( q_t \) is oriented in the same way as \( W_+^s \), while for \( t < t_0 \) that curve segment is oriented the opposite direction.

This implies that orientation of \( C_t \) is positive for \( t > t_0 \), while it is negative for \( t < t_0 \). Since by Assumption \( \Omega(A) = 0 \), Proposition 3.3.5 implies for all \( t > t_0 \) that

\[
v(q_{t-1}(\xi)) > v(q_t(\xi)),
\]

while for \( t < t_0 \),

\[
v(q_{t-1}(\xi)) < v(q_t(\xi)).
\]

As \( v(q_t(\xi)) \rightarrow V_-(\xi) \) as \( t \rightarrow -\infty \), it follows that

\[
v(q_{t_0}(\xi)) > V_-(\xi),
\]

and consequently that the optimal state trajectory starting at \( \xi \) will tend to \( x_+ \).

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Figure 3.11: Intersections of $W^+_t$ with the line $x = x_-$ (solid) and the line $x = \xi$ (dashed).

Given a value $\tau$ of $t_0(\xi)$, there are $\xi_1^\tau$ and $\xi_2^\tau$ such that

$$t_0(\xi) = \tau \quad \text{if} \quad \xi_1^\tau < \xi < \xi_2^\tau$$

and that

$$q_{\tau-1}(\xi_1^\tau) = p_{\tau-1} \quad \text{and} \quad q_{\tau}(\xi_2^\tau) = p_{\tau}.$$

The area rule as illustrated in Figure 3.12 implies that

$$v (q_{\tau-1}(\xi_1^\tau)) - v (q_{\tau}(\xi_1^\tau)) < 0 \quad (3.8)$$

and

$$v (q_{\tau-1}(\xi_2^\tau)) - v (q_{\tau}(\xi_2^\tau)) > 0. \quad (3.9)$$

Recall that for $t > \tau$

$$v (q_{t-1}(\xi)) - v (q_{t}(\xi)) > 0 \quad (3.10)$$
and that for $t < \tau$

$$v(q_{r-1}(\xi)) - v(q_t(\xi)) < 0.$$ \hfill (3.11)

If $\xi = \xi_1^*$, using equations (3.8), (3.10) and (3.11) it is obtained that

$$\cdots < v(q_{r-2}) < v(q_{r-1}) < v(q_r) > v(q_{r+1}) > \cdots .$$

Moreover, if $\xi = \xi_2^*$, equations (3.9), (3.10) and (3.11) imply that

$$\cdots < v(q_{r-2}) < v(q_{r-1}) > v(q_r) > v(q_{r+1}) > \cdots .$$

That is, if $\xi = \xi_1^*$ then $v(q_r(\xi))$ is maximal, whereas if $\xi = \xi_2^*$, then $v(q_{r-1}(\xi))$ is maximal. Consequently there is $\xi_1^* < \xi_2^* < \xi_3^*$ such that

$$v(q_r(\xi_1^*)) = v(q_{r-1}(\xi_2^*)).$$

For the final claim of the theorem, note that $t_0 = t_0(\xi)$ decreases towards $-\infty$ as $\xi \to x_-$. \hfill $\blacksquare$

Figure 3.12: Situations $\xi = \xi_2^*$ and $\xi = \xi_1^*$. 
Chapter 4

Numerics of invariant manifolds, indifference points, and indifference-attractor bifurcations

In Chapter 3 the genesis of indifference thresholds in an indifference-attractor bifurcation is studied for a class of non-convex infinite horizon discrete-time optimal control problems. This chapter presents details of the computation of optimisers and bifurcation curves.

As in Chapter 3 a geometrical point of view is taken. Unlike the well-known iterative methods of numerical dynamic programming (see Judd (1998)), the algorithms presented in this chapter are based on the direct computation of stable manifolds. It is usually impossible to determine analytic expressions for these invariant manifolds, so numerical computations are needed.
Details are given of a simple method which accurately computes arbitrarily long segments of an invariant manifold. This information is used to compute the locus of the indifference-attractor bifurcation points of the problem.

The chapter is organised as follows. In Section 4.1 the optimisation problem is recalled and the main results obtained in Chapter 3 are sketched briefly. In Section 4.2 the numerical method for computing stable manifold is given. Based on this in Section 4.3 methods to determine optimal solutions, value functions and thresholds as well as indifference points are given. In Section 4.4 a method to compute indifference-attractor bifurcations is presented.

4.1 The family of optimisation problems

A family of problems of maximising an objective

\[ J_\mu(x, u) = \sum_{t=1}^{\infty} g(x_{t-1}, u_t; \mu) e^{-\rho t}. \]  

is considered. The family depends on a system parameter \( \mu \). The discount rate \( \rho \) is assumed to be positive. The maximisation is performed over all pairs of sequences \( x = \{x_t\} \) and \( u = \{u_t\} \) such that \( x_t \in \mathcal{X}, u_t \in \mathcal{U} \) and such that for all \( t = 1, 2, \ldots \),

\[ x_t = f(x_{t-1}, u_t; \mu). \]  

The parameter \( \mu \) takes values in the parameter space \( \mathcal{P} \subset \mathbb{R} \), which is an open and bounded interval. Also \( \mathcal{X} \subset \mathbb{R} \) and \( \mathcal{U} \subset \mathbb{R} \) are open intervals. The initial state \( x_0 \) is assumed to be given. The functions \( f \) and \( g \) are assumed to be smooth, i.e. \( C^\infty \), in the interior of \( \mathcal{X} \times \mathcal{U} \), and their derivatives are assumed to be bounded there. Moreover, it is assumed that \( f(x, u; \mu) \in \mathcal{X} \).
for all \((x, u; \mu)\), and that the partial derivative \(f_u(x, u; \mu)\) does not vanish anywhere.

Recall definition (2.8) of the costate variable \(y\) and of the phase map \(\varphi\) of the problem (see Section 2.1.4). In terms of these, the necessary first order conditions of the problem are formulated as the following boundary value problem: if \((x, y)\) is maximising and \(\delta\)–interior, for \(\delta > 0\), then

\[
(x_t, y_t) = \varphi(x_{t-1}, y_{t-1}),
\]

\[
x_0 = \alpha \quad \text{and} \quad \lim_{t \to \infty} e^{-\rho t} y_t = 0.
\]

Recall moreover that if an optimal state trajectory \(x\) converges to some steady state, this state corresponds to a fixed point of the phase map \(\varphi\).

The main result of Chapter 3 has been obtained in the following setting. The parametrised family \(\varphi = \varphi_\mu\) of phase maps associated to the optimisation problem (4.1) depends smoothly on a parameter \(\mu \in \mathcal{P}\). Every map \(\varphi = \varphi_\mu\) is assumed to have at least two saddle fixed points \(z_{\pm} = (x_{\pm}, y_{\pm})\), with \(x_- < x_+\).

Recall from Chapter 3 that the family \(\varphi_\mu\) is said to go through a heteroclinic bifurcation scenario involving \(W^u_+\) and \(W^s_-\), if there is a parameter interval \([\mu_1, \mu_2] \subset \mathcal{P}\) such that for \(\mu < \mu_1\) and \(\mu > \mu_2\), the manifolds \(W^u_+\) and \(W^s_-\) have no points in common, while for \(\mu \in [\mu_1, \mu_2]\) there is at least one heteroclinic orbit. Recall moreover that all heteroclinic orbits are necessarily tangencies at \(\mu = \mu_1\) and \(\mu = \mu_2\). Assumptions 3.1–3.5 imply that the family \(\varphi_\mu\) of phase maps goes through a heteroclinic bifurcation scenario.

Theorem 3.1.1 states that there is a parameter value \(\mu_c \in (\mu_1, \mu_2)\) such that for \(\mu < \mu_c\), the steady state \(x_+\) is globally optimal. If \(\mu > \mu_c\), both
steady states are locally optimal, and there is a state \( x_- < x_s < x_+ \) such that \( x_s \) is initial state to two optimal solutions of which one converges to \( x_- \) and the other converges to \( x_+ \). This is summarised by saying that at \( \mu_c \) the solution structure exhibits an indifference-attractor bifurcation.

### 4.2 Invariant Manifolds

Optimal solutions correspond to trajectories on one of the stable manifolds. In order to compute these solutions the manifolds need to be approximated numerically. This is achieved by first computing a local stable manifold, and then extending it iteratively.

Consider therefore a saddle fixed point \( \bar{z} \) of the map \( \varphi \). Let \( E^s \) and \( E^u \) be the stable and unstable eigenspaces of \( D\varphi(\bar{z}) \) corresponding to \( \lambda^s \) and \( \lambda^u \) respectively. A linear change of coordinates is performed such that \( \bar{z} \) corresponds to the origin, \( E^s \) to the \( x \)-axis and \( E^u \) to the \( y \)-axis. Then

\[
\varphi(z) = \begin{pmatrix} \lambda^s & 0 \\ 0 & \lambda^u \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + O(|z|^2),
\]

where \( z = \begin{pmatrix} x \\ y \end{pmatrix} \) and \( 0 < \lambda^s < 1 < \lambda^u \).

Recall that \( W^s \) is a smooth manifold, tangent to \( E^s \) at the fixed point. Locally around the origin \( W^s \) can therefore be parametrised as \( y = w(x) \) with \( w(0) = 0 \) and \( w'(0) = 0 \). Moreover, since \( W^s \) is invariant under \( \varphi \), the function \( w \) has to satisfy the functional equation

\[
w(\varphi_1(x, w(x))) = \varphi_2(x, w(x)). \tag{4.3}
\]

Finding a local stable manifold amounts to solving (4.3) numerically.
4.2.1 Computing the local stable manifold. For the computation of invariant manifolds there are several approaches: a naive approach is to take points on \( E^s \) and obtain a discrete approximation of \( W^s \) by computing backward iterations of these points. A more sophisticated approach takes the graph transform as its starting point, obtaining the invariant manifold as a fixed point of a discrete graph transform operator. Based on this, Homburg et al. (1995) derived a numerical algorithm for the computation of invariant manifolds of hyperbolic fixed points using invariant foliations. Yet other methods are based on obtaining the coefficients of an expansion \( w(x) = \sum a_n x^n \) (cf. Simó (1989)). The approach proposed in this section was suggested by F. Takens; while less sophisticated than the methods referred to, it still provides accurate approximations to the stable manifolds. It is however restricted to two-dimensional systems.

For a given point \((\hat{x}, 0) \in E^s\) the algorithm will determine an approximation \((\hat{x}, \hat{w})\) to \((\hat{x}, w(\hat{x})) \in W^s\). For \( \delta > 0 \), define \( R = [0, \hat{x}] \times [-\delta, \delta] \). Set \( z_0 = (\hat{x}, y_0) \) with \(-\delta \leq y_0 \leq \delta\) and consider the orbit \( z = \{z_t\} \) of \( \varphi \) starting at \( z_0 \). There are two possibilities: if \( y_0 = w(\hat{x}) \), then \( z_t \) converges towards the origin as \( t \to \infty \). In particular this means that \( z_t \in R \) for all \( t > 0 \). If however \( y_0 \neq w(\hat{x}) \), then the orbit will leave the rectangle \( R \) eventually; that is, there is \( T > 0 \) such that \( z_t \in R \) if \( 0 \leq t < T \) and \( z_T \notin R \). Based on the linear approximation \( \varphi(x) \approx D\varphi(0) x \), the following approximation is obtained

\[
T \approx \left[ \frac{-\log |y_0 - w(\hat{x})| + \log \delta}{\log \lambda^u} \right].
\]

Note moreover that if \( y_0 > w(\hat{x}) \) then \( y_T > \delta \); if \( y_0 < w(\hat{x}) \) then \( y_T < -\delta \). The algorithm is based on this observation.
Recall the notation $B_r(\tilde{z}) = \{ z \in \mathbb{R}^2 : ||z - \tilde{z}|| < r \}$ for a ball with radius $r$ around the point $\tilde{z}$. The algorithm takes as input a point $\hat{x} \in \mathcal{X}$ near the origin, that is $|\hat{x}| < \varepsilon_0$ with $\varepsilon_0 \ll 1$, the rectangle $R$, and a given tolerance $\varepsilon$. It then determines an approximation $\hat{w}$ of $w(\hat{x})$. Let $\zeta_1^m = (\hat{x},0)$ and $\hat{z} = (\hat{x},w(\hat{x}))$. If $\delta > 0$ is sufficiently large then $\hat{z} \in B_\delta(\zeta_1^m)$. The ball $B_\delta(\zeta_1^m)$ intersects the line $x = \hat{x}$ in two points, namely $\zeta_1^1 = (\hat{x}, b_1)$ and $\zeta_1^1 = (\hat{x}, a_1)$, where $a_1 < b_1$. This situation is illustrated in Figure 4.1.

The algorithm uses the standard bisection method. Fix a time horizon

$$T_0 > \frac{\log (\delta/\varepsilon)}{\log \lambda^u}. \quad (4.4)$$

The point $\zeta_1^m = (\hat{x}, c_1) = (\hat{x}, 0)$ is the midpoint of the line segment $\zeta_1^1 \zeta_1^0$. Let $\tau > 0$ be the smallest integer such that $(x_\tau, y_\tau) = \varphi^\tau(\zeta_1^m) \notin R$. Compute the
iterates \((x_t, y_t) = \varphi'(\zeta^1_m)\) for \(0 \leq t \leq T\), where \(T = \min\{T_0, \tau\}\).

If \(-\delta \leq y_T \leq \delta\) then \(T = T_0, z_{T_0} \in R\) and \(\hat{w} = c_1\), and the algorithm stops with

\[|\lambda^u(T_0)|\hat{w} - w(\hat{x})| \lesssim \delta\]

which implies, using (4.4), that

\[|\hat{w} - w(\hat{x})| \lesssim \varepsilon.\]

Otherwise, if \(y_T > \delta\) then \(c_1 > w(\hat{x})\); set therefore \(a_2 = a_1\), and \(b_2 = c_1\); if \(y_T < -\delta\) set \(b_2 = b_1\) and \(a_2 = c_1\). Take \(\zeta^2_m = (\hat{x}, c_2)\) as the midpoint of the line segment \(\zeta^2\), where \(\zeta^2 = (\hat{x}, a_2)\) and \(\overline{c_2} = (\hat{x}, b_2)\). In this way sequences \(\{\zeta^k\}\) and \(\{\overline{c^k}\}\) are obtained such that \(w(\hat{x}) \in [a_k, b_k]\). The algorithm stops if \(|b_k - a_k| \leq \varepsilon\), with output

\[\hat{w} = \frac{b_k - a_k}{2}.\]

Note that then \(|\hat{w} - w(\hat{x})| \leq \varepsilon/2\).

4.2.2 Computing the global stable manifold. To obtain a discretisation of the stable manifold \(W_s\), set \(\tilde{z} = \varphi(\tilde{z})\) and \(I_0 = [\tilde{x}, \tilde{x}]\). An equidistant grid \(\{x_0^1, ..., x_0^k\}\) on the interval \([\tilde{x}, \tilde{x}]\) is introduced such that

\[\tilde{x} = x_0^1 < x_0^2 < ... < x_0^k = \hat{x}.\]

By the algorithm described above, approximations \(w_0^k\) to \(w(x_0^k)\) are computed.

Set \(z_0^i = (x_0^i, w_0^i)\); the set

\[\mathcal{W}_0 = \{z_0^1, ..., z_0^k\}\]

is a discrete representation of the local stable manifold.
To obtain a discrete representation of the global manifold $W^s$ the set $W_0$ is iterated. Define

$$z^i_t = \varphi^t(z^i_0), \quad \text{where } i = 1, ..., k \quad \text{and } t = 1, 2, ..., N.$$  

Then the set

$$W = \{ z^1_0, z^2_0, ..., z^k_0, ..., z^k_{k-1}, z^k_N \}$$  \hspace{1cm} (4.5)

is a discrete representation of $W^s$.

### 4.3 The value functions

Recall the two kinds of value functions introduced in Chapter 2: the orbit value function $v$ (cf. equation (2.23)) and the value function $\bar{V}$ (cf. equation (2.24)).

In Section 2.2 it is shown that if the stable manifold of $\bar{z}$ can be represented as the graph of a function $y = w(x)$, then the local value function $\tilde{V}$ associated to $\bar{z}$ (cf. equation (2.22)) satisfies

$$\frac{d\tilde{V}}{dx} = w,$$  \hspace{1cm} (4.6)

at least locally around $\bar{z}$. To recover the value function from $w$ requires an integration. To find the values at other points in the state space the Bellman equation is used. It follows from equation (4.6) that

$$\tilde{V}(x) = \tilde{V}(0) + \int_0^x w(\xi) \, d\xi.$$  

As $w(0) = 0$, an approximation $\tilde{V}_\sim$ of $\tilde{V}$ is obtained by replacing $w$ by its first order Taylor approximation

$$\tilde{V}_\sim(x) = \tilde{V}(0) + \int_0^x w'(0) \xi \, d\xi.$$  \hspace{1cm} (4.7)
Figure 4.2: The plot on the left shows a part of the stable manifold $W^s$ parametrised by a parameter $\sigma \in \mathbb{R}$ as $z(\sigma) = (x(\sigma), y(\sigma))$. In the right hand plot, the orbit value function $v$, which associates the value $v(\sigma) = v(z(\sigma))$ to a point $z(\sigma) \in W^s$, is shown. The value function $\tilde{V}$ associated to $W^s$ is indicated by a black thick line. The state $x = x_0$ corresponds to an indifference point; at this state, the value function $\tilde{V}$ has a kink.

Note that $|\tilde{V}_z(x) - \hat{V}(x)| \leq cx^3, c > 0$. A more accurate approximation could be achieved by a higher order approximation of the integral.

4.3.1 The value function and indifference points. In this section values for points of $W^s$ are computed.

Suppose that $z = \{z_t\}$ is an orbit starting at $z_0 \in W^s$. As seen in Chapter 2, the orbit value function $v(z_0)$ associated to $W^s$ is defined as

$$v(z_0) = \bar{J}(z) = \sum_{t=1}^{\infty} G(z_{t-1}) e^{-\rho t},$$

where $G$ is given in (2.27). Note that it may happen that there are several
$z_0^{(1)}, z_0^{(2)}, \ldots \in W^s$ such that $x_0^{(i)} = x_0$. In this setting the definition of the value function $\bar{V}$ associated to $W^s$ (cf. (2.24)) read as

$$\bar{V}(x_0) = \sup \left\{ v(z_0^{(i)}) : x_0^{(i)} = x_0 \right\}$$

(recall that $\sup \emptyset = -\infty$). The orbit value function $v$ is approximated using the discrete representation $\mathcal{W}$ of $W^s$. For every point $z_l^i \in \mathcal{W}$ and for every $1 \leq l \leq k$ a value $v_l^i$ is computed as follows. For $t = 0$, the approximation (4.7) is used to compute $v_0^i = \tilde{V}_\infty(x_0^i)$. For $t \geq 1$ the value $v_l^i$ is computed using the Bellman equation

$$v_l^i = G(x_l^i, y_l^i) + e^{-\rho} v_{l-1}^i.$$

The set

$$\mathcal{V} = \{v_0^k, \ldots, v_N^{k-1}, v_N^k\}$$

is then a discrete representation of the orbit value function associated to $W^s$.

In order to obtain a discretisation of the value function $\bar{V}$ associated to $W^s$, an equidistant grid $\{x_1, x_2, \ldots, x_n\}$, $n > 0$, is introduced on $\mathcal{X}$. The elements of $\mathcal{W}$ are relabelled to obtain a sequence $\tilde{z} = \{\tilde{z}_s\}_{s=1}^S$ in such a way that

$$\tilde{z}_{jk+i} = z_j^i, \quad 1 \leq i \leq k \quad \text{and} \quad 0 < j \leq N.$$

In the same way a sequence $\tilde{v} = \{\tilde{v}_s\}_{s=1}^S$ of values is defined, such that $\tilde{v}_{jk+i} = v_j^i$.

Let $\pi : T^* \mathcal{X} \to \mathcal{X}$ be given as $\pi(x, y) = x$. A turning point of $W^s$ is a point $z \in W^s$ such that $T_z W^s \subset \ker D\pi$, that is a point where $W^s$ is tangent to the vertical direction. A discrete turning point is a point $\tilde{z}_s$ such that

$$(\tilde{x}_s - \tilde{x}_{s-1})(\tilde{x}_{s+1} - \tilde{x}_s) \leq 0.$$
Figure 4.3: Discrete turning points $\zeta_1$ and $\zeta_2$ of $W$.  

Let $\zeta_1, \zeta_2, \ldots, \zeta_L$ be the sequence of discrete turning points of $W$ (see Figure 4.3); that is $\zeta_i = \tilde{z}_{s_i}$ for $s_1 < s_2 < \ldots < s_L$. Introduce the notation

$$[a, b] = \{x| \min(a, b) \leq x \leq \max(a, b)\}.$$

If $x^n \in [\tilde{x}_i, \tilde{x}_{i+1}]$ then there is $\vartheta = \vartheta_i(x^n)$ such that $0 \leq \vartheta \leq 1$ and

$$x^n = (1 - \vartheta) \tilde{x}_i + \vartheta \tilde{x}_{i+1}.$$

Determine $y^n$ and $\bar{v}^n$ by linear interpolation

$$y^n = (1 - \vartheta) \bar{y}_i + \vartheta \bar{y}_{i+1},$$
$$\bar{v}^n = (1 - \vartheta) \bar{v}_i + \vartheta \bar{v}_{i+1},$$

where $\vartheta$ is as before. Note that $x^n$ may be contained in several segments $S_\ell = [\zeta_\ell, \zeta_{\ell+1}]$. In this case, values for $y^n_\ell$ and $\bar{v}^n_\ell$ corresponding to each
segment are obtained. Let $\bar{v}_x^n = -\infty$ if $x^n \notin S_\ell$. Now $\bar{v}_x^n$ is defined for all $1 \leq \ell \leq L$.

The approximation of $\bar{V}(x^n)$ obtained is then given as

$$\bar{V}_z^n = \max_{\ell} \{ \bar{v}_x^n \}.$$ 

If the maximum is obtained for $\ell = \ell^*$, then $(x^n, y^n_\ell)$ is an approximation to the initial point of the trajectory tending to $\bar{z}$ that has value $\bar{V}(x^n)$.

Finally indifference points are located numerically: these are states $x \in X$ for which two different points $(x, y_1) = z_1 \in W^s$ and $(x, y_2) = z_2 \in W^s$ have both the same associated values $v(z_1) = v(z_2) = \bar{V}(x)$. A decision maker is indifferent between the trajectories starting at $z_1$ and $z_2$. To obtain an approximation to the indifference points, first the segment value function $V_\ell : X \to \mathbb{R}$ corresponding to the segment $S_\ell$ is constructed. The $V_\ell$’s are piecewise linear functions, obtained by linear interpolation between points $(x^n_\ell, V^n_\ell)$ and $(x^{n+1}_\ell, V^{n+1}_\ell)$. A point $\hat{x}$ such that $V_{\ell_1}(\hat{x}) = V_{\ell_2}(\hat{x})$ and $V_{\ell_1}(\hat{x}) \geq V_\ell(\hat{x})$ for all $\ell$ is a numerical approximation of an indifference point. A bisection method is used to find these points.

### 4.3.2 Indifference thresholds.

Recall the assumption that the phase map $\varphi$ has two saddle fixed points $z_- = (x_-, y_-)$ and $z_+ = (x_+, y_+)$, with $x_- < x_+$, and associated stable manifolds $W^s_-$ and $W^s_+$. Let the value function corresponding to the stable manifold $W^s_j$ be denoted by $\bar{V}_j$, $j \in \{-, +\}$.

Recall that an indifference threshold is an indifference point for which the originating optimal solutions converge to different steady states (see Chapter
3). To find an approximation of such a threshold the stable manifolds $W^-_s$ and $W^+_s$ and their associated value functions $\bar{V}_-$ and $\bar{V}_+$ are computed by the methods which have been described in the previous sections. For a grid $(x^1, ..., x^n)$ in $\mathcal{R}$, piecewise linear functions $\bar{V}_j$, $j \in \{-, +\}$ are computed by linear interpolation between points $(x^n, \bar{V}_j(x^n))$ and $(x^{n+1}, \bar{V}_j(x^{n+1}))$. Intersection points $(\bar{x}, \bar{y})$ of the graphs of these functions are then computed by the bisection method. The state component $\bar{x}$ of this point is a numerical approximation of the indifference threshold.

4.4 Determining indifference-attractor bifurcations numerically

This section deals with the numerical analysis of the indifference-attractor bifurcation curves. Consider the parametrised family $\varphi = \varphi_\mu$ of phase maps associated to the optimisation problem (4.1).

Indifference thresholds can be generated in an indifference-attractor bifurcation (see Chapter 3). In such a bifurcation, a semi-stable optimal state emerges as a parameter is varied and splits into a locally stable attractor and an indifference threshold. More precisely, there is a parameter value $\mu_c$ such that for $\mu < \mu_c$, the point $x_+$ is a global attractor for the optimal dynamics. For $\mu > \mu_c$, there is an indifference threshold $x_s$, and both $x_-$ and $x_+$ are local attractors of the optimal dynamics. If $\mu = \mu_c$, then the orbit $x = x_-$ is semistable: all orbits starting to the left of it converge to $x_-$, while all orbits starting to the right converge to $x_+$. The bifurcation value $\mu_c$ is determined by a geometric criterion which is recalled next and which is used to compute the indifference-attractor bifurcation curve numerically.
4.4.1 Determination of the bifurcation parameter value. Assume that the parts of $W_u^-$ and $W_s^+$ that interact in the heteroclinic bifurcation are parametrised by arclength, starting from the respective fixed points. That is, the parametrisations $\gamma_s(\sigma) = (x_s(\sigma), y_s(\sigma))$ and $\gamma_u(\sigma) = (x_u(\sigma), y_u(\sigma))$ satisfy $\gamma_u(0) = z_-$ and $\gamma_s(0) = z_+,$ and $||\gamma_u'(\sigma)|| = ||\gamma_s'(\sigma)|| = 1.$ Note that these parametrisations determine the orientations of $W_s^+$ and $W_u^-.$ Using these parametrisations, the intersection number of a heteroclinic point $p = \gamma_s(\sigma_1) = \gamma_u(\sigma_2)$ is defined as $\text{sgn} \left( \det (\gamma_s'(\sigma_1)\gamma_u'(\sigma_2)) \right),$ where

$$\text{sgn}(x) = \begin{cases} 
1 & \text{if } x > 0, \\
0 & \text{if } x = 0, \\
-1 & \text{if } x < 0
\end{cases} \quad (4.8)$$

A heteroclinic orbit is upward if the intersection number of one point (and hence all points) of the orbit is +1 (cf. Section 3.3). Let $p = \{p_t\}_{t=-\infty}^\infty$ be an upward heteroclinic orbit including $W_s^+$ and $W_u^-.$ For simplicity assume that $p$ is the only upward orbit. Moreover, assume that $p_0$ is such that smooth curves $c^u, c^s$ as postulated in Assumption 3.1.6 exist, connecting $p_0$ to $p_{-1}.$ The closed curve $c = c^s + c^u$ is then the boundary of a region $A;$ the positively and negatively oriented components of $A$ are defined by $A^+$ and $A^-$ respectively (see Figure 4.4). The designation $A_\mu$ is used for $A$ to stress the dependence on the parameter $\mu.$ As the family $\varphi_\mu$ goes through a heteroclinic bifurcation scenario, there is a parameter interval $[\mu_1, \mu_2] \subset \mathcal{P}$ such that for $\mu < \mu_1$ and $\mu > \mu_2,$ the manifolds $W_u^-$ and $W_s^+$ have no points in common, and for $\mu \in [\mu_1, \mu_2]$ there is at least one heteroclinic orbit. Moreover all heteroclinic orbits are necessarily tangencies at $\mu = \mu_1$ and $\mu = \mu_2.$
Figure 4.4: The region $A$, bounded by the curve $c = c^s + c^u$

Let

$$\Delta : (\mu_1, \mu_2) \to \mathbb{R}$$

be the function that assigns to each parameter $\mu_1 < \mu < \mu_2$ the area difference between $A^-_\mu$ and $A^+_\mu$, that is

$$\Delta(\mu) = \Omega(A_\mu) = \text{area}(A^-_\mu) - \text{area}(A^+_\mu),$$

where $\Omega$ is as equation (3.4). Recall from Theorem 3.2.3 that at the indifference-attractor bifurcation

$$\text{area}(A^+_\mu) = \text{area}(A^-_\mu);$$

that is, the bifurcation parameter value $\mu_c$ is a zero of $\Delta(\mu)$. Therefore, determination of $\mu_c$ amounts to solving the equation $\Delta(\mu) = 0$.

Let $c_1^s$ and $c_1^u$ be the segments of $c^s$ and $c^u$ which connects $p_0$ to $r$; similarly
$c_2^s$ and $c_2^u$ be the segments of $c^s$ and $c^u$ which connects $r$ to $p_{-1}$ respectively. The curves $c_1 = c_1^s + c_1^u$ and $c_2 = c_2^s + c_2^u$, which are the boundaries of the regions $A^+$, $A^-$ receptively, are simple closed curves. Therefore given the orientation of $A^+$ and $A^-$, the areas of these regions are obtained as follows

\[
\text{area}(A^+) = \int \int_{A^+} dx \, dy = \int_{c_1^u} y \, dx - \int_{c_1^s} y \, dx,
\]

\[
\text{area}(A^-) = \int \int_{A^-} dx \, dy = \int_{c_2^u} y \, dx - \int_{c_2^s} y \, dx.
\]

Hence

\[
\Delta(\mu) = \text{area}(A^+) - \text{area}(A^-) = \int_{c_u} y \, dx - \int_{c_s} y \, dx.
\]

Note that this formula is general, i.e. it does not depend on the assumption of $p$ being the only upward orbit. If there are several upward orbits $p^{(k)}$, $k = 1, 2, ..., K$, with $p^{(1)} = p$ then differences $\Delta^{(k)}(\mu)$ have to be computed for each of them. The indifference-attractor bifurcation then satisfies $\Delta^{(1)}(\mu_c) = 0$ and $\Delta^{(k)}(\mu_c) > 0$ for $k > 1$ (cf. Theorem 3.2.3).

Recall the parametrisation of $W^s$ and $W^u$. There exist $\sigma_i$ and $\bar{\sigma}_i$, $i = 1, 2$, such that $p_0 = \gamma_s(\sigma_1) = \gamma_u(\sigma_2)$ and $p_{-1} = \gamma_s(\bar{\sigma}_1) = \gamma_u(\bar{\sigma}_2)$. The line integrals in the expression of $\Delta(\mu)$ are rewritten as follows

\[
\Delta(\mu) = \int_{\sigma_2}^{\sigma_1} y_u(\sigma) \, x_u'(\sigma) \, d\sigma - \int_{\bar{\sigma}_2}^{\bar{\sigma}_1} y_s(\sigma) \, x_s'(\sigma) \, d\sigma. \quad (4.9)
\]

In this way a numerical approximation of $\Delta$ is obtained using the discrete representation of invariant manifolds computed in the previous section. The secant root finding method (see Atkinson (1989)) is used to solve the equation $\Delta(\mu) = 0$ and to obtain an approximation of the indifference-attractor bifurcation parameter value $\mu_c$. 

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4.4.2 Bifurcation diagram in parameter space. In the previous section it was assumed that the phase map $\varphi$ depends on a scalar parameter $\mu$. However, in applications the phase map may depend on vector-valued parameters. The method generalises easily to this more complex case, yielding bifurcation manifolds instead of bifurcation points.

In the special case that $\mathcal{P} \subset \mathbb{R}^2$ the equation $\Delta(\mu) = 0$ determines a curve in the parameter space, which can be computed by a continuation method.
Chapter 5

Application to the lake system

In Mäler et al. (2003) the economics of a lake pollution problem is analysed as an optimal management problem, and in the situation that the lake is a common property resource, as a differential game. The social planner, or in the game context, the players, weigh the conflicting interests of farmers, who indirectly pollute the lake through agricultural activities, and lake users, who are affected as ecosystem services of the lake decrease.

Initially most lakes are in a clear water state. However, due to heavy use of fertilisers by farmers, at some point lakes flip from a clear state to a turbid state that is caused by a dominance of phytoplankton (Carpenter and Cottingham 1997; Scheffer 1997). Lakes are hard to restore to the clear water state in the sense that the nutrient loads have to be reduced far below the level where the flip occurred before the lake returns to a clear state.

When the lake flips to a turbid state, the value of the ecological services of the lake decreases, but there is a high level of agricultural activities. It
depends, of course, on the relative weight attached to these welfare com-ponents whether it is better to keep the lake clear or to use it as a waste dump. The complexity of the lake optimal management problem derives from the non-linear dynamics of the lake that leads to a non-convex optimal control problem featuring several system parameters. In such problems, depending on the values of these parameters, there may exist multiple steady states that are the long-run outcome of an optimal management policy. Also, the structure of optimal solutions may change if parameters are varied. The bifurcation analysis developed in Chapter 3 is an appropriate tool to classify the qualitative characteristics of the set of optimal solutions for different values of the model parameters.

5.1 Lake dynamics

Nature is often expected to respond to gradual changes in a smooth way. However, studies of lakes, coral reefs, oceans, forests and arid lands have shown that smooth change can be interrupted by sharp (or catastrophic) shifts to different regimes (Scheffer et al. 2001; Carpenter 2003). One of the best-studied catastrophic shifts is the sudden loss of transparency and vegetation observed in shallow lakes, i.e. lakes with a depth less that 3 meters, as a result of human activities. Initially shallow lakes have clear water and a rich submerged vegetation. However, nutrient loading may change this. For instance nutrients arrive in the lake as a result of the use of artificial fertilisers on surrounding land; they are washed into the lake by rainfall.

Initially water clarity in the lake seems to be hardly affected by the increased amount of nutrients until a critical threshold is reached, at which the lake shifts abruptly from a clear state to a turbid state characterised by
a dominance of phytoplankton. With this increase in turbidity, submerged plants largely disappear, releasing the lake sediment, which causes resuspension of nutrients in the lake. Now, even an instant reduction in nutrient loading is not enough for water plants to regrow easily, as the turbidity of the lake remains high. Longer periods of reduction or an alternative treatment is needed to reduce the nutrient load and hence the turbidity in the lake sufficiently, as the load has to be reduced far below the level where the flip occurred before the lake can flip back to a clear state. The lake is said to show hysteresis. In some cases the turbidity of the lake is even irreversible.

The lake model that is used in the following gives a very simplified representation of these complex ecological feedback mechanisms that are active in a shallow lake. The lake model can also be viewed as a metaphor for general ecological systems with tipping points, so that the analysis developed here will have a wider applicability (cf. Scheffer et al. (2001)).

5.1.1 The lake model. The dynamics of a shallow lake, which was described above, can be modelled as a single non-linear difference equation

$$x_t = u_t + (1 - b)x_{t-1} + h(x_{t-1}).$$

(5.1)

Here $x_t$ is the concentration of phosphorus, one of the main nutrients, in the lake. Artificial fertilisers containing phosphorus are used on the fields surrounding the lake. The phosphorus is washed into the lake by rainfall, yields a net inflow $u_t$ of phosphorus. The parameter $b$ denotes the sedimentation rate at which phosphorus leaves the water column and enters the sediment at the bottom of the lake. The last term models the internal production of phosphorus in the lake, e.g. through resuspension of the sediment, and is
assumed to be an S-shape function that has its inflection point at the point $x = 1$:

$$h(x) = \frac{x^q}{1 + x^q}.$$  

The exponent $q$, the *responsiveness* of the lake, is proportional to the steepness of $h$ at $x = 1$.

For a constant pollution loading $u_t = u$ for all $t$, the fixed points of the lake are solutions of the equation

$$u = g(x) = bx - \frac{x^q}{1 + x^q},$$

which is illustrated for $b = 0.6$, and $q = 2$ and $q = 4$ in Figure 5.1.

![Figure 5.1](image-url)

(a) Weakly responsive lake.  
(b) Strongly responsive lake.

**Figure 5.1**: Location of fixed points for constant pollution streams $u_t = u$ for all $t$, plotted for $b = 0.6$, and for (a) weakly ($q = 2$) and (b) strongly responsive lakes ($q = 4$). Indicated are stable (solid) and unstable fixed points (dashed).

For both values of $q$ there is a range of $u$-values such that there are multiple steady states. However, the range is bigger for $q = 4$ than for $q = 2$. If the system starts in a low pollution steady state, and if $u$ is then raised very slowly past the first critical value (i.e. the local maximum of $g$) it switches
to a high pollution steady state. A small subsequent decrement of $u$ will not move the system back to the clean branch of steady states. For this, the pollution flow has to be lowered significantly, below the second critical value (the local minimum of $g$).

There is a value $b = b^*$ such that for $b < b^*$ the lake can be trapped in the high pollution steady state of phosphorus. This happens if the first flip, which occurs at $u = \bar{u}$, is irreversible. The critical value is $b^* = \sqrt{27}/4 \approx 0.57$ for $q = 4$ and $b^* = 0.5$ for $q = 2$ (see Figure 5.2 for the case $q = 4$).

![Figure 5.2: Irreversibility; location of fixed points for constant pollution streams $u_t = u$ for all $t$, plotted for $b \approx 0.57$ and $q = 4$. Indicated are stable (solid) and unstable fixed points (dashed).]

5.1.2 Optimal pollution management in lakes. In the lake pollution management problem, a social manager has to weigh the interest of the farmers that derive income from the use of artificial fertilisers against that of the lake users that suffer from pollution damage to the lake. Following
Mäler et al. (2003), the social utility functional is modelled as

\[ J = \sum_{t=1}^{\infty} \left( \log u_t - cx_{t-1}^2 \right) e^{-\rho t}. \]

Here \( c \) is the social preference parameter, and \( \rho > 0 \) the discount rate.

The social manager tries to optimally manage the phosphorus pollution stream

\[ u = \{u_t\}_{t=1}^{\infty} \]

that originates from the use of artificial fertilisers given that the concentration \( x_t \) of phosphorus in the lake follows the lake dynamic (5.1). The optimisation problem is to maximise

\[ J = \sum_{t=1}^{\infty} \left( \log u_t - cx_{t-1}^2 \right) e^{-\rho t}, \] (5.2)

subject to

\[ x_t = u_t + (1-b)x_{t-1} + \frac{x_{t-1}^q}{1+x_{t-1}^q}. \] (5.3)

State space and control space are given as \( \mathcal{X} = \mathcal{U} = (0, \infty) \) respectively.

The discrete Pontryagin function is

\[ P = \log u - cx^2 + y \left( u + (1-b)x + \frac{x^q}{1+x^q} \right). \]

Note that \( P_{uu} < 0 \) for all \( u > 0 \). The necessary condition \( P_u = 0 \) takes the form

\[ 0 = P_u = \frac{1}{u} + y. \]

Solving for \( u \) yields that

\[ u = U(x, y) = -1/y. \]
Substituting out \( u \), the discrete Hamilton function is obtained as
\[
H = -\log(-y) - cx^2 - 1 + y \left( (1 - b)x + \frac{x^q}{1 + x^q} \right).
\]
Since \( H_{yy} = y^{-2} > 0 \) and
\[
H_{xy} = 1 - b + q \frac{x^{q-1}}{(1 + x^q)^2} > 0,
\]
Assumption 3.1.1 is satisfied.

The necessary conditions read as
\[
\begin{align*}
x_t &= H_y = -\frac{1}{y_t} + (1 - b)x_{t-1} + \frac{x_{t-1}^q}{1 + x_{t-1}^q}, \\
e^\rho y_{t-1} &= H_x = -2cx_{t-1} + y_t \left( (1 - b) + q \frac{x_{t-1}^{q-1}}{(1 + x_{t-1}^q)^2} \right).
\end{align*}
\]
Solving the second equation for \( y_t \) and substituting into the first yields the phase map
\[
\varphi(x, y) = \left( -\frac{(1 - b) + q \frac{x^{q-1}}{1 + x^q}}{e^\rho y + 2cx} + (1 - b)x + \frac{x^q}{1 + x^q} \left( (1 - b) + q \frac{x^{q-1}}{(1 + x^q)^2} \right) \right).
\]
Using
\[
g(x) = (1 - b)x + x^q/(1 + x^q),
\]
this expression can be written as
\[
\varphi(x, y) = \left( -\frac{g'(x)}{e^\rho y + 2cx} + g(x), \frac{e^\rho y + 2cx}{g'(x)} \right).
\]

### 5.2 Indifference-attractor bifurcations

In the rest of the chapter, the value of \( \rho \) is fixed to \( \rho = 0.03 \). For \( b = 0.6 \) and \( q = 4 \), in Figure 5.3 fixed points and their stable and unstable manifolds are plotted for a range of values of \( c \); for all these values, the phase map has two saddle fixed points \( z_- \) and \( z_+ \).
Figure 5.3: Subfigures (a) and (b) show affecter-friendly configurations (low values of $c$), and subfigure (c) depicts an enjoyer-friendly configuration (high value of $c$). Solid lines indicate stable manifolds, dotted lines unstable manifolds; optimal solutions are marked by thick lines. Note that $y < 0$ throughout, so that the $x$-axis is at the top of the figure. On the $x$-axis, the optimal dynamics are indicated; attractors are marked by a circle, the indifference threshold by a diamond.
It can be shown that Assumption 3.1.3 is satisfied. Taking \( \mu = c \) and accepting the geometric evidence from the plots in Figure 5.3, the intermediate value theorem implies that Assumption 3.1.4 is satisfied as well. At least at \( c = 0.1541 \), geometric evidence also supports Assumptions 3.1.5 and 3.1.6. Granting the assumptions, Theorem 3.1.1 applies.

It should be noted that for \( c = 0.14 \), that is, in a situation where the interest of farmers weighs relatively heavily, it is for every initial state \( x_0 \in \mathcal{X} \) optimal to steer the lake to the high pollution state \( x_+ \). There is a single upward orbit \( p \) such that the value \( c = c_{IA} \approx 0.1541 \) corresponds to the case \( \Omega(A_p) = 0 \) (cf. Section 3.2); it follows from the Main Theorem that then for \( x_0 \leq x_- \) the optimal policy steers the lake to the low pollution state \( x_- \), while if \( x_0 > x_- \), it is optimal to end at the high pollution state \( x_+ \).

Moreover, Theorem 3.1.1 implies that for \( c = c_{IA} \) there is a countable infinity of indifference states. Recall that indifference states are initial states to two distinct optimal policies. At these points the policy function jumps; two of these jumps can be seen in Figure 5.4.

Finally, for \( c = 0.17 \), both \( x_- \) and \( x_+ \) are locally optimal, and their basins of attraction are separated by an indifference threshold.

At first, the shape of the optimal costate rule shown in Figure 5.4, which is related to the shape of the optimal policy function by the transformation

\[
    u = -1/y,
\]

is completely counter-intuitive, as this function is varying wildly over a relatively small range of state values. However, it should be born in mind that the saddle manifold of \( z_+ \), of which the graph of the optimal costate function is
Figure 5.4: For \( c = 0.1541 \), the equality \( \Omega(A) = 0 \) holds, and consequently there is an infinity of indifference points.

a part, consists of many orbits. Consider for instance the rightmost indifference point in Figure 5.4. At that point, two different policies are equivalent, one corresponding to relatively high agricultural activity, high pollution and fast convergence to the steady state \( z_+ \), the other characterised by lower pollution and slower convergence to \( z_+ \). Though the asymptotic steady state is equal in both cases, the policies that reach it are vastly different.

5.2.1 The bifurcation diagram. Setting \( q = 4 \) the indifference-attractor bifurcation diagram of \( \varphi \) in Figure 5.5 is plotted. Furthermore, the bifurcation diagrams corresponding to \( q = 2 \) and \( q = 4 \) in Figure 5.6 are depicted and the effect of the change of the responsiveness of the lake on the optimal solution is studied.
Figure 5.5: Bifurcation diagram of the highly responsive lake ($q = 4$).
Figure 5.5(d) shows the bifurcation diagram of the lake system in the 
\((b, c)\)-parameter space for \(q = 4\) and \(\rho = 0.03\). The dashed curve represents 
saddle-node bifurcations, separating the region of values of the parameters for 
which the phase map has a fixed point from the region of multiple fixed points. 
Solid lines indicate indifference-attractor bifurcation curves, separating three 
parameter regions: (i) the low pollution region for which the clean steady 
state is globally optimal, (ii) the high pollution region for which the turbid 
steady state is globally optimal, and (iii) the dependent on the initial state 
region for which both the clean steady state and turbid steady state are 
locally optimal.

Phase portraits of the phase map \(\varphi\) and of the optimal map \(\varphi^o\) are given 
for \(b = 0.7\) in Figure 5.5(b) and \(b = 0.6\) for (c) – (f) and for \(b = 0.7\) in Figures 
5.5 (a) and (b). The graphs of the optimal costate maps are represented by 
thick curves; these graphs are subsets of either \(W^-\) or \(W^+\). The other parts of 
the stable manifolds are indicated by solid curves, and the unstable manifolds 
are given as dotted curves. Note that \(y < 0\) throughout, so that the \(x\)-axis is 
at the top of the figure. On the \(x\)-axis, the optimal dynamics are indicated; 
attractors are marked by a circle, indifference thresholds by diamonds.

For the values of the parameters \(b\) and \(c\) in the unique equilibrium region 
the phase map \(\varphi\) has a unique fixed point. This is a saddle, see Figure 
5.5(a). The optimal orbits are always situated on its stable manifold. The 
long run pollution level depends then on the values of the parameters \(c\) and 
\(b\), changing within the region.

If the pair \((b, c)\) corresponds to a point of the dependent on the initial state 
region, the phase map \(\varphi\) has always two saddle fixed points characterised by
respectively low pollution and high pollution. The clear state of the lake corresponds to a high level of water services and a low level of agricultural activities, whereas the turbid state corresponds to a high level of agricultural activities and a low level of water services. Depending on the initial pollution load, the social planner steers the lake to the clear or to the turbid steady state.

If the pair \((b, c)\) is in the low pollution region the optimal policy steers the lake to the clean steady state independently of the initial state of the lake; the clear state of the lake is globally optimal (see Figure 5.5(a)). Note that the optimal policy imposes very small values of \(u\) for intermediate values of \(x\), until the clear steady state is reached, in order to get rid of the turbidity (Figure 5.5(a)).

If \((b, c)\) is in the high pollution region, see Figure 5.5(e) and (f), the optimal orbit lies on the stable manifold of the polluted equilibrium \(W^*_s z_+\). Regardless of the initial state of the lake, the optimal policy steers the lake to the turbid state, that is the turbid steady state is globally optimal. Note in Figure 5.5(e) again the equivalence of different policies both leading to \(z_+\). In Figure 5.5(f) the optimal policy prescribes lower values of \(u\) for intermediate values of \(x\), in order to keep the pollution level low as long as is optimally possible.

For a pair \((b, c)\) in the dependent on the initial state region, there exist an indifference threshold, see Figure 5.5(c). If the initial state is below the threshold then the clean steady state is optimal, whereas if the initial state is above the threshold then the turbid steady state is optimal. An indifference threshold separates two basins of attraction of the optimal state map: the
states below the threshold form the basin of attraction of the clear state, and the states above that threshold constitute the basin of attraction of the turbid state.

Therefore, for a pair \((b, c)\) in the dependent on the initial state region the lake is steered to the clear state only if it is initially not very polluted, otherwise it is steered to the turbid state. Note that at the indifference threshold, two different policies are radically different and non-equivalent, one corresponding to high agricultural activity, high pollution and convergence to the steady state \(z_+\), whereas the other is characterised by lower pollution and convergence to \(z_-\).

5.2.2 Effects of the responsiveness of the lake. Recall that the responsiveness of a lake \(q\) is given as the maximum steepness of the function \(h\). Note moreover that the lake system (5.1) can have multiple stable states only if

\[
\max h'(x) > b.
\]

Thus, higher values of \(q\) corresponds to more sudden shifts between the clean and the turbid regimes. This means that the regime shift will be more pronounced in a lake with a higher responsiveness, whereas this shift will be more gradual in a lake with relatively low responsiveness. To illustrate this in Figure 5.6 two bifurcation diagrams in the \((b, c)\)-parameter space are given: for a weakly responsive lake in Figure 5.6(a), and for a strongly responsive lake in Figure 5.6(b).
Figure 5.6: Figure 6(a) and 6(b) show the bifurcation diagram of the discrete time lake system in the \((b,c)\)-parameter space for \(q = 2\) and \(q = 4\) respectively. The dashed curve represents saddle-node bifurcations of the state-costate system, separating the region of values of the parameters for which the phase map has a fixed point from the region of multiple fixed points. Solid lines indicate indifference-attractor bifurcation curves, separating four regions of values of the parameters: (i) the low pollution region for which the clean steady state is globally optimal, (ii) the high pollution region for which the turbid steady state is globally optimal, (iii) the dependent on the initial state region for which both the clean steady state and turbid steady state are locally optimal, and (iv) the unique equilibrium region.

The main difference between these diagrams is that the high pollution region is much smaller for the strongly responsive lake, whereas the low pollution region is much larger. Put differently, for a given value of the physical parameter \(b\) the minimal value of the economic weight \(c\) of the lake for which it is always optimal to steer the lake to the clean steady state is much lower in the strongly responsive lake. This was to be expected, as the
impact of regime shift towards the turbid regime would be felt much earlier in time in the strongly responsive case, hence it is optimal to avoid such a shift for a much larger range of values of $c$. 

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Chapter 6

Summary

In this thesis a class of autonomous discrete time infinite horizon optimal control problems with non-convex state dynamics is studied. Methodologically this thesis contributes to the geometrical or phase space analysis of discrete time optimal control problems. Contrary to the situation in the theory of continuous time problems, these methods are not particularly popular in the discrete setting. The main reason for this seems to be that some powerful instruments of the continuous time theory are not readily available.

In Chapter 2 the necessary first order conditions of this class of optimal control problems are formulated in terms of a boundary value problem of the associated phase map $\varphi$. If this map possesses a saddle point, then all orbits on its stable manifold will solve the boundary value problem. In the situation that there are several saddle points whose stable manifold cover overlapping parts of the state space of the problem, or in the situation that there are several points on a single stable manifold that project down to the same point in state space, the values of the extremal orbits originating at
these points have to be compared. The chapter provides a number of results that allow to make such comparisons.

Systems where the phase map \( \varphi \) has a unique saddle point are encountered in many places in the economic literature (e.g. Ramsey (1928)). Usually in these systems the saddle corresponds to an optimal steady state which is such that all solutions, regardless of the initial state of the system, tend towards it. Whenever there is more than one saddle point present (c.f. Dechert and Nishimura (1983)), or when there is an additional optimal trajectory tending towards infinity (c.f. Hinloopen et al. (2011)), the solution structure is more complicated. If, in the two-saddle case, both correspond to an optimal steady state, there is an indifference threshold (Skiba state) that is initial state to two different optimal trajectories.

In analogous continuous time problems, whether or not a Skiba state occurs in a system depends on the relative position of the stable and unstable manifolds of the saddle equilibria of the phase flow. In particular, the shift from one type of solution to another is characterised by occurrence of a heteroclinic connection, where the stable manifold of one saddle equilibrium coincides with the unstable manifold of the other saddle. In Chapter 3 it has been shown that in discrete time problems the situation is analogous but more complex, due to the fact that, unlike in continuous time, stable and unstable manifolds do not automatically coincide once they have a single point in common. More precisely, the genesis of indifference points in a so-called indifference-attractor bifurcation is linked to heteroclinic bifurcations of the family of phase maps \( \varphi \), and the consequences for the optimal solutions are analysed. In particular, the bifurcation value at which an indifference threshold appears is characterised by a geometric condition, and it is found
that at the bifurcation there are countably infinitely many indifference points that are not indifference thresholds.

In most applications it is impossible to determine analytic expressions for invariant manifolds, so numerical methods are needed: these are discussed in Chapter 4. A simple algorithm is described to compute invariant manifolds numerically. This information is used to determine the locus of the indifference-attractor bifurcation points.

The results and methods developed in this thesis are applied to the lake pollution management problem of Mäler, Xepapadeas and de Zeeuw (Mäler et al. (2003)). In ecological systems such as lakes, internal positive feedbacks may trigger catastrophic shifts. In the lake problem the economics of lake pollution is analysed in terms of tradeoffs between the benefits of agricultural activities, which are responsible for the release of nutrients, and the costs of a polluted lake.

In Chapter 5 a bifurcation analysis of the lake model is performed. The resulting bifurcation diagram summarises the joint effect of the (physical) robustness of the lake and the (economic) importance of the lake on the form of the optimal policy. The diagram is partitioned into four parameter regions: unique steady state, low pollution, high pollution, and dependent on the initial state. In the first region, there is a single fixed point of the phase map that corresponds to a globally attracting steady state of the optimal state dynamics. In the other regions, the phase map has two saddle fixed points that can be distinguished as corresponding to a clean or a polluted steady state. The regions correspond to the situations that either the clean steady state or the polluted steady state are globally attracting, or both
are locally attracting under the optimal state dynamics. These parameter regions are separated by indifference-attractor and saddle-node bifurcation curves.

In the low pollution region, characterised by high robustness and great economic importance of the lake, the optimal policy always steers the lake to the clean steady state independently of the initial state. In the high pollution region – low economic importance of the lake – the polluted steady state is eventually reached under optimal management, irrespective of the initial state. For lakes in the dependent on the initial state region – lakes that are fragile and of medium to high economic importance fall in this category – the outcome of optimal management is dependent on the initial state: if it is sufficiently low, the clean steady state is reached, otherwise the polluted state results. The two regions in state space are separated by an indifference point.

In Chapter 5 also the ‘stiffness’ or responsiveness of the lake is varied. A strongly responsive lake exhibits more sudden shifts between clean and polluted regimes. It is found that for a strongly responsive lake the high pollution region is much smaller compared to a weakly responsive lake, whereas the low pollution region is much larger. That is, it is optimal to avoid regime shift towards the polluted regime if the lake is less economically important, but strongly responsive. Hence, in the pollution management of strongly responsive ecosystems, it is more likely that the optimal policy is ‘green’.
Appendix

Differential forms

Recall from the theory of differential forms the following facts (taken from Spivak, 1965). As the thesis is concerned with 1-forms and 2-forms, the following results are restricted to that case.

A differential 0-form $f$ on $\mathbb{R}^n$ is a real-valued function $f : \mathbb{R}^n \to \mathbb{R}$. A differential 1-form $\eta$ on $\mathbb{R}^n$ is a linear form

$$\eta = \eta_x = \sum_{i=1}^{n} \eta_i(x) \, dx_i,$$

on $\mathbb{R}^n$ with coefficients $\eta_i(x)$ that are differentiable functions; that is, if $v \in \mathbb{R}^n$, then

$$\eta_x(v) = \sum_{i=1}^{n} \eta_i(x)v_i.$$

A differential 2-form

$$w = \sum_{i<j} w_{ij} \, dx_i \wedge dx_j$$

is an anti-symmetric bilinear form:

$$\omega_x(v, w) = \sum_{i<j} \omega_{ij}(x)(v_i w_j - v_j w_i)$$

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The derivative of a 0-form \( f \) is given as \( d f = \sum_i \frac{\partial f}{\partial x_i} \, dx_i \); the derivative of a 1-form \( \eta \) is given as
\[
d \eta = \sum_{i<j} \left( \frac{\partial \eta_j}{\partial x_i} - \frac{\partial \eta_i}{\partial x_j} \right) \, dx_i \wedge dx_j.
\]
Second derivatives are always zero: \( d^2 \omega = 0 \) for any \( k \)-form \( \omega \).

If \( \varphi : \mathbb{R}^m \to \mathbb{R}^n \) is a differentiable map, the pull-back \( \varphi^* \eta \) of a 1-form \( \eta \) is given as
\[
(\varphi^* \eta)_x(v) = \eta_{\varphi(x)}(D\varphi(x)v) = \sum_i \eta_i(\varphi(x))(D\varphi(x)v)_i.
\]
This can be written more suggestively as
\[
\varphi^* \eta = \sum_i \eta_i(\varphi(x)) \, d\varphi_i(x)
\]
Likewise, the pull-back \( \varphi^* \omega \) of a 2-form takes the form
\[
(\varphi^* \omega)_x(v, w) = \omega_{\varphi(x)}(D\varphi(x)v, D\varphi(x)w).
\]
A singular \( k \)-cube \( c \) in \( \mathcal{M} \) is a continuous map \( c : [0, 1]^k \to \mathcal{M} \); a singular \( k \)-chain is a formal finite sum of singular \( k \)-cubes with integer coefficients. The standard \( k \)-cube is \( I^k : [0, 1]^k \to \mathbb{R}^k \) given by \( I^k(x) = x \). The definition of boundary is given only for 2-chains. Define the edges of \( I^2 \) as the singular 1-cubes \( I^2_{(i,0)} \) and \( I^2_{(i,1)} \), \( i = 1, 2 \) for which
\[
I^2_{(1,0)}(x) = I^2(x, 0), \quad I^2_{(1,1)}(x) = I^2(x, 1), \\
I^2_{(2,0)}(x) = I^2(0, x), \quad I^2_{(2,1)}(x) = I^2(1, x).
\]
Define the boundary as the singular 1-chain
\[
\partial I^2 = I^2_{(1,0)} - I^2_{(1,1)} + I^2_{(2,0)} - I^2_{(2,1)}.
\]
For a general singular 2-cube $c$ define first the edges $c_{(i,j)}$, $i = 1, 2$, $j = 0, 1$ by setting

$$c_{(i,j)} = c \circ I_{(i,j)};$$

then define

$$\partial c = c_{(1,0)} - c_{(1,1)} + c_{(2,0)} - c_{(2,1)}.$$

The integral of a differential $k$-form $\omega = f(x) \, dx_1 \wedge \cdots \wedge dx_k$ on $[0, 1]^k$ is given as

$$\int_{[0,1]^k} \omega = \int_{[0,1]^k} f(x) \, dx_1 \cdots dx_k.$$

The integral of a $k$-form on $\mathbb{R}^n$ over a differentiable singular $k$-cube $c$ is given as

$$\int_c \omega = \int_{[0,1]^k} c^* \omega$$

The integral over a singular $k$-chain $c = \sum_i \gamma_i c_i$ is defined as

$$\int_c \omega = \sum_i \gamma_i \int_{c_i} \omega.$$

For instance, if $\omega = P(x, y) \, dx + Q(x, y) \, dy$ is a 1-form in $\mathbb{R}^2$, and $c : [0, 1] \to \mathbb{R}^2$ is given as $c(t) = (c_1(t), c_2(t))$, then

$$\int_c \omega = \int_0^1 \left( P(c(t)) c_1'(t) + Q(c(t)) c_2'(t) \right) \, dt.$$

This is usually called the line integral of $\omega$ over $c$.

The central result about the integration of differentiable forms is Stokes’ theorem, which holds for arbitrary $k + 1$-forms $\omega$ and $k$-chains $c$, and which states that

$$\int_{\partial c} \omega = \int_c \partial \omega.$$
An approximation result

Proposition 6.0.1. Let $a$ be a summable sequence of positive real numbers. Assume that there is a sequence $z^{(1)}, z^{(2)}, \cdots$ of orbits of the phase map $\varphi$, such that

$$\left| G(z^{(k)}_t) e^{-\rho t} \right| \leq a_t$$

for all $k \geq 1$ and all $t \geq 1$. Assume moreover that

$$z^{(k)}_0 \to z^{(\infty)}_0 \text{ as } k \to \infty.$$

Then

$$\tilde{J}(z^{(k)}) \to \tilde{J}(z^{(\infty)}).$$

Proof. Choose $\varepsilon > 0$. Then there is an $T > 0$ such that for any $k$ we have

$$\left| \tilde{J}(z^{(k)}) - \tilde{J}_T(z^{(k)}) \right| = \left| \sum_{t=T+1}^{\infty} G(z_{t-1}) e^{-\rho t} \right| \leq \sum_{t=T+1}^{\infty} a_t < \varepsilon/3.$$

Moreover, if $z$ is an orbit of $\varphi$, note that $J_T(z)$ only depends on the initial segment

$$z_T = (z_0, \cdots, z_T) = (z_0, \varphi(z_0), \varphi^2(z_0), \cdots, \varphi^T(z_0)),$$

which is a continuous function of $z_0$. Therefore, there is a constant $\delta > 0$ such that

$$|z^{(k)}_0 - z^{(\infty)}_0| < \delta \implies |\tilde{J}_T(z^{(k)}) - \tilde{J}_T(z^{(\infty)})| < \varepsilon/3.$$

Take now $N > 0$ such that $|z^{(n)}_0 - z^{(\infty)}_0| < \delta$ for all $n \geq N$. Then

$$|\tilde{J}(z^{(k)}) - \tilde{J}(z^{(\infty)})| \leq |\tilde{J}(z^{(k)}) - \tilde{J}_T(z^{(k)})| + |\tilde{J}_T(z^{(k)}) - \tilde{J}_T(z^{(\infty)})|$$

$$+ |\tilde{J}_T(z^{(\infty)}) - \tilde{J}(z^{(\infty)})|$$

$$\leq \varepsilon.$$

This proves the claim of the proposition.  

\[\Box\]
Bibliography


Samenvatting

In dit proefschrift wordt een klasse van autonome discrete-tijd oneindigehorizon optimale besturingsproblemen met niet-convexe toestandsdynamica bestudeerd. Methodologisch draagt dit proefschrift bij aan de meetkundige of ‘faseruimte’ analyse van dit soort problemen. In tegenstelling tot het geval waar tijd continu is, is deze aanpak niet bijzonder populair in het discrete-tijd geval. De belangrijkste reden hiervoor lijkt te zijn dat een aantal krachtige instrumenten uit de continue-tijd theorie niet beschikbaar zijn.

In hoofdstuk 2 worden de noodzakelijke eerste-orde voorwaarden van deze klasse van optimale besturingsproblemen geformuleerd in termen van een randwaardeprobleem van de bijbehorende fase-afbeelding $\varphi$. Als deze afbeelding beschikt over een zadelpunt, dan zijn alle banen op de stabiele variëteit een oplossing van het randwaardeprobleem. In het geval met meerdere zadelpunten waarvan de projecties van de stabiele variëteiten op de toestandsruimte elkaar overlappen, of in het geval dat verschillende punten van één stabiele variëteit op hetzelfde punt in de toestandsruimte worden ge-projecteerd, worden de waarden van de banen vergeleken die in deze punten beginnen. Het hoofdstuk geeft een aantal resultaten die deze vergelijking mogelijk maken.
Systemen waarin de faseafbeelding $\varphi$ een uniek zadelpunt heeft, worden op veel plaatsen aangetroffen in de economische literatuur (bijv. Ramsey (1928)). Meestal komt in deze systemen het zadelpunt overeen met een dekpunt van de optimale dynamica, welk zodanig is dat alle oplossingen, ongeacht de oorspronkelijke toestand van het systeem, naar dit dekpunt toegaan. Wanneer er meer dan één zadelpunt aanwezig is (vgl. Dechert and Nishimura (1983)), of wanneer er een tweede optimale baan is die naar oneindig gaat (vgl. Hinloopen et al. (2011)), dan is de oplossingstructuur ingewikkelder. Indien in het geval met twee zadelpunten beide corresponderen met een dekpunt van de optimale dynamica, dan is er een onverschilligheidsdrempel (Skibapunt) die de begintoestand is van twee verschillende optimale banen.

In vergelijkbare continue-tijd problemen hangt het bestaan van Skiba-punten af van de relatieve positie van de stabiele en onstabiele variëteiten van de zadeldekpunten van de fasedynamica. In het bijzonder wordt de bifurcatie waarin een onverschilligheidsdrempel ontstaat gekenmerkt door het optreden van een heterocliene verbinding, waarbij de stabiele variëteit van een zadeldekpunt samenvalt met de instabiele variëteit van een andere zadel. In hoofdstuk 3 is aangetoond dat in discrete-tijd problemen de situatie analog is, maar complexer, als gevolg van het feit dat in tegenstelling tot het continue-tijd geval stabiele en onstabiele variëten niet automatisch samenvallen als ze eenmaal een enkel punt gemeen hebben. In het hoofdstuk wordt gepreciseerd hoe het ontstaan van onverschilligheidspunten in een zogenaamde onverschilligheid-attractor bifurcatie gekoppeld is aan het optreden van heterocliene bifurcaties van de familie van fase-afbeeldingen $\varphi$, en wat voor gevolgen dit heeft voor de optimale oplossingen. In het bijzonder wordt de bi-
furcatiewaarde waarvoor de onverschilligheidsdrempel verschijnt gekenmerkt door een meetkundige voorwaarde. Bovendien wordt er aangetoond dat er op moment van bifurcatie oneindig veel onverschilligheiddpunten optreden die geen onverschilligheidsdrempels zijn.

In de meeste toepassingen is het onmogelijk om analytische uitdrukkingen voor invariante variëteiten te bepalen en zijn er numerieke methoden nodig: deze worden besproken in hoofdstuk 4. Een eenvoudig algoritme wordt beschreven om numeriek invariante variëteiten te berekenen. Deze informatie wordt gebruikt om de locus van de onverschilligheid-attractor bifurcatiepunten te bepalen.

De resultaten en methoden ontwikkeld in dit proefschrift worden toegepast op het probleem van het beheer van verontreinigde meren (Måler et al. (2003)). In ecologische systemen, zoals meren, kunnen interne positieve terugkoppelingen leiden tot catastrofale verschuivingen in het vervuilingsniveau. De economische afweging van een beleidsmaker in dit geval vergelijkt de voordelen van landbouwactiviteiten, die verantwoordelijk zijn voor de vervuiling van het meer, en de kosten van de vervuiling.

In hoofdstuk 5 wordt een bifurcatieanalyse van dit model uitgevoerd. Het resulterende bifurcatiediagram geeft een overzicht van het gezamenlijke effect van de (fysieke) robuustheid van het meer en het (economische) belang van het meer op de vorm van het optimale beleid. Het diagram is verdeeld in vier parametergebieden: uniek dekpunt, geringe vervuiling, hoge vervuiling, en begintoestandafhankelijkheid. In het eerste gebied is er een uniek zadelpunt van de fase-afbeelding dat overeenkomt met een globale aantrekker van de optimale toestandsdynamica. In de andere gebieden heeft de fase-afbeelding twee
zadelpunten die corresponderen met een schoon of een vervuild mogelijk dek-
punt van de toestanddynamica. Zowel het schone als het vervuilde mogelijke
dekpunt kan een globale aantrekker zijn: in dat geval wordt het alternatieve
mogelijke dekpunt niet gerealiseerd door de optimale dynamica. Het is ook
mogelijk dat beide dekpunten gerealiseerd worden; dan correspondeert elk
met een lokale attractor. Deze parametergebieden worden gescheiden door
onverschilligheid-attractor- en zadelknoop-bifurcatiekrommen.

In het *geringe vervuiling* gebied, dat gekenmerkt wordt door een hoge
robuustheid en groot economisch belang van het meer, stuurt het optimale
beleid het meer altijd naar het schone dekpunt, onafhankelijk van de oor-
spronkelijke toestand. In het *hoge vervuiling* gebied, waar het economische
belang van het meer relatief laag is, zal het vervuilde dekpunt uiteindelijk
bereikt worden onder optimaal beheer ongeacht de oorspronkelijke toestand.
Voor meren in het *begintoestandafhankelijke* gebied - meren die fragiel zijn
en van gemiddeld tot hoog economisch belang vallen onder deze categorie -
is het resultaat van een optimaal beleid afhankelijk van de begintoestand:
as het initiële vervuilingsniveau voldoende laag is, wordt het schone dek-
punt bereikt, anders het vervuilde. De twee gebieden in de toestandsruimte
worden gescheiden door een onverschilligheidspunt.

In hoofdstuk 5 wordt ook de ‘stijfheid’ of reactiesnelheid van het meer
gevarieerd. Een sterk-reagerend meer vertoont meer plotselinge verschuivin-
gen tussen schone en vervuilde regimes. Het is gebleken dat voor een sterk-
reagerend meer het gebied met *hoge vervuiling* veel kleiner is vergeleken met
een zwak-reagerend meer, terwijl het gebied met *lage vervuiling* veel groter
is. Dat wil zeggen, het is optimaal een regimeverschuiving in de richting van
het vervuilde evenwicht te vermijden als het meer een laag economisch belang

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heeft, maar sterk reageert. In het beheer van sterk reagerende ecosystemen is het waarschijnlijker dat het optimale beleid 'groen' is.
The Tinbergen Institute is the Institute for Economic Research, which was founded in 1987 by the Faculties of Economics and Econometrics of the Erasmus University Rotterdam, University of Amsterdam and VU University Amsterdam. The Institute is named after the late Professor Jan Tinbergen, Dutch Nobel Prize laureate in economics in 1969. The Tinbergen Institute is located in Amsterdam and Rotterdam. The following books recently appeared in the Tinbergen Institute Research Series:


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